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CONTINUATION AND DIRECT SOLUTION OF THE FLUTTER EQUATION

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Abstract—A new formulation of the flutter equation allowing efficient solutions both by a continuation and a direct method is herewith presented.

The continuation method differentiates the flutter equation with respect to the speed giving rise to a system of differential equations whose solution permits an easy and efficient tracking of the aeroelastic modes, frequencies and approximate dampings.

If only the flutter point, i.e. its mode, speed and frequency, is required, a direct solution of the nonlinear algebraic system of the flutter equation is performed.

Some examples of practical applications are briefly presented and discussed.

NOMENCLATURE

- $[M]$ generalized mass matrix
- $[C]$ generalized damping matrix
- $[K]$ generalized stiffness matrix
- $\{F_a\}$ vector of generalized aerodynamic forces
- $\{\bar{q}\}$ vector of generalized coordinates
- $\{q\}$ amplitude of $\{\bar{q}\}$
- ρ air density
- V asymptotic speed
- $[A]$ generalized aerodynamic stiffness
- ω circular frequency
- s damping coefficient
- c mean aerodynamic reference chord
- a speed of sound
- $\mathcal{M}$ Mach number
- k reduced frequency
- $Re(\ )$ real part of a complex number
- $Im(\ )$ imaginary part of a complex number
- $(\dot{\ })$ indicates derivation with respect to time
- $(')$ indicates derivation with respect to V

1. INTRODUCTION

Aeroelastic constraints face the designer of modern aircraft from the very beginning of the design process. Therefore the engineer must be able to deal in an efficient way with all the aspects of aeroelasticity, and particularly flutter, in order to proceed with the design layout quickly and economically.

Usually from the start the aeroelastic analysis takes into account several distinct structural configurations and the interaction of servocontrols with the deformation modes of the aircraft, according to the scheme indicated in Fig. 1.

The phases involved in this scheme can be combined in a completely integrated procedure of optimum structural design under aeroelastic constraints or linked together by an engineering team that interacts with the computer and controls the looping as the design goes on[1].

This paper presents a method of calculation for solving the portion of the flutter design that more directly concerns the search of the stability conditions of the differential equations constituting the mathematical model of the flutter phenomenon.

This method gives no limitation to the modelling of the flutter problem since it can take into account all the equations coming from the full application of the steps indicated in Fig. 1. The determination of the flutter condition results from the tracking at increasing speed, of dampings, frequencies and aeroelastic modes in a very efficient computational process.

The ability of determining the aeroelastic modes is stressed as this can give a physical insight into the way in which energy is exchanged among the modes themselves, a feature that may be useful to the designer, should he wish to modify the system being analyzed[2].

Moreover the scheme of solution easily leads to a formulation that provides the flutter speed, frequency and mode directly, thus avoiding the costly tracking if this is not needed.

2. FLUTTER EQUATION AND SOLUTION METHODS

2.1 *Problem formulation*

If $\{\bar{q}\}$ are the n generalized coordinates, a union of structural degrees of freedom and variables inherent to servocontrols, the equation of motion, using a matrix notation, can be written as

$$[M]\{\ddot{\bar{q}}\}+[C]\{\dot{\bar{q}}\}+[K]\{\bar{q}\}=\{F_a\} \tag{1}$$

where $[M]$ defines the inertia characteristics, $[C]$ is the matrix of damping, $[K]$ is the stiffness matrix and $\{F_a\}$ is the vector representing the generalized aerodynamic forces.

Following a sufficiently general scheme, the unsteady aerodynamic forces, due to a harmonic motion, are linearized with respect to $\{\bar{q}\}$, i.e.

$$\{F_a\}=\frac{\rho V^2}{2}[A]\{\bar{q}\} \tag{2}$$

where ρ indicates the air density, V the asymptotic speed of the free stream and $\{\bar{q}\}$ is harmonic, i.e.

$$\{\bar{q}\}=\{q\}\,e^{j\omega t} \tag{3}$$

where ω is the angular frequency and t is the time. The coefficients of the matrix $[A]$ are functions only of the Mach number, $\mathcal{M}$ and of the reduced frequency $k=$

