1. Quaternion Definition

Euler angles are commonly used in aircrafts to track their yaw, pitch, and roll angles. However, Euler angles can potentially experience gimbal lock, in which one rotation axis is accidentally aligned with another rotation axis. This potential singularity is fine for aircrafts, since their motion are often very limited. However, a satellite rigid body motion has no limits. As a result, quaternions are used to track the attitude, or orientation, of the satellite with respect to the inertial frame. Quaternions, unlike Euler angles, have the benefit of being singularity-free. A quaternion is defined as:

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{\rho} \\ q_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{1}$$

Where **e** is the axis of rotation and θ is the angle of rotation from inertial to body frame. A quaternion in nature is a three-dimensional hyper-complex number with the form:

$$\boldsymbol{q} = q_1 i + q_2 j + q_3 k + q_4 \tag{2}$$

Just as one-dimensional complex number multiplication represents rotation in 2D, quaternion multiplication represents rotations in 3D. The rules of multiplication, discovered by Hamilton, are:

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \\ ij = k; jk = i; ki = j \end{gathered} \tag{3}$$

This is the right-hand rule for multiplication. Quaternion multiplication is represented by the symbol $\otimes$. To rotate a vector $\boldsymbol{v}$ about an axis $\boldsymbol{e}$ with an angle θ, the following equation is used:

$$\begin{bmatrix} \boldsymbol{v_{new}} \\ 0 \end{bmatrix} = \boldsymbol{q} \otimes \begin{bmatrix} \boldsymbol{v} \\ 0 \end{bmatrix} \otimes \boldsymbol{q^{-1}} \tag{4}$$

Note this operation is the only way to transform a pure quaternion into another pure quaternion. The conjugate quaternion (as well as the inverse quaternion) is defined as:

$$\boldsymbol{q^{-1}} = \boldsymbol{q}^* = \begin{bmatrix} -\boldsymbol{\rho} \\ q_4 \end{bmatrix} \tag{5}$$

Another interpretation is that equation (4) transforms a vector from body coordinates to inertial coordinates. Also, the inverse of a quaternion must obey the rule:

$$(\boldsymbol{p} \otimes \boldsymbol{q})^{-1} = \boldsymbol{q}^{-1} \otimes \boldsymbol{p}^{-1} \tag{6}$$

A quaternion essentially tracks the transformation between two different frames of reference. In the application of satellite attitude control, the quaternion represents the transformation between inertial frame and body frame, which is a frame attached to the satellite's rigid body. Another property of quaternions is the negative of a quaternion represents the same rigid body rotation:

$$\boldsymbol{q} \leftrightarrow -\boldsymbol{q} \tag{7}$$

Quaternions must obey the norm constraint of 1 in order to act as a rotation operator. This can be easily satisfied if the spin axis $\boldsymbol{e}$ is a unit vector.

