JOHNS HOPKINS
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Robust Analysis of Consensus Algorithms

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01/14/2020

Support from the NSF is gratefully acknowledged

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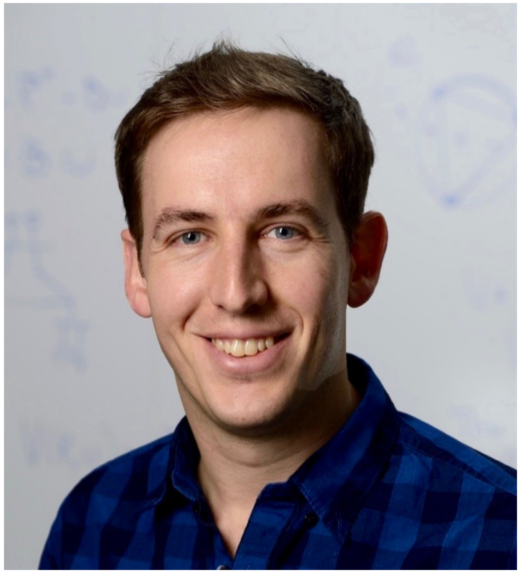
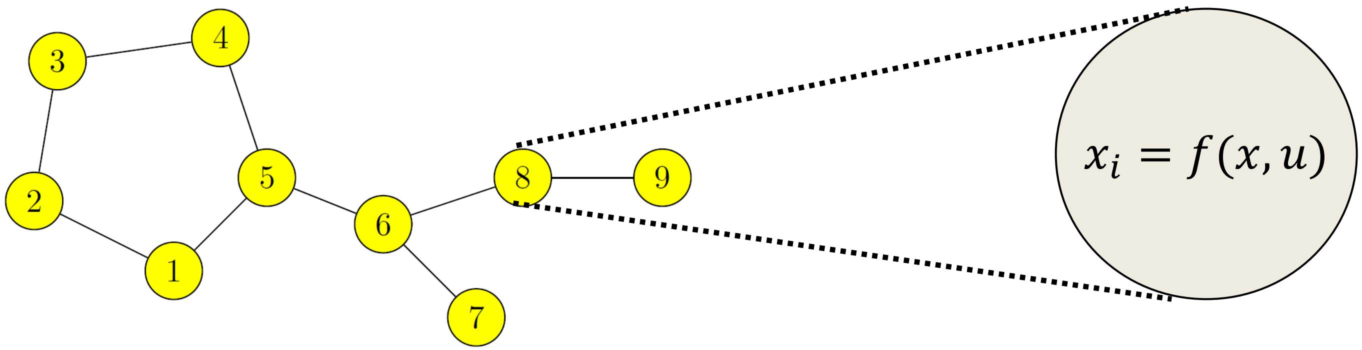


Enrique Mallada

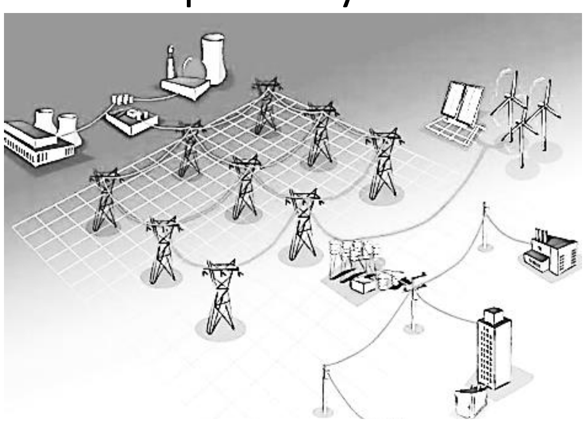
Marin Kobilarov

Network of dynamical systems

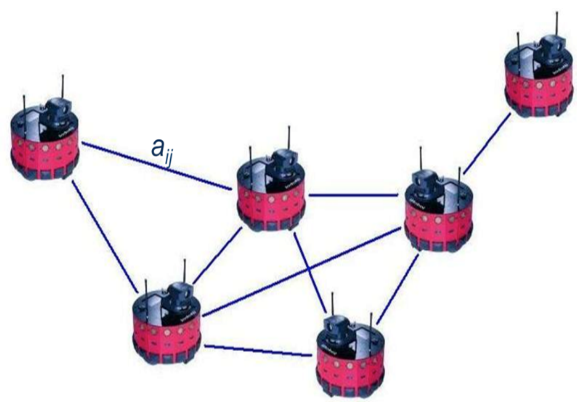
- Many systems can be abstracted as collection of heterogeneous dynamical systems connected over a graph



