GAMES101 Lecture Notes

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2025 年 1 月 17 日

目录

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2 Review of Linear Algebra

Assume 3-dimensional space. Dot product can be written as:

$$
\begin{aligned}
\vec{a} \cdot \vec{b} &= \vec{a}^{\mathrm{T}} \vec{b} \\
&= \begin{pmatrix} x_a & y_a & z_a \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \\
&= x_a x_b + y_a y_b + z_a z_b
\end{aligned}
$$

Cross product can be written as:

$$
\vec{a} \times \vec{b} = A^* \vec{b} = \underbrace{\begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix}}_{\text{dual matrix of } \vec{a}} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}
$$

3 Transformation

3.1 Homogeneous Coordinates

2D point is represented as $(x, y, 1)^{\mathrm{T}}$. 2D vector is represented as $(x, y, 0)^{\mathrm{T}}$. This ensures

- Vector + Vector = Vector
- Point − Point = Vector
- Point + Vector = Point

We further define $(x, y, w)^{\mathrm{T}} = (x/w, y/w, 1)^{\mathrm{T}}, w \neq 0$.

3.2 2D Transformation

Below are some common 2D transformations in homogeneous coordinates:

- Scale

$$S(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Translation

$$T(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix is orthogonal, i.e. $R^{\mathrm{T}} = R^{-1}$.

3.3 3D Transformation

Rotation matrix around x, y, z axis:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rodrigues' rotation formula: Rotation by angle α around axis $\vec{n}$

$$R(\vec{n}, \alpha) = \cos\alpha \mathbf{I} + (1 - \cos\alpha)\vec{n}\vec{n}^{\mathrm{T}} + \sin\alpha \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$

Define a camera:

- Position: $\vec{e}$
- Look-at / gaze direction: $\hat{g}$
- Up direction: $\hat{t}$

We always transform the camera to

- the origin
- looking at $-z$ axis
- up at y axis

So the view/camera transformation matrix is $M_{\text{view}} = R_{\text{view}} T_{\text{view}}$, where

$$T_{\text{view}} = \begin{pmatrix} 1 & 0 & 0 & -x_e \\ 0 & 1 & 0 & -y_e \\ 0 & 0 & 1 & -z_e \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and (rotate $\hat{g}$ to $-z$, $\hat{t}$ to y, $\hat{g} \times \hat{t}$ to x)

$$R_{\text{view}} = (R_{\text{view}}^{-1})^{\mathrm{T}} = \begin{pmatrix} x_{\hat{g}\times\hat{t}} & x_t & x_{-g} & 0 \\ y_{\hat{g}\times\hat{t}} & y_t & y_{-g} & 0 \\ z_{\hat{g}\times\hat{t}} & z_t & z_{-g} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} x_{\hat{g}\times\hat{t}} & y_{\hat{g}\times\hat{t}} & z_{\hat{g}\times\hat{t}} & 0 \\ x_t & y_t & z_t & 0 \\ x_{-g} & y_{-g} & z_{-g} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.4 Orthographic Projection

1. Center cuboid by translation
2. Scale into canonical cube

$$M_{\text{ortho}} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.5 Perspective Projection

1. Squish the frustum into a cuboid
2. Do orthographic projection

$$M_{\text{pers->ortho}} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix}$$