STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 3: Point Estimation II

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Methods of Point Estimation

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Now we introduce some methods to construct point estimators:

- Method of Moments (MoM)
- Maximum Likelihood Estimation (MLE)

Method of Moments (MoM)

Definition (Moments)

Let $X_1, \ldots, X_n$ be a random sample from a population with pmf or pdf $f(x)$. For $k = 1, 2, \ldots$, the **kth population moment** or **kth moment of the distribution** $f(x)$, is $E(X^k)$. The **kth sample moment** is

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k.$$

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$$\frac{1}{n}\sum_{i=1}^{n} X_i^k.$$

- kth moment of a distribution is the expected value of X^k.
- kth sample moment is the sample average of X^k.
- When $n \to \infty$, the two moments are equal (by Law of Large Numbers).

Method of Moments (MoM)

Let $X_1, \ldots, X_n$ be a random sample from a population with pmf or pdf $f(x; \theta_1, \ldots, \theta_m)$, where $\theta_1, \ldots, \theta_m$ are the unknown parameters we want to estimate.

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The **method of moments estimator** of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = g_1(\theta_1, \ldots, \theta_m)$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X^2) = g_2(\theta_1, \ldots, \theta_m)$$

$$\vdots$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^m = E(X^k) = g_m(\theta_1, \ldots, \theta_m)$$

Method of Moments (MoM)

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The **method of moments estimator** of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations:

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} X_i &= E(X) = g_1(\theta_1, \ldots, \theta_m) \\
\frac{1}{n}\sum_{i=1}^{n} X_i^2 &= E(X^2) = g_2(\theta_1, \ldots, \theta_m) \\
&\vdots \\
\frac{1}{n}\sum_{i=1}^{n} X_i^m &= E(X^k) = g_m(\theta_1, \ldots, \theta_m)
\end{aligned}
$$

In short: **MoM matches the sample moments with the population moments.**

Example

Let $X_1, \ldots, X_n$ be a random sample from a population with unknown mean μ and unknown variance σ^2.

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MoM matches the first two moments:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = \mu$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X^2) = E(X)^2 + \mathrm{Var}(X) = \mu^2 + \sigma^2$$

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The solution, the MoM estimator, is

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \overline{X^2} - (\bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Example (Textbook 6.13)

Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with parameters α and β. The pdf is

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}.$$

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The first two moments of the Gamma distribution are

$$E(X) = \alpha\beta, \quad E(X^2) = \alpha(\alpha+1)\beta^2.$$

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The MoM estimator of α and β are the solutions to the following equations:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = \alpha\beta$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X^2) = \alpha(\alpha+1)\beta^2$$

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The solutions are

$$\hat{\alpha} = \frac{\bar{X}^2}{\overline{X^2} - \bar{X}^2}$$
$$\hat{\beta} = \frac{\overline{X^2} - \bar{X}^2}{\bar{X}}$$

Method of Moments

- MoM only requires the first few moments of the distribution. (Not the explicit pmf or pdf)
- If the first m moments do not give a unique solution, we can use more moments.
- MoM estimator is approximately normal if the sample size is large enough (by CLT).

Maximum Likelihood Estimation (MLE)

Definition (Likelihood Function)

Let $X_1, \ldots, X_n$ be a random sample from a population with pmf or pdf $f(x; \theta_1, \ldots, \theta_m)$. The **likelihood function** is

$$L(\theta_1, \ldots, \theta_m) = \prod_{i=1}^{n} f(x_i; \theta_1, \ldots, \theta_m).$$

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- The likelihood function is a function of the parameters $\theta_1, \ldots, \theta_m$.
- though it has exactly the same formula as the joint pmf or pdf of the sample.
- In order to compute the likelihood, we need to know the pmf or pdf explicitly. (compare it to MoM)

Important Clarificaitons on Likelihood

- The likelihood function is **the probability of observing the sample** given the parameters.
- **NOT** the probability of the parameters given the sample.

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- The likelihood function is **the probability of observing the sample** given the parameters.
- **NOT** the probability of the parameters given the sample.
- That is

$$L(\theta_1, \ldots, \theta_m) \neq p(\theta_1, \ldots, \theta_m \mid X_1, \ldots, X_n)$$

- The r.h.s. of above is

$$p(\theta_1, \ldots, \theta_m \mid X_1, \ldots, X_n) = \frac{p(X_1, \ldots, X_n \mid \theta_1, \ldots, \theta_m) p(\theta_1, \ldots, \theta_m)}{p(X_1, \ldots, X_n)}$$

but we assume $\theta_1, \ldots, \theta_m$ to be fixed.

Example (Textbook 6.15)

Suppose 10 email accounts are randomly sampled and the 1st, 3rd and 10th accounts are found to have strong passwords. We want to estiamte p: the proportion of email accounts with strong passwords.

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The likelihood function is

$$L(p) = f(x_1, \ldots, x_{10}; p) = p(1-p)p(1-p)\cdots p = p^3(1-p)^7.$$

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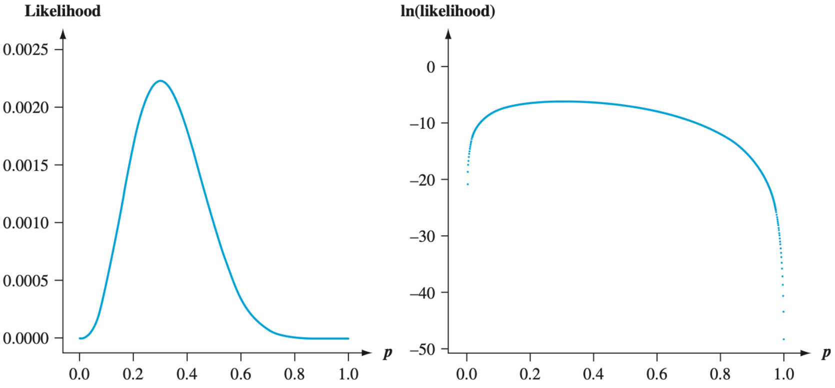
The logarithm of the likelihood function is called the **log-likelihood function**:

$$\ell(p) := \log L(p) = 3\log p + 7\log(1-p).$$

Example (Textbook 6.15)



The intuitively best guess of p is the value that maximizes the likelihood function.

