HW3

第九組

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Problem 1

(a)(b)小題

$$\delta(x)=\lim_{\alpha\to 0}\delta_\alpha(x)=\lim_{\alpha\to 0}\frac{1}{\sqrt{2\pi\alpha^2}}e^{-\frac{1}{2\alpha^2}x^2}$$

(a)

pf: $$\int_{-\infty}^{\infty}\delta(x)\,dx=\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\alpha^2}}e^{-\frac{1}{2\alpha^2}x^2}dx$$

$$=\lim_{\alpha\to 0}\left(\frac{1}{\sqrt{2\pi\alpha^2}}\cdot\sqrt{\frac{\pi}{\frac{1}{2\alpha^2}}}\right)\rightarrow \text{By 高斯積分}$$

$$=\lim_{\alpha\to 0}1=1$$

$$\Rightarrow \int_{-\infty}^{\infty}\delta(x)\,dx=1$$

(b)

pf: $$\int_{-\infty}^{\infty}\delta(cx)\,dx=\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\alpha^2}}e^{-\frac{c^2}{2\alpha^2}x^2}dx$$

$$=\lim_{\alpha\to 0}\left(\frac{1}{\sqrt{2\pi\alpha^2}}\cdot\sqrt{\frac{\pi}{\frac{c^2}{2\alpha^2}}}\right)\rightarrow \text{By 高斯積分}$$

$$=\lim_{\alpha\to 0}\frac{1}{\sqrt{c^2}}=\frac{1}{|c|}$$

$$\Rightarrow \int_{-\infty}^{\infty}\delta(cx)\,dx=\frac{1}{|c|}\ ,\ c\in\mathbb{R}$$

(c)小題

(c)

pf: $f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$

$= f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = \lim_{\alpha\to0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\alpha^2}} e^{-\frac{1}{2\alpha^2}(x-x_0)^2} f(x)\, dx$ ，令 $\begin{cases} y = x - x_0 \\ a = \frac{1}{2\alpha^2}\end{cases}$

$\Rightarrow \lim_{\alpha\to0}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\alpha^2}} e^{-ay^2}\left[f(x_0)+\sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} y^n\right] dy$

$= f(x_0)\lim_{\alpha\to0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\alpha^2}} e^{-\frac{1}{2\alpha^2}y^2}dy + \lim_{\alpha\to0} \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}\cdot\frac{1}{\sqrt{2\pi\alpha^2}}\int_{-\infty}^{\infty} y^n e^{-ay^2}dy$ —①

※

令 $I = \int_{-\infty}^{\infty} e^{-ay^2}dy = \sqrt{\frac{\pi}{a}} = \sqrt{\pi}\, a^{-\frac12}$ ……高斯積分

$\frac{\partial I}{\partial a} = \int_{-\infty}^{\infty} -y^2 e^{-ay^2} dy = -\frac{\sqrt\pi}{2} a^{-\frac32} \xrightarrow{a\to\infty} 0$ $\left(\alpha\to0 \Rightarrow a = \frac{1}{2\alpha^2}\to\infty\right)$

⋮

$\frac{\partial^k I}{\partial a^k} = \int_{-\infty}^{\infty} (-1)^k y^{2k} e^{-ay^2}dy = c\cdot a^{-\frac12-k} \xrightarrow{a\to\infty} 0$

$\Rightarrow \int_{-\infty}^{\infty} y^n e^{-ay^2} dy = 0$ $\begin{cases} n\in\text{even 積分值為0} \\ n\in\text{odd 為奇函數積分值仍為0} \end{cases}$

代入① $\Rightarrow \int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0) \lim_{\alpha\to0} \frac{1}{\sqrt{2\pi\alpha^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}x^2} dx$

$= f(x_0)\cdot \lim_{\alpha\to0} \frac{1}{\sqrt{2\pi\alpha^2}} \cdot \frac{\sqrt{\pi}}{\sqrt{\frac{1}{2\alpha^2}}} = f(x_0)$