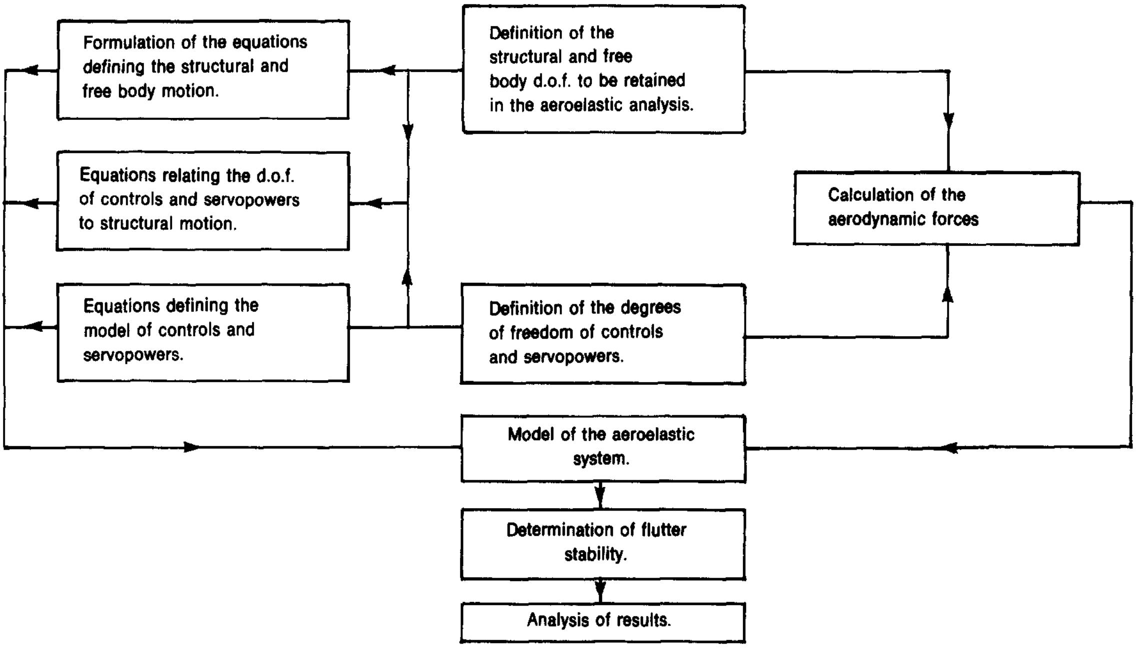


Fig. 1. Aeroelastic analysis scheme.

$\omega c/V$, where c stands for the aerodynamic reference chord.

This can be strictly applied only at the limit of the dynamic stability of the system, that is at the critical flutter speed, while as a rule the motions are damped† when the speed V varies within the flight envelope.

Although a general linearized theory, which describes the unsteady aerodynamic forces due to generally damped motions, is now foreseen[3, 4], it is considered sufficient to formulate $\{F_a\}$, having assumed $\{\bar{q}\} = \{q\}\,e^{pt}$, as

$$\{F_a\} = \frac{\rho V^2}{2}[A(\mathcal{M}, k)]\{q\}\,e^{pt} \tag{4}$$

where p is generally complex $p = s + j\omega$ and the coefficients of $[A]$ remain those of a pure harmonic motion of circular frequency ω. This hypothesis is quite acceptable when the damping is small or zero, that is near to the stability limit, while it is believed a sufficient approximation far from the flutter condition.

Therefore it is possible to search solutions of the eqn (1) within those of the type $\{\bar{q}\} = \{q\}\,e^{pt}$. Substituting we obtain

$$\left([M]p^2 + [C]p + [K] - \frac{\rho V^2}{2}[A]\right)\{q\} = 0. \tag{5}$$

The various matrices appearing in (5) are obtained from previous procedures (Fig. 1) and their structure depends on the methods employed to obtain them, such as numerical models, vibration tests, different damping idealizations and the technique adopted to represent control and servo-power systems. Therefore they can be of different kind.

In particular the algebraic equations relating structural motion to controls degrees of freedom are included by means of Lagrange multipliers. Servocontrols transfer functions can be simply represented by an expansion in quadratic polynomials of p, eventually through the introduction of dummy generalized coordinates. Each polynomial constitutes a further equation of (5)[5].

If we call $[F]$ the matrix of the coefficients of $\{q\}$, eqn (5) becomes

$$\{r\} = [F]\{q\} = 0. \tag{6}$$

The expression (6), which is a flutter equation, is a linear homogenous system in $\{q\}$ and nonlinear in p. In order that the system can have solutions of $\{q\}$ different from zero, it is well known that the determinant of $[F]$ must be zero. The stability of the system will be determined by the sign of the real part s, which we call "damping".

From this formulation the solution of the flutter equation may be considered as a non standard eigenvalue problem of a complex non hermitian matrix with a nonlinearity in p, which is also present in the aerodynamic terms. This approach to the problem is called the p-k method[6].

The direct application of the several standard methods which determine the eigenvalues and eigenvectors, is prevented in any scheme by the coupling of the terms of p with V in the aero-dynamic stiffness, unless $[C] = 0$[7–9]. This condition greatly limits the generality of the method in view of the presentation here made, since servocontrols transfer functions and any kind of viscous dampers are not allowed.

Anyway, in such a case, flutter equation can be rewritten in a form that, once the reduced frequency parameter has been fixed and then the aerodynamic matrix calculated, a standard eigenvalue problem is solved. Each calculated eigenvalue along with the previously assumed reduced frequency makes possible to compute a point in the speed-damping plane. But as the computations are repeated at various reduced frequencies and since it is

†The expression "damped motions" is here used to denote oscillating motions with an amplitude that varies exponentially. So they are properly called damped vibrations or diverging ones according to the sign of damping.

not certain that the order of extraction of the eigenvalues is always the same, the problem of tracking them automatically, in a continuous manner, as function of speed, arises. Furthermore this technique is not directly usable when variation of Mach number has to be taken into account, i.e. when the tracking of damping and speed must be performed at constant density-altitude.