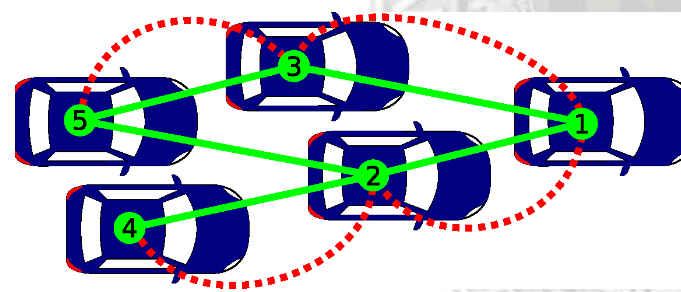
Electric power systems



Robotic networks

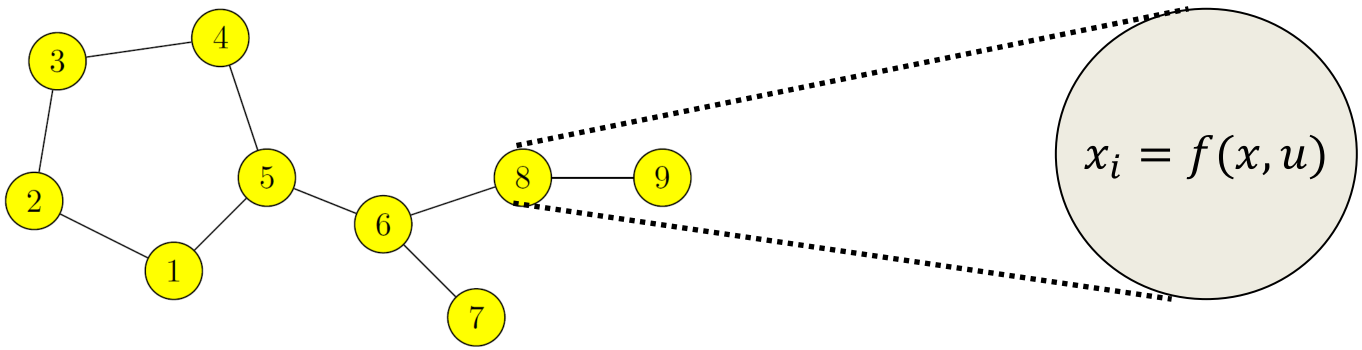


Vehicular platoons



Network of dynamical systems

- Many systems can be abstracted as collection of heterogeneous dynamical systems connected over a graph



Computer systems

Social networks

Fireflies

Network of dynamical systems

- In this talk, we focus on analyzing the system's performance through control theory.

- Why not FANCY learning theories?

Computer systems

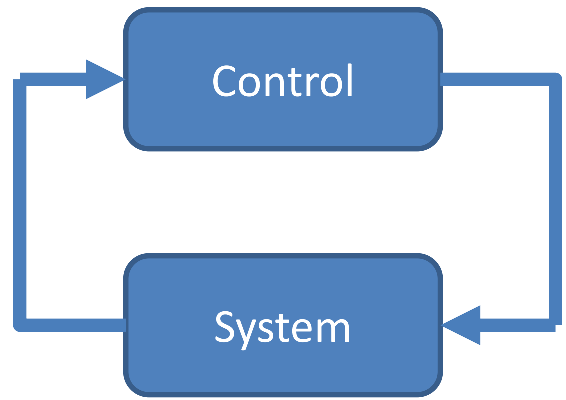
Social networks

Fireflies

An unicycle

An ideal control strategy:

- identify desired position
- Least control effort
- Shortest path



We need a bridge that links the control input with the unicycle movement.

