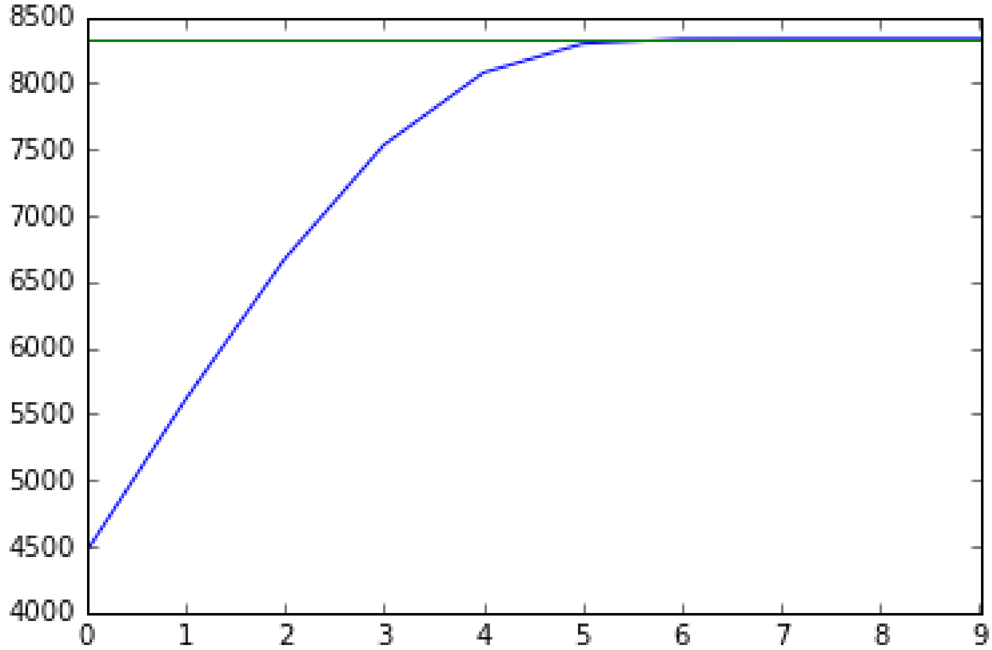
2.(f)

The plot of the log-likelihood vs iteration is as follows:

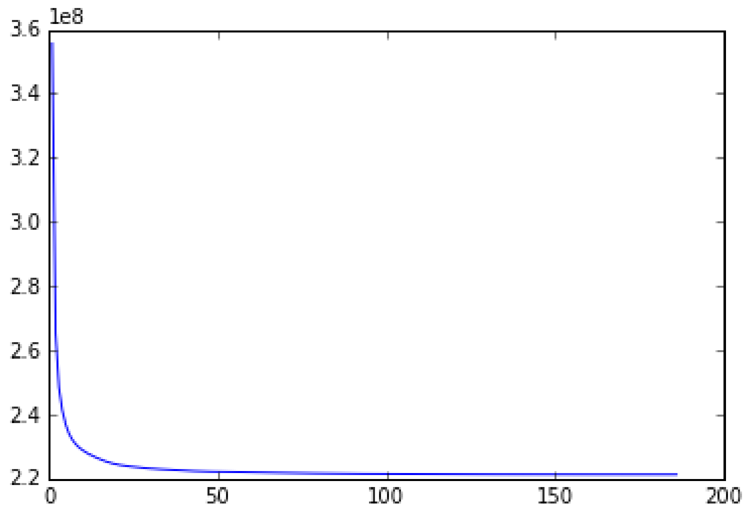


The estimated model parameters are:

9.96979256, 4.90985978, 14.86165346, 19.66495057, 48.97431779

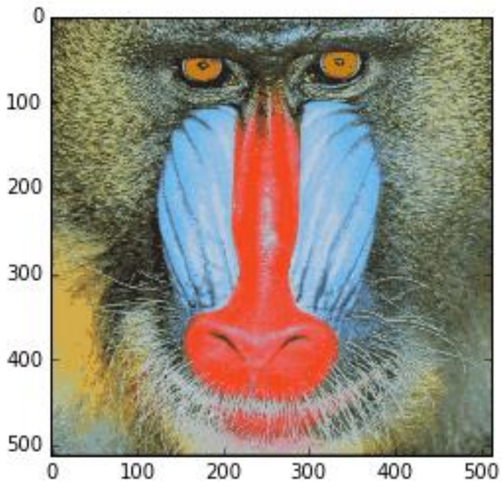
4.(a)

