Projection method for 3D flow

Zi-Hsuan Wei

NATIONAL TAIWAN UNIVERSITY OF SCIENCE AND TECHNOLOGY

May 13, 2020

Contents

1. Governing Equations
2. flowchat
3. Large Eddy Simulation
4. Projection Method
5. DFIB
6. Internship record

Contents

1. Governing Equations
2. flowchat
3. Large Eddy Simulation
4. Projection Method
5. DFIB
6. Internship record

- The momentum equation, when surface tension is neglected, there is no body force except gravity, and the viscosity of both fluids is the same, is

$$\frac{\partial u}{\partial t} + \triangledown \cdot uu = -\frac{1}{\rho} \triangledown p + \nu \triangledown^2 u + f \tag{1.1}$$

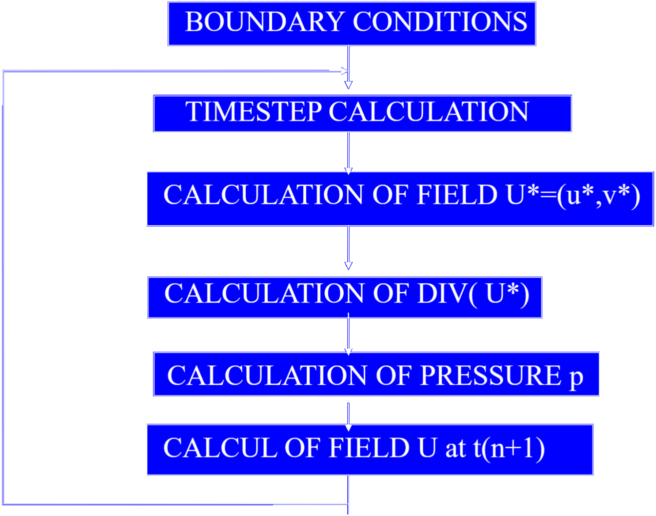
- The conservation of mass equation for incompressible flow is unchanged.

$$\triangledown \cdot u = 0 \tag{1.2}$$

Contents

1. Governing Equations
2. flowchat
3. Large Eddy Simulation
4. Projection Method
5. DFIB
6. Internship record

flow chat


BOUNDARY CONDITIONS
TIMESTEP CALCULATION
CALCULATION OF FIELD U*=(u*,v*)
CALCULATION OF DIV(U*)
CALCULATION OF PRESSURE p
CALCUL OF FIELD U at t(n+1)

Contents

1. Governing Equations
2. flowchat
3. Large Eddy Simulation
4. Projection Method
5. DFIB
6. Internship record

Large Eddy Simulation

N-S equations for LES

- $$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u_j u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + 2\frac{\partial}{\partial x_j}[(\nu + \nu_t)\overline{S}_{ij}] \tag{3.1}$$

  which $\overline{S}_{ij} = \frac{1}{2}(\partial_i \overline{u_j} + \partial_j \overline{u_i})$

- $$\nu_t = (C_s \overline{\Delta})^2)|\overline{S}| \tag{3.2}$$

  which $|\overline{S}|$ is strain rate tensor scale, $(\overline{2 s_{ij}} \overline{s_{ij}})^{1/2}$

- $C_s = 0.18$ Reference Yang Zhiyin, Large-eddy simulation:Past, present and the future

Contents

1. Governing Equations
2. flowchat
3. Large Eddy Simulation
4. Projection Method
5. DFIB
6. Internship record

Projection Method

In the first stage, an intermediate velocity that does not satisfy the incompressibility constraint is computed at each time step.

$$\frac{u^* - u^n}{\Delta t} = -A^n + D^n + f \tag{4.1}$$

Where n is the time level index. A_n is a discrete approximation of the advection term. D_n is an approximation of the diffusion term.

In the second, the pressure is used to project the intermediate velocity onto a space of convergence velocity field to get the next update of velocity and pressure.

$$\frac{u^{n+1} - u^*}{\Delta t} = \frac{-\nabla p^{n+1}}{\rho} \tag{4.2}$$

Where the new vector u^* is the intermediate velocity.