An efficient method consists in following one eigenvalue at a time by using, at several predetermined speeds, an iterative process which obtains the solution, starting from the preceding one, through, by example, the search of the roots of $\det[F]=0$ [6, 9], or by the inverse power method. The $[A]$ matrix is calculated at every step in correspondence to the imaginary part of the approximated eigenvalue. Such formulation gives no limitations to the matrices involved.

A new algorithm for dealing with this problem was proposed in an authors' previous paper [10], and consisted in solving the system (6) directly as a nonlinear system.

As a matter of fact, any component of $\{q\}$ could be fixed *a priori* so that system (6) becomes a system of n nonlinear equations with $n-1$ unknowns in $\{q\}$ and a further one in p [11]. Unfortunately, in doing so, we could have fixed the value of a component of $\{q\}$ which can turn out to be zero or nearly zero, thus running in a very ill conditioned system of equations. Then we prefer to add to (6) an equation of normalization of the vector $\{q\}$ in order to obtain a nonlinear system with $n+1$ unknowns, p and $\{q\}$, which may be solved in different ways at different fixed speeds, obtaining eigenvalues and eigenvectors all at once.

The method, being presented now, retains the basic idea but devises a more straightforward scheme of solution, allowing an easier and more efficient use of the method itself.

Furthermore if the above nonlinear system is written at the fluttering speed that is at $s=0$, we can solve it directly for obtaining $\{q\}$, ω and V, without any tracking of p and $\{q\}$ for various speeds. Thus the fluttering mode, speed and frequency are determined all at once. It is very useful in order to determine variations in flutter, occurring owing to small changes in design and/or configuration of a given airplane and is very suitable in order to be incorporated in an automatic design optimization procedure [1].

The possibility of devising, within the same scheme, both an efficient tracking procedure and an easy direct solution of the flutter equation is a rather unique feature of this formulation.

2.2 *Solution by a continuation method*

Let us write the system (6) adding an equation of normalization which we choose in the form

$$N=\frac{1}{2}\{q\}^T[W]\{q\}-1=0 \tag{7}$$

where $[W]$ is a suitable weighting symmetric matrix.

If a value $\bar{V}$ is fixed, the following system of nonlinear equations in p and $\{q\}$ is obtained

$$\begin{cases}\{r\}=[F(p,\bar{V})]\{q\}=0\\ N(\{q\})=0.\end{cases} \tag{8}$$

Now, instead of fixing V, we consider the speed as a parameter and differentiate (8) with respect to it; thus falling into a system of nonlinear differential equations in the form [12].

$$\begin{cases}\dfrac{\mathrm{d}\{r\}}{\mathrm{d}V}=\dfrac{\partial\{r\}}{\partial V}+\dfrac{\partial\{r\}}{\partial p}\dfrac{\mathrm{d}p}{\mathrm{d}V}+\dfrac{\partial\{r\}}{\partial\{q\}}\dfrac{\mathrm{d}\{q\}}{\mathrm{d}V}=0\\[2ex] \dfrac{\mathrm{d}N}{\mathrm{d}V}=\dfrac{\partial N}{\partial V}+\dfrac{\partial N}{\partial\{q\}}\dfrac{\mathrm{d}\{q\}}{\mathrm{d}V}=0\end{cases} \tag{9}$$

which can be written explicitly as

$$\begin{bmatrix} Re([F]) & -Im([F]) & Re(\{v\}) & (-Im(\{v\})+Re(\{w\}))\\ Im([F]) & Re([F]) & Im(\{v\}) & (Re(\{v\})+Im(\{w\}))\\ Re(\{n\}^T) & -Im(\{n\}^T) & 0 & 0\\ Im(\{n\}^T) & Re(\{n\}^T) & 0 & 0\end{bmatrix} \times\begin{Bmatrix}Re(\{q'\})\\ Im(\{q'\})\\ Re(p')\\ Im(p')\end{Bmatrix}=\begin{Bmatrix}Re(\{t\})\\ Im(\{t\})\\ 0\\ 0\end{Bmatrix} \tag{10}$$

or

$$[G]\{d'\}=\{z\}$$

where $Re(\,)$ indicates the real part and $Im(\,)$ the imaginary one of a complex number and where

$$\begin{aligned}\{v\}&=(2p[M]+[C],\{q\}\\ \{w\}&=\frac{1}{2}\rho c\bar{V}\frac{\partial[A]}{\partial k}\{q\}\\ \{t\}&=\left(\rho\bar{V}[A]-\frac{1}{2}\rho\omega c\frac{\partial[A]}{\partial k}+\frac{1}{2}\frac{\rho\bar{V}^2}{a}\frac{\partial[A]}{\partial\mathcal{M}}\right)\{q\}\\ \{n\}&=[W]\{q\}\end{aligned} \tag{11}$$

a being the speed of sound at the altitude-density corresponding to ρ.