The plot of objective function is

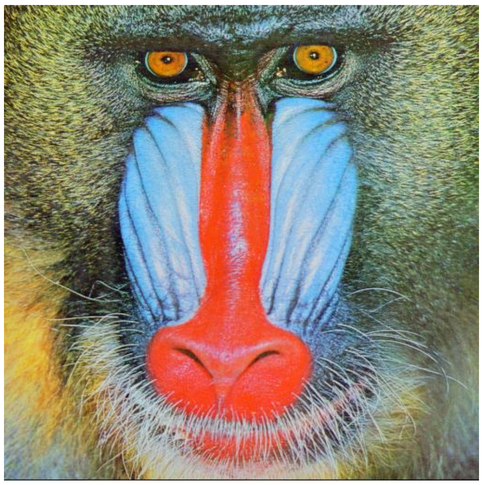


Put the two pictures together:

Compressed

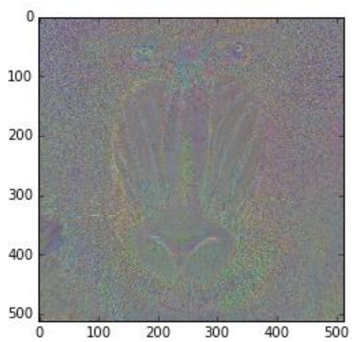


Original



Comparing the compressed picture with the original one, we can find the large parts with single color are best preserved. For the border parts where the color changes from one to another, they are generally not preserved well.

The picture of the difference:



The compression ratio is :

$$\mathrm{r} = \frac{24 * 2^2 * 64 + 216^2 * log_2 64}{24 * 512^2} = 6.3477\%$$

The relative mean absolute error of the compresses image is: 0.05

4.(b)

The number of 24 bits per pixel needed:

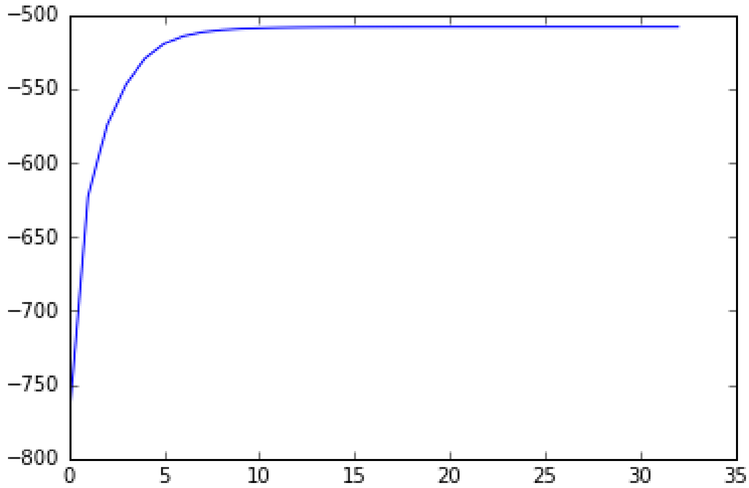
$$\mathrm{bpp} = \frac{24 * M^2 * K + \left(\frac{N}{M}\right)^2 * log_2 K}{N^2}$$

the compress ratio is:

$$\mathrm{ratio} = \mathrm{bpp}/24 = \frac{24 * M^2 * K + \left(\frac{N}{M}\right)^2 * log_2 K}{24 N^2}$$

5.(e)

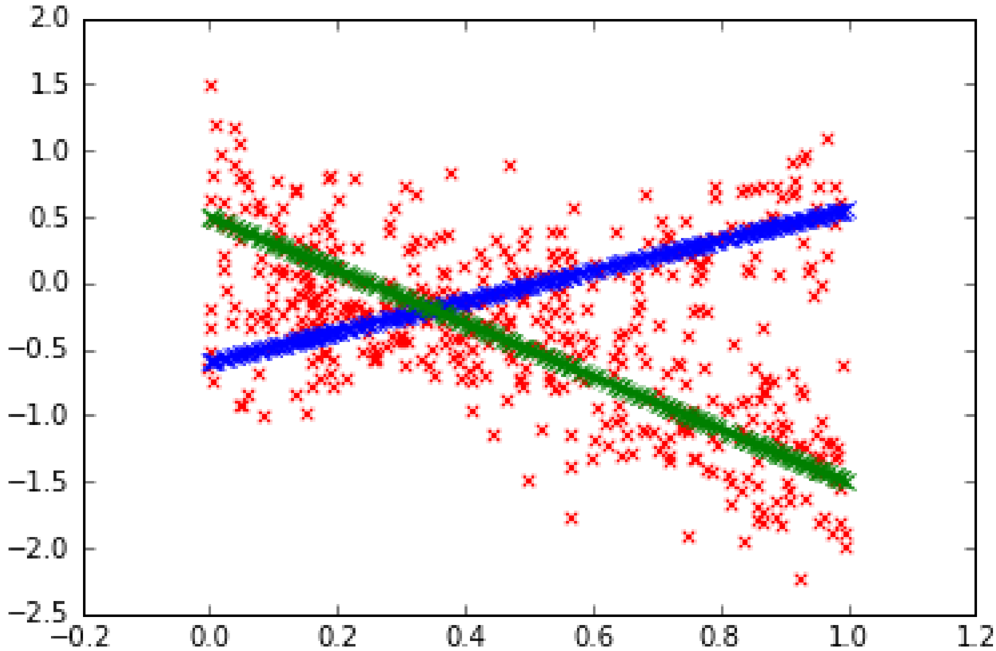
The plot of the log-likelihood is :



