一、求极限

1) 有界无穷小：$\lim\limits_{x\to\infty}(\frac{2}{x}\sin x + x\sin\frac{2}{x}) = \lim\limits_{x\to\infty}\frac{2}{x}\sin x + \lim\limits_{x\to\infty} x\sin\frac{2}{x} = 2$-----

2) 等价无穷小：$\lim\limits_{x\to 0}\frac{\sqrt{1+x\tan x}-1}{(e^{2x}-1)\ln(1-3x)} = \lim\limits_{x\to 0}\frac{\frac{1}{2}x\tan x}{2x(-3x)} = -\frac{1}{12}$-----

3) 重要极限：$\lim\limits_{x\to\infty}(\frac{x+2a}{x-a})^x = 8$ $\lim\limits_{x\to\infty}(1+\frac{3a}{x-a})^{\frac{x-a}{3a}\frac{3a}{x-a}x} = e^{3a}$ $==>a = \ln 2$-----

1^∞

4) 夹逼准则：$\lim\limits_{n\to\infty}(\frac{1}{n^2+\pi}+\frac{2}{n^2+2\pi}+\cdots+\frac{n}{n^2+n\pi})$

$$\frac{1+2+\cdots+n}{n^2+n\pi} < x_n < \frac{1+2+\cdots+n}{n^2+\pi}$$

$\frac{1}{2} = \frac{\frac{n(n+1)}{2}}{n^2+n\pi} < x_n < \frac{\frac{n(n+1)}{2}}{n^2+\pi} = \frac{1}{2}$-----

5) 定积分定义：$\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} = \lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{1}{n}}{1+(\frac{i}{n})^2} = \int_0^1\frac{1}{1+x^2}dx = \arctan x\,|_0^1 = \frac{\pi}{4}$-----

$\lim\limits_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n})$ $==>\sum\to\int$ $\quad\frac{1}{n}\to dx$ $\frac{i}{n}\to x$

6) 洛必达：$\lim\limits_{x\to 0}(\frac{1}{x^2}-\frac{1}{x\tan x}) = \lim\limits_{x\to 0}\frac{\tan x - x}{x^2\tan x} = \lim\limits_{x\to 0}\frac{\sec^2 x - 1 = \tan^2 x}{3x^2} = \frac{1}{3}$-----

$\lim\limits_{x\to 1}(1-x)\tan\frac{\pi x}{2} = \lim\limits_{x\to 1}(1-x)\frac{\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}} = \lim\limits_{x\to 1}\frac{-1}{-\frac{\pi x}{2}\sin\frac{\pi x}{2}} = \frac{2}{\pi}$-----

$\lim\limits_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\operatorname{arccot} x} = \lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{\operatorname{arccot} x} = \lim\limits_{x\to+\infty}\frac{-x^{-2}}{-\frac{1}{1+x^2}} = 1$-----

$\lim\limits_{x\to 1}\frac{\int_1^x e^{t^2}dt}{\ln x} = \lim\limits_{x\to 1}\frac{e^{x^2}}{\frac{1}{x}} = e$-----

$\lim\limits_{x\to+\infty}\frac{\int_1^x\left[t^2\left(e^{\frac{1}{t}}-1\right)\right]}{x^2\ln\left(1+\frac{1}{x}\right)} = \lim\limits_{x\to\infty}\frac{\int_1^x\left[t^2\left(e^{\frac{1}{t}}-1\right)-t\right]}{x} = \lim\limits_{x\to\infty}\frac{x^2\left(e^{\frac{1}{x}}-1\right)-x}{1}\ \frac{1}{x} \underset{===}{=} t\ \lim\limits_{t\to 0^+}\frac{(e^t-1)-t}{t^2} = \lim\limits_{t\to 0^+}\frac{e^t-1}{2t} = \frac{1}{2}$-----

7) 运用导数的定义：$\lim\limits_{n\to\infty} n\left[f\left(1+\frac{1}{n}\right)-f\left(1-\frac{2}{n}\right)\right]$ $\quad f'(1) = 3$

$\lim\limits_{n\to\infty} 3\cdot\frac{\left[f\left(1+\frac{1}{n}\right)-f\left(1-\frac{2}{n}\right)\right]}{\frac{3}{n}} = \lim\limits_{n\to\infty} 3\cdot f'(1) = 9$-----

二、极限与连续性

$$f(x) = \begin{cases}\frac{\sin x}{x},\ x>0\\ 3,\ x=0\\ e^{\frac{1}{x}}+2,\ x<0\end{cases}\ ,\ \text{求}\lim_{x\to 0} f(x)$$