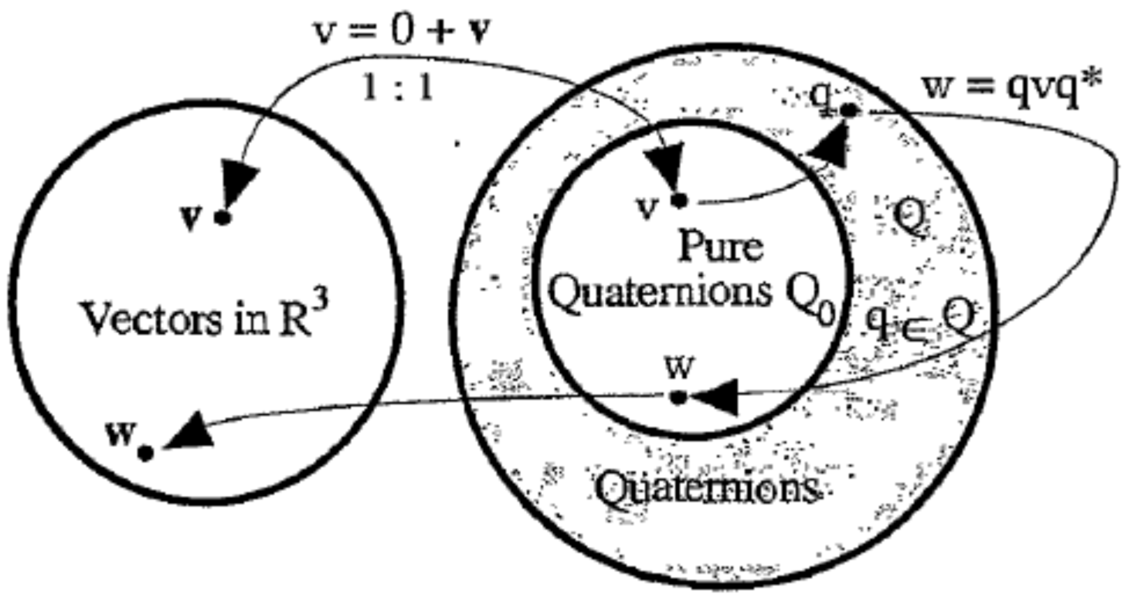

Figure 1. Quaternion Rotation [Kuipers, Quaternions and Rotation Sequences]

2. Vector Interpretation

The quaternion product can be broken down into vector cross product and dot product:

$$p \otimes q = \begin{bmatrix} \boldsymbol{p} \\ p_4 \end{bmatrix} \otimes \begin{bmatrix} \boldsymbol{q} \\ q_4 \end{bmatrix} = \boxed{\begin{bmatrix} p_4\boldsymbol{q} + q_4\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q} \\ p_4 q_4 - \boldsymbol{p} \bullet \boldsymbol{q} \end{bmatrix}} \tag{8}$$

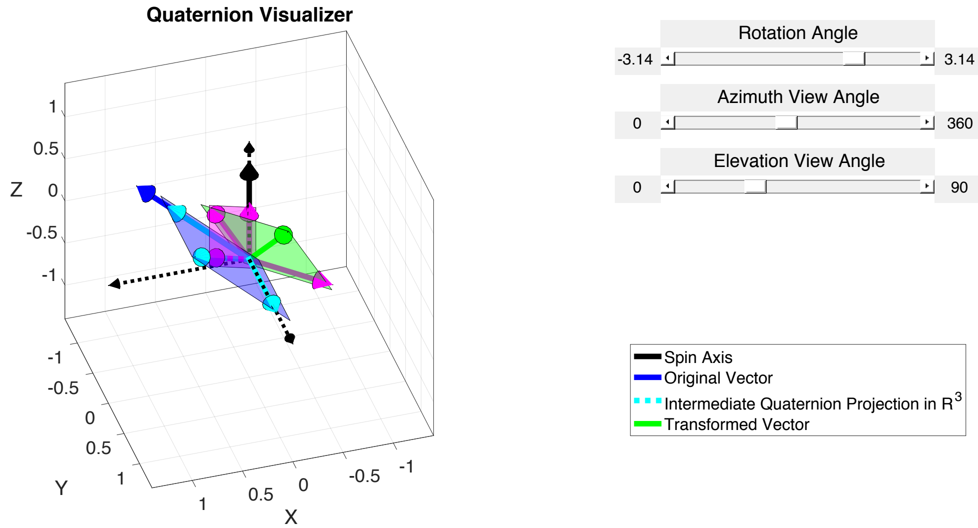

Figure 2. Quaternion Visualizer

Figure 1. visualizes equation (8) in 3D space. The standard blue vector represents the original vector being operated on. The black vector represents the spin axis. The green vector is the resultant vector after the rotation. Both the blue and green vectors are pure quaternions (where the scalar part of the quaternion). The dashed light blue quaternion represents the intermediate quaternion (after the multiplication $q \otimes \begin{bmatrix} \boldsymbol{v} \\ 0 \end{bmatrix}$) projected in 3D space. It is shorter because quaternion multiplication preserves norm and the scalar part of this intermediate quaternion is nonzero. This projection in 3D space is halfway between the original and the resultant vector, albeit being smaller in length due to its projection from 4D space. The plane formed by this projection and the spin axis serves as a reflector between the original vector and the transformed vector. Using equation (8), equation (4) can be expressed as:

$$\boldsymbol{v_{new}} = \boxed{(\boldsymbol{\rho} \bullet \boldsymbol{v})\boldsymbol{\rho} + q_4(q_4\boldsymbol{v} + \boldsymbol{\rho} \times \boldsymbol{v}) - (q_4\boldsymbol{v} + \boldsymbol{\rho} \times \boldsymbol{v}) \times \boldsymbol{\rho}} \tag{9}$$