The estimated model parameters:

$$\pi = (0.293, 0.707), \quad (\mathrm{w}, \mathrm{b}) = \begin{pmatrix} 0.960 & -0.474 \\ -2.063 & 0.492 \end{pmatrix}, \qquad \sigma^2 = 0.1778$$

The plot showing the data and estimated lines is :



Codes:

```
#2.f
import numpy as np
from matplotlib import pyplot as plt
from scipy.special import gammaln, polygamma
from __future__ import division
N = 1000
m = 5
alpha = np.array([10, 5, 15, 20, 50])
P = np.random.dirichlet(alpha, N)
t=np.sum(np.log(P),axis=0)/N
t=t.reshape((5,1))
U=np.ones((5,1))
alpha0=np.ones((5,1))
loglihd=np.zeros((1000,1))
for i in range(1000):
    dalpha1=N*(t+polygamma(0,np.sum(alpha0))*U-polygamma(0,alpha0));
    g1=-N*polygamma(1,alpha0)
    Q=np.diag(np.diag(np.ones((5,5))*g1))
    C=N*polygamma(1,np.sum(alpha0))
    Q1=np.linalg.inv(Q)
    alphan=alpha0-(Q1-
(Q1.dot(C*U.dot(U.T)).dot(Q1))/(1+C*U.T.dot(Q1).dot(U))).dot(dalpha1)
    loglihd[i]=N*((alphan-1).T.dot(t)+gammaln(np.sum(alphan))-
np.sum(gammaln(alphan)))
    if i==0:
        alpha0=alphan
    elif i > 0 and np.abs(loglihd[i]-loglihd[i-1])>0.0001:
        alpha0=alphan
    else:
        alphafnl=alphan
        break
lstd=N*((alpha-1).T.dot(t)+gammaln(np.sum(alpha))-np.sum(gammaln(alpha)))
x=np.linspace(0,9,10);y=loglihd[0:10];stdl=lstd*np.ones((10,1))
plt.plot(x,y);plt.plot(x,stdl)
```

```
#4.a
import numpy as np
from matplotlib import pyplot as plt
from __future__ import division
from scipy.ndimage import imread
mandrill = imread('mandrill.png', mode='RGB').astype(float)
N = int(mandrill.shape[0])
```

```
M = 2;k = 64
X = np.zeros((N**2//M**2, 3*M**2))
for i in range(N//M):
    for j in range(N//M):
        X[i*N//M+j,:] = mandrill[i*M:(i+1)*M,j*M:(j+1)*M,:].reshape(3*M**2)
Jf=np.zeros((300,1))
#calculating tagfet value J
def calcJ(data,centers):
    diffsq=(centers[:,np.newaxis,:]-data)**2
    return np.sum(np.min(np.sum(diffsq,axis=2),axis=0))
#implement k means
def kmeans(data,k):
#initializing centers and list J
    centers=data[np.random.choice(range(data.shape[0]),k,replace=False),:]
    J=[];
#closest center for each sample
    for itera in range(300):
        sqdistances=np.sum((centers[:,np.newaxis,:]-data)**2,axis=2)
        closest=np.argmin(sqdistances,axis=0)
#calculate J and append to list
        J.append(calcJ(data,centers))
        Jf[itera]=calcJ(data,centers)
#update clusters
        for i in range(k):
            centers[i,:]=data[closest==i,:].mean(axis=0)
#decide whether stopping
        if itera>0 and np.abs(Jf[itera]-Jf[itera-1])==0:
            X=centers[closest,:]
            break
        else:
            continue
    J.append(calcJ(data,centers))
    return J,centers,closest

JF,centersf,closestf=kmeans(X,64)
