必背公式

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求导公式

$$(\sin x)^{(n)} = \sin(x + n\,\frac{\pi}{2})$$

$$(\cos x)^{(n)} = \cos(x + n\,\frac{\pi}{2})$$

$$[\ln(1+x)]^{(n)} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(\csc x)' = -\csc x \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(arccotx)' = -\frac{1}{1+x^2}$$

$(uv)^{(n)}=\sum_{k=0}^{n}C_n^k u^{(k)}v^{(n-k)}$，其中$u^{(0)}=u$，$v^{(0)}=v$

等效无穷小替换

当$x\ \to 0$时

$$\sin x\ \sim\ x\quad \tan x\ \sim\ x$$

$$\arctan x \sim\ x$$

$$\arcsin x\ \sim\ x$$

$$e^x\ -1\ \sim\ x$$

$$\ln{(1+x)}\ \sim\ x$$

$$(1+\beta x)^\alpha-1\ \sim\ \alpha\beta x$$

$$1-\cos^\alpha x\ \sim\ \frac{\alpha}{2}x^2$$

$$\log_a(1+x)\ \sim\ \frac{x}{lna}$$

$$x-\sin x\ \sim\ \frac{1}{6}x^3$$

$$x-\arctan x\ \sim\ \frac{1}{3}x^3$$

$$\tan x-\sin x\ \sim\ \frac{1}{2}x^3$$

$$x-\ln{(1+x)}\ \sim\ \frac{1}{2}x^2$$

$$\ln{\left(x+\sqrt{1+x^2}\right)} \sim\ x$$

$$a^x-1\ \sim\ x\ln a$$

常用的麦克劳林公式

$$e^x=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\dots$$

$$\ln{(1+x)}=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$

$$(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+\frac{a(a-1)(a-2)}{3!}x^3+\dots$$

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots$$

$$\frac{1}{1+x}=1-x+x^2-x^3+\dots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$$

$$\arcsin x = x + \frac{1}{2} * \frac{x^3}{3} + \frac{1*3}{2*4} * \frac{x^5}{5} + \frac{1*3*5}{2*4*6} * \frac{x^7}{7} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - x - \frac{1}{2} * \frac{x^3}{3} - \frac{1*3}{2*4} * \frac{x^5}{5} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

不定积分公式

$$\int \tan x dx = -\ln|\cos x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{1}{2} x \sqrt{a^2 - x^2} + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx (= \int_0^{\frac{\pi}{2}} \cos^n x \, dx)$$
$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ \cdots \ \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n\text{为正偶数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ \cdots \ \cdot \frac{4}{5} \cdot \frac{2}{3}, & n\text{为正奇数} \end{cases}$$

$$\int_0^{+\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

积分常用转换

一般的对于$\sin^{2k} \cos^{2l} x$型函数，可利用三角恒等式：$\sin^2 x = \frac{1}{2}(1 - cos2x)$，$\cos^2 x = \frac{1}{2}(1 + \cos^2 x)$化为$\cos 2x$的多项式

三角代换式:

$$\sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

万能替换:

令$u = \tan \dfrac{x}{2}$，
则:

$$\begin{cases} dx = \dfrac{2}{1+u^2} du \\ \sin x = \dfrac{2u}{1+u^2} \\ \cos x = \dfrac{1-u^2}{1+u^2} \end{cases}$$

定积分的应用

由连续曲线$r = r(\theta)$与矢径$\theta = \alpha$，$\theta = \beta$围成的图形面积

$$A = \frac{1}{2} \int_\alpha^\beta r^2(\theta) d\theta$$

光滑曲线$y = f(x) \ \ (a \le x \le b)$的弧长为

$$l = \int_a^b \sqrt{1 + [f' \ (x\}]^2} \, dx$$

光滑曲线$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \ \ (\alpha \le t \le \beta)$的弧长为:

$$l = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt$$

光滑曲线$r=r(\theta)$，$\varphi_0\le\theta\le\varphi_1$的弧长:

$$l=\int_{\varphi_0}^{\varphi_1}\sqrt{r^2+r'^2}d\theta$$

伽马公式

$$\Gamma(s)=\int_0^{+\infty}e^{-x}x^{s-1}\,dx\ \ (s>0)$$

1. $\Gamma(s+1)=s\Gamma(s)$
2. $\Gamma(n+1)=n!$

一阶线性微分方程

$$\frac{dy}{dx}+p(x)y=Q(x)$$

$$y=e^{-\int p(x)dx}(\int Q(x)e^{\int p(x)dx}dx+C)$$

欧拉公式

$$e^{ix}=(\cos x+i\sin x)$$

多元函数的微分

全微分:

$$dz=A\Delta x+B\Delta y=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

方向导数:

$$grad\ f(x_0,y_0)=\nabla f(x_0,y_0)=f_x\cdot\vec{i}+f_y\cdot\vec{j}$$

多元函数的极值:

1. $\begin{vmatrix}f_{xx}&f_{yx}\\f_{xy}&f_{yy}\end{vmatrix}>0$时，具有极值，当$A<0$时有极大值，当$A>0$时有极小值
2. $\begin{vmatrix}f_{xx}&f_{yx}\\f_{xy}&f_{yy}\end{vmatrix}<0$时，没有极值
3. $\begin{vmatrix}f_{xx}&f_{yx}\\f_{xy}&f_{yy}\end{vmatrix}=0$时，还需另做讨论

条件极值，拉格朗日乘数法(将条件极值转换为无条件极值):

要找出函数$z=f(x,y)$在条件$\varphi(x,y)=0$下的可能极值点，先做拉格朗日函数:

$$L(x,y)=f(x,y)+\lambda\varphi(x,y)$$

在对其x与y求偏导，使之为0

$$\begin{cases} f_x(x,y)+\lambda\varphi_x(x,y)=0 \\ f_y(x,y)+\lambda\varphi_y(x,y)=0 \\ \varphi(x,y)=0 \end{cases}$$

重积分计算

坐标变换公式：

$$\iint\limits_{D} f(x,y)\,dxdy=\iint\limits_{D'} f(u,v)\cdot\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}dudv$$

$$\text{其中，}\begin{cases} x(u,v) \\ y(u,v) \end{cases}$$

二重积分坐标变换：

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases},\quad d\sigma=dxdy=\rho\,d\rho d\theta$$

三重积分球坐标系变换：

$$\begin{cases} x=R\sin\varphi\cos\theta \\ y=R\sin\varphi\sin\theta \\ z=R\cos\varphi \end{cases},\quad dV=dxdydz=\rho^2\sin\varphi\,d\rho d\varphi d\theta$$

格林公式（二重积分与曲线积分的联系）：

$$\oint_L Pdx+Qdy=\iint\limits_{D}(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\,dxdy$$

格林公式推论：

$$A=\iint\limits_{D} dxdy=\frac{1}{2}\oint_L x\,dy-y\,dx$$

积分与路径无关的条件：

$$\oint_C P\,dx+Q\,Qdy=0$$
$$\exists\,U(x,y),\ du=Pdx+Qdy$$
$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

高斯公式（三重积分与曲面积分的联系）：

$$\begin{aligned}&\iiint_\Omega(\,\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\,)\,dxdydz\\ =&\oiint_\Sigma P\,dydz+Q\,dzdx+R\,dxdy\\ =&\oiint_\Sigma(\,P\cos\alpha+Q\cos\beta+R\cos\gamma\,)\,dS\end{aligned}$$

高斯公式推论：

$$
\begin{aligned}
V &= \iint_{\Omega} dV \\
&= \frac{1}{3} \oiint_{\Sigma} x\, dydz + y\, dzdx + z\, dxdy
\end{aligned}
$$

闭曲面积分为零的充要条件：

$$
\begin{aligned}
&\oiint_{\Sigma} P\, dydz + Q\, dzdx + R\, dxdy \\
&\Longleftrightarrow \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0
\end{aligned}
$$

斯托克斯公式（曲面积分与曲线积分的联系）：

$$
\begin{aligned}
&\oint_{\Gamma} Pdx + Qdy + Rdz \\
&= \iint_{\Sigma} \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \iint_{\Sigma} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS
\end{aligned}
$$

重积分的应用

曲面面积：

$$
\begin{aligned}
\cos\gamma &= \frac{1}{\sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}} \\
A &= \iint_D \frac{d\sigma}{\cos\gamma} \\
&= \iint_D \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}\, d\sigma
\end{aligned}
$$

求质心：

$$
\bar{x} = \frac{1}{A} \iint\limits_D x\, d\sigma, \quad \bar{y} = \frac{1}{A} \iint\limits_D y\, d\sigma
$$

$$
\text{其中，} A = \iint\limits_D d\sigma
$$

求转动惯量：

$$
I_x = \iint\limits_D y^2 \mu(x,y)\, d\sigma, \quad I_y = \iint\limits_D x^2 \mu(x,y)\, d\sigma
$$

对弧长的曲线积分（第一类曲线积分）：

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} (\alpha \le t \le \beta),$$

$$\int_L f(x,y)\, ds = \int_\alpha^\beta f\left[\, \varphi(t),\ \psi(t)\,\right] \sqrt{\varphi'^{\,2}(t) + \psi'^{\,2}(t)}\, dt$$

$$\Delta s_i = \sqrt{\varphi'^{\,2}(\tau_i') + \psi'^{\,2}(\tau_i')}\, \Delta t_i$$

