Robotics Project: Part 1

- ## Interface

Using MATLAB R2019a to finish the project. Open the 'forward_kinematics.m' and 'inverse_kinematics.m' and run the script. Command window will show the instruction for the user to enter the parameters.

- ## Program Architecture

  - Inverse kinematics:

    Define the T_6 matrix and the [n o a p] from user's input.

    Calculate the joint valuables $[\theta_1 \ \ \theta_2 \ \ d_3 \ \ \theta_4 \ \ \theta_5 \ \ \theta_6]$ one by one.

    Check the joint limit.

    Print the result.

  - Forward kinematics:

    Define the T_6 matrix from user's input.

    Calculate (x, y, z, ψ, θ, ϕ).

    Print the result.

- ## Mathematics

  - Inverse kinematics:

    input: Cartesian point (n, o, a, p).

    output: the corresponding joint variables.

    To solve inverse kinematics of Stanford Arm, we first obtain the transformation matrix A_1 to A_6, the process is in the forward kinematics below.

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_1 & 0 \\ s_2 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We first look at the first three joints of robot arm,

$$T_3 = A_1 * A_2 * A_3 = \begin{bmatrix} | & | & | & c_1 s_2 d_3 - s_1 d_2 \\ | & | & | & s_1 s_2 d_3 + c_1 d_2 \\ | & | & | & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The equation above, there are no obvious relationships that isolate a single joint variable $(\theta_1, \theta_2, d_3)$, so we rearrange the equation to let the variable be isolated.

$$A_1^{-1} * T_3 = \begin{bmatrix} | & | & | & c_1p_x + s_1p_y \\ | & | & | & -p_z \\ | & | & | & -s_1p_x + c_1p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} | & | & | & s_2d_3 \\ | & | & | & -c_2d_3 \\ | & | & | & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_2 * A_3$$

We can now equate the third element of the fourth column of $A_1^{-1} * T_3$ and $A_2 * A_3$,

$$-s_1p_x + c_1p_y = d_3$$

Now define variables $r = \sqrt{p_x^2 + p_y^2}$ and $\Psi = atan2(p_y, p_x)$, $p_x = rc\phi$, $p_y = rs\phi$, substituting this into the equation above, we can obtain $\sin(\phi - \theta_1) = \frac{d_2}{r}$. Using the fact that

$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$, $\cos(\phi - \theta_1) = \pm\frac{\sqrt{r^2 - d_2^2}}{r}$, $\tan(\phi - \theta_1) = \pm\frac{d_2}{\sqrt{r^2 - d_2^2}}$.

$$\theta_1 = atan_2(p_y, p_x) - atan_2(d_2, \pm\sqrt{r^2 - d_2^2}).$$

For θ_2, we equate the (1,4) and (2,4) elements of $A_1^{-1} * T_3$ and $A_2 * A_3$,

$$s_2d_3 = c_1p_x + s_1p_y$$
$$c_2d_3 = p_z$$
$$tan\theta_2 = \frac{s_2d_3}{c_2d_3} = \frac{c_1p_x + s_1p_y}{p_z}$$
$$\theta_2 = atan_2(c_1p_x + s_1p_y, p_z).$$

To find d_3, multiply $s_2d_3 = c_1p_x + s_1p_y$ by s_2 and multiply $c_2d_3 = p_z$ by c_2, then add the these two equations.

$$d_3 = s_2(c_1p_x + s_1p_y) + c_2p_z$$

After solving the three joints, we can take off the effect of them by multiply T_6 by $A_3^{-1} * A_2^{-1} * A_1^{-1}$,

$$A_3^{-1} * A_2^{-1} * A_1^{-1} * T_6 = \begin{bmatrix} c_1c_2 & c_2s_1 & -s_2 & 0 \\ -s_1 & c_1 & 0 & -d_2 \\ c_1s_2 & s_1s_2 & c_2 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -s_4s_6 - c_4c_5c_6 & c_4s_5 & 0 \\ c_4s_6 + s_4c_5c_6 & c_4c_6 - s_4c_5c_6 & s_4s_5 & 0 \\ -s_5c_6 & s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_4 * A_5 * A_6$$

From (1,3) and (2,3) of the matrix, we obtain two equations to solve θ_4

$$c_4s_5 = a_xc_1c_2 + a_yc_2s_1 - a_zs_2$$
$$s_4s_5 = -a_xs_1 + a_yc_1$$
$$\theta_4 = atan(\frac{s_4s_5}{c_4s_5}) = atan(\frac{-a_xs_1 + a_yc_1}{a_xc_1c_2 + a_yc_2s_1 - a_zs_2})$$

From (3,1) and (3,2) of the matrix, we obtain two equations to solve θ_6

$$s_5 c_6 = -(n_x c_1 s_2 + n_y s_1 s_2 + n_z c_2)$$
$$s_5 s_6 = o_x c_1 s_2 + o_y s_1 s_2 + o_z c_2$$
$$\theta_6 = atan\left(\frac{s_5 s_6}{s_5 c_6}\right) = atan\left(\frac{o_x c_1 s_2 + o_y s_1 s_2 + o_z c_2}{-(n_x c_1 s_2 + n_y s_1 s_2 + n_z c_2)}\right)$$