$f(0^-) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\left(e^{\frac{1}{x}}+2\right) = 2$

$f(0^+) = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\frac{\sin 2x}{x} = 2$-----

$f(x) = \begin{cases}\frac{ax+b}{\sqrt{1+3x}-\sqrt{x+3}},\ x\neq 1\\ 4,\ x=1\end{cases}$ 在$x=1$连续，求a，b

$$f(1^+) = f(1^-) = \lim_{x\to 1}\frac{ax+b}{\sqrt{1+3x}-\sqrt{x+3}} = \lim_{x\to 1}\frac{a}{\frac{3}{2\sqrt{1+3x}}-\frac{1}{2\sqrt{x+3}}} = 2a = 4$$

又因为$a+b=0$，所以$\begin{cases}a=2\\ b=-2\end{cases}$-----

$f(x)=\begin{cases}x^2,\ x<1\\ax+b,\ x>1\end{cases}$处处可导，求$a,b,f'(x)$

$a+b=1$

$f(1^-)=f(1^+)=\lim\limits_{x\to1^-}2x=\lim\limits_{x\to1^+}a=2$

所以$\begin{cases}a=2\\b=-1\end{cases}$，$f'(x)=\begin{cases}2x,\ x<1\\a,\ x>1\end{cases}$-----

$f(x)=\begin{cases}\frac{1-\sqrt{1-x}}{x},\ x<0\\a+bx,\ x\ge0\end{cases}$处处可导，求$a,b,f'(x)$

$f(0^-)=\lim\limits_{x\to0^-}\dfrac{1-\sqrt{1-x}}{x}=\dfrac{1}{2}=f(0^+)=\lim\limits_{x\to0^+}(a+bx)=a$

$f'(0^+)=b=f'(0^-)=\lim\limits_{\Delta x\to0^-}(\frac{f(\Delta x)-f(0)}{\Delta x})=\lim\limits_{\Delta x\to0^-}\frac{\frac{1-\sqrt{1-\Delta x}}{\Delta x}-\frac{1}{2}}{\Delta x}=\lim\limits_{\Delta x\to0^-}\frac{2-2\sqrt{1-\Delta x}-\Delta x}{2\Delta x^2}=\frac{1}{8}$-----

三、参数方程

$\begin{cases}x=e^{-t}\\y=e^{2t}\end{cases}$ 求$t=0$处的切线

$\dfrac{dy}{dx}=\dfrac{2e^{2t}}{-e^{-t}}=-2e^{3t},t=0,\dfrac{dy}{dx}=-2,\begin{cases}x=1\\y=1\end{cases}$

切线：$2x+y-3=0$-----

$\begin{cases}x=\arctan t\\y=\ln\sqrt{1+t^2}\end{cases}$在$t=1$处的切线

$\dfrac{dy}{dx}=\dfrac{\frac{2t}{2(1+t^2)}}{\frac{1}{1+t^2}}=t,t=1,\dfrac{dy}{dx}=1,\begin{cases}x=\dfrac{\pi}{4}\\y=\dfrac{1}{2}\ln2\end{cases}$

切线方程：$x-y-\frac{\pi}{4}+\frac{1}{2}\ln2$-----

四、隐函数求导

$e^y+6xy+x^2-1=0$，求$y',y'',y''(0)$

$y'e^y+6y+6xy'+2x=0 ==> y'=-\dfrac{2x+6y}{6x+e^y}$

$y''e^y+y'^2e^y+6y'+6y'+6xy''+2=0 ==> y''=-\dfrac{y'^2e^y+12y'+2}{e^y+6x}$

$x=0,\begin{cases}y=0\\y'=0\\y''=-2\end{cases}$ -----

$x+y^2+\int_0^1\arctan t^2\,dt=\int_0^{y-x}e^{-t^2}dt$，求$y'(x)$

$1+2yy'=(y'-1)e^{-(y-x)^2} ==> y'=\frac{1+e^{-(y-x)^2}}{e^{-(y-x)^2}-2y}$-----

五、中值定理

$f(x)$在$[0,\pi]$连续，$(0,\pi)$可导，证明：$\exists\xi\in(0,\pi)$，使$f'(\xi)\sin\xi+f(\xi)\cos\xi=0$

$\dfrac{f'(x)}{f(x)}=-\dfrac{\cos x}{\sin x} ==> [\ln f(x)]'=-[\ln\sin x]' ==> (\ln f(x)+\ln\sin x)'=0$

令$F(x)=f(x)\sin x$

$F'(x)=f'(x)\sin x+f(x)\cos x$

$F(0)=0=F(\pi)$

由罗尔中值定理:

$\exists\xi\in(0,\pi)$，使$f'(\xi)\sin\xi+f(\xi)\cos\xi=0$成立-----

$a>0$，$f(x)$在$[a,b]$上连续，(a,b)上可导，证明：存在两点$\xi,\eta\in(a,b)$，使$f'(\xi)=\frac{(b+a)f'(\eta)}{2\eta}$

$$\begin{cases}\frac{f(b)-f(a)}{b^2-a^2}=\frac{f'(\eta)}{2\eta}\\ f(b)-f(a)=(b-a)f'(\xi)\end{cases}==>f'(\xi)=\frac{(b+a)f'(\eta)}{2\eta}-----$$

设$f(x)$在$[0,3]$连续，$(0,3)$内连续可导，且$f(0)+f(1)+f(2)=3$，$f(3)=1$，证明：$\exists\xi\in(0,3)$，使$f'(\xi)=0$

$\exists\eta\in(0,2)$，使得$f(\eta)=1=f(3)$

由罗尔中值定理:

$\exists\xi\in(\eta,3)$，使得$f(\xi)=0$-----

$f(x)$在$[0,1]$连续，$(0,1)$处可导，且$f(0)=0$，$f(1)=1$

证明：1)$\exists\xi\in(0,1)$使$f(\xi)=1-\xi$

2)存在两个不同点$\eta,\mu\in(0,1)$，使$f'(\eta)\cdot f'(\mu)=1$

1)令$F(x)=f(x)+x-1$，$F(0)=-1$，$F(1)=1$

所以，$\exists\xi\in(0,1)$，使得$F(\xi)=0$

2)$0<\mu<\xi<\eta<1$

$$\begin{cases}\frac{f(\xi)-f(0)}{\xi-0}=f'(\mu)\\ \frac{f(1)-f(\xi)}{1-\xi}=f'(\eta)\end{cases}==>f'(\eta)\cdot f'(\mu)=\frac{f(\xi)-f(0)}{\xi-0}\cdot\frac{f(1)-f(\xi)}{1-\xi}=1------$$

六、泰勒公式

$f'(x)>0$，求证：$x_i\in(a,b)$，$f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\le\frac{1}{n}\sum_{i=1}^{n}f(x_i)$

$$f(x)\ge f(x_0)+f'(x_0)(x-x_1)+\frac{f''(x_0)}{2}(x-x_1)^2$$

$$f(x_1)\ge f(x_0)+f'(x_0)(x_1-x_0)+\frac{f''(x_0)}{2}(x_1-x_0)^2$$

$$f(x_2)\ge f(x_0)+f'(x_0)(x_2-x_0)+\frac{f''(x_0)}{2}(x_2-x_0)^2$$

$$f(x_n)\ge f(x_0)+f'(x_0)(x_n-x_0)+\frac{f''(x_0)}{2}(x_n-x_0)^2$$

$$\sum_{i=1}^{n}f(x_i)\ge nf(x_0)+f(x_1+x_2+x_3+\cdots+x_n)$$

七、不定积分

$\int\frac{dx}{1+\sqrt{2x}}$ 令 $\sqrt{2x}=t$ ， $\int\frac{tdt}{1+t}=\int td(\ln(1+t)=t\ln(1+t)-\int\ln(1+t)\,dt=t-\ln(1+t)+C=\sqrt{2x}-\ln\left(1+\sqrt{2x}\right)+C$-----

$\int\frac{dx}{e^x+e^{-x}}$，令$e^x=t$，$\int\frac{dt}{t^2+1}=\arctan t+C=\arctan e^x+C$-----

$$\int\ln(1+x^2)dx=x\ln(1+x^2)-\int\frac{2x^2dx}{1+x^2}=x\ln(1+x^2)-2x+2\arctan x+C$$

$\int e^{\sqrt{3x+9}}dx$，令$\sqrt{3x+9}=t$，$\int\frac{2e^tt}{3}dt=\frac{2}{3}(t-1)e^t+C=\frac{2}{3}\left(\sqrt{3x+9}-1\right)e^{\sqrt{3x+9}}+C$-----

$\int x^3\cos x^2\,dx=\frac{1}{2}\int x^2\cos x^2\,dx^2=\frac{1}{2}\int x^2d\sin x^2=\frac{1}{2}(x^2\sin x^2-\int\sin x^2\,dx^2)=\frac{1}{2}x^2\sin x^2+\frac{1}{2}\cos x^2+C$----

$\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}$，令$x=\tan t$，$\int\frac{\sec^2 t dt}{\sec^3 t}=\sin t+C=\frac{x}{\sqrt{1+x^2}}+C$-----