In this way we put the search for the solution of (6) into a continuation form in a natural and straightforward manner. This greatly enhances the solution process of the flutter equation. Indeed if we are interested in tracking frequencies, dampings and modes numerically, at increasing speeds, from an initial known solution, the speed intervals should be, for reasons of economy, as great as possible within the limits required by a meaningful interpolation of the quantities needed at the flutter conditions, within an acceptable error.

The numerical method of solution adopted must match these requirements limiting the intervention of the user to the determination of the speed steps and range suitable to his problems.

For this reason a predictor-corrector scheme of integration presents itself as the most appropriate method since it is able to control automatically the error made in the calculation and then to reduce the speeds intervals to the suitable length.

Out of the many trials performed, the best technique has proved to be the subsequent one

(a) a Heun predictor in the form

$$\bar{V}=V+\frac{3}{4}\Delta V$$

$$\bar{p}=p_{(V)}+\frac{3}{4}\Delta Vp'_{(V)}$$

$$\{\bar{q}\} = \{q\}_{(V)} + \frac{3}{4}\Delta V\{q'\}_{(V)}$$

$$\begin{Bmatrix}\{q\}\\ p\end{Bmatrix}_{(V+\Delta V)} = \begin{Bmatrix}\{q\}\\ p\end{Bmatrix}_{(V)} + \Delta V\left(\frac{1}{3}\begin{Bmatrix}\{q'\}\\ p'\end{Bmatrix}_{(V,p,\{q\})} + \frac{2}{3}\begin{Bmatrix}\{q'\}\\ p'\end{Bmatrix}_{(\bar{V},\bar{p},\{\bar{q}\})}\right) \quad (12)$$

where ΔV is the speed increment.

(b) a corrector constituted by a fixed number of modified Newton–Raphson solutions of the system (6) plus eqn (7), i.e.

$$\begin{Bmatrix} Re(\{q\}) \\ Im(\{q\}) \\ Re(p) \\ Im(p) \end{Bmatrix}^{(i+1)}_{(V+\Delta V)} = \begin{Bmatrix} Re(\{q\}) \\ Im(\{q\}) \\ Re(p) \\ Im(p) \end{Bmatrix}^{(i)}_{(V+\Delta V)} - [G]^{-1}_{(\bar{V})} \begin{Bmatrix} Re(\{r\}) \\ Im(\{r\}) \\ Re(N) \\ Im(N) \end{Bmatrix}_{(V+\Delta V)} \quad (13)$$

starting from the predicted values of point (a) and using for the matrix $[G]$ the one used to calculate the derivatives for the predictor at the 3/4 of the increment of the speed interval.

It should be noted that in order to maintain the maximum efficiency, $[G]^{-1}$ is never obtained explicitly but only the triangular factors of $[G]$ are calculated. Furthermore even if in (10), $[G]$ is written for convenience as a real matrix, all the calculations on its part, containing $[F]$ as a submatrix, are performed in complex arithmetics. This to speed up the procedure as indicated in [10].

If the required precision is not achieved after the application of a fixed number of corrector steps, the interval is halved; if the precision is better, the interval is doubled never exceeding the one indicated by the user. This is easily implemented since the formulation of the predictor is a forward integration formula, which requires no tracking of the past values.

It should be noted that usually only one or two iterations for a single step of corrector are applied, as experience shows no gain in applying more iterations. Doubling of the intervals is made if the error is many times (50–100) lower than the allowed one in order to avoid the continuous switching from halving to doubling and viceversa.

The organization of the computing procedure is summarized in the flow-chart of Fig. 2.

We must note that in this method it is essential to start from a true initial solution at a given initial speed, but generally only a guess, the free vibration modes and frequencies, for most cases, are given. So before beginning the process the real solution is determined through a Newton–Raphson iteration.

By successive application of the integration process, all the speed range is explored; we can follow the behaviour of the different modes, one by one, at increasing speeds, thus the stability of the system can be fully ascertained. If flutter occurs, the flutter speed at which s is zero in anyone of the modes, is obtained by interpolating among the first values of V at which "damping" becomes positive and the preceding ones at which s is still negative. The same procedure enabled us to find the flutter frequency and mode.

It is noted that if it is thought, on the grounds of appropriate hypotheses or previous analyses, that some modes, which are significant for the formulation of the model, do not cause flutter, their evolution with the speed may be neglected with an obvious saving in calculating time.