Projection Method

Final velocity field is convergence at the end of the time step and satisfy the discrete version of equation.

$$\nabla \cdot u^{n+1} = 0 \tag{4.3}$$

We get a Poisson equation for the pressure:

$$\nabla^2 p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot u^* \tag{4.4}$$

Projection Method

Using the staggered mesh and (4.1)

$$u^*_{i+1/2,j,k} = u^n_{i+1/2,j,k} + \Delta t \left\{ -(A_x)^n_{i+1/2,j,k} + (D_x)^n_{i+1/2,j,k} \right\} \tag{4.5}$$

$$v^*_{i,j+1/2,k} = v^n_{i,j+2/1,k} + \Delta t \left\{ -(A_y)^n_{i,j+1/2,k} + (D_y)^n_{i,j+1/2,k} \right\} \tag{4.6}$$

$$w^*_{i,j,k+1/2} = w^n_{i,j,k+1/2} + \Delta t \left\{ -(A_z)^n_{i,j,k+1/2} + (D_z)^n_{i,j,k+1/2} \right\} \tag{4.7}$$

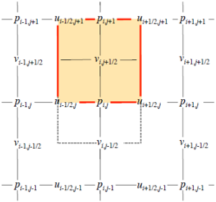


Figure: v-velocity

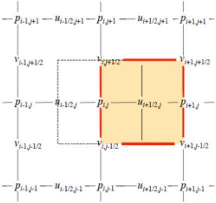


Figure: u-velocity

Projection Method

Using the staggered mesh and (4.2)

$$u^{n+1}_{i+1/2,j,k} = u^{*}_{i+1/2,j,k} - \frac{\Delta t}{\rho} \nabla p \tag{4.8}$$

$$v^{n+1}_{i,j+1/2,k} = v^{*}_{i,j+1/2,k} - \frac{\Delta t}{\rho} \nabla p \tag{4.9}$$

$$w^{n+1}_{i,j,k+1/2} = w^{*}_{i,j,k+1/2} - \frac{\Delta t}{\rho} \nabla p \tag{4.10}$$

Define the average value over a control volume and then write the volume integral as a surface integral using the divergence theorem:

$$A = \frac{1}{\forall} \int_{\forall} \nabla \cdot uu \mathrm{d}\forall = \frac{1}{\forall} \oint_{S} u(u \cdot n) \mathrm{d}S \tag{4.11}$$

Using (4.11)

$$\begin{aligned}(A_x)_{i+1/2,j,k} = & \frac{1}{\Delta x \Delta y \Delta z} \{[(uu)_{i+1,j,k} - (uu)_{i,j,k}]\Delta y \Delta z + \\ & [(vu)_{i+1/2,j+1/2,k} - (vu)_{i+1/2,j-1/2,k}]\Delta x \Delta z + \\ & [(wu)_{i+1/2,j,k+1/2} - (wu)_{i+1/2,j,k-1/2}]\Delta x \Delta y\end{aligned} \tag{4.12}$$

Using (4.11)

$$
\begin{aligned}
(A_y)_{i,j+1/2,k} = & \frac{1}{\Delta x \Delta y \Delta z} \{ [(uv)_{i+1/2,j+1/2,k} - (uv)_{i-1/2,j+1/2,k}] \Delta y \Delta z + \\
& [(vv)_{i,j+1,k} - (vv)_{i,j,k}] \Delta x \Delta z + \\
& [(wv)_{i,j+1/2,k+1/2} - (wv)_{i,j+1/2,k-1/2}] \Delta x \Delta y
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
(A_z)_{i,j,k+1/2} = & \frac{1}{\Delta x \Delta y \Delta z} \{ [(uw)_{i+1/2,j,k+1/2} - (uw)_{i-1/2,j,k+1/2}] \Delta y \Delta z + \\
& [(vw)_{i,j+1/2,k+1/2} - (vw)_{i,j-1/2,k+1/2}] \Delta x \Delta z + \\
& [(ww)_{i,j,k+1} - (ww)_{i,j,k}] \Delta x \Delta y
\end{aligned}
\tag{4.14}
$$