对坐标的曲线积分（第二类曲线积分）：

$$\begin{aligned} &\int_L P(x,y)dx + Q(x,y)dy \\ &= \int_\alpha^\beta \{P[\, \varphi(t),\ \psi(t)\,]\, \varphi'(t) + Q\,[\, \varphi(t),\ \psi(t)\,]\, \psi'(t)\}\, dt \end{aligned}$$

对面积的曲面积分（第一类曲面积分）[1]：

$$dS = \sqrt{1 + z_x^2(x,y) + z_y^2(x,y)}\, dxdy$$

$$\iint\limits_{\Sigma} f(x,y,z)\, dS = \iint\limits_{Dxy} f\left[\, x,\ y,\ z(x,y)\,\right] \sqrt{1 + z_x^2(x,y) + z_y^2(x,y)}\, dxdy$$

$$Dxy\text{是曲面在}xOy\text{面的的投影}$$

球面方程 $x^2+y^2+z^2=r^2$，则 $z=\sqrt{r^2-x^2-y^2}$，则 $\sqrt{1+z_x^2+z_y^2}=\dfrac{r}{\sqrt{r^2-x^2-y^2}}$

球面坐标系上的面积微元:

$$\begin{cases} x = R\sin\varphi\cos\theta \\ y = R\sin\varphi\sin\theta \\ z = R\cos\varphi \end{cases}$$

$$dS = R^2 \sin\varphi\, d\varphi d\theta$$

对坐标的曲面积分（第二类曲面积分）[2]：

$$\begin{aligned} &\iint_\Sigma P\, dydz + Q\, dzdx + R\, dxdy \\ &= \pm \iint_{Dyz} P\left[\, x(y,\ z),\ y,\ z\,\right] dydz \pm \iint_{Dzx} Q\left[\, x,\ y(z,\ x),\ y,\ z\,\right] dzdx \pm \iint_{Dxy} R\left[\, x,\ y,\ z(x,\ y)\,\right] dxd \end{aligned}$$

当面积为上侧时，取$+$，为下侧时取$-$

$$
\begin{cases}
\cos\alpha = \dfrac{z_x'}{\sqrt{1+{z_x'}^{\ 2}+{z_y'}^{\ 2}}} \\
\cos\beta = \dfrac{z_y'}{\sqrt{1+{z_x'}^{\ 2}+{z_y'}^{\ 2}}} \\
\cos\gamma = \dfrac{1}{\sqrt{1+{z_x'}^{\ 2}+{z_y'}^{\ 2}}}
\end{cases}
$$

$$
\begin{aligned}
&\iint_\Sigma P\,dydz + Q\,dzdx + R\,dxdy \\
=&\iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS
\end{aligned}
$$

$$
\vec{A} = (p, Q, R), \quad \vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)
$$

$$
\vec{dS} = \vec{n}\,dS = (dydz,\ dzdx,\ dxdy)
$$

$$
\iint_\Sigma \vec{A}\cdot\vec{dS} = \iint_\Sigma \vec{A}\cdot\vec{n}\,dS
$$

$$
\begin{aligned}
A_n =&\vec{A}\cdot\vec{n} \\
&= \iint_\Sigma A_n\,dS
\end{aligned}
$$

投影合一法:

$$
I = \iint_\Sigma (\ P\frac{\cos\alpha}{\cos\gamma} + Q\frac{\cos\beta}{\cos\gamma} + R\ )\,dxdy
$$

梯度:

$$
\operatorname{grad} u = \frac{\partial u}{\partial x}\,\vec{i} + \frac{\partial u}{\partial y}\,\vec{j} + \frac{\partial u}{\partial z}\,\vec{k}
$$

通量:

$$
\vec{v}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}
$$

$$
\begin{aligned}
\Phi &= \iint_\Sigma P\,dydz + Q\,dzdx + R\,dxdy \\
&= \iint_\Sigma (\ P\cos\alpha + Q\cos\beta + R\cos\gamma\ )\,dS \\
&= \iint_\Sigma \vec{v}\cdot\vec{n}\,dS
\end{aligned}
$$

散度:

$$
\operatorname{div}\vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
$$

环流度:

$$
\Gamma = \oint_\Gamma Pdx + Qdy + Rdz
$$

无穷级数

等比级数:

$$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}, \quad (0<q<1)$$

其他

$$[f^{-1}(x)]^{'} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \to 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{f^{'}(y)}$$

设$f(x)$在$[1,\ 0]$上连续，则:

$$(1)\ \int_0^{\frac{\pi}{2}} f\left(\sin x\right)\,dx = \int_0^{\frac{\pi}{2}} f\left(\cos x\right)\,dx$$

$$(2)\ \int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\ \ (n \in N)$$

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1. 一投 二代 三微变 ↩

2. 一投 二代 三定号 ↩