Xnew=centersf[closestf,:]
mandrillnew=np.zeros(512*512*3)
mandrillnew=mandrillnew.reshape(512,512,3)
for i in range(256):
    for j in range(256):
        mandrillnew[i*2:(i+1)*2,j*2:(j+1)*2,:]=Xnew[i*256+j,:].reshape(2,2,3)
plt.imshow(mandrillnew/255)
plt.show()
mandrillgrey=mandrillnew-mandrill+128*np.ones(512*512*3).reshape(512,512,3)
```

```
plt.imshow(mandrillgrey/255)
plt.show()
Jf1=Jf[Jf>0]
x1=np.linspace(1,186,186)
plt.plot(x1,Jf1)
error=np.sum(np.abs(mandrillnew-mandrill))/(3*255*512**2)


#5.e
from __future__ import division
import numpy as np
from matplotlib import pyplot as plt
from scipy.stats import norm as norm
# Generate the data according to the specification in the homework description

N = 500
x = np.random.rand(N)

pi0 = np.array([0.7, 0.3])
w0 = np.array([-2, 1])
b0 = np.array([0.5, -0.5])
sigma0 = np.array([.4, .3])

y = np.zeros_like(x)
for i in range(N):
    k = 0 if np.random.rand() < pi0[0] else 1
    y[i] = w0[k]*x[i] + b0[k] + np.random.randn()*sigma0[k]

ccpiest = np.array([0.5, 0.5]);cwest = np.array([1, -1])
cbest = np.array([0, 0]);sigmaest = np.array([np.std(y), np.std(y)])
cwest2 = np.array([cwest,cbest]).T;x2 = np.array([x,np.ones(N)])
p1 = ccpiest[0]*norm(cwest2[0].dot(x2),sigmaest[0]).pdf(y)
p2 = ccpiest[1]*norm(cwest2[1].dot(x2),sigmaest[1]).pdf(y)
r1 = p1/(p1+p2);r2 = p2/(p1+p2);r = np.array([r1,r2])
Q1 = np.sum(r1*np.log(ccpiest[0]*norm(cwest2[0].dot(x2),sigmaest[0]).pdf(y)))
Q2 = np.sum(r2*np.log(ccpiest[1]*norm(cwest2[1].dot(x2),sigmaest[1]).pdf(y)))
Q = Q1+Q2;diff = 1;ll = [];ite = [];count = 0

for i in range(100):
    ll.append(Q);ite.append(count);tmp = Q
    ccpiest = np.array([np.mean(r1),np.mean(r2)])
    w1 = np.linalg.inv(x2.dot(np.diag(r1)).dot(x2.T)).dot(x2).dot(np.diag(r1)).dot(y)
    w2 = np.linalg.inv(x2.dot(np.diag(r2)).dot(x2.T)).dot(x2).dot(np.diag(r2)).dot(y)
    cwest2 = np.array([w1,w2])
```

```
    sigmaest1 = ((np.sum(r1*(y-cwest2[0].dot(x2))**2))/np.sum(r1))**0.5
    sigmaest2 = ((np.sum(r2*(y-cwest2[1].dot(x2))**2))/np.sum(r2))**0.5
    sigmaest = np.array([sigmaest1,sigmaest2])
    p1 = ccpiest[0]*norm(cwest2[0].dot(x2),sigmaest[0]).pdf(y)
    p2 = ccpiest[1]*norm(cwest2[1].dot(x2),sigmaest[1]).pdf(y)
    r1 = p1/(p1+p2);r2 = p2/(p1+p2);r = np.array([r1,r2])
    Q1 = np.sum(r1*np.log(ccpiest[0]*norm(cwest2[0].dot(x2),sigmaest[0]).pdf(y)))
    Q2 = np.sum(r2*np.log(ccpiest[1]*norm(cwest2[1].dot(x2),sigmaest[1]).pdf(y)))
    Q = Q1+Q2;diff = Q-tmp;count = count+1
    if diff>=1e-4:
        continue
    else:
        break
plt.plot(x, cwest2[0].dot(x2), c='b', marker='x')
plt.plot(x, cwest2[1].dot(x2), c='g', marker='x')
plt.scatter(x, y, c='r', marker='x')
plt.plot(ite,ll)
```