$\Rightarrow \int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)$ #

(d)小題

(d)

$$\int_{-\infty}^{\infty} \delta'(x) f(x) dx$$

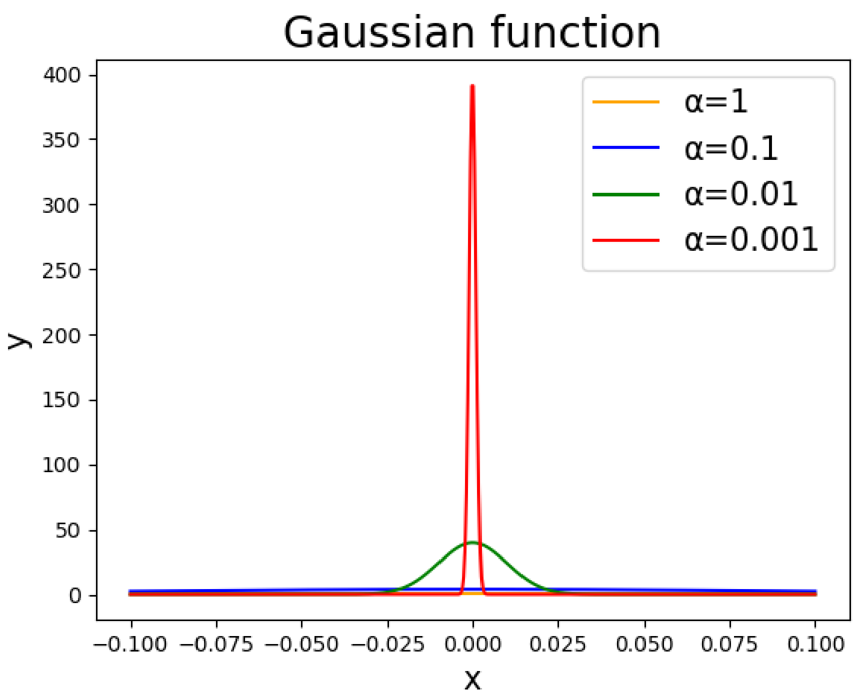
$$= \delta(x) f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) f'(x) dx$$

$$= \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha^2}} e^{-\frac{1}{2\alpha^2}x^2} f(x) \Big|_{-\infty}^{\infty} - f'(0) \leftarrow \text{By上一題}$$

$$= \lim_{\alpha \to 0} 0 - f'(0) = -f'(0)$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta'(x) f(x) = -f'(0) \ \#$$

(e)小題



從圖中可以看出當 α 越小，高斯函數愈接近 Dirac delta function

Ps.程式詳見 homework.py

Problem 2 (5.1)

1. $P(x) = A\exp(-2x)$

$\int_0^\infty p(x)\, dx = 1$

$\frac{1}{2}A = 1$

$A = 2$

$\int_1^\infty 2\exp(-2x)\, dx$

$= e^{-2}$ ※

2. disagree. 此函數發散

3. $p(y) = \sqrt{\frac{c}{y}}$

$\int_0^1 p(y)\, dy = 1$

$2\sqrt{c} = 1$

$c = \frac{1}{4}$

agree. $c = \frac{1}{4}$ ※

4. (a) $\int_0^1 \int_0^{1-y} Dxy\, dx\, dy = 1$

$D\int_0^1 \frac{(1-y)^2}{2} y\, dy = 1$

$\frac{1}{2}D\left(\frac{3}{4} - \frac{2}{3}\right) = 1$

$D = 24$ ※

(b)

$\int_0^1 \int_0^{1-y} 24xy\, dx dy - \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1-y} 24xy\, dx dy$

$= 1 - \frac{5}{16} = \frac{11}{16}$ ※

(c) $p(x<\frac{1}{2} \wedge y<\frac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} Dxy\, dx dy = \frac{3}{8}$

$P(y<\frac{1}{2}) = \frac{11}{16}$

$\frac{11}{16} \times \frac{11}{16} = \frac{121}{256} \neq \frac{3}{8}$

not independent

Problem 3 (5.2)

No.

