# STAT 423/523 Statistical Methods for Engineers and Scientists

## Lecture 7: The Analysis of Variance II

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# Example: One-way ANVOA

The fracture load of a certain type of material was measured under three different distances from the center.

| | | | | | | $x_{i\cdot}$ |
|---|---|---|---|---|---|---|
| | 42 mm: | 2.62 | 2.99 | 3.39 | 2.86 | 11.86 |
| Distance | 36 mm: | 3.47 | 3.85 | 3.77 | 3.63 | 14.72 |
| | 31.2 mm: | 4.78 | 4.41 | 4.91 | 5.06 | 19.16 |
| | | | | | | $x_{\cdot\cdot} = 45.74$ |

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- The factor: distance from the center.
- Three levels: 1, 2, 3.
- Each level has 4 observations.
- $x_{i\cdot}$ and $x_{\cdot\cdot}$ are the sums of observations.

# Example: One-way ANVOA

- The mean observation at each level is

$$\bar{x}_{1\cdot} = x_{1\cdot}/4 = 2.965, \quad \bar{x}_{2\cdot} = x_{2\cdot}/4 = 3.680, \quad \bar{x}_{3\cdot} = x_{3\cdot}/4 = 4.790.$$

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- The grand mean is

$$\bar{x}_{\cdot\cdot} = x_{\cdot\cdot}/12 = 3.812$$

# Example: One-way ANVOA

- The sample variance within each level is

$$
\begin{aligned}
s_1^2 &= \frac{1}{4-1}\left[(2.62-2.965)^2+(2.99-2.965)^2+(3.39-2.965)^2+(2.86-2.965)^2\right]\\
&= 0.1038\\
s_2^2 &= \frac{1}{4-1}\left[(3.47-3.680)^2+(3.85-3.680)^2+(3.77-3.680)^2+(3.63-3.680)^2\right]\\
&= 0.0279\\
s_3^2 &= (4.78-4.790)^2+(4.41-4.790)^2+(4.91-4.790)^2+(5.06-4.790)^2\\
&= 0.0773
\end{aligned}
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- The SSE is

$$SSE = (4-1)(s_1^2+s_2^2+s_3^2) = 0.6267$$

# Example: One-way ANVOA

- The SSTr is

$$
\begin{aligned}
SSTr &= 4\left[(2.965 - 3.812)^2 + (3.680 - 3.812)^2 + (4.790 - 3.812)^2\right] \\
&= 6.7653
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$$

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$$SSTr = 4\left[(2.965 - 3.812)^2 + (3.680 - 3.812)^2 + (4.790 - 3.812)^2\right]$$
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- The SST is

$$SST = SSTr + SSE = 7.3920$$

# Example: One-way ANVOA

- The MSE is

$$MSE = \frac{SSE}{IJ - I} = \frac{0.6267}{12 - 3} = 0.0696$$

- The MSTr is

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- The $F$-statistic is

$$F = \frac{MSTr}{MSE} = \frac{3.3826}{0.0696} = 48.60$$

- The $p$-value is

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- The conclusion is that the distance from the center has a significant effect on the fracture load.

# ANOVA Table

All the SS and MS information as well as F-statistic can be organized in an **ANOVA table**.

| Source | df | SS | MS | F |
|---|---|---|---|---|
| Treatment | $I-1$ | SSTr | MSTr | $F = \frac{MSTr}{MSE}$ |
| Error | $IJ-I$ | SSE | MSE | |
| Total | $IJ-1$ | SST | | |

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- "Treatment" can be replaced by the factor name, and "Error" can be replaced by "Residual".
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- MS is computed per source of variation by dividing SS by df.
- The $F$-statistic is the ratio of MS for Treatment to MS for Error.

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- Decomposition 2: Total SS = Treatment SS + Error SS.
- MS is computed per source of variation by dividing SS by df.
- The $F$-statistic is the ratio of MS for Treatment to MS for Error.
- The "Total" row is sometimes omitted.

# Example

The ANOVA table for the fracture load example is

| Source | df | SS | MS | F | p-value |
|---|---|---|---|---|---|
| Treatment | 2 | 6.7653 | 3.3826 | 48.60 | $< 0.001$ |
| Error | 9 | 0.6267 | 0.0696 | | |
| Total | 11 | 7.3920 | | | |

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Output from R:

```
            Df Sum Sq Mean Sq F value  Pr(>F)
level        2  6.765   3.383   48.58 1.5e-05 ***
Residuals    9  0.627   0.070
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

# Multiple Comparison

Suppose we want to compare the mean of one treatment level to another.

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The two sample t-test (Pooled) gives the following CI for $\mu_i - \mu_{i'}$:

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We only need check whether the CI contains 0.

- We have $I(I-1)/2$ pairs of comparisons.
- If all tests are independent with significance level $\alpha$, the overall Type I error rate is $1 - (1-\alpha)^{I(I-1)/2}$, whcih could be large when $I$ is large.
- We need to adjust the significance level for each test to control the overall Type I error rate.

# Multiple Comparison — Tukey's Method

The **simultaneous confidence interval** for $\mu_i - \mu_{i'}$ is

$$\bar{x}_i - \bar{x}_{i'} \pm Q_{\alpha,I,I(J-1)} \sqrt{\frac{MSE}{J}}$$

where $Q_{\alpha,I,I(J-1)}$ is the $\alpha$-quantile of the **Studentized range distribution** with $I$ and $I(J-1)$ degrees of freedom.

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where $Q_{\alpha,I,I(J-1)}$ is the $\alpha$-quantile of the **Studentized range distribution** with $I$ and $I(J-1)$ degrees of freedom.

- There is at least $1-\alpha$ probability that the interval contains $\mu_i - \mu_{i'}$ for **every** pair of $i$ and $i'$.
- The Studentized range distribution is a generalization of the Student's t-distribution for multiple comparisons.
- The quantile $Q_{\alpha,I,I(J-1)}$ can be found by `qtukey` in R.

# Tukey's Procedure in Identifying Significant Differences

- Compute

$$w = Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}$$

- Order the treatment means from smallest to largest.
- Underscore maximal consecutive treatment means such that the difference of the maximum and minimum of the underscored means is less than $w$.

# Tukey's Procedure in Identifying Significant Differences

Suppose we have $I = 5$ treatment levels with sample means:

$$\bar{x}_{1\cdot} = 14.5,\ \bar{x}_{2\cdot} = 13.8,\ \bar{x}_{3\cdot} = 13.3,\ \bar{x}_{4\cdot} = 14.3,\ \bar{x}_{5\cdot} = 13.1$$

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We order the means:

$$\bar{x}_{5\cdot} = 13.1,\ \bar{x}_{3\cdot} = 13.3,\ \bar{x}_{2\cdot} = 13.8,\ \bar{x}_{4\cdot} = 14.3,\ \bar{x}_{1\cdot} = 14.5$$

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Suppose we compute $w = 0.4$.

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Suppose we compute $w = 0.4$. Then we underscore the means:

$$\begin{array}{ccccc} \bar{x}_{5\cdot} & \bar{x}_{3\cdot} & \bar{x}_{2\cdot} & \bar{x}_{4\cdot} & \bar{x}_{1\cdot} \\ \multicolumn{2}{c}{\underline{13.1 \quad 13.3}} & \multicolumn{3}{c}{\underline{13.8 \quad 14.3 \quad 14.5}} \end{array}$$