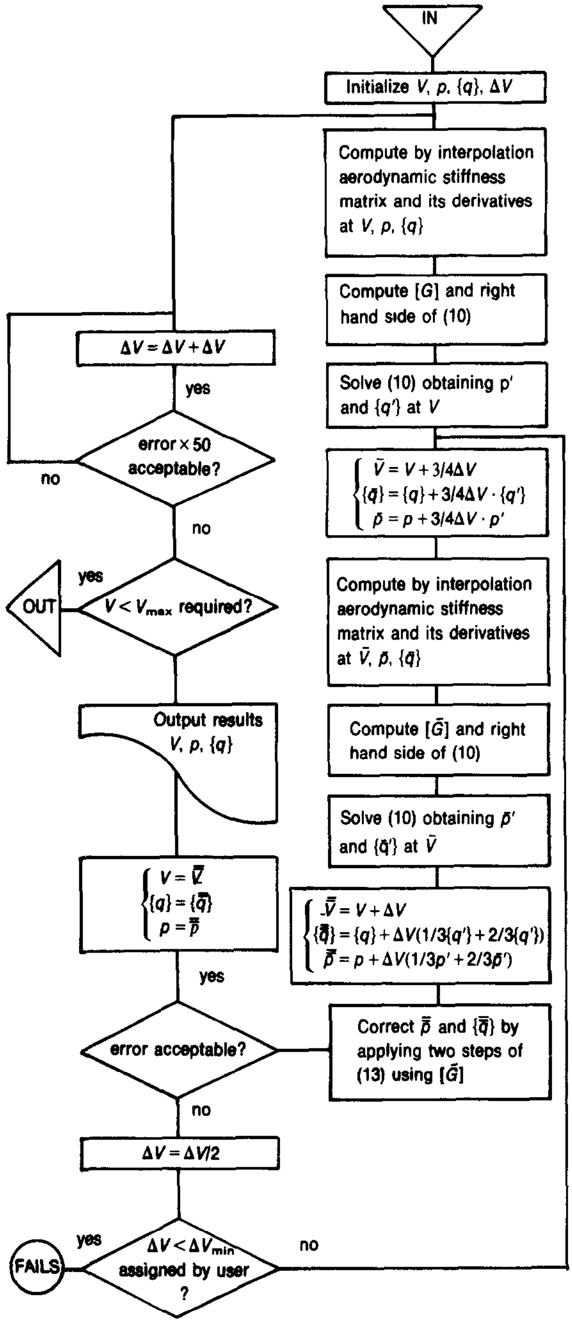


Fig. 2. Flow-chart of the tracking method used for each mode of interest.

The terms of the aerodynamic forces and those of their derivatives with respect to k and ω, must be known at any value of frequency and Mach number. On the contrary only the terms of the matrix $[A]$ are known at few values of $\mathcal{M}$ and ω, their calculation being one of the most expensive parts of a flutter integrated computer program.

In general the matrix $[A]$ and its derivatives, at any k or ω, can be obtained with a very good approximation, through a cubic spline method of interpolation [13, 14], since the variation of the aerodynamic matrices with the two parameters $\mathcal{M}$ and ω is smooth and well behaved.

Two methods of flutter calculation can be included: (a) tracking at constant Mach number; (b) tracking at constant density-altitude. The former requires the interpolation of the aerodynamic matrix only against the reduced frequency and is more efficient if a complete knowledge of the variations of the flutter quantities at various altitudes is needed. The latter requires the interpolation both with respect to the Mach number and to the reduced frequency thus making the interpolation itself more onerous, but it is readily applicable when the flutter behaviour has to be known only at few altitudes.

2.3 *Direct search of the flutter condition*

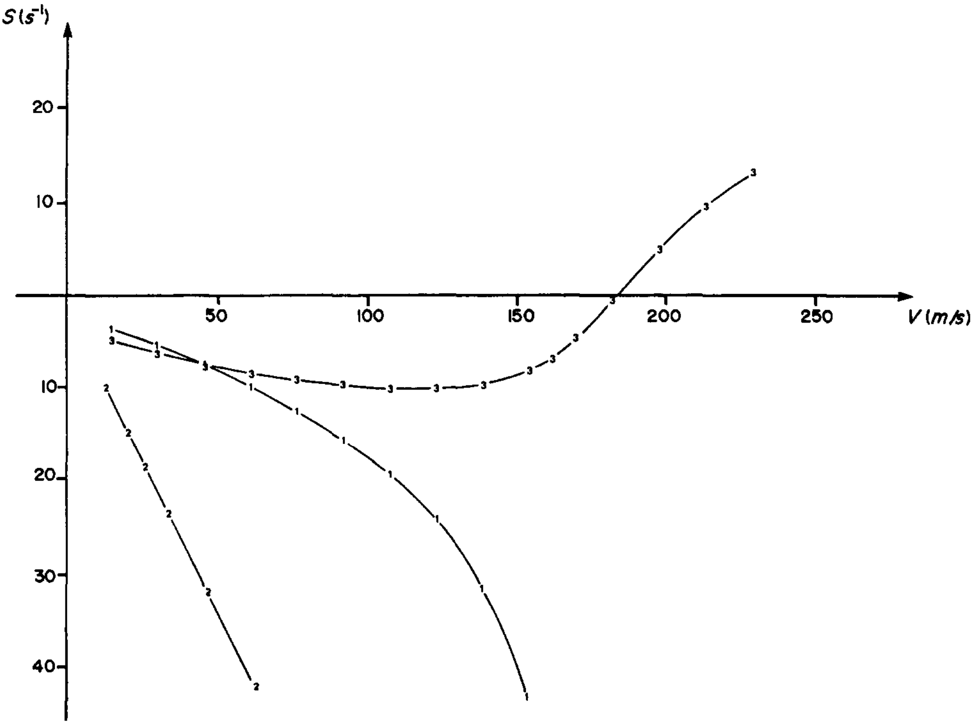
When flutter mode, speed and frequency are directly required, it is possible to avoid the tracking procedure previously described. Indeed at the flutter condition ($s = 0$) we can write (5) as

$$\left(-[M]\omega^2 + j[C]\omega + [K] - \frac{\rho V^2}{2}[A]\right)\{q\} = 0$$