3. Euler Axis-Angle Representation

The Euler Axis-Angle representation has a direct correlation with the quaternion formulation. It can be derived geometrically without invoking complex numbers. First, the original and transformed vectors are decomposed into vectors parallel to the spin axis and vectors perpendicular to the spin axis:

$$\boldsymbol{v} = \boldsymbol{v}_{\perp} + \boldsymbol{v}_{\parallel} \tag{10}$$
$$\boldsymbol{v_{new}} = \boldsymbol{v_{new}}_{\perp} + \boldsymbol{v_{new}}_{\parallel}$$

By definition, the parallel vectors remain constant under rotation and can be defined as:

$$\boldsymbol{v}_{\parallel} = \boldsymbol{v_{new}}_{\parallel} = (\boldsymbol{v} \bullet \boldsymbol{e})\boldsymbol{e} = \boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v} \tag{11}$$

The transformed perpendicular vector can be further decomposed into a vector parallel to the original perpendicular vector and another vector parallel to $\boldsymbol{e} \times \boldsymbol{v}$. The transformed perpendicular vector must have the same length as the original perpendicular vector:

$$\boldsymbol{v}_{\boldsymbol{new}_\perp} = \|\boldsymbol{v}_\perp\| cos\theta \frac{\boldsymbol{v}_\perp}{\|\boldsymbol{v}_\perp\|} + \|\boldsymbol{v}_\perp\| sin\theta \frac{\boldsymbol{e} \times \boldsymbol{v}_\perp}{\|\boldsymbol{e} \times \boldsymbol{v}_\perp\|} \tag{12}$$

Equation (12) can be simplified into:

$$\boldsymbol{v}_{\boldsymbol{new}_\perp} = \boldsymbol{v}_\perp cos\theta + (\boldsymbol{e} \times \boldsymbol{v}_\perp) sin\theta = (\boldsymbol{v} - \boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v}) cos\theta + (\boldsymbol{e} \times \boldsymbol{v}) sin\theta \tag{13}$$

Using equations (10) and (13), we can then form the Euler axis-angle representation of rotation:

$$\boldsymbol{v}_{\boldsymbol{new}} = \boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v} + (\boldsymbol{e} \times \boldsymbol{v}) sin\theta + (I_{3x3} - \boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}})\boldsymbol{v} cos\theta \tag{14}$$

Equation (14) can be shown to be equivalent to equation (9) by plugging in the definition of quaternion from equation (1):

$$\begin{gathered} \boldsymbol{v}_{\boldsymbol{new}} = \left(\boldsymbol{e} \sin\left(\tfrac{\theta}{2}\right) \bullet \boldsymbol{v}\right) \boldsymbol{e} \sin\left(\tfrac{\theta}{2}\right) + \cos\left(\tfrac{\theta}{2}\right)\left(\cos\left(\tfrac{\theta}{2}\right)\boldsymbol{v} + \boldsymbol{e}\sin\left(\tfrac{\theta}{2}\right) \times \boldsymbol{v}\right) - \\ \left(\cos\left(\tfrac{\theta}{2}\right)\boldsymbol{v} + \boldsymbol{e}\sin\left(\tfrac{\theta}{2}\right) \times \boldsymbol{v}\right) \times \boldsymbol{e}\sin\left(\tfrac{\theta}{2}\right) \\ = \sin^2\left(\tfrac{\theta}{2}\right)\boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v} + \cos^2\left(\tfrac{\theta}{2}\right)\boldsymbol{v} + 2\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)\boldsymbol{e} \times \\ \boldsymbol{v} - \sin^2\left(\tfrac{\theta}{2}\right)(\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{e}\boldsymbol{v} - \boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v}) = (1 - cos\theta)\boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v} + cos\theta\boldsymbol{v} + (\boldsymbol{e} \times \\ \boldsymbol{v}) sin\theta = \boxed{\boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{v} + (\boldsymbol{e} \times \boldsymbol{v}) sin\theta + (I_{3x3} - \boldsymbol{e}\boldsymbol{e}^{\boldsymbol{T}})\boldsymbol{v} cos\theta} \end{gathered} \tag{15}$$

Through the help of well-known trigonometric identities:

$$\begin{gathered} sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ 2\sin^2\left(\frac{\theta}{2}\right) = 1 - cos\theta \\ cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \end{gathered} \tag{16}$$