Define the average value over a control volume and then write the volume integral as a surface integral using the divergence theorem:

$$D = \frac{\nu}{\forall}\int_{\forall}\nabla^2 u \mathrm{d}\forall = \frac{\nu}{\forall}\int_{\forall}\nabla\cdot\nabla u \mathrm{d}\forall = \frac{\nu}{\forall}\oint_{S}\nabla u\cdot n \mathrm{d}S \tag{4.15}$$

Using (4.15)

$$\begin{aligned}(D_x)_{i+\frac{1}{2},j,k} = &\frac{\nu}{\Delta x\Delta y\Delta z}\Big\{[(\frac{\partial u}{\partial x})_{i+1,j,k} - (\frac{\partial u}{\partial x})_{i,j,k}]\Delta y\Delta z + \\ &[(\frac{\partial u}{\partial y})_{i+\frac{1}{2},j+\frac{1}{2},k} - (\frac{\partial u}{\partial y})_{i+\frac{1}{2},j-\frac{1}{2},k}]\Delta x\Delta z + \\ &[(\frac{\partial u}{\partial z})_{i+\frac{1}{2},j,k+\frac{1}{2}} - (\frac{\partial u}{\partial z})_{i+\frac{1}{2},j,k-\frac{1}{2}}]\Delta x\Delta y\end{aligned} \tag{4.16}$$

Using (4.15)

$$
\begin{aligned}
(D_y)_{i,j+\frac{1}{2},k} =& \frac{\nu}{\Delta x \Delta y \Delta z} \Big\{ [(\frac{\partial v}{\partial x})_{i+\frac{1}{2},j+\frac{1}{2},k} - (\frac{\partial v}{\partial x})_{i-\frac{1}{2},j+\frac{1}{2},k}]\Delta y \Delta z + \\
& [(\frac{\partial v}{\partial y})_{i,j+1,k} - (\frac{\partial v}{\partial y})_{i,j,k}]\Delta x \Delta z + \\
& [(\frac{\partial v}{\partial z})_{i,j+\frac{1}{2},k+\frac{1}{2}} - (\frac{\partial v}{\partial z})_{i,j+\frac{1}{2},k-\frac{1}{2}}]\Delta x \Delta y
\end{aligned} \tag{4.17}
$$

$$
\begin{aligned}
(D_z)_{i,j,k+\frac{1}{2}} =& \frac{\nu}{\Delta x \Delta y \Delta z} \Big\{ [(\frac{\partial w}{\partial x})_{i+\frac{1}{2},j,k+\frac{1}{2}} - (\frac{\partial w}{\partial x})_{i-\frac{1}{2},j,k+\frac{1}{2}}]\Delta y \Delta z + \\
& [(\frac{\partial w}{\partial y})_{i,j+\frac{1}{2},k+\frac{1}{2}} - (\frac{\partial w}{\partial y})_{i,j-\frac{1}{2},k+\frac{1}{2}}]\Delta x \Delta z + \\
& [(\frac{\partial w}{\partial z})_{i,j,k+1} - (\frac{\partial w}{\partial z})_{i,j,k}]\Delta x \Delta y
\end{aligned} \tag{4.18}
$$

Using the center differencing scheme.

$$
\begin{aligned}
(D_x)_{i+\frac{1}{2},j,k} = & \nu[(\frac{u_{i+\frac{3}{2},j,k} - 2u_{i+\frac{1}{2},j,k} + u_{i-\frac{1}{2},j,k}}{\Delta x^2})+ \\
& (\frac{u_{i+\frac{1}{2},j+1,k} - 2u_{i+\frac{1}{2},j,k} + u_{i+\frac{1}{2},j-1,k}}{\Delta y^2})+ \\
& (\frac{u_{i+\frac{1}{2},j,k+1} - 2u_{i+\frac{1}{2},j,k} + u_{i+\frac{1}{2},j,k-1}}{\Delta z^2})]
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
(D_y)_{i,j+\frac{1}{2},k} = & \nu[(\frac{v_{i+1,j+\frac{1}{2},k} - 2v_{i,j+\frac{1}{2},k} + v_{i-1,j+\frac{1}{2},k}}{\Delta x^2})+ \\
& (\frac{v_{i,j+\frac{3}{2},k} - 2v_{i,j+\frac{1}{2},k} + v_{i,j-\frac{1}{2},k}}{\Delta y^2})+ \\
& (\frac{v_{i,j+\frac{1}{2},k+1} - 2v_{i,j+\frac{1}{2},k} + v_{i,j+\frac{1}{2},k-1}}{\Delta z^2})]
\end{aligned}
\tag{4.20}
$$