1. (a) $$I(X,Y) = \sum_y \sum_x p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

$$= \sum_y \sum_x p(x,y) \left[\log \frac{p(x,y)}{p(x)} - \log p(y)\right]$$

$$= \sum_y \sum_x p(x,y) \left[\log p(y|x) - \log p(y)\right]$$

$$= \sum_y \sum_x p(x,y) (-\log p(y)) + \sum_y \sum_x p(x,y) \log p(y|x)$$

$$= -\sum_y \log p(y) \left(\sum_x p(x,y)\right) - \left[-\left(\sum_y \sum_x p(y|x) p(x) \log p(y|x)\right)\right]$$

$$= -\sum_y \log p(y) \cdot p(y) - \left[-\sum_x p(x) \sum_y p(y|x) \log p(y|x)\right]$$

$$= H(Y) - \sum_x p(x) H(Y|X=x) = H(Y) - H(Y|X)$$

Since $I(X,Y)$ is symmetric for X, Y, Similarly we can get $I(X,Y) = H(X) - H(X|Y)$

(b) From (a) we know, $I(X,Y) = H(X) - H(X|Y)$, $\Rightarrow$

$I(X,Y) = H(X) + \sum_x p(x) \sum_y p(y|x) \log p(y|x)$. Since $X = f(Y)$, then

$p(y|x) = 0$ or 1, for each case, $\sum_x p(x) \sum_y p(y|x) \log p(y|x) = 0$, then

$I(X,Y) = H(X) + 0 = H(X)$. Similarly, $I(X,Y) = H(Y)$

(c) $$\min_\theta D_{KL}(\hat{p}\|q) = H(\hat{p},q) - H(\hat{p}) \propto H(\hat{p},q)$$

(Noticing only q contains θ here)

$$H(\hat{p},q) = -\sum_x \hat{p}(x) \log q(x|\theta)$$

$$= -\sum_x \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{x=x_i\}} \log q(x|\theta)$$

$$= -\frac{1}{N} \sum_{i=1}^N \log(q(x_i|\theta))$$

Thus $$\min_\theta D_{KL}(\hat{p}\|q) \Leftrightarrow \min_\theta -\frac{1}{N} \sum_{i=1}^N \log q(x_i|\theta)$$

$$\Leftrightarrow \max_\theta \sum_{i=1}^N \log q(x_i|\theta)$$

$$\Leftrightarrow \max_\theta \prod_{i=1}^N q(x_i|\theta) \longrightarrow \text{likelyhood}$$

Thus the minimum Kullback-Leibler divergence is obtained by the maximum likelihood.

(d) For continuous variable,

we have $H(x) = -\int p(x) \ln p(x)\, dx$,

From the question, it has three constraint:

$$\text{s.t.} \begin{cases} \int p(x)\, dx = 1 \\ \int x\, p(x)\, dx = \mu \\ \int (x-\mu)^2 p(x)\, dx = 6^2 \end{cases}$$

Use Lagrange multipliers to max $H(x)$:

$$L = -\int p(x) \ln p(x) dx + \lambda_1 \left(\int p(x) dx - 1\right) + \lambda_2 \left(\int x\, p(x) dx - \mu\right) + \lambda_3 \left(\int (x-\mu)^2 p(x) dx - 6^2\right)$$

$$\frac{dL}{dp(x)} = -\ln p(x) - 1 + \lambda_1 + \lambda_2 x + \lambda_3 (x-\mu)^2 = 0$$

$$\Rightarrow p(x) = e^{-1+\lambda_1 + \lambda_2 x + \lambda_3 (x-\mu)^2}$$

with
$$\begin{cases} \int e^{-1+\lambda_1+\lambda_2 x+\lambda_3(x-\mu)^2} dx = 1 \\ \int x e^{-1+\lambda_1+\lambda_2 x+\lambda_3(x-\mu)^2} dx = \mu \\ \int (x-\mu)^2 e^{-1+\lambda_1+\lambda_2 x+\lambda_3(x-\mu)^2} dx = 6^2 \end{cases} \Rightarrow p(x) = \frac{1}{(2\pi 6^2)^{\frac{1}{2}}} e^{-\frac{(x-\mu)^2}{26^2}}$$

$\therefore$ $p = N(\mu, 6^2)$ is the solution of max $H(x)$

$\Leftrightarrow$ $H(q) \leq H(p)$ for q be any probability density with mean μ and variance 6^2.