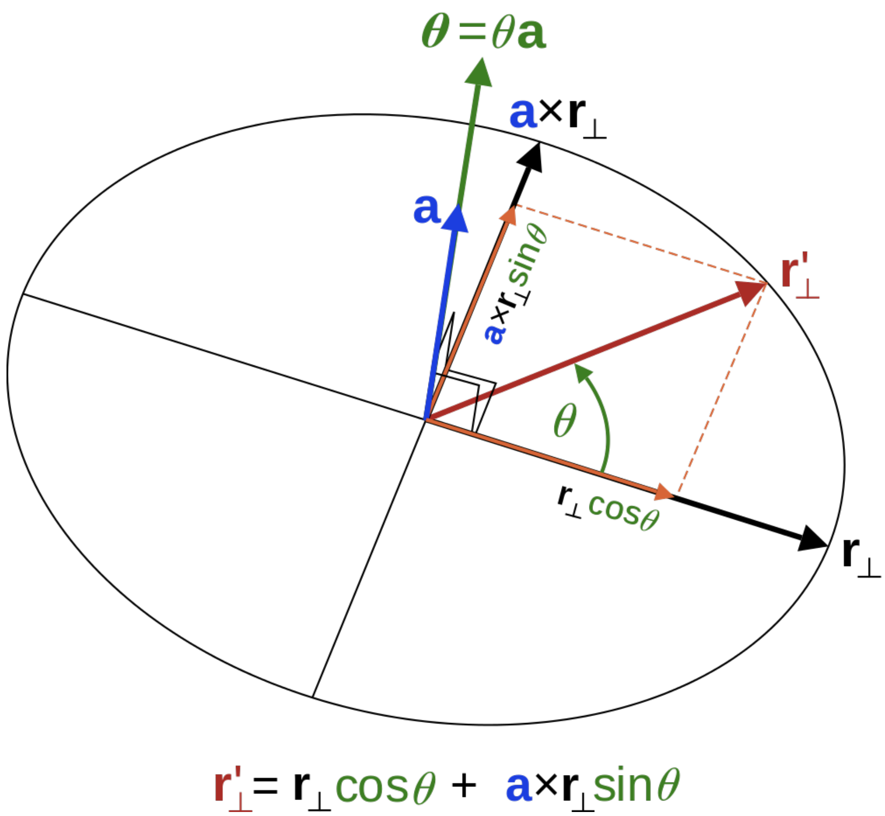

Figure 3. Euler Axis-Angle Representation

4. Complex Matrix Representation

2D rotations can be represented by either complex numbers of dimension 1x1 or real number matrices of dimension 2x2. To show this in a right-handed coordinate system, a counter clockwise rotation of θ can be operated using the rotation matrix:

$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \tag{17}$$

A general complex number encodes a scaling action and a rotation action. A complex number of unit length (any point on the unit circle in a complex plane) will only cause a rotation. Using Euler's formula:

$$e^{i\theta} = cos\theta + isin\theta \tag{18}$$

Multiplying a unit complex number to any other complex number will rotate the original complex number by θ in the counterclockwise direction in the complex plane. To extend normal complex numbers to 2x2 real matrices, we can redefine the complex number i as:

$$i \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = I; \quad 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U \tag{19}$$

The definitions in equation (19) satisfy the original relation for imaginary numbers:

$$i^2 \mapsto I^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -U \mapsto -1 \tag{20}$$

Which allows complex numbers to have 1-to-1 mapping to 2x2 real matrices in terms of all possible operations. Using equations (18) and (19), the Euler's formula can be re-expressed as:

$$e^{i\theta} \mapsto cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} = R \tag{21}$$

Which is the same expression as equation (17)! Similarly, 3D rotations are represented by a quaternion, which is a hyper-complex number with three imaginary numbers, defined using the right-handed rule in equation (3). Each quaternion corresponds to either a 2x2 complex matrix or a 4x4 real matrix. For the complex matrix case, each imaginary number can be mapped to:

$$i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = I; \quad j \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J; \quad k \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = K \quad 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U \tag{22}$$

Equation (22) satisfies the quaternion multiplication rule in equation (3):

$$I^2 = J^2 = K^2 = IJK = -U \tag{23}$$

We can then re-express the typical quaternion as:

$$\boldsymbol{q} \mapsto q_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + q_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + q_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + q_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} q_4 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_4 - iq_1 \end{bmatrix} = Q \tag{24}$$

The matrix Q can be thought of as being part of the special unitary group of dimension 2, SU(2). The complex-equivalent of a matrix transpose is the Hermitian transpose, which can be found by taking the transpose of the complex matrix and then taking the complex conjugate of each entry:

$$Q^\dagger \equiv \overline{Q^T} = \begin{bmatrix} q_4 - iq_1 & -q_2 - iq_3 \\ q_2 - iq_3 & q_4 + iq_1 \end{bmatrix} \tag{25}$$