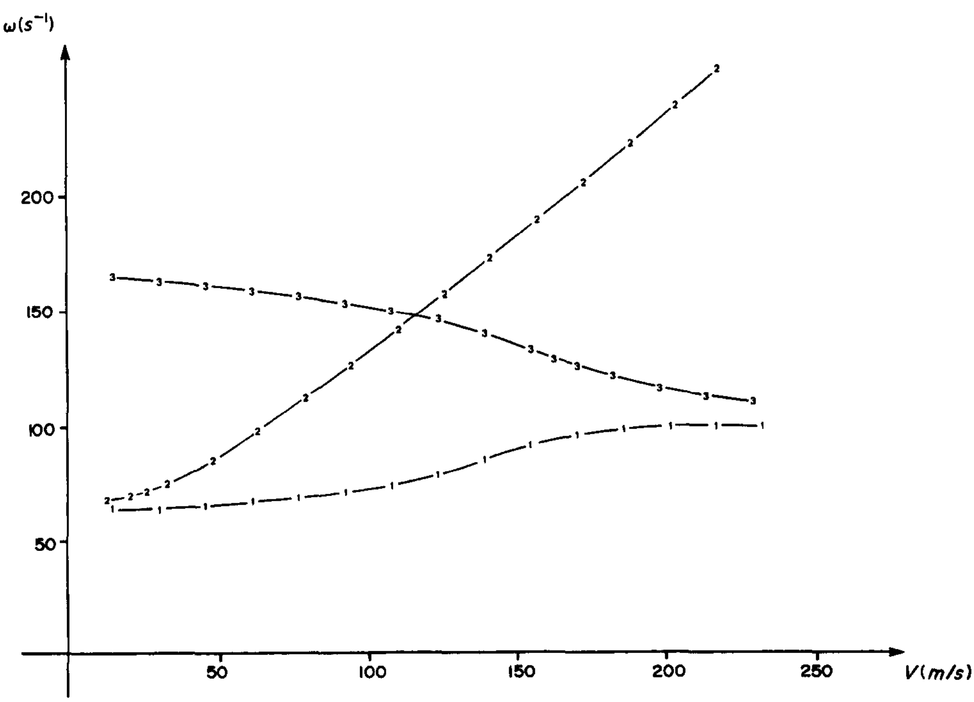
or

$$\{r\} = [F]\{q\} = 0. \tag{14}$$

This equation along with the normalizing one (7) takes

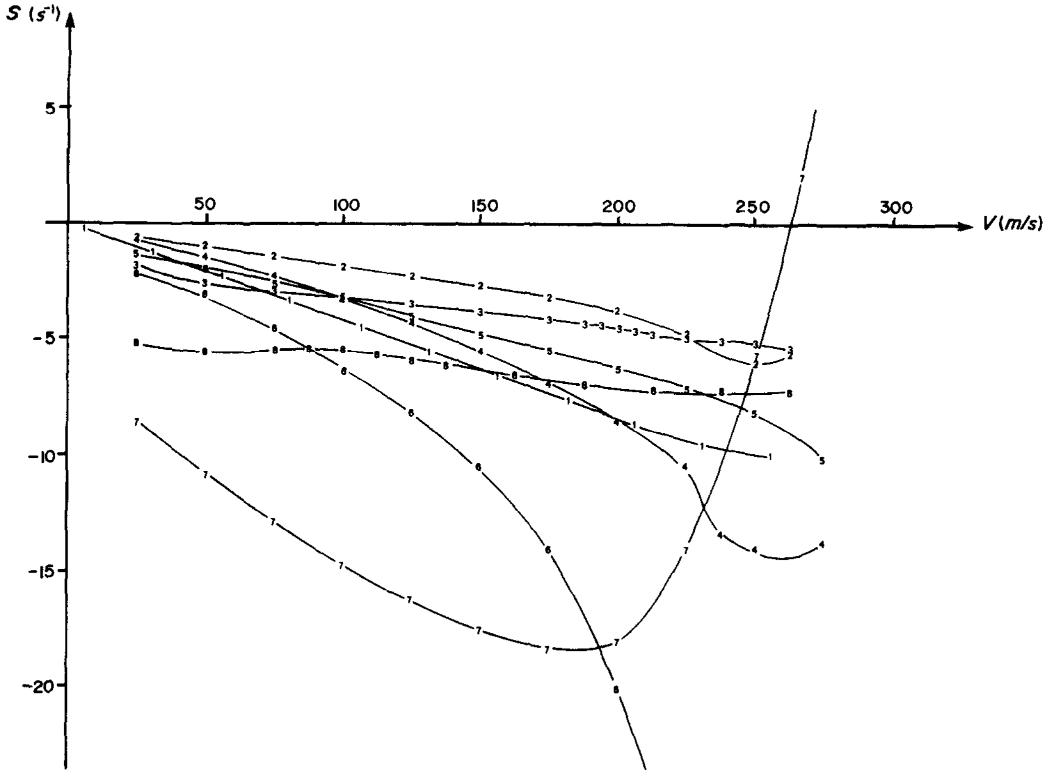


(a)

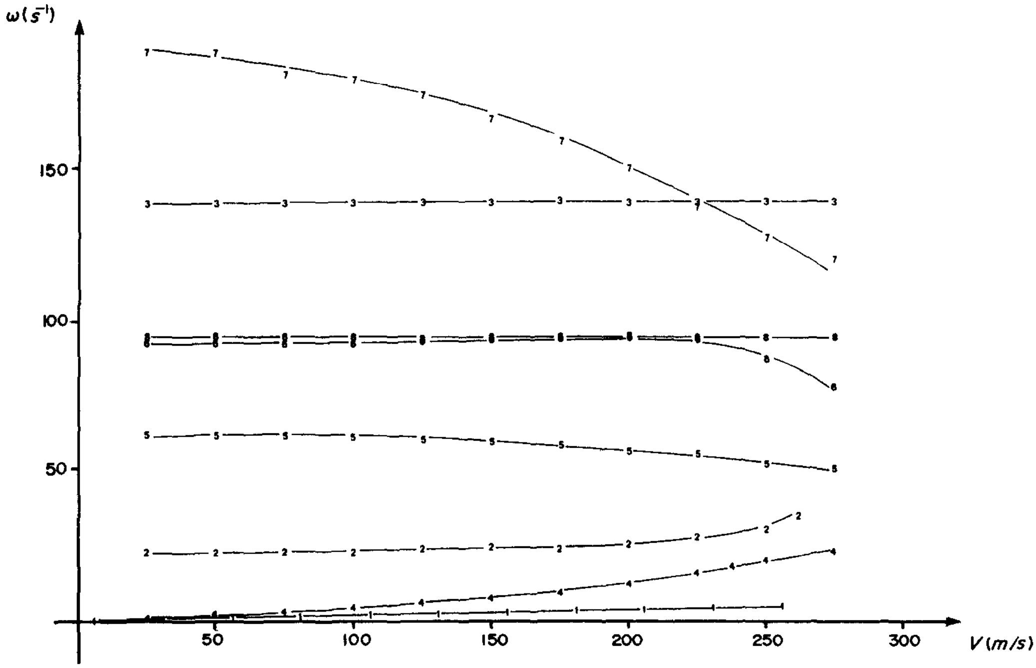


(b)

Fig. 3. Variation of s and ω vs V (3 d.o.f system).



(a)



(b)

Fig. 4. Variation of s and ω vs V (8 d.o.f system).

the place of the system (8) but the unknowns are now $\{q\}$, ω and V. Starting from an approximated solution, the iterative process calculates

$$\begin{aligned} \{q\}^{(i+1)} &= \{q\}^{(i)} - \Delta\{q\} \\ \omega^{(i+1)} &= \omega^{(i)} - \Delta\omega \\ V^{(i+1)} &= V^{(i)} - \Delta V \end{aligned} \tag{15}$$

by solving

$$\begin{bmatrix} Re([F]) & -Im([F]) & Re(\{v\}) & Re(\{t\}) \\ Im([F]) & Re([F]) & Im(\{v\}) & Im(\{t\}) \\ Re(\{n\}^T) & -Im(\{n\}^T) & 0 & 0 \\ Im(\{n\}^T) & Re(\{n\}^T) & 0 & 0 \end{bmatrix} \times \begin{Bmatrix} Re(\{\Delta q\}) \\ Im(\{\Delta q\}) \\ \Delta\omega \\ \Delta V \end{Bmatrix} = \begin{Bmatrix} Re(\{r\}) \\ Im(\{r\}) \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

where

$$\{v\} = \left(-2[M]\omega + j[C] + \frac{\rho}{2} c V \frac{\partial[A]}{\partial k}\right)\{q\}$$

$$\{t\} = \left(-\frac{\rho V^2}{2a}\frac{\partial[A]}{\partial \mathcal{M}} - \rho V[A] + \frac{1}{2}\rho\omega c\frac{\partial[A]}{\partial k}\right)\{q\} \tag{17}$$

until the increments (Δ) are as small as acceptable.

It can be noted that generally even a rough choice of V and ω along with a randomly generated $\{q\}$ gives full convergence.

If more flutter conditions are thought to exist it is useful to start from various V and ω within the field of the feasible ones[15].