2. (a) $\text{Dirichlet}(p|\alpha) = \frac{\Gamma(\sum_1^m \alpha_k)}{\prod_{k=1}^m \Gamma(\alpha_k)} \prod_{k=1}^m p_k^{\alpha_k - 1}$

$$= \exp\left(\log \frac{\Gamma(\sum_1^m \alpha_k)}{\prod_{k=1}^m \Gamma(\alpha_k)} \prod_{k=1}^m p_k^{\alpha_k-1}\right)$$

$$= \exp\left[\log\Gamma(\sum_1^m \alpha_k) + \sum_1^m (\alpha_k - 1)\log p_k - \sum_1^m \log\Gamma(\alpha_k)\right]$$

$$= \exp\left[\sum_1^m (\alpha_k - 1)\log p_k - \left(\sum_1^m \log\Gamma(\alpha_k) - \log\Gamma(\sum_1^m \alpha_k)\right)\right]$$

Thus, set $\eta(\alpha) = \begin{pmatrix} \alpha_1 - 1 \\ \alpha_2 - 1 \\ \vdots \\ \alpha_m - 1 \end{pmatrix}$, $T(p) = \begin{pmatrix} \log p_1 \\ \log p_2 \\ \vdots \\ \log p_m \end{pmatrix}$,

$A(\alpha) = \sum_1^m \log\Gamma(\alpha_k) - \log\Gamma(\sum_1^m \alpha_k)$. Then $D(p|\alpha) = \exp[\eta(\alpha)^T T(p) - A(\alpha)]$

(b) The Dirichlet log-likehood function $F(\alpha) = \log p(D|\alpha) = \sum_{j=1}^N \log D(p^j|\alpha)$

$$\therefore F(\alpha) = \sum_{j=1}^N \left[\sum_{k=1}^m (\alpha_k - 1)\log p_k^j + A(\alpha)\right]$$

$$= \sum_{k=1}^m (\alpha_k - 1)\sum_{j=1}^N \log p_k^j + N A(\alpha)$$

$$= N\sum_{k=1}^m (\alpha_k - 1)\frac{1}{N}\sum_{j=1}^N \log p_k^j + N\left[\log\Gamma(\sum_{k=1}^m \alpha_k) - \sum_{k=1}^m \log\Gamma(\alpha_k)\right]$$

$$= N\left[\sum_{k=1}^m (\alpha_k - 1)\hat{t}_k + \log\Gamma(\sum_{k=1}^m \alpha_k) - \sum_{k=1}^m \log\Gamma(\alpha_k)\right], \quad \hat{t}_k = \frac{1}{N}\sum_{j=1}^N \log p_k^j$$

(c) $$\frac{\partial F}{\partial \alpha_k} = N\left(\hat{t}_k + \frac{\partial \log\Gamma(\sum_{k=1}^m \alpha_k)}{\partial \sum_{k=1}^m \alpha_k}\cdot 1 - \frac{\partial \log\Gamma(\alpha_k)}{\partial \alpha_k}\right)$$

$$= N\left(\hat{t}_k + \Psi(\sum_{k=1}^m \alpha_k) - \Psi(\alpha_k)\right), \quad k = 1, \cdots m$$

$$\therefore \frac{\partial F}{\partial \alpha} = \left(\frac{\partial F}{\partial \alpha_1}, \frac{\partial F}{\partial \alpha_2}, \cdots \frac{\partial F}{\partial \alpha_m}\right)$$

(d) $\frac{\partial^2 F}{\partial \alpha_k^2} = N\left(\Psi'(\sum_{k=1}^m \alpha_k) - \Psi'(\alpha_k)\right)$, $\frac{\partial^2 F}{\partial \alpha_k \alpha_j} = N\Psi'(\sum_{k=1}^m \alpha_k)$

Thus set $q_{kk} = -N\Psi'(\alpha_k)$, $c = N\Psi'(\sum_{k=1}^m \alpha_k)$.

then $\nabla_\alpha^2 F(\alpha) = \left(\frac{\partial F}{\partial \alpha_k \alpha_j}\right)_{k,j=1,\cdots m} = Q + c\mathbf{1}\mathbf{1}^T$

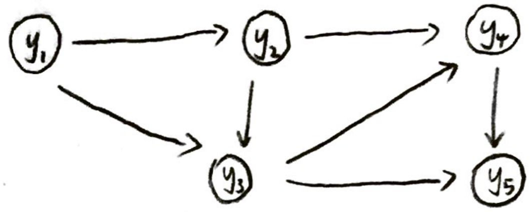
(e) $$\alpha^{new} = \alpha^{old} - [H_F(\alpha^{old})]^{-1}\nabla F(\alpha^{old})$$

$$= \alpha^{old} - [Q + c\mathbf{1}\mathbf{1}^T]^{-1}\cdot\nabla F(\alpha^{old})$$