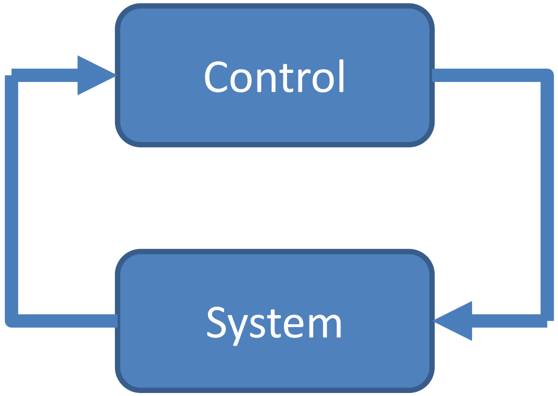
An unicycle

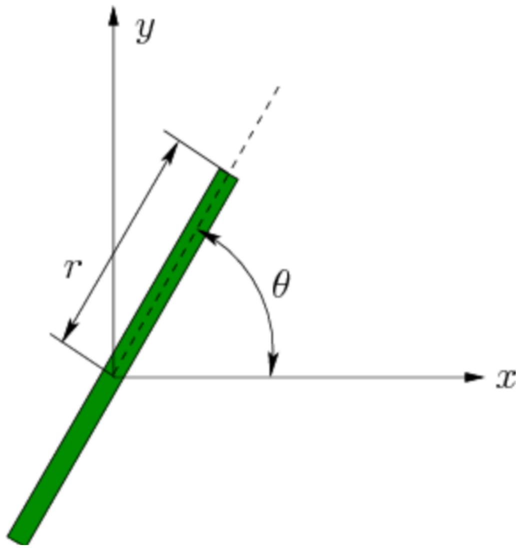
An ideal control strategy:

- Least control effort
- Shortest path
- Reach desired position

$$u = \text{argmin} \int_{t_0}^{t_f} u^2 + \dot{x}^2 + \dot{y}^2 dt$$

$$s.t.\ x_{tf} = x^*$$





System:

$$\dot{x} = u\ cos\theta$$

$$\dot{y} = u\ sin\theta$$

$$\dot{\theta} = \tau$$

- x direction acceleration
- y direction acceleration
- rotation

Back to our lovely unicycle

If you were a baby ......

1. Push the pedal, move forward
2. Push harder, move faster
3. Rotate your waist, "artfully," direction change
4. Wave hands, nothing happens, no reward
5. Try process 1-4 a thousand times

Control vs learning

The **control** deals with continuously operating dynamical systems to:

- Stabilize a system or achieve certain movement goal
- Mitigate uncertainty
- ……

