STAT 576 Bayesian Analysis

Lecture 5: Hierarchical Models

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Hierarchical Models

- Many statistical applications involve multiple parameters that are **related**.
- Example: (Multi-center study on the effectiveness of a drug)
  - Hospitals $j = 1, \ldots, J$.
  - Patients in hospital j has a probability of recovering of θ_j.
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- If we observe y_{ij}, $i = 1, \ldots, n_j$ for hospital $j = 1, \ldots, J$.
- we can use the data y_{ij} to estimate aspects of the population distribution of θ.
- If furthermore, we approximation the population distribution by a parametric family, the corresponding parameters are called **hyperparameters**.

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- The corresponding posterior is $\theta \mid y = 4 \sim \text{Beta}(\alpha + 4, \beta + 10)$.
- So far, the values for α and β are arbitrary.
- If we have a historical records of previous experiments, we can have better choices for α and β if we interpret the prior distribution as the population distribution.

Example: Analysis using Historical Data

Previous experiments:

0/20	0/20	0/20	0/20	0/20	0/20	0/20	0/19	0/19	0/19
0/19	0/18	0/18	0/17	1/20	1/20	1/20	1/20	1/19	1/19
1/18	1/18	2/25	2/24	2/23	2/20	2/20	2/20	2/20	2/20
2/20	1/10	5/49	2/19	5/46	3/27	2/17	7/49	7/47	3/20
3/20	2/13	9/48	10/50	4/20	4/20	4/20	4/20	4/20	4/20
4/20	10/48	4/19	4/19	4/19	5/22	11/46	12/49	5/20	5/20
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- For $j = 1, \ldots, 70$ experiments, we observed y_j out of n_j rats with the symptom.
- The estimated mean and standard deviation for y_j/n_j are 0.136 and 0.103.

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- For $j = 1, \ldots, 70$ experiments, we observed y_j out of n_j rats with the symptom.
- The estimated mean and standard deviation for y_j/n_j are 0.136 and 0.103.
- We may choose the hyperparameters (α, β) by (**Variance is overestimated!!**)

$$\frac{\alpha}{\alpha+\beta} = 0.135, \quad \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.103^2.$$

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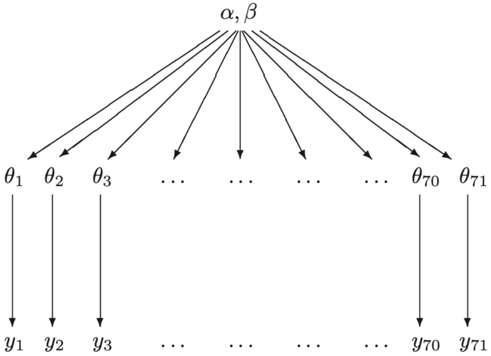

Note: the calculation demonstrated here is not a Bayesian calculation!

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We may elaborate those factors into a more complicated model.

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- The data is used twice: (1) estimating the prior (2) computing each's posterior.
- We ignored the uncertainty in estimating α and β. (In oppose to Bayesian inference, where the posterior measures uncertainty.)
- The prior distributions should be known **before** observing any data. Shall we really estimate them?

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- It definitely makes sense to estimate the population distribution from all the data than to estiamte them separately.
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- In order to retain the advantage of the hierarchical model and to get rid of the aforementioned trouble, we will build a full probability model for all parameters.
- The analysis using the data to estimate the prior parameters, which is sometimes called **empirical Bayes**, can be viewed as an approximation to the complete hierarchical Bayesian analysis.

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- If there is no additional information other than the observations y_j's, we assume the **exchangeability** of the parameters, that is

$$p(\theta_1, \ldots, \theta_J) \sim p(\theta_{\pi(1)}, \ldots \theta_{\pi(J)}),$$

for any permutation $\pi : \{1, \ldots, J\} \to \{1, \ldots, J\}$.

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- Or in other words,

$$p(\theta_1, \ldots, \theta_J) = \int \left(\prod_{j=1}^{J} p(\theta_j \mid \phi) \right) p(\phi) d\mu(\phi)$$

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Back to the rat tumor example.

