STAT 576 HOMEWORK 1

DUE SEPT. 18, 2024 (WEDNESDAY), 11:59 PM PST

NOTES. **NO** late submission will be accepted except for approval from the instructor. Answers should be either scanned or typed. For the coding part (if any), you may either prepare an R notebook or put the code in a separate file.

1 (Single Parameter Bayesian Inference). Suppose $X_1, X_2, \ldots$, be an i.i.d. sequence of Bernoulli(θ) trials with unknown parameter $\theta \in (0, 1)$. Let Y_k be the number of trials before k-th failures for some fixed $k \geq 1$. That is

$$Y_k = \min \left\{ n : \sum_{i=1}^{n} (1 - X_i) \geq k \right\}$$

with the convention that $\min \varnothing = +\infty$.

(a) Show that $\mathbb{P}[Y_k < \infty] = 1$.
   Hint: First show that $Y_k = Z_1 + Z_2 + \cdots + Z_k$ with $Z_1, \ldots, Z_k \sim$ Geometric($1 - \theta$). Then for any constant c, show that $\mathbb{P}[Y_k \geq c] \leq c^{-1}\mathbb{E}[Y_k]$, which converges to 0 when $c \to \infty$.
(b) Compute $p(y_k \mid \theta)$ and relate it to the negative-binomial distribution.
(c) Find the conjugate prior on θ for the distribution in (b). Identify the hyperparameters.
(d) Use the prior in part (c) to get the posterior distribution.
(e) What are the MAP estimator and the posterior mean estimator for θ?
(f) Compute the Jeffreys prior and figure out whether it belongs to the conjugate prior family.

2 (Textbook Problems). Finish Problems 5 and 12 in Chapter 2 of the textbook.

3 (Normal with Precision Parameter).

(a) Consider the **Normal-Gamma** distribution with density

$$p(x, \tau \mid \mu, \lambda, \alpha, \beta) = \frac{\beta^\alpha \sqrt{\lambda}}{\Gamma(\alpha)\sqrt{2\pi}} \tau^{\alpha - 1/2} e^{-\beta\tau} e^{-\lambda\tau(x-\mu)^2/2}$$

   Find the conditional distritions $p(x \mid \tau)$ and $p(\tau \mid x)$, and the marginal distributions $p(x)$ and $p(\tau)$.
(b) Consider a normal distribution with precision parametrization, that is,

$$p(y \mid \mu, \tau) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{\tau(y-\mu)^2}{2}}$$

   Use the Normal-Gamma distribution as a conjugate prior and find the posterior distribution.
(c) Find the marginal posterior distributions for μ and τ correspondingly.

4 (Textbook Problems). Finish Problems 10 in Chapter 3 of the textbook.