Appendix

proof 1: Symmetry.
Because

$$D_{SRP}(RPS_1 \| RPS_2) = \frac{D_{RP}(RPS_1 \| RPS_2) + D_{RP}(RPS_2 \| RPS_1)}{2}$$

And

$$D_{SRP}(RPS_2 \| RPS_1) = \frac{D_{RP}(RPS_2 \| RPS_1) + D_{RP}(RPS_1 \| RPS_2)}{2}$$

Therefore

$$D_{SRP}(RPS_1 \| RPS_2) = D_{SRP}(RPS_2 \| RPS_1)$$

proof 2: Unboundedness.
Suppose $\Gamma = \{\gamma_1, \gamma_2\}$ is a fixed set, and given two RPSs as follows

$$\begin{aligned} RPS_1 &= \{\langle(\gamma_1, \gamma_2), 1-a\rangle, \langle(\gamma_2, \gamma_1), a\rangle\} \\ RPS_2 &= \{\langle(\gamma_1, \gamma_2), a\rangle, \langle(\gamma_2, \gamma_1), 1-a\rangle\} \end{aligned}$$

where $a \to 1$, then we can know that $\mathscr{M}_1(\gamma_1, \gamma_2) \to 0$ and $\mathscr{M}_2(\gamma_2, \gamma_1) \to 0$.
Therefore

$$\begin{aligned} &\lim_{a \to 1} D_{SRP}(RPS_1 \| RPS_2) \\ &= \frac{1}{\beta - 1} \log \left[\frac{\mathscr{M}_1(\gamma_1, \gamma_2)^{\beta}}{\mathscr{M}_2(\gamma_1, \gamma_2)^{\beta-1}} + \frac{\mathscr{M}_1(\gamma_2, \gamma_1)^{\beta}}{\mathscr{M}_2(\gamma_2, \gamma_1)^{\beta-1}} \right] \\ &= \infty \end{aligned}$$

proof 3: Semi-compatibility.

$$\begin{aligned} &\lim_{\beta \to 1} D_{SRP}(RPS_1 \| RPS_2) \\ &= \lim_{\beta \to 1} \frac{\frac{\log \sum_{A_{ij} \in PES(\Gamma)} \frac{\mathscr{M}_1(A_{ij})^{\beta}}{\mathscr{M}_2(B_{ij})^{\beta-1}}}{\beta-1} + \frac{\log \sum_{A_{ij} \in PES(\Gamma)} \frac{\mathscr{M}_2(A_{ij})^{\beta}}{\mathscr{M}_1(B_{ij})^{\beta-1}}}{\beta-1}}{2} \\ &= \lim_{\beta \to 1} \frac{\frac{\sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \mathscr{M}_2(B_{ij}) \ln\left[\frac{\mathscr{M}_1(A_{ij})}{\mathscr{M}_2(B_{ij})}\right] \left[\frac{\mathscr{M}_1(A_{ij})}{\mathscr{M}_2(B_{ij})}\right]^{\beta}}{\ln 2 \sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \left[\frac{\mathscr{M}_1(A_{ij})}{\mathscr{M}_2(B_{ij})}\right]^{\beta-1}} + \frac{\sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \mathscr{M}_1(B_{ij}) \ln\left[\frac{\mathscr{M}_2(A_{ij})}{\mathscr{M}_1(B_{ij})}\right] \left[\frac{\mathscr{M}_2(A_{ij})}{\mathscr{M}_1(B_{ij})}\right]^{\beta}}{\ln 2 \sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \left[\frac{\mathscr{M}_2(A_{ij})}{\mathscr{M}_1(B_{ij})}\right]^{\beta-1}}}{2} \\ &= \frac{\frac{\sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \mathscr{M}_1(A_{ij}) \ln\left[\frac{\mathscr{M}_1(A_{ij})}{\mathscr{M}_2(B_{ij})}\right]}{\ln 2} + \frac{\sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \mathscr{M}_2(A_{ij}) \ln\left[\frac{\mathscr{M}_2(A_{ij})}{\mathscr{M}_1(B_{ij})}\right]}{\ln 2}}{2} \\ &= \frac{\sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \mathscr{M}_1(A_{ij}) \log\left[\frac{\mathscr{M}_1(A_{ij})}{\mathscr{M}_2(B_{ij})}\right] + \sum_{A_{ij}, B_{ij} \in PES(\Gamma)} \mathscr{M}_2(A_{ij}) \log\left[\frac{\mathscr{M}_2(A_{ij})}{\mathscr{M}_1(B_{ij})}\right]}{2} \\ &= \frac{D_{KL}(RPS_1 \| RPS_2) + (RPS_2 \| RPS_1)}{2} \end{aligned}$$

proof 4: Non-negativity.

