

STAT 576 Bayesian Analysis

Lecture 10: State-space Models and Sequential Monte Carlo I

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State-space Models

The **state-space model** is a general framework for modeling time series data. It consists of two components:

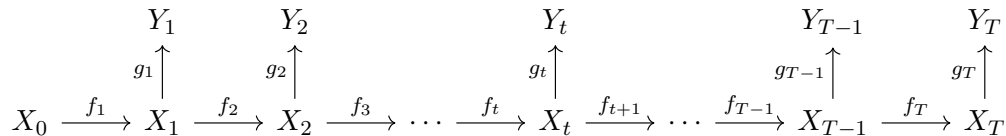
- ▶ The **state equation**: describes the evolution of the latent state variables over time.
- ▶ The **observation equation**: describes the relationship between the latent state variables and the observed data.

State-space Models

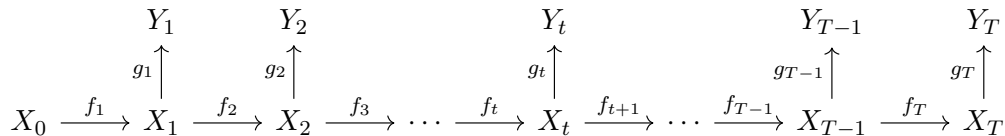
The **state-space model** is a general framework for modeling time series data. It consists of two components:

- ▶ The **state equation**: describes the evolution of the latent state variables over time.
- ▶ The **observation equation**: describes the relationship between the latent state variables and the observed data.
- ▶ The state-space model is also known as the **hidden Markov model (HMM)** when the state space is finite and the process is Markovian.

State-space Models

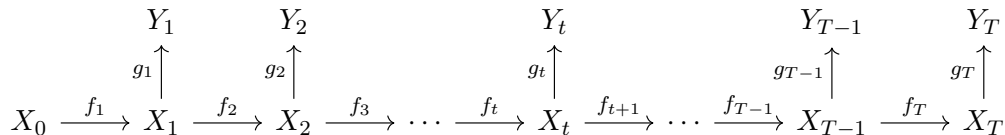


State-space Models



- Observed data: $\mathbf{Y} = (Y_1, \dots, Y_T)$
- Latent states: $\mathbf{X} = (X_0, X_1, \dots, X_T)$

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- Latent states: $\mathbf{X} = (X_0, X_1, \dots, X_T)$
- The state equation:

$$p(X_0) = f_0(X_0), \quad p(X_t | \mathbf{X}_{t-1}) = f_t(X_t | \mathbf{X}_{t-1})$$

- The observation equation:

$$p(Y_t | \mathbf{X}_t) = g_t(Y_t | X_t)$$

State-space Model

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- ▶ The (Markovian) state-space model is **linear** if

$$\mathbb{E}[X_t \mid X_{t-1}] = \mathbf{A}_t X_{t-1}$$

and

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for some matrices \mathbf{A}_t and \mathbf{B}_t .

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- ▶ The (Markovian) state-space model is **linear Gaussian** if

$$X_t \mid X_{t-1} \sim \mathcal{N}(\mathbf{A}_t X_{t-1}, \boldsymbol{\Sigma}_t) \text{ and } Y_t \mid X_t \sim \mathcal{N}(\mathbf{A}_t X_t, \mathbf{R}_t)$$

Example: Object Tracking

- ▶ Consider the problem that tracks the position of an object moving in a 2D plane.
- ▶ The data contains the observed positions (with noise) of the object at different time points. $Y_t = (a_t, b_t)^T$.

Example: Object Tracking

- ▶ Consider the problem that tracks the position of an object moving in a 2D plane.
- ▶ The data contains the observed positions (with noise) of the object at different time points. $Y_t = (a_t, b_t)^T$.
- ▶ We can assume the latent states $X_t = (x_t, y_t)$, the true positions of the object.
- ▶ The observation equation is

$$Y_t = X_t + \epsilon_t$$

where $\epsilon_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$. \mathbf{R} is the accuracy of the sensor.

- ▶ For the latent states X_t , we can assume a linear Gaussian model (random walk):

$$X_t = X_{t-1} + \eta_t,$$

where $\eta_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$ and Σ is the process noise.

Example: Object Tracking

The previous model has a continuous path, but quite stochastic velocities. We can add a velocity component to the model to stabilize the dynamics.