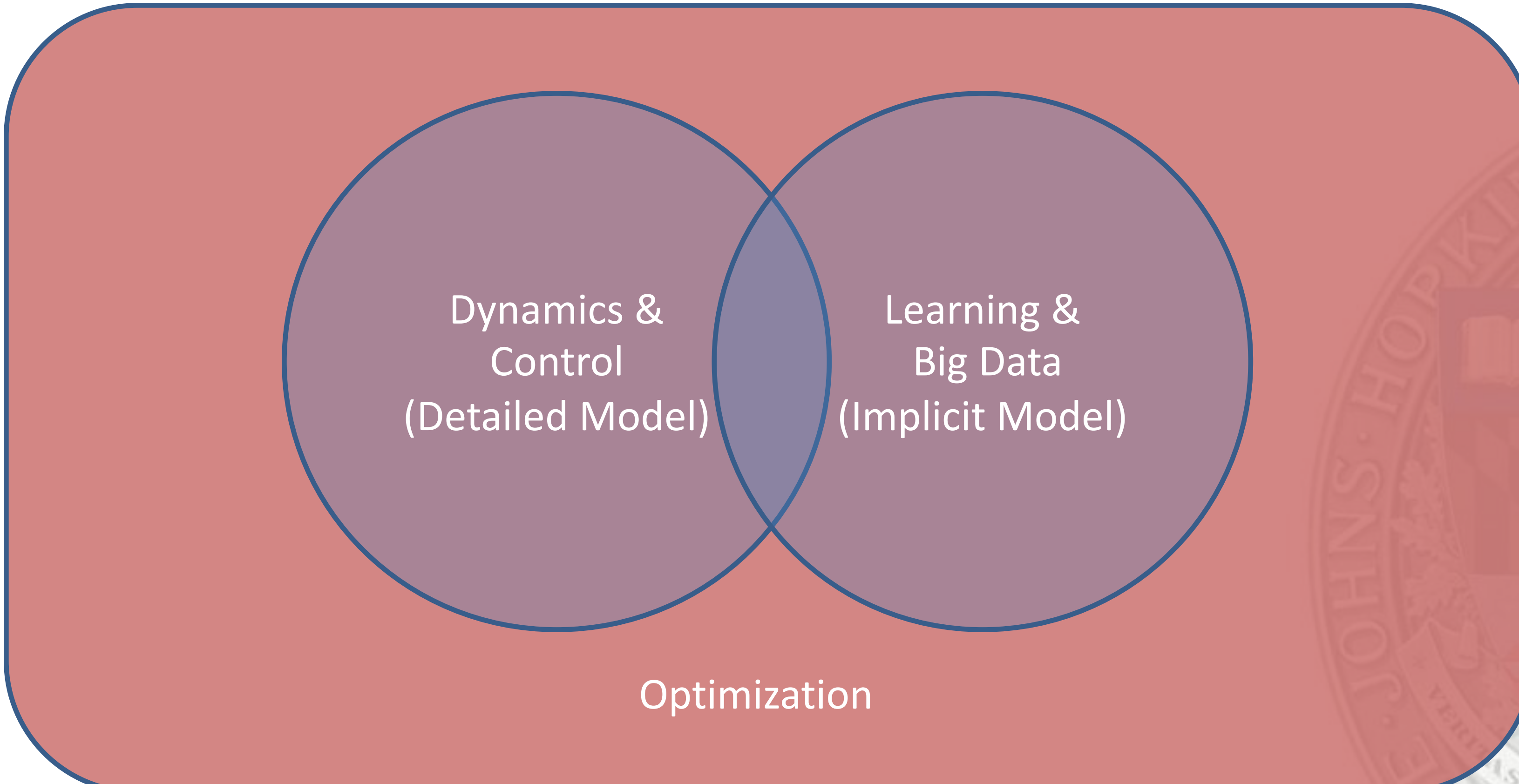
Key: A detailed system model

With **learning**, we can:

- Simulate the system behavior
- Mitigate uncertainty
- ……

Key: An implicit system model

Control vs learning



Preliminaries

Consensus

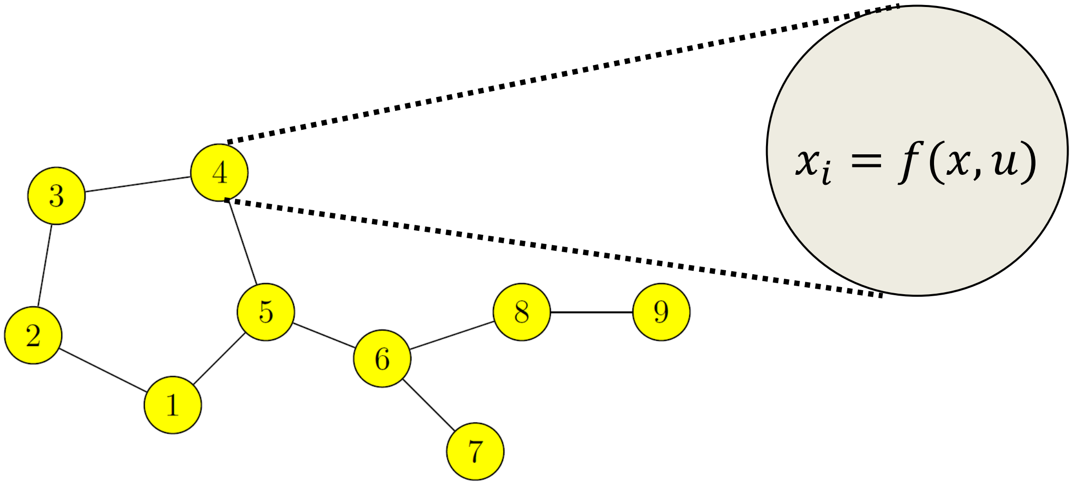
Why?

- Vehicle Position coordination
- Phase synchronization
- Information coherency

Definition (Weighted Consensus State)

A dynamical system is referred to as achieving consensus state if each node share the same state and that state is determined by the initial state x_0 and a set of positive weighting parameters $w = \{w_1 \cdots w_n\} > 0$, i.e., $x = 1_n w^T x_0$. [Ji et. al. 2019]

Consensus algorithm (1st order)



Dynamics of each oscillator i (associated with vertex i)

$$\dot{m}_i \dot{x}_i = \sum_{\{i,j\}\in E} a_{ij}\left(x_i - x_j\right) + w_i \quad \Rightarrow \quad \dot{x} = -\frac{1}{m} L x + \frac{1}{m} w$$

