STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 10: Simple Linear Regression

Chencheng Cai

Washington State University

Simple Linear Regression

Regression is a statistical method for estimating the relationships among variables. THe simpest form of regression is **simple linear regression**:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$

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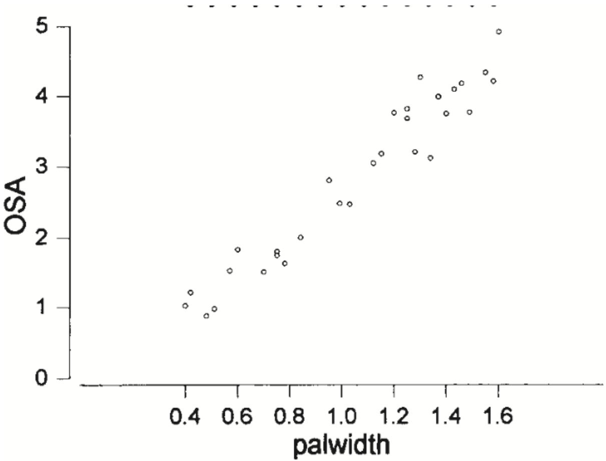
- y_i is the response variable (dependent variable).
- x_i is the predictor variable (independent variable).
- β_0 is the intercept.
- β_1 is the slope.
- ϵ_i is the error term.

Example

- y: ocular surface area
- x: width of the palprebal fissure



Assumptions

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$

- Linearity: The relationship between x and y is linear.
- Independence: The errors are independent.
- Normality: The errors are normally distributed.
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For short, the LINE assumptions give:

$$y_i = \beta_0 + \beta_1 x_i + N(0, \sigma^2) \quad \forall i$$

Violations of Assumptions

- Linearity: Nonliear regression model.
- Independence: Structural equation model (SEM) in econometrics.
- Normality: ϵ_i could have a heavy-tailed distribution.
- Equal variance: Heteroscedasticity.

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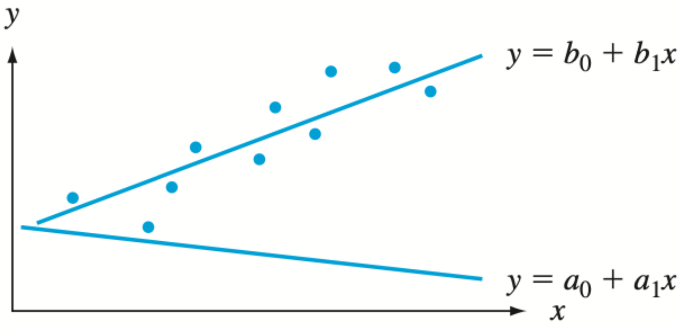
If we get the estimated coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$,

- The **fitted value** for y_i is $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$.
- The **residual** for y_i is $\hat{\epsilon}_i = y_i - \hat{y}_i$.

Estimation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$

Given the data points



we want to find the line that **best fits** the data points.

Ordinary Least Squares

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- The OLS estimates are the values of β_0 and β_1 that minimize the RSS:

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{\beta_0, \beta_1} \ \mathrm{RSS}(\beta_0, \beta_1)$$

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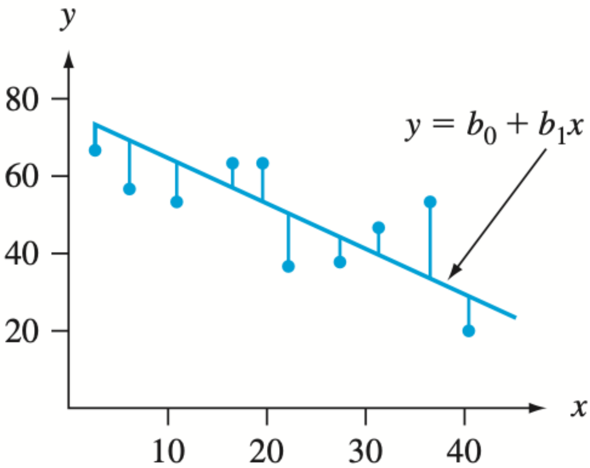

OLS

In order to minimize the RSS, we first compute its partial derivatives.

$$\mathrm{RSS}(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial \mathrm{RSS}}{\partial \beta_0} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = -2 \sum_{i=1}^{n} y_i + 2N\beta_0 + 2\beta_1 \sum_{i=1}^{n} x_i$$

$$\frac{\partial \mathrm{RSS}}{\partial \beta_1} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) x_i = -2 \sum_{i=1}^{n} y_i x_i + 2\beta_0 \sum_{i=1}^{n} x_i + 2\beta_1 \sum_{i=1}^{n} x_i^2$$

To find the minimum, we set the partial derivatives to zero.