- ▶ The latent states $X_t = (x_t, y_t, v_t, u_t)$, where (x_t, y_t) is the position and (v_t, u_t) is the velocity.
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- ▶ The state equation is

$$x_t = x_{t-1} + v_{t-1}$$

$$y_t = y_{t-1} + u_{t-1}$$

$$v_t = v_{t-1} + \eta_t$$

$$u_t = u_{t-1} + \xi_t,$$

where $\eta_t, \xi_t \sim \mathcal{N}(0, \sigma^2)$.

Example: Object Tracking

The previous model is a linear Gaussian model. We can write it in the matrix form:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \boldsymbol{\eta}_t$$

$$\mathbf{Y}_t = \mathbf{B}\mathbf{X}_t + \boldsymbol{\epsilon}_t,$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$$

$$\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}).$$

The Probabilities

The state-space model is a full probabilistic model.

- The joint distribution of the latent states and the observed data is

$$p(\mathbf{X}, \mathbf{Y}) = p(X_0) \prod_{t=1}^T p(X_t \mid \mathbf{X}_{t-1}) p(Y_t \mid X_t) = f_0(X_0) \prod_{t=1}^T f_t(X_t \mid \mathbf{X}_{t-1}) g_t(Y_t \mid X_t)$$

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- ▶ The joint distribution of the observed data is

$$p(\mathbf{Y}) = \int p(\mathbf{X}, \mathbf{Y}) d\mathbf{X} = \int f_0(X_0) \prod_{t=1}^T f_t(X_t | \mathbf{X}_{t-1}) g_t(Y_t | X_t) d\mathbf{X}$$

Bayesian Framework

- The prior:

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- The posterior:

$$p(\mathbf{X} \mid \mathbf{Y}) = \frac{p(\mathbf{X}, \mathbf{Y})}{p(\mathbf{Y})} \propto p(\mathbf{X}, \mathbf{Y}) = f_0(X_0) \prod_{t=1}^T f_t(X_t \mid \mathbf{X}_{t-1}) g_t(Y_t \mid X_t)$$

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Direct sampling from this posterior distribution can be difficult. We need to utilize the **sequential** structure of the model.

The Sequential Structure

Suppose we are at time t .

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- ▶ The **sequential** posterior for the latent states up to time t is (also called the **filtering** distribution)

$$p(\mathbf{X}_t \mid \mathbf{Y}_t) \propto f_t(X_0) \prod_{s=1}^t f_s(X_s \mid \mathbf{X}_{s-1}) g_t(Y_t \mid X_t)$$

The Sequential Structure

At time t ,

- The **predictive** distribution for the latent state at time $t + 1$ is

$$p(X_{t+1} \mid \mathbf{Y}_t) = \int p(X_{t+1} \mid X_t) p(X_t \mid \mathbf{Y}_t) dX_t$$

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- ▶ The joint distribution of the latent states up to time $t + 1$ is

$$\begin{aligned} p(\mathbf{X}_{t+1} \mid \mathbf{Y}_t) &= p(X_{t+1} \mid \mathbf{Y}_t) p(\mathbf{X}_t \mid \mathbf{Y}_t) \\ &\propto f_{t+1}(X_{t+1} \mid \mathbf{X}_t) f_t(X_0) \prod_{s=1}^t f_s(X_s \mid \mathbf{X}_{s-1}) g_t(Y_t \mid X_t) \end{aligned}$$

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- ▶ The filtering distribution for the latent states up to time $t + 1$ is

The Sequential Structure

The sequential structure of the state-space model allows us to update the latent states one by one.

- ▶ $p(\mathbf{X}_{t+1} \mid \mathbf{Y}_t)$ is the prior
- ▶ $p(Y_{t+1} \mid X_{t+1})$ is the likelihood
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A rudiment of sequential Monte Carlo:

- ▶ If we have a sample from $\mathbf{X}_t \mid \mathbf{Y}_t$.
- ▶ We can draw a sample from $\mathbf{X}_{t+1} \mid \mathbf{Y}_t$ by drawing X_{t+1} from $p(X_{t+1} \mid \mathbf{X}_t)$.
- ▶ We can update the sample to $\mathbf{X}_{t+1} \mid \mathbf{Y}_{t+1}$ by adjusting its weight according $p(Y_{t+1} \mid X_{t+1})$.

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- ▶ We can update the sample to $\mathbf{X}_{t+1} \mid \mathbf{Y}_{t+1}$ by adjusting its weight according $p(Y_{t+1} \mid X_{t+1})$.

Remark:

- ▶ The distribution $p(\mathbf{X}_t \mid \mathbf{Y}_t)$ is called the **filtering** distribution.
- ▶ The distribution $p(\mathbf{X}_t \mid \mathbf{Y})$ is called the **smoothing** distribution.