Maximum Likelihood Estimation (MLE)

The **maximum likelihood estimator** of $\theta_1, \ldots, \theta_m$ is the value of $\theta_1, \ldots, \theta_m$ that maximizes the likelihood function $L(\theta_1, \ldots, \theta_m)$.

The **log-likelihood function** is

$$\ell(\theta_1, \ldots, \theta_m) = \log L(\theta_1, \ldots, \theta_m).$$

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In many cases, the MLE of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations: The MLE of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations:

$$\frac{\partial \ell}{\partial \theta_1} = 0, \ldots, \frac{\partial \ell}{\partial \theta_m} = 0.$$

The first order derivatives are called the **score functions**. The MLE is a zero of the score functions.

Example (Textbook 6.15) Cont.

Continue the example of passwords. The score function is

$$\frac{d\ell(p)}{dp} = \frac{d(3\log p + 7\log(1-p))}{dp} = \frac{3}{p} - \frac{7}{1-p}.$$

Example (Textbook 6.15) Cont.

Continue the example of passwords. The score function is

$$\frac{d\ell(p)}{dp} = \frac{d(3\log p + 7\log(1-p))}{dp} = \frac{3}{p} - \frac{7}{1-p}.$$

The MLE is the solution to

$$\frac{3}{p} - \frac{7}{1-p} = 0.$$

The solution is $\hat{p} = 3/10$.

Example

Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter $\lambda > 0$. The pdf is

$$f(x; \lambda) = \lambda e^{-\lambda x}.$$

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The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i}.$$

The log-likelihood function is

$$\ell(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} X_i.$$

Example

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The log-likelihood function is

$$\ell(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} X_i.$$

The score function is

$$\frac{d\ell(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i.$$

Example

The MLE is the solution to

$$\frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0.$$

The solution is $\hat{\lambda} = n / \sum_{i=1}^{n} X_i = 1/\bar{X}$.

Example

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The solution is $\hat{\lambda} = n / \sum_{i=1}^{n} X_i = 1/\bar{X}$.

- $\hat{\lambda}$ is biased because (Jensen's inequality)

$$E\left(\frac{1}{\bar{X}}\right) > \frac{1}{E(\bar{X})} = \lambda.$$

- The MoM estimator for λ is $\hat{\lambda} = 1/\bar{X}$ as well (based on the first moment).

Example

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The likelihood function is

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_i - \mu)^2}{2\sigma^2}\right) = \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} \exp\left(-\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{2\sigma^2}\right)$$

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The log-likelihood function is

$$\ell(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2.$$

The score functions are

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu), \qquad \frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(X_i - \mu)^2.$$

Example

The MLE is the solution to

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)=0, \quad -\frac{n}{2\sigma^2}+\frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(X_i-\mu)^2=0.$$

The solution is

$$\hat{\mu}=\bar{X}, \quad \hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

The MLE of μ is the sample mean, and the MLE of σ^2 is the sample variance.

Example

The MLE is the solution to

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The solution is

$$\hat{\mu}=\bar{X}, \quad \hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

The MLE of μ is the sample mean, and the MLE of σ^2 is the sample variance.

- $\hat{\mu}$ is unbiased.
- $\hat{\sigma}^2$ is biased.

Maximum Likelihood Estimation

- MLE is not unique. (The likelihood function may have multiple maxima.)
- MLE is not always the solution to the score functions. (The score functions may not have zeros.)
- When the sample size is large enough and the MLE is a zero of the score functions, the MLE is approximately normal (by CLT).
- When the sample size is large enough, the MLE is approximately unbiased (by Law of Large Numbers).
- When the sample size is large enough, the MLE is approximately efficient (with smallest variance).
- MLE is transformation invariant. (If $\hat{\theta}$ is the MLE of θ, then $\phi(\hat{\theta})$ is the MLE of $\phi(\theta)$ for any function ϕ).

Example: Domain-related Distribution

Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[0, \theta]$. The pdf is

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The likelihood function is

$$L(\theta) = \frac{1}{\theta^n} I(\max(X_1, \ldots, X_n) \leq \theta).$$

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The likelhood function is monotone decreasing in $\theta > X_{max}$. The MLE is $\hat{\theta} = X_{max}$.

- $\hat{\theta}$ is biased. (because $E(X_{max}) < \theta$)
- When the sample size is large enough, $\hat{\theta}$ is **not** approximately normal.

Example: Domain-related Distribution

If we consider MoM for the same problem. The MoM estimator is the solution to

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = \frac{\theta}{2}.$$

The MoM estimator is $\hat{\theta} = 2\bar{X}$.

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The MoM estimator is $\hat{\theta} = 2\bar{X}$.

- $\hat{\theta}$ is unbiased.
- When the sample size is large enough, $\hat{\theta}$ is approximately normal.
- However, it could happen that $\hat{\theta} < X_{max}$.