STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 2: Point Estimation

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Sample Mean

Proposition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean μ and standard deviation σ. Then

- $E(\bar{X}) = \mu$
- $Var(\bar{X}) = \frac{\sigma^2}{n}$

In addition, with $T = X_1 + \cdots + X_n$, we have

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Interpretation:

The sample mean's expectation is the population mean, and its variance is the population variance divided by the sample size.

Sample Mean — Concepts

- **Population**: In statistics, a population is the entire pool from which a statistical sample is drawn. It is the complete set of individuals or objects that we are interested in.
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- **Random Sample**: A random sample is a sample in which each individual or object in the population has an equal chance of being selected.
- An alternative expression is
  $X_1, X_2, \ldots, X_n$ *are independent and identically distributed (i.i.d.) random variables with mean* μ *and variance* σ^2*.*

Sample Mean — Justification

- By linearity of expectation, we have

$$E(T) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = n\mu.$$

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- Since $\bar{X} = T/n$, we have

$$E(\bar{X}) = E(T/n) = E(T)/n = \mu$$

and

$$Var(\bar{X}) = Var(T/n) = Var(T)/n^2 = \sigma^2/n.$$

Example: Bernoulli and Binomial

Suppose we have an unfair coin whose probability of landing heads is p.
We toss the coin n times and let X_i be the indicator of the i-th toss.

- $X_i = 1$ if the i-th toss is head
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X_i follows a **Bernoulli distribution** with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$.

- $E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = p$
- $Var(X_i) = E(X_i^2) - [E(X_i)]^2 = E(X_i) - [E(X_i)]^2 = p - p^2 = p(1-p)$

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Similary, let $\bar{X} = T/n$ be the proportion of heads from n tosses. Then

- $E(\bar{X}) = E(X_i) = p$
- $Var(\bar{X}) = Var(X_i)/n = p(1-p)/n$

Normal Population Distribution

Proposition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a **normal** *distribution with mean μ and standard deviation σ. Then for any n, $\bar{X}$ is normally distributed with mean μ and variance σ^2/n.*

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A random variable X is said to have a normal distribution with mean μ and variance σ^2, denoted by $N(\mu, \sigma^2)$, if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

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If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$c_1 X_1 + c_2 X_2 \sim N(c_1\mu_1 + c_2\mu_2, c_1^2\sigma_1^2 + c_2^2\sigma_2^2).$$

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$$E(T) = 12 \times 53 = 636, \quad Var(T) = 12 \times 0.3^2 = 1.08.$$

The probability that the total weight of the dozen eggs is between 635 and 640 is

$$P(635 < T < 640) = P\left(\frac{635 - 636}{\sqrt{1.08}} < Z < \frac{640 - 636}{\sqrt{1.08}}\right) = P(-0.96 < Z < 3.85) = 0.8315,$$

where $Z \sim N(0, 1)$ follows the **standard normal distribution**.

Central Limit Theorem

Theorem (Central Limit Theorem (CLT))

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2. Then if n is sufficiently large, $\bar{X}$ has approximately a normal distribution with mean μ and variance σ^2/n, and T also has approximately a normal distribution with mean $n\mu$ and variance $n\sigma^2$. The larger the value of n, the better the approximation.

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A shorter version:

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as } n \to \infty$$

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- **Proof**: The proof of the Central Limit Theorem is beyond the scope of this course. It is a result from the characteristic function and the Lévy's convergence theorem.
- **Rule of Thumb**: $n \geq 30$ is often considered as a sufficiently large sample size.

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From the central limit theorem,

$$\bar{X} \approx N(p, p(1-p)/n) \sim N(0.5, 0.0025).$$

Therefore, $T = n\bar{X} \sim N(50, 25)$.

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Therefore, $T = n\bar{X} \sim N(50, 25)$.
We have

$$P(40 < Y < 60) = P(40 < T < 60) \approx P\left(\frac{40-50}{\sqrt{25}} < Z < \frac{60-50}{\sqrt{25}}\right) = P(-2 < Z < 2)$$

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