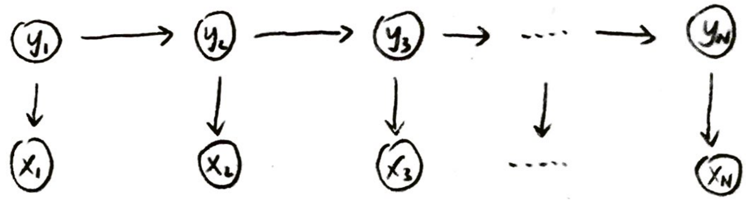
$$= \alpha^{old} - \left(Q^{-1} - \frac{Q^{-1}c\mathbf{1}\mathbf{1}^T Q^{-1}}{1 + \mathbf{1}^T Q^{-1} c\mathbf{1}}\right)\nabla F(\alpha^{old})$$

3. (a)

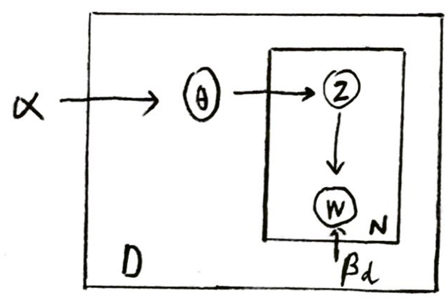
(i) $p(y_1, y_2, y_3, y_4, y_4, y_5) = p(y_1)p(y_2) \prod_{k=3}^{5} p(y_k \mid y_{k-1}, y_{k-2})$



(ii) $P(x_1, \cdots x_N, y_1, \cdots y_N) = P(y_1) \prod_{k=2}^{N} P(y_k \mid y_{k-1}) \prod_{k=1}^{N} P(x_k \mid y_k)$



(b)



(c) ①: $p(r, \theta, \phi, z) = \prod_{i=1}^{M} \left[p(r_i)\, p(\theta_i \mid r_i) \prod_{j=1}^{N} p(\phi_{ij})\, p(z_{ij} \mid \phi_{ij}) \right]$

②: $p(z, W) = \prod_{i=1}^{M} \left[p(z_i) \prod_{j=1}^{N} p(W_{ij} \mid z_i) \right]$

5. (a) $f(y|x;\theta) = \sum_{k=1}^{K} \pi_k \phi(y; w_k^T x + b_k, \sigma_k^2)$

$$\Rightarrow f(\underline{y}|\underline{x};\theta) = \prod_{i=1}^{N} \sum_{k=1}^{K} \pi_k \phi(y_i; w_k^T x_i + b_k, \sigma_k^2)$$

$$\therefore \log f(\underline{y}|\underline{x};\theta) = \sum_{i=1}^{N} \log \sum_{k=1}^{K} \pi_k \phi(y_i; w_k^T x_i + b_k, \sigma_k^2)$$

(b) $f(y|x, z=k;\theta) = \phi(y; w_k^T x + b_k, \sigma_k^2)$,

$f(y,z|x;\theta) = \pi_z \phi(y; w_z^T x + b_z, \sigma_z^2)$, Let $\Delta_{ik} = 1_{\{z_i=k\}}$

Then $\log f(y,z|x;\theta) = \sum_{k=1}^{K} \sum_{i=1}^{N} 1_{\{z_i=k\}} \log \pi_{z_i} \phi(y_i; w_{z_i}^T x_i + b_{z_i}, \sigma_{z_i}^2)$

$$:= \sum_{k=1}^{K} \sum_{i=1}^{N} \Delta_{ik} \log \pi_k \phi(y_i; w_k^T x_i + b_k, \sigma_k^2)$$

(c) $Q(\theta, \theta^{old}) = E_z[\log f(y,z|x;\theta) | y, x; \theta^{old}]$

Let $\gamma_{ik} = p(z_i=k | y_i, x_i; \theta^{old})$

$$\gamma_{ik} = \frac{f(z_i, y_i | x_i; \theta^{old})}{f(y_i | x_i; \theta^{old})} = \frac{\pi_k^{old} \phi(y_i, w_k^{T\,old} x_i + b_k^{old}, \sigma_k^{old})}{\sum_{i=1}^{K} \pi_k^{old} \phi(y_i, w_k^{T\,old} x_i + b_k^{old}, \sigma_k^{old})}$$

$$Q(\theta, \theta^{old}) = \sum_{i=1}^{N} \sum_{k=1}^{K} E_q[1_{\{z_i=k\}} \log \pi_k \phi(y, w_k^T x + b_k, \sigma_k^2]$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{K} \gamma_{ik} \log \pi_k \phi(y, w_k^T x + b_k, \sigma_k^2)$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{K} \gamma_{ik} \log \pi_k + \sum_{i=1}^{N} \sum_{k=1}^{K} \gamma_{ik} \log \phi(y, w_k^T x + b_k, \sigma_k^2)$$