L is the graph Laplacian

$$[L]_{ij} = \begin{cases} -a_{ij}, i \neq j \\ \sum_{i\neq j} a_{ij}, i = j \end{cases}$$

Consensus algorithm (1st order)

Graph Laplacian:

$$[L]_{ij} = \begin{cases} -a_{ij}, i \neq j \\ \sum_{i \neq j} a_{ij}, i = j \end{cases}$$

$\rightarrow$

$\dot{x} = Ax, \rightarrow x = e^{At} x_0$

$\lim_{t \rightarrow \infty} x(t) \rightarrow w^T x_o 1_n$

w is the unit left eigenvector

Note: (Special properties of graph Laplacian)

- The graph Laplacian L satisfies, $L1_n = 0 \times 1_n$ (Special eigenvector and eigenvalue)
- Undirected graph, if connected, only one zero eigenvalue
- Directed graph, if only one globally reachable node, only one zero eigenvalue

Consensus algorithm (1st order)

Definition (Traditional Consensus Algorithm)

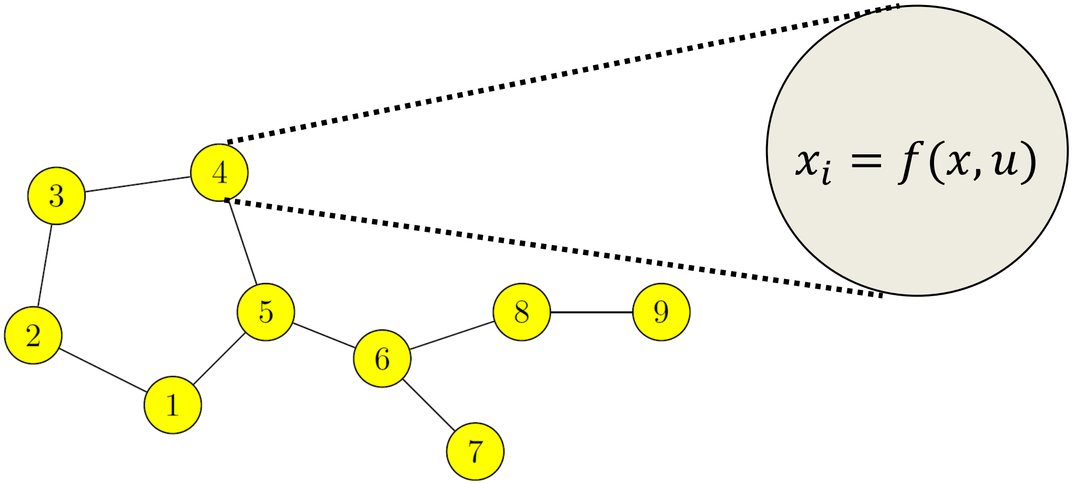
A (first order) consensus algorithm is referred to as traditional consensus algorithm if the algorithm only takes the relative measurements as the control feedback.

- $x^k = x^{k-1} + u^{k-1}, u^{k-1} = Lx^{k-1}$ (Discrete Systems)
- $\dot{x} = u, u = Lx$ (Continuous Systems)
- F is the graph Laplacian matrix that forms the relative measurement.

Note: (Special properties of graph Laplacian)

- The graph Laplacian L satisfies, $L1_n = 0 \times 1_n$ (Special eigenvector and eigenvalue)
- Undirected graph, if connected, only one zero eigenvalue
- Directed graph, if only one globally reachable node, only one zero eigenvalue

Consensus algorithm (2nd order)



Dynamics of each oscillator i (associated with vertex i)

$$m_i \ddot{x}_i + \beta_i \dot{x}_i = -\sum_{\{i,j\}\in \mathrm{E}_B} b_{ij}\left(x_i - x_j\right) - \sum_{\{i,j\}\in \mathrm{E}_D} d_{ij}\left(\dot{x}_i - \dot{x}_j\right) + w_i$$

Example: If $d_{ij}=0$ linearized swing equations subject to stochastic forcing (a Kron reduced power network or wide area power system model)

Consensus Performance

System performance

Here we study the performance by defining the system output

$$\mathbf{y} = C\mathbf{x}$$

Such that $\|G\|_{\mathrm{H}_2}$ quantifies an input-output performance

Bamieh et al. 2012, Jovanović

Why H2 norm?

Interpretation of $\mathrm{y} = C\mathrm{x}$?

