拉格朗日乘数法

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理论

$$f(x,y) \qquad \varphi(x_0,y_0)=0$$

$$\begin{cases} f_x+\lambda\varphi_x=0 \\ f_y+\lambda\varphi_y=0 \end{cases}$$

$z=f(x,y)$在$\varphi(x,y)=0$下的极值点

1. $L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$
2. $L_x=L_y=L_\lambda=0$
3. 解方程组所得(x_0,y_0)就是可能的极值点

推广到三元

$u=f(x,y,z)$在$\varphi=(x,y,z)=0$下的极值

1. $L(x,y,z,\lambda)=f+\lambda\varphi$
2. $L_x=L_y=L_z=L_\lambda=0$

推广到多条件

$u=f(x,y,z)$在$\begin{cases} \varphi(x,y,z)=0 \\ \varphi(x,y,z)=0 \end{cases}$

1. $L(x,y,z,\lambda,\mu)=f+\lambda\varphi+\mu\varphi$

2. $L_x = L_y = L_z = L_\lambda = L_\mu = 0$

例题

消去λ

1. 求$f(x,y,z) = xyz$在条件$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} \quad (x,y,z,a>0)$条件下的最小值

   设拉式函数

$$L(x,y,z,\lambda) = xyz + \lambda(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a})$$

$$\begin{cases} L_x = yz - \frac{\lambda}{x^2} = 0 \\ L_y = yz - \frac{\lambda}{y^2} = 0 \\ L_z = yz - \frac{\lambda}{z^2} = 0 \\ L_\lambda = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a} = 0 \end{cases}$$

$$x = y = z = 3a$$

2. 求$u = xy + 2yz$在$x^2 + y^2 + z^2 = 10$下的最大

   作拉式函数

$$L(x,y,z,\lambda) = xy + 2yz - \lambda(x^2 + y^2 + z^2 - 10)$$

$$\begin{cases} L_x = y - 2\lambda x = 0 \\ L_y = x + 2z - 2\lambda y = 0 \\ L_z = 2y - 2\lambda z = 0 \\ L_\lambda = x^2 + y^2 + z^2 - 10 = 0 \end{cases}$$

$$\lambda = \frac{y}{2x} = \frac{x+2y}{2y} = \frac{y}{z}$$

   解得:

$$(1,\sqrt{5},2) \quad (-1,\sqrt{5},-2) \quad (1,-\sqrt{5},2)$$
$$(-1,-\sqrt{5},-2) \quad (2\sqrt{2},0,-\sqrt{2}) \quad (-2\sqrt{2},0,\sqrt{2})$$

$$u_{max} = 5\sqrt{5}, u_{min} = -5\sqrt{5}$$

3. 求$u = x^2 + y^2 + z^2$在$z = x^2 + y^2$和$x + y + z = 4$下的最大值和最小值

   作拉式函数:

$$L(x,y,z,\lambda,\mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - z) - \mu(x + y + z - 4)$$

$$\begin{cases} L_x = 2(1-\lambda)x - \mu = 0 \\ L_y = 2(1-\lambda)y - \mu = 0 \\ L_z = 2z + \lambda - \mu = 0 \\ L_\lambda = x^2 + y^2 - z = 0 \\ L_\mu = x + y + z - 4 \end{cases}$$

$$(x-y)(1-\lambda) = 0$$

因为$\lambda = -1$时$\mu = 0$，所以$x = y$

所以解得:$(1,1,2) \quad (-2,-2,8)$

利用对称性(容易漏解)

1. 求$f(x,y) = x^2 + y^2 - x - y$在条件$x^2 + y^2 = 1$下的最值

作拉式函数:

$$L(x,y,\lambda) = x^2 + y^2 - x - y - \lambda(x^2 + y^2 - 1)$$

$$L_x = 2(1-\lambda)x - 1 = 0$$
$$L_y = 2(1-\lambda)y - 1 = 0$$
$$L_\lambda = x^2 + y^2 - 1 = 0$$

观察并猜测$x = y$

解得:

$$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$$

利用齐次性解λ(需要技巧)

- 如果**目标函数和约束条件**是关于x和y的**齐次函数**，那么可以凑$x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} + z\frac{\partial L}{\partial z}$得到目标函数与$\lambda$之间的关系

1. 求中心在坐标原点的椭圆$x^2 - 4xy + 5y^2 = 1$的长半轴和短半轴

问题归结为求$f(x,y) = x^2 + y^2$在条件$x^2 - 4xy + 5y^2 = 1$下的最大值和最小值

作拉式函数:

$$L(x,y,\lambda) = x^2 + y^2 - \lambda(x^2 - 4xy + 5y^2 - 1)$$

$$\begin{aligned} L_x &= 2(1-\lambda)x + 4\lambda y &= 0 \quad (1) \\ L_y &= 2(1-5\lambda)y - 4\lambda x &= 0 \quad (2) \\ L_\lambda &= x^2 - 4xy + 5y^2 - 1 &= 0 \quad (3) \end{aligned}$$

$(1)\times\frac{1}{2}+(2)\times\frac{y}{2}$得:

$$x^2+y^2-\lambda(x^2-4xy+5y^2)=0 \tag{4}$$

将(3)代入(4)中得:

$$x^2+y^2=\lambda$$

(1)(2)式得:

已知x,y有非零解，即:

$$\begin{vmatrix}1-\lambda & 2\lambda \\ -2\lambda & 1-5\lambda\end{vmatrix}=0$$

$$\lambda=3\pm2\sqrt{2}$$

故:

$$a=\sqrt{3+2\sqrt{2}}=1+\sqrt{2},\quad b=\sqrt{3-2\sqrt{2}}=1-\sqrt{2}$$

- **欧拉定理**:

  设$f(x,y)$是k次齐次函数，即$\forall\lambda\neq0$，总有$f(\lambda x,\lambda y)=\lambda^k f(x,y)$，且$f(x,y)$有一阶偏导，那么$xf'_x+yf'_y=kf(x,y)$

转换为无条件极值

- 把条件代入目标函数当中，降低维度，减少未知数个数，转化为无条件极值

1. 求$f(x,y)=x^2+2y^2-x^2y^2$在条件$x^2+y^2=4$下的最大值和最小值

将$y^2=4-x^2$代入$f(x,y)$中，得:

$$\begin{aligned}h(x)&=x^2+2(4-x^2)-x^2(4-x^2)\\&=x^4-5x^2+8\end{aligned}$$

$$h'(x)=4x^3-10x$$

驻点为:

$$(0,\pm2)\qquad(\pm\sqrt{\frac{5}{2}},\pm\sqrt{\frac{3}{2}})$$