Finally, we multiply T_6 by $A_4^{-1} * A_3^{-1} * A_2^{-1} * A_1^{-1}$,

$$A_4^{-1} * A_3^{-1} * A_2^{-1} * A_1^{-1} * T_6$$
$$= \begin{bmatrix} c_1c_2c_4 - s_1s_4 & c_1s_4 + c_2c_4s_1 & -c_4s_2 & -d_2s_4 \\ -c_1s_2 & -s_1s_2 & -c_2 & d_3 \\ -c_4s_1 - c_1c_2s_4 & c_1c_4 - c_2s_1s_4 & s_2s_4 & -d_2c_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_5c_6 & -c_5s_6 & s_5 & 0 \\ c_6s_5 & -s_5s_6 & -c_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_5 * A_6$$
$$s_5 = a_x(c_1c_2c_4 - s_1s_4) + a_y(c_1s_4 + c_2c_4s_1) + a_z(-c_4s_2)$$
$$c_5 = -(a_x(-c_1s_2) + a_y(-s_1s_2) + a_z(-c_2))$$
$$\theta_5 = atan\left(\frac{s_5}{c_5}\right) = atan\left(\frac{a_x(c_1c_2c_4 - s_1s_4) + a_y(c_1s_4 + c_2c_4s_1) + a_z(-c_4s_2)}{-(a_x(-c_1s_2) + a_y(-s_1s_2) + a_z(-c_2))}\right)$$

θ_1	$atan_2(p_y, p_x) - atan_2(d_2, \pm\sqrt{r^2 - d_2^2})$
θ_2	$atan_2(c_1p_x + s_1p_y, p_z)$
d_3	$s_2(c_1p_x + s_1p_y) + c_2p_z$
θ_4	$atan(\frac{-a_xs_1 + a_yc_1}{a_xc_1c_2 + a_yc_2s_1 - a_zs_2})$
θ_5	$atan\left(\frac{a_x(c_1c_2c_4 - s_1s_4) + a_y(c_1s_4 + c_2c_4s_1) + a_z(-c_4s_2)}{-(a_x(-c_1s_2) + a_y(-s_1s_2) + a_z(-c_2))}\right)$
θ_6	$atan\left(\frac{o_xc_1s_2 + o_ys_1s_2 + o_zc_2}{-(n_xc_1s_2 + n_ys_1s_2 + n_zc_2)}\right)$

- Forward kinematics:
  input: joint variables
  output: Cartesian point (n, o, a, p) and (x, y, z, ψ, θ, ϕ)

  Using D-H model to establish the relationship between two coordinate systems. First, find A_n transformation matrix between coordinate frames n-1 and n.

$$A_n = Rot(z,\theta_n) * Trans(0,0,d_n) * Trans(a_n,0,0) * Rot(x,\alpha_n)$$

$$= \begin{pmatrix} c\theta_n & -s\theta_n c\alpha_n & s\theta_n s\alpha_n & a_n c\theta_n \\ s\theta_n & c\theta_n c\alpha_n & -c\theta_n s\alpha_n & a_n s\theta_n \\ 0 & s\alpha_n & c\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Obtaining the D-H parameters from the kinematic table below,

$Joint$	$d(in)$	$a(in)$	α	θ
1	0	0	-90°	θ_1
2	6.375	0	90°	θ_2
3	d_3	0	0°	0
4	0	0	-90°	θ_4
5	0	0	90°	θ_5
6	0	0	0°	θ_6

By substituting the parameters into the transformation matrix, A_1 to A_6 will be

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_1 & 0 \\ s_2 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, from $T_N = A_1 * A_2 * A_3 * A_4 * A_5 * A_6$, we get the transformation matrix T_6 from base of robot to the tool which is at the end of robot. T_6 is look like the form below.

$$T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can further divide T_6 into linear transformation and Euler angle transformation, the vector $[p_x \;\; p_y \;\; p_z]^T$ is the position of end effector $[x \;\; y \;\; z]^T$. Then we need to solve the Euler angles ψ, θ, ϕ.

$$Euler(\phi,\theta,\psi) = T_{6,without\ \ translation}$$

$$\begin{pmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta & 0 \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From the solving process of ϕ, θ, ψ in the lecture note, we can obtain the solution:

$$\phi = \tan^{-1}(\frac{a_y}{a_x}) \text{ or } \phi = \tan^{-1}(\frac{a_y}{a_x}) + 180^{\circ}$$

$$\theta = \tan^{-1}\left(\frac{s\theta}{c\theta}\right) = \tan^{-1}(\frac{c\phi a_x + c\phi n_y}{a_z})$$

$$\psi = \tan^{-1}(\frac{s\psi}{c\psi}) = \tan^{-1}(\frac{-s\phi n_x + c\phi n_y}{-s\phi o_x + c\phi o_y})$$

$$x = p_x$$

$$y = p_y$$

$$z = p_z$$

- **Bonus**
  - **Algebraic method**

    It is complicated for increasing degree of freedoms.

    Involving some complicated matrix multiplication.

    The algebraic method may occur the problems like heavy computational burden, high time-consuming computation, and difficult symbolic expansion.

    To keep numerical errors small, we may try to simpler the system of equations by some method.

    Does not guarantee a solution for general structure and even if it exists.

  - **Geometric method**

    Need the intuition of robot physical construction in 3-D space.

    The geometric method cannot solve the solutions of 4, 5, 6 axis.

    Can obtain solution through less computation and phases.