Where the overline denotes complex conjugation. Equation (4) can then be expressed using 2x2 complex matrices:

$$V_{new} = QVQ^\dagger; \quad V_{new} \mapsto \boldsymbol{v_{new}} \tag{26}$$

Then V_{new} can be mapped back to quaternion to obtain the transformed vector (pure quaternion). The same principle can be applied to mapping of imaginary numbers to 4x4 real matrices. The determinant of equation (24) can be shown to be:

$$\det(Q) = \left\| \begin{bmatrix} q_4 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_4 - iq_1 \end{bmatrix} \right\| = q_1^2 + q_2^2 + q_3^2 + q_4^2 = \|\boldsymbol{q}\|_2^2 = 1 \tag{27}$$

A determinant of 1 is a requirement for rotation operators (3D space is not scaled, only rotated). It is also a requirement for SU(2).

5. Quaternion Time Derivatives

The quaternion derivative can be defined using the definition of a derivative:

$$\dot{\boldsymbol{q}} = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}(t + \Delta t) - \boldsymbol{q}(t)}{\Delta t} \tag{28}$$

The limit can be reformulated into a small change in angle about a spin axis fixed on the body frame. This "infinitesimal" rotation can be modeled as another quaternion that left multiplies original quaternion

$$\dot{\boldsymbol{q}} = \frac{\begin{bmatrix} \boldsymbol{n} \sin\left(\frac{\Delta\theta}{2}\right) \\ \cos\left(\frac{\Delta\theta}{2}\right) \end{bmatrix} \otimes \boldsymbol{q}(t) - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes \boldsymbol{q}(t)}{\Delta t} \tag{29}$$

Using the small angle approximation, equation (29) can be re-expressed as:

$$\dot{\boldsymbol{q}} \approx \frac{\begin{bmatrix} \boldsymbol{n} \frac{\Delta\theta}{2} \\ 1 \end{bmatrix} \otimes \boldsymbol{q}(t) - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes \boldsymbol{q}(t)}{\Delta t} = \frac{1}{2} \begin{bmatrix} \boldsymbol{n} \frac{\Delta\theta}{\Delta t} \\ 0 \end{bmatrix} \otimes \boldsymbol{q}(t) \tag{30}$$

Using the definition of angular velocity vector of a body frame expressed in body frame coordinates:

$$\boldsymbol{\omega} = \boldsymbol{n} \frac{\Delta\theta}{\Delta t} \tag{31}$$

We can re-express equation (30) into the final form of the quaternion derivative:

$$\boxed{\dot{\boldsymbol{q}}(t) = \frac{1}{2} \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \boldsymbol{q}(t)} \tag{32}$$

Note the time derivative is an inertial time derivative; however, the angular velocity vector is expressed in body coordinates according to equation (31), since the "infinitesimal" rotation is defined in the body frame. This is convenient for aerospace applications, since gyroscope measurements are often expressed in body frame coordinates. We can take the time derivative of equation (33) to get the quaternion acceleration:

$$\begin{aligned}\ddot{\boldsymbol{q}}(t) &= \frac{1}{2}\left(\begin{bmatrix}\boldsymbol{\alpha}\\0\end{bmatrix} \otimes \boldsymbol{q}(t) + \begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix} \otimes \dot{\boldsymbol{q}}(t)\right) \\ &= \frac{1}{2}\left(\begin{bmatrix}\boldsymbol{\alpha}\\0\end{bmatrix} \otimes \boldsymbol{q}(t) - \frac{1}{2}\boldsymbol{\omega} \bullet \boldsymbol{\omega}\begin{bmatrix}0\\1\end{bmatrix} \otimes \boldsymbol{q}(t)\right)\end{aligned} \tag{33}$$

Equation (33) can be simplified to:

$$\boxed{\ddot{\boldsymbol{q}}(t) = \frac{1}{2}\begin{bmatrix}\boldsymbol{\alpha}\\ -\frac{1}{2}\boldsymbol{\omega}^{\boldsymbol{T}}\boldsymbol{\omega}\end{bmatrix} \otimes \boldsymbol{q}(t)} \tag{34}$$

Where $\boldsymbol{\alpha}$ is the angular acceleration vector in body coordinates. Equations (32) and (34) are useful for specifying reference tracking and slew maneuver objectives.