This method can fruitfully complement the full tracking plot and is valuable in evaluating changes in flutter behaviour corresponding to small modifications in design and/or configuration since it converges very quickly in this case and it can be really useful in optimization codes with flutter constraints[1, 16].

In these cases the starting approximation can be given by exploiting the full tracking of p and $\{q\}$ for various V only once at the beginning.

The two capabilities, ρ = const. or $\mathcal{M}$ = const., apply to this method, too, along with the comments of the previous paragraph. Moreover if the procedure at $\mathcal{M}$ = const. is applied at increasing Mach numbers, a plot of the variations of the flutter speed with altitude is quickly obtained once the previous solution is adopted as starting point of the new one.

3. EXAMPLES OF CALCULATION

Two applications of the method are presented by way of examples.

The matrices $[M]$ and $[K]$ come from the analysis of a suitable structure and $[K]$ is retained complex by assuming a structural damping. $[C]$ is zero and $[A]$ is obtained by means of a doublet lattice method[17].

Figure 3 represents the variation of the frequency and damping with speed, for a three degrees of freedom system (wing bending, torsion, aileron modes). The system flutters in the classical bending-torsion mode. It is noted that two of the initial values present very close frequencies, nevertheless the method has no trouble in carrying on the solution without mixing the modes or jumping from one to another.

The second example (Fig. 4) retains eight degrees of freedom, six of them are symmetric vibration modes (fuselage bending, two wing bendings, wing torsion, bending and torsion of the stabilizer), and two are rigid body motions (heave and pitch). The fluttering mode in this case involves mainly bending and torsion of the stabilizer and the way in which these degrees of freedom are coupled at various speeds is shown in Fig. 5. The inclusion of rigid body motions causes no particular difficulties if the procedure is adequately started via a Newton–Raphson iteration as previously stated.

The flutter quantities were calculated for these two

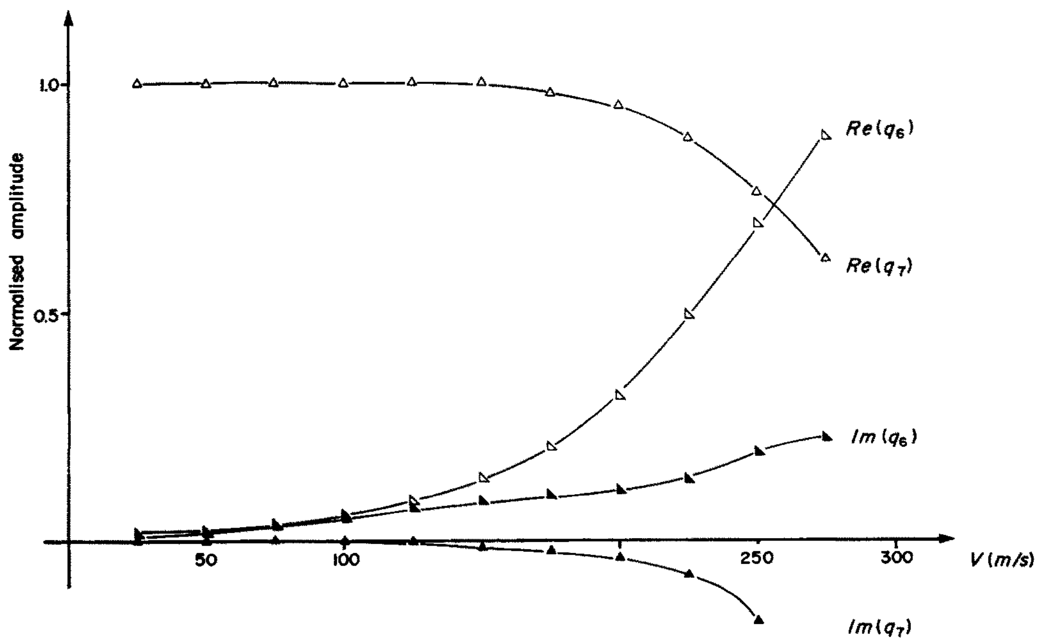


Fig. 5. Mixing of significant degrees of freedom vs V.

examples by the direct method starting with a frequency amid the minimum and maximum of the vibration modes chosen as degrees of freedom, a speed equal to the maximum allowed speed and a randomly generated aeroelastic mode. The results matched fully those coming from the tracking calculations.

4. CONCLUSIONS

The adoption of a continuation technique for the solution of the flutter stability problem, cast in the form of a non-linear system of equations, shows qualities of simplicity and straightforwardness in establishing the behaviour of the system, proceeding mode after mode among those of interest.

The predictor-corrector method assures a reliable solution through an efficient speed step control, allowing a completely automatic and economic formulation of the aeroelastic procedure.

The added direct search method may be helpful when only flutter frequency and speed are needed and their values are approximately known.

Within the limits of the approximation of the aerodynamic forces which have been adopted, different type of damping idealization as well as the inclusion of a control system or servo-power can easily be implemented.

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