(d) $\theta^{new} = \arg\max_\theta Q(\theta, \theta^{old})$, s.t. $\sum_{k=1}^{K} \pi_k = 1$, $\forall \pi_k > 0$ for $k=1, \cdots K$

let $f = Q(\theta, \theta^{old}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) - \sum_{k=1}^{K} \alpha_k \pi_k$, $\alpha_k = 0$, $\forall k = 1, \cdots K$

$$\frac{\partial f}{\partial \pi_k} = \sum_{i=1}^{N} \frac{\gamma_{ik}}{\pi_k} + \lambda = 0 \Rightarrow \pi_k = \frac{\sum_{i=1}^{N} \gamma_{ik}}{-\lambda}$$

$$\frac{\partial f}{\partial \lambda} = \sum_{k=1}^{K} \pi_k - 1 = 0 \Rightarrow \sum_{k=1}^{K} \sum_{i=1}^{N} \gamma_{ik} / -\lambda - 1 = 0 \Rightarrow \lambda = -N \Rightarrow \pi_k = \frac{\sum^{N} \gamma_{ik}}{K}$$

$$\frac{\partial f}{\partial w_k} = \sum_{i=1}^{N} \gamma_{ik}^2 \left(\frac{\log \phi(y, w_k^T x + b_k, \sigma_k^2)}{\partial w_k} \right) = \sum_{i=1}^{N} \gamma_{ik} \frac{\partial \left(\log \frac{1}{\sqrt{2\pi}\sigma_k} + \frac{y_i - (w_k^T x_i + b_k))^2}{2\sigma_k} \right)}{\partial w_k}$$

$$= \sum_{i=1}^{N} \gamma_{ik} \frac{2(y_i - w_k^T x_i + b_k)}{2\sigma_k} (-x_i) = 0$$

$$\frac{\partial f}{\partial b_k} = \sum_{i=1}^{N} \gamma_{ik} \frac{\partial \left(\log \frac{1}{\sqrt{2\pi}\sigma_k} + \frac{(y_i - (w_k^T x_i + b_k))^2}{2\sigma_k} \right)}{\partial b_k} = \sum_{i=1}^{N} \gamma_{ik} \frac{y_i - (w_k^T x_i + b_k)}{\sigma_k} \cdot (-1) = 0$$

Set $\tilde{w}_k = \begin{bmatrix} w_k \\ b_k \end{bmatrix}$, $\tilde{x} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} = [x, 1]$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$

then we can combine $\frac{\partial f}{\partial w_k}$ & $\frac{\partial f}{\partial b_k}$ to get $\sum_{i=1}^{N} \gamma_{ik} (y_i - \tilde{w}_k^T \tilde{x}_i) \tilde{x}_i = 0$

Use closed form to represent the whole question, we can set the objective function as

$\sum_{i=1}^{N} r_{ik}(y_i - \tilde{w}_k^T \tilde{x}_i)^2$ can be written as $(\tilde{X}\tilde{w}_k - y)^T R(\tilde{X}\tilde{w}_k - y)$

where $R = \mathrm{diag}(r_{1k}, r_{2k}, \cdots r_{Nk})$

$$\therefore \sum_{i=1}^{N} r_{ik}(y_i - \tilde{w}_k^T \tilde{x}_i)\tilde{x}_i = 2X^T R X w - 2X^T R y = 0$$

$$\Rightarrow \tilde{w}_k = (X^T R X)^{-1} X^T R y$$

$$\frac{\partial f}{\partial \sigma_k^2} = \frac{\partial \sum_{i=1}^{N} r_{ik}\left(\log \frac{1}{\sqrt{2\pi\sigma_k^2}} + \left(-\frac{1}{2\sigma_k^2}(y_i - \tilde{w}_k^T \tilde{x}_i)^2\right)\right)}{\partial \sigma_k^2}$$

$$= \sum_{i=1}^{N} r_{ik} + \frac{1}{\sigma_k^2} \sum_{i=1}^{N} (y_i - \tilde{w}_k^T \tilde{x}_i)^2 r_{ik} = 0$$

$$\Rightarrow \sigma_k^2 = \frac{1}{\sum_{i=1}^{N} r_{ik}} \sum_{i=1}^{N} (y_i - \tilde{w}_k^T \tilde{x}_i)^2 r_{ik}$$

$$= \frac{1}{\sum_{i=1}^{N} r_{ik}} (\tilde{X}\tilde{w}_k - y)^T R(\tilde{X}\tilde{w}_k - y)$$