OLS

The **estimating equations** for OLS are:

$$0 = -2\sum_{i=1}^{n} y_i + 2n\beta_0 + 2\beta_1 \sum_{i=1}^{n} x_i \quad (1)$$

$$0 = -2\sum_{i=1}^{n} y_i x_i + 2\beta_0 \sum_{i=1}^{n} x_i + 2\beta_1 \sum_{i=1}^{n} x_i^2 \quad (2)$$

OLS

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Compute $(1) \times \sum_i x_i - (2) \times n$:

$$0 = 2n\sum_i x_i y_i - 2\sum_i x_i \sum_i y_i + \left(\left(\sum_i x_i \right)^2 - n\sum_i x_i^2 \right) \beta_1 .$$

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$$\implies \hat{\beta}_1 = \frac{\sum_i x_i y_i - n^{-1}\sum_i x_i \sum_i y_i}{\sum_i x_i^2 - n^{-1}\left(\sum_i x_i\right)^2}.$$

OLS

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- The numerator is

$$\sum_i x_i y_i - n^{-1} \sum_i x_i \sum_i y_i = S_{xy} = \sum_i (y_i - \bar{y})(x_i - \bar{x})$$

- The denominator is

$$\sum_i x_i^2 - n^{-1} \left(\sum_i x_i \right)^2 = S_{xx} = \sum_i (x_i - \bar{x})^2$$

OLS

Therefore,

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}$$

with

$$S_{xy} = \sum_i (y_i - \bar{y})(x_i - \bar{x}) = \sum_i y_i x_i - n^{-1} \sum_i x_i \sum_i y_i$$

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From Eq. (1), we can get $\hat{\beta}_0$:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

OLS

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A quick formula in computing $\mathrm{RSS}(\hat{\beta}_0, \hat{\beta}_1)$ is

$$\mathrm{RSS}(\hat{\beta}_0, \hat{\beta}_1) = S_{yy} - \hat{\beta}_1 S_{xy} = S_{yy} - \hat{\beta}_1^2 S_{xx},$$

where

$$S_{yy} = \sum_i (y_i - \bar{y})^2 = \sum_i y_i^2 - n^{-1} \left(\sum_i y_i \right)^2.$$

OLS

Summary for OLS estimators:

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\sigma}^2 = \frac{\mathrm{RSS}(\hat{\beta}_0, \hat{\beta}_1)}{n-2} = \frac{S_{yy} - \hat{\beta}_1 S_{xy}}{n-2}$$

Example (Textbook Example 12.8)

x	12	30	36	40	45	57	62	67	71	78	93	94	100	105
y	3.3	3.2	3.4	3.0	2.8	2.9	2.7	2.6	2.5	2.6	2.2	2.0	2.3	2.1

Some statistics:

$$n = 14 \qquad \sum x_i = 890 \qquad \sum x_i^2 = 67182$$

$$\sum y_i = 37.6 \qquad \sum y_i^2 = 103.54 \qquad \sum x_i y_i = 2234.30$$

Example (Textbook Example 12.8)

We can compute the following statistics:

$$S_{xx} = 10603.43, \quad S_{xy} = -155.99, \quad S_{yy} = 2.557$$

The estimators are

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{-155.99}{10603.43} = -0.0147$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \frac{37.6}{14} - (-0.0147) \times \frac{890}{14} = 3.62$$

$$\hat{\sigma}^2 = \frac{S_{yy} - \hat{\beta}_1 S_{xy}}{n-2} = \frac{2.557 - (-0.0147) \times (-155.99)}{14-2} = 0.022$$

Properties of OLS Estimators

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$$S_{xy} = \sum x_i y_i - n^{-1} \sum x_i \sum y_i = \sum_i \left[(x_i - \bar{x})\, y_i \right]$$

The highlighted y_i's are the only random variables and we have

$$y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2),$$

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- Now we have

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \sim N(\beta_1, \sigma^2 S_{xx}^{-1})$$

Properties of OLS Estimators

- For the intercept estimator, we have

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- For the variance estimator, we have

$$E(\hat{\sigma}^2) = \sigma^2.$$

Properties of OLS Estimators

Summary:

- All OLS estimators are **unbiased**:

$$E(\hat{\beta}_0) = \beta_0$$
$$E(\hat{\beta}_1) = \beta_1$$
$$E(\hat{\sigma}^2) = \sigma^2$$

- The estimated **standard errors (se)** of the estimators are:

$$\widehat{se}(\hat{\beta}_0) = \sqrt{(n^{-1} + \bar{x}^2 S_{xx}^{-1})\hat{\sigma}^2}$$
$$\widehat{se}(\hat{\beta}_1) = \sqrt{S_{xx}^{-1}\hat{\sigma}^2}$$
$$\widehat{se}(\hat{\sigma}^2) = \sqrt{\frac{2\hat{\sigma}^4}{n-2}}$$