Date

1. $P(n|p) = C^N_n \, p^n (1-p)^{N-n}$

$$P_N(p|n) = \frac{P_N(n|f)P(f)}{P(n)} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$P_N(p|n) \sim p^n (1-p)^{N-n}$$

$$\frac{\partial}{\partial p} \ln P(p|n) = \frac{\partial}{\partial p}\left(n \ln p + (N-n)\ln(1-p)\right)$$

$$= \frac{n}{p} - \frac{(N-n)}{1-p}\Bigg|_{p=p_{max}} = 0$$

$$\frac{n}{P_{max}} - \frac{N-n}{1-P_{max}} = 0 \Rightarrow P_{max} = \frac{n}{N}$$

2. $$-\frac{1}{\sigma^2} = \frac{d^2}{dp^2} \ln P_N(p|n) = \frac{d}{dp}\left(\frac{n}{p} - \frac{N-n}{1-p}\right)\Bigg|_{P_{max}=\frac{n}{N}}$$

$$= \frac{-n}{\left(\frac{n}{N}\right)^2} - \frac{N-n}{\left(1-\frac{n}{N}\right)^2} = \frac{-N^2}{n} - \frac{N^2}{(N-n)} = -N\frac{1}{\frac{n}{N}} - \frac{N}{\left(1-\frac{n}{N}\right)}$$

$$= -N\frac{1}{P_{max}} - \frac{N}{1-P_{max}} = \frac{-N}{P_{max}(1-P_{max})}$$

$$\Rightarrow \sigma = \frac{\sqrt{P_{max}(1-P_{max})}}{\sqrt{N}}$$

Problem 4 (5.3)

5.3 $\int_0^\infty A e^{-ax} dx = 1 \Rightarrow A \cdot \frac{e^{-a0}}{a} = 1 \Rightarrow A = a$

$\int_1^\infty B x^{-3} dx = 1 \Rightarrow B \cdot \frac{1}{2 \cdot 1^2} = 1 \Rightarrow B = 2$

1. mean: $\int_0^\infty a x e^{-ax} dx = \frac{1}{a}$

var: $\int_0^\infty (x^2) a e^{-ax} dx - \frac{1}{a^2} = \frac{1}{a^2}$

standard Deviation: $\sqrt{\frac{1}{a^2}} = \frac{1}{a}$

2. mean: $\int_1^\infty 2x^{-2} dx = 2$

var: $\int_1^\infty x^2 \cdot 2x^{-3} dx - 4 = \infty$

Standard Deviation: $\sqrt{\int_1^\infty \frac{1}{x} dx - 4} = \infty$

culture

Problem 5 (5.4)   Problem 6 (5.5)

5-4.

1. $= \frac{y^4}{|P(y)|} + \frac{(-y)^4}{|P(-y)|} = y^3$

2. $\sin x = 0$, $x = n\pi$, $n = 0, 1, 2 \cdots$

$$= \sum_{n=0}^{\infty} \frac{e^{-n\pi}}{|\cos(n\pi)|} = \sum_{n=0}^{\infty} e^{-n\pi} = \frac{1}{1-e^{-\pi}}$$

3. $P(z) = \int_0^1 dx \int_0^{1-x} dy\, P(x,y)\, \delta(z-(x+y))$

$$= 24\int_0^z dx \cdot x \cdot (z-x) = 24\int_0^z dx(-x^2+zx)$$

$$= 24\left(-\frac{1}{3}z^3 + \frac{1}{2}z^3\right) = 4z^3$$

5-5.

1. $\int_0^{\infty}\int_0^1 A e^{-x^2-2y}\, dx\, dy = 1$

$$\Rightarrow A\left[\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{1}{2} - \frac{e^{-2}}{2}\right)\right] = 1$$

$$\Rightarrow A = \frac{1}{\frac{\sqrt{\pi}}{4} - \frac{e^{-2}}{4}\sqrt{\pi}} = \frac{4}{\sqrt{\pi} - e^{-2}\sqrt{\pi}}$$

2. $P(x) = \int_0^1 P(x,y) = Be^{-x^2}$, $P(y) = \int_0^{\infty} P(x,y) = Ce^{-2y}$ 令 $(B \cdot C) = A$

$\Rightarrow P(x,y) = P(x)P(y) = Be^{-x^2}Ce^{-2y} = Ae^{-x^2-2y}$ ∴兩者獨立

3. $z = x^2 + 2y$

$p(z) = P(z(x,y))$

$$= A \cdot e^{-z} = \frac{4}{\sqrt{\pi} - e^{-2}\sqrt{\pi}} \cdot e^{-z}$$