Suppose $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a fixed set, and given two RPSs as follows

$$RPS_1 = \{\langle A_1, p_1\rangle, \langle A_2, p_2\rangle, \ldots, \langle A_n, p_n\rangle\}$$
$$RPS_2 = \{\langle A_1, q_1\rangle, \langle A_2, q_2\rangle, \ldots, \langle A_n, q_n\rangle\}$$

where $A_i \in PES(\Gamma)$, and $\sum\limits_{i=1}^{n} p_i = \sum\limits_{i=1}^{n} q_i = 1$. Because $D_{SRP}(RPS_1 \| RPS_2) = \frac{D_{RP}(RPS_1 \| RPS_2) + D_{RP}(RPS_2 \| RPS_1)}{2}$, we only need to prove that $D_{RP}(RPS_1 \| RPS_2) \geq 0$ and $D_{RP}(RPS_2 \| RPS_1) \geq 0$. Firstly, we rewrite the expression for D_{SRP} as follows

$$\begin{aligned}
&D_{RP}(RPS_1 \| RPS_2) \\
&= \frac{1}{\beta - 1} \log \sum_{A_i \in PES(\Gamma)} \frac{\mathscr{M}_1(A_i)^{\beta}}{\mathscr{M}_2(A_i)^{\beta-1}} \\
&= \frac{1}{\beta - 1} \log \sum_{A_i \in PES(\Gamma)} \mathscr{M}_1(A_i) \left[\frac{\mathscr{M}_1(A_i)}{\mathscr{M}_2(A_i)}\right]^{\beta-1} \\
&= \frac{1}{\beta - 1} \log \sum_{i=1}^{n} p_i \left(\frac{p_i}{q_i}\right)^{\beta-1}
\end{aligned}$$

where $\beta = \max(|A_{ij}|)$, and through the definition of Equation **??**, we can know $\beta \geq 1$. Then, we prove it in the following two situations.

(1) $\beta = 1$.

In this case, according to the semi-compatibility property of D_{SRP} and proof 3, $D_{SRP}(RPS_1 \| RPS_2) = D_{KL}(RPS_1 \| RPS_2)$. Next, perform the following transformations

$$\begin{aligned}
&D_{RP}(RPS_1 \| RPS_2) = D_{KL}(RPS_1 \| RPS_2) \\
&= \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i - \log \frac{q_i}{p_i}
\end{aligned}$$

Since $-\log(x)$ is a convex function, according to Jensen inequality, we have

$$\sum_{i=1}^{n} p_i - \log \frac{q_i}{p_i} \geq -\log \sum_{i=1}^{n} p_i \frac{q_i}{p_i} = -\log \sum_{i=1}^{n} p_i = -\log 1 = 0$$

Therefore, if $\beta = 1$, we can proof that

$$D_{RP}(RPS_1 \| RPS_2) = \sum_{i=1}^{n} p_i - \log \frac{q_i}{p_i} \geq 0$$

(2) $\beta > 1$. In this case

$$D_{RP}(RPS_1 \| RPS_2) = \frac{1}{\beta - 1} \log \sum_{i=1}^{n} p_i \left(\frac{p_i}{q_i}\right)^{\beta-1}$$

Because $\beta > 1$, $\frac{1}{\beta-1} > 0$. Therefore, we only need to prove that $\log \sum\limits_{i=1}^{n} p_i \left(\frac{p_i}{q_i}\right)^{\beta-1} \geq 0$. Since $\log(x)$ is a concave function, according to Jensen inequality, we have

$$\frac{1}{\beta - 1} \log \sum_{i=1}^{n} p_i \left(\frac{p_i}{q_i}\right)^{\beta-1} \geq \frac{1}{\beta - 1} \sum_{i=1}^{n} p_i \log \left(\frac{p_i}{q_i}\right)^{\beta-1} = \sum_{i=1}^{n} p_i - \log \frac{q_i}{p_i}$$

This inequality is the same as the first case. Therefore, if $\beta > 1$, we can proof that

$$D_{RP}\left(RPS_1 \,\|RPS_2\right) = \sum_{i=1}^{n} p_i - \log\frac{q_i}{p_i} \geq 0$$

Similarly, we can prove that $D_{RP}\left(RPS_2 \,\|RPS_1\right) \geq 0$. Finally, based on the above derivation process, we proof that

$$D_{SRP}\left(RPS_1 \,\|RPS_2\right) = \frac{D_{RP}\left(RPS_1 \,\|RPS_2\right) + D_{RP}\left(RPS_2 \,\|RPS_1\right)}{2} \geq 0$$