Using the center differencing scheme.

$$
\begin{aligned}
(D_z)_{i,j,k+\frac{1}{2}} =& \nu[(\frac{w_{i+1,j,k+\frac{1}{2}} - 2w_{i,j,k+\frac{1}{2}} + w_{i-1,j,k+\frac{1}{2}}}{\Delta x^2})+ \\
& (\frac{w_{i,j+1,k+\frac{1}{2}} - 2w_{i,j,k+\frac{1}{2}} + w_{i,j-1,k+\frac{1}{2}}}{\Delta y^2})+ \\
& (\frac{w_{i,j,k+\frac{3}{2}} - 2w_{i,j,k+\frac{1}{2}} + w_{i,j,k-\frac{1}{2}}}{\Delta z^2})]
\end{aligned} \tag{4.21}
$$

The pressure equation

Under discrete form,we obtain:

$$(\frac{P^{n+1}_{i+1,j,k} - 2P^{n+1}_{i,j,k} + P^{n+1}_{i-1,j,k}}{\Delta x^2}) + (\frac{P^{n+1}_{i,j+1,k} - 2P^{n+1}_{i,j,k} + P^{n+1}_{i,j-1,k}}{\Delta y^2}) + (\frac{P^{n+1}_{i,j,k+1} - 2P^{n+1}_{i,j,k} + P^{n+1}_{i,j,k-1}}{\Delta z^2}) =$$
$$\frac{1}{\Delta t}(\frac{u*_{i+\frac{1}{2},j,k} - u*_{i-\frac{1}{2},j,k}}{\Delta x} + \frac{v*_{i,j+\frac{1}{2},k} - v*_{i,j-\frac{1}{2},k}}{\Delta y} + \frac{w*_{i,j,k+\frac{1}{2}} - w*_{i,j,k-\frac{1}{2}}}{\Delta z}) \tag{4.22}$$

Biconjugate gradient stabilized method

Resolution of pressure equation

1. Solving pressure equation : use the subroutine BiCG in the file solver.f90

Contents

1. Governing Equations
2. flowchat
3. Large Eddy Simulation
4. Projection Method
5. DFIB
6. Internship record

Direct forcing immersed boundary method

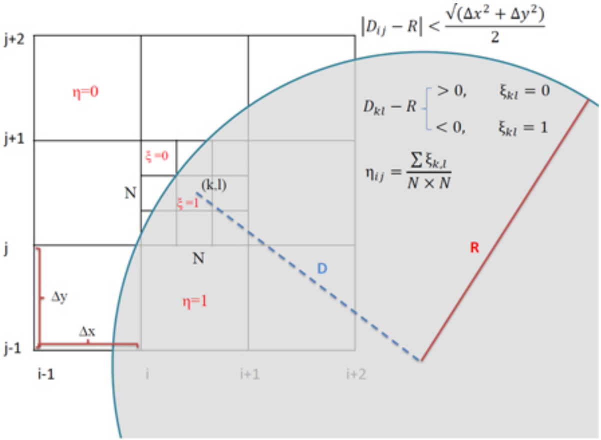


Figure: Illustration of direct forcing immersed boundary.

Contents

1 Governing Equations

2 flowchat

3 Large Eddy Simulation

4 Projection Method

5 DFIB

6 Internship record

Projection method for 3D flow

Zi-Hsuan Wei

TAIWAN TECH
NATIONAL TAIWAN UNIVERSITY OF SCIENCE AND TECHNOLOGY

May 13, 2020