H2 norm

H2 norm measures the energy of the impulse response:

$$\|G\|_{\mathcal{L}_2}^2 := \sum_i \sum_j \int_0^{+\infty} \|g_{ij}(t)\|_2^2 \, dt = \int_0^{+\infty} \|G(t)\|_F^2 \, dt$$

$$= \mathrm{Tr}\left[\int_0^{+\infty} G(t)' G(t) \, dt\right] = \frac{1}{2\pi} \mathrm{Tr}\left[\int_{-\infty}^{+\infty} G(j\omega)' G(j\omega) \, d\omega\right] =: \|G\|_{\mathcal{H}_2}^2.$$

Alternate interpretation: The energy of the response to initial conditions $x_0 = 1_n$, and the control input as $u(t) = u_0 \delta(t)$

H2 norm

H2 norm measures the energy of the impulse response:

The (expected) power of the response to white noise:

$$
\begin{aligned}
&\mathbb{E}\left[\lim_{T\to+\infty}\frac{1}{T}\mathrm{Tr}\left(\int_0^T y(t)y(t)'\,dt\right)\right]\\
&=\lim_{T\to+\infty}\frac{1}{T}\mathrm{Tr}\left(\int_0^T \mathbb{E}\left[\int_0^t\int_0^t G(t-\tau_1)u(\tau_1)u(\tau_2)'G(t-\tau_2)'\,d\tau_1 d\tau_2\right]\,dt\right)\\
&\qquad=\lim_{T\to+\infty}\frac{1}{T}\mathrm{Tr}\left(\int_0^T\int_0^t G(t-\tau)G(t-\tau)'\,d\tau\,dt\right)\\
&\quad=-\lim_{T\to+\infty}\mathrm{Tr}\left[\int_0^T G(T-\tau)G(T-\tau)'\,d(T-\tau)\right]=\|G\|^2_{\mathcal{L}_2}=\|G\|^2_{\mathcal{H}_2}.
\end{aligned}
$$

Total transient power losses

Let $x_i = \theta_i$ and $\dot{x}_i = \omega_i$ be the phase angle and frequency of generator i

Power flow over a line (Ohm's law): $P_{ik}^{loss} := g_{ik}\left|V_i - V_k\right|^2$

Total losses (linearized):

$$P^{loss} := \sum_{i\sim j} g_{ik}\left|x_i - x_j\right|^2 = \mathbf{x}^* L_G \mathbf{x}$$

Define system output as: $\mathbf{y} = \begin{bmatrix} L_G^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}$

Bamieh & Gayme 2013, Tegling et al 2015

which gives: $\mathbf{y}^*\mathbf{y} = \mathbf{x}^* L_G \mathbf{x} = P^{loss}$

Input-output norm $\| G\|_{H_2}^2 = \lim_{t\to\infty} E\left\{\mathbf{y}^*(t)\mathbf{y}(t)\right\}$

Nodal performance measure

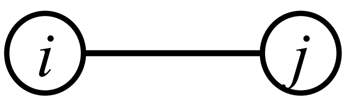
Define the nodal performance with respect to nodes i and j as

$$P_{ij} = \lim_{t \to \infty} E\left[\left(x_i - x_j\right)^2\right]$$

Gaussian input vector $\mathbf{w}(t)$ each element unit strength

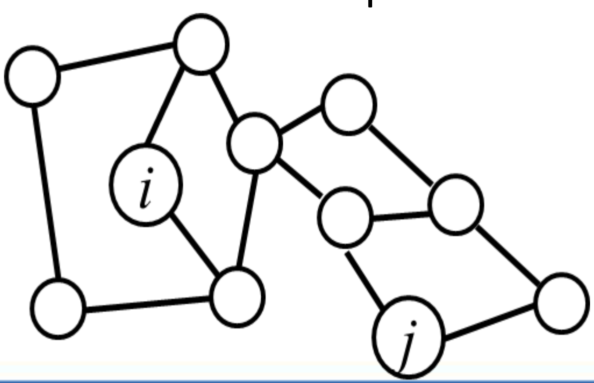
Interpretations

Adjacent nodes



Local measure of how much this edge is used t
maintain stability

Nodes are far apart



Quantifies the lack of coherence the subnetwork connecting i and j in terms of long range disorder Bamieh et al. 2012

[Grunberg & Gayme 2015, in press]