$\int \frac{dx}{x(x-1)^2} = \int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx = \ln|x| - \ln(x-1) - \frac{1}{x-1} + C$-----

$\int \frac{1-x}{x^2+2x+2} dx = \int \frac{-\frac{1}{2}(2x+2)+2}{x^2+2x+2} dx = -\frac{1}{2}\ln(x^2+2x+2) + 2\arctan(1+x) + C$-----

$f(x)$的一个原函数是$\frac{\sin x}{x}$，求$\int x f'(x) dx$

$f(x) = \left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}$，$\int x df(x) = xf(x) - \int f(x) dx = \frac{x\cos x - \sin x}{x} - \frac{\sin x}{x} + C = \frac{x\cos x - 2\sin}{x} + C$-----

八、定积分

$f(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ e^{2x}, & 1 \le x \le 3 \end{cases}$，求$\int_0^3 f(x) dx$

$\int_0^3 f(x) dx = \int_0^1 (1-x) dx + \int_1^3 e^{2x} dx = \frac{1}{2} + \frac{1}{2}e^6 + \frac{1}{2}e^2$-----

$\int_0^\pi \sqrt{\sin x - \sin^3 x}\, dx = \int_0^\pi \sqrt{\sin x}\sqrt{1-\sin^2 x}\, dx = \int_0^\pi \sin^{\frac{1}{2}} x\, |\cos x|dx = 2\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x\, d\sin x = \frac{4}{3}\sin^{\frac{3}{2}} x\,|_0^{\frac{\pi}{2}} = \frac{4}{3}$-----

$\int_1^9 \frac{\sqrt{x}}{1+\sqrt{x}} dx$，令$\sqrt{x} = t$，$\int_1^3 \frac{2t^2}{1+t} dt = \int_1^3 \frac{2(t+1)(t-1)+2}{1+t} dt = 2\int_1^3 \left(t - 1 + \frac{1}{1+t}\right) dt = 2\left[\frac{1}{2}t^2 - t + \ln(1+t)\right]|_1^3 = 4 + 2\ln 2$-----

$\int_1^2 \frac{e^{\frac{1}{x}}}{x^3} dx$，令$\frac{1}{x} = t$，$\int_{\frac{1}{2}}^1 te^t dt = (t-1)e^t|_{\frac{1}{2}}^1 = \frac{\sqrt{e}}{2}$-----

$\int_1^e x\ln x\, dx = \frac{1}{2}\int_1^e \ln x\, dx^2 = \frac{1}{2}\left(x^2\ln x\,|_1^e - \int_1^e x dx\right) = \frac{1}{2}\left(x^2\ln x - \frac{1}{2}x^2\right)|_1^e = \frac{e^2}{4} - \frac{1}{4}$-----

$\int_0^\pi x\sin^5 x\, dx = \frac{\pi}{2}\int_0^\pi \sin^5 x\, dx = \pi\int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \pi\frac{4}{5}\cdot\frac{2}{3} = \frac{8\pi}{15}$-----

$\int_{-1}^1 \left[x^3\cos x + x^2\sqrt{1-x^2}\right] dx = 2\int_0^1 x^2\sqrt{1-x^2} dx$，令 $x = \sin t$，$2\int_0^{\frac{\pi}{2}} \sin^2 t\,(1-\sin^2 t) dt = 2(\int_0^{\frac{\pi}{2}} \sin^2 x\, dx - \int_0^{\frac{\pi}{2}} \sin^4 x\, dx) = \frac{\pi}{8}$-----

$f(x)$连续，$F(x) = \int_0^{\sin x} (\sin x - t) f(t) dt$，求$F'(x)$，$F'(0)$

$F(x) = \sin x \int_0^{\sin x} f(t) dt - \int_0^{\sin x} t f(t) dt$

$F'(x) = \cos x \int_0^{\sin x} f(t) dt + \sin x\cos x\, f(\sin x) - \sin x\cos x\, f(\sin x) = \cos x \int_0^{\sin x} f(t) dt$

$F'(0) = 0$-----

$f(x) = \int_0^x \frac{\sin t}{\pi - t} dt$，求$\int_0^\pi f(x) dx$

$f'(x) = \frac{\sin x}{\pi - x}$

$\int_0^\pi f(x) dx = xf(x)|_0^\pi - \int_0^\pi x df(x) = \pi f(\pi) - \int_0^\pi x\frac{\sin x}{\pi - x} dx = \pi\int_0^\pi \frac{\sin x}{\pi - x} dx - \int_0^\pi x\frac{\sin x}{\pi - x} dx = \int_0^\pi \sin x\, dx = 2$