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- If each experiment j is equiped with covariate x_j, we assume

$$\phi \sim p(\phi), \quad \theta_1, \dots, \theta_{70} \mid \phi, x_1, \dots, x_{70} \sim \prod_{j=1}^{70} p(\theta_j \mid \phi, x_j)$$

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- Now the complete model is

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- We often adopt a hybrid approach both analytically and numerically to conduct Bayesian inference.
- **Step 1 (analytic)**: get the marginal posterior distribution for ϕ.
- **Step 2 (numerical)**: draw samples of $(\phi, \theta_1, \ldots, \theta_J)$ from the joint posterior distribution.

Hierarchical Modeling

Step 1 Procedure:

1. Get the posterior in proportinal form:

$$p(\phi, \theta_1, \ldots, \theta_J \mid y_1, \ldots, y_J) \propto p(\phi) \prod_{j=1}^{J} p(y_i \mid \theta_j) p(\theta_j \mid \phi)$$

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2. Determine the conditional posterior distribution of $(\theta_1, \ldots, \theta_J)$:

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3. Determine the marginal posterior distribution of ϕ by

$$p(\phi \mid y_1, \ldots, y_n) = \frac{p(\phi, \theta_1, \ldots, \theta_J \mid y_1, \ldots, y_n)}{p(\theta_1, \ldots, \theta_J \mid \phi, y_1, \ldots, y_n)} \propto [A(\phi, y_1, \ldots, y_J)]^{-1}$$

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Step 1 is analytical because $p(\theta_j \mid \phi)$ is chosen conjugate to $p(y_j \mid \theta_j)$.

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2. Draw samples of $(\theta_1, \ldots, \theta_J)$ from the conditional distribution $p(\theta_1, \ldots, \theta_J \mid \phi, y_1, \ldots, y_J)$. This step can be done coordinate-wise because

$$p(\theta_1, \ldots, \theta_J \mid \phi, y_1, \ldots, y_J) \propto \prod_{j=1}^{J} p(y_i \mid \theta_j) p(\theta_j \mid \phi)$$

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  - to predict a new observation for experiment j: draw new $\tilde{y}_j$ given a sample of θ_j.

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  - to predict a new observation for experiment j: draw new $\tilde{y}_j$ given a sample of θ_j.
  - to predict a new observation for a new experiment:
    (1) draw a new $\tilde{\theta}$ given a sample of ϕ
    (2) draw a new $\tilde{y}$ given $\tilde{\theta}$.

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- Then the marginal posterior distribution for α, β is

$$\begin{aligned} p(\alpha, \beta \mid y_1, \ldots, y_J) &= \frac{p(\alpha, \beta, \theta_1, \ldots, \theta_J \mid y_1, \ldots, y_J)}{p(\theta_1, \ldots, \theta_J \mid \alpha, \beta, y_1, \ldots, y_J)} \\ &\propto p(\alpha, \beta) \prod_{j=1}^{J} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+y_j)\Gamma(\beta+n-y_j)}{\Gamma(\alpha+\beta+n)} \end{aligned}$$

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- It is difficult to calculate the Fisher information matrix for α, β. Therefore, we choose the prior in an ad-hoc way:

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- Remark: assigning a uniform prior on the natural transformed scale results in an improper posterior distribution

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- Let $y = \{y_{ij} : i = 1, \dots, n_j; j = 1, \dots, J\}$, $\theta = (\theta_1, \dots, \theta_J)$.
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- The joint posterior is

$$p(\mu, \tau, \theta \mid y) \propto p(\tau)\tau^{-J} \exp\left\{-\frac{1}{2}\sum_{j=1}^{J}\frac{(\mu - \bar{y}_j)^2}{\tau^2 + \sigma_j^2}\right\} \prod_{j=1}^{J} \exp\left\{-\frac{1}{2V_j}(\theta_j - \hat{\theta}_j)^2\right\}$$

$$\hat{\theta}_j = \frac{\frac{\mu}{\tau^2} + \frac{\bar{y}_j}{\sigma_j^2}}{\frac{1}{\tau^2} + \frac{1}{\sigma_j^2}}, \quad V_j = \frac{1}{\frac{1}{\tau^2} + \frac{1}{\sigma_j^2}}$$

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- The conditional posterior for θ is therefore

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- Hence, the marginal posterior for μ, τ is

$$p(\mu, \tau \mid y) = \frac{p(\mu, \tau, \theta \mid y)}{p(\theta \mid \mu, \tau, y)} \propto p(\tau) \left(\prod_{j=1}^{J} (\tau^2 + \sigma_j^2)\right)^{-1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{J} \frac{(\mu - \bar{y}_j)^2}{\tau^2 + \sigma_j^2}\right\}$$

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