Nodal performance measure

Define the nodal performance with respect to nodes i and j as

$$P_{ij} = \lim_{t\to\infty} E\left[\left(x_i - x_j\right)^2\right]$$

Gaussian input vector $\mathbf{w}(t)$ each element unit strength

Define system output as: $y = \left(e_i - e_j\right)^T x$

e_i is the vectors with zeros everywhere excepts the i^{th} element

Consensus performance

Define the consensus performance with respect to x as

$$P = \lim_{t\to\infty} ||x - x^T 1_n 1_n / n||$$

Interpretations: the deviation from the consensus state

Define system output as: $y = (I - \frac{1}{n} 1_n^T 1_n)x$

Note: $\frac{1}{n} 1_n^T x 1_n$ is the averaging process of x

Undirected graph

- Undirected graph Laplacian, the analytical representation of the H2 norm is **easy** to compute.
  - Unitary diagonalizable, i.e., exists unitary matrix that forms the eigen-matrix of the system
- Directed graph Laplacian, the analytical representation H2 norm is **hard** to obtain.
- Calculate bounds of the H2 norm
  - Singular value
  - Resistance graph
  - ……

[Bamieh, Bullo, Chertkov, Dörfler, Fardad, Jovanović, Mitra, Motee, Patterson …]

Undirected graph

First order

$$\dot{m}_i \dot{x}_i = \sum_{\{i,j\} \in E_B} a_{ij}(x_i - x_j) + w_i$$

$$P = \frac{1}{2} tr(\mathrm{C}^2 L_B^{\dagger})$$

Second order

$$m_i \ddot{x}_i = -\sum_{\{i,j\} \in \mathrm{E}_B} b_{ij}\left(x_i - x_j\right) - \sum_{\{i,j\} \in \mathrm{E}_D} d_{ij}\left(\dot{x}_i - \dot{x}_j\right) + w_i$$

$$P = \frac{1}{2} tr(C^2 L_B^{\dagger} L_D^{\dagger})$$

[Bamieh, Bullo, Chertkov, Dörfler, Fardad, Jovanović, Mitra, Motee, Patterson …]

Directed graph (2nd order system)

A more general representation

$$A = \begin{bmatrix} \mathbf{0}_{n\times n} & I_{n\times n} \\ -(\alpha L_a + (1-\alpha)aI) & -(\beta L_b + (1-\beta)bI) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & I \\ G & F \end{bmatrix}$$

<u>Note:</u>
Current method dealing with second order systems mainly revolves calculating the bounds.

Example: bounds from resistance graph

$$\frac{1}{2\beta} R_{Bij}^{min} \sum_{\{i,j\}\in E} g_{ij} \le \|G\|_{H_2}^2 \le \frac{1}{2\beta} R_{Bij}^{max} \sum_{\{i,j\}\in E} g_{ij}$$

Directed graph (2nd order system)

A more general representation

$$A = \begin{bmatrix} \mathbf{0}_{n\times n} & I_{n\times n} \\ -(\alpha L_a + (1-\alpha)aI) & -(\beta L_b + (1-\beta)bI) \end{bmatrix} \rightarrow \begin{bmatrix} 0 & I \\ G & F \end{bmatrix}$$

<u>Lemma</u>

If matrix G and F are **simultaneously diagonalizable**

i.e., $G = T\Lambda_G T^{-1}$ and $F = T\Lambda_F T^{-1}$; then A is diagonalizable.

Directed graph (2nd order system)

Lemma

If matrix G and F are **simultaneously diagonalizable**

i.e., $G = T\Lambda_G T^{-1}$ and $F = T\Lambda_F T^{-1}$, then A is diagonalizable.

Proof: $A = \hat{T}\Delta\hat{T}^{-1}$ with $\hat{T} = T_{diag}EKR$ where

$$E = \begin{bmatrix} e_1 & e_{n+1} & \cdots & e_i & e_{i+n} & \cdots & e_n & e_{2n} \end{bmatrix} \qquad T_{diag} = \begin{bmatrix} T & \\ & T \end{bmatrix}$$

$$R = \mathrm{diag}\left\{ \begin{bmatrix} -\frac{\lambda_i^F + \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2\sqrt{\lambda_i^G}} & -\frac{\lambda_i^F - \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2\sqrt{\lambda_i^G}} \\ 1 & 1 \end{bmatrix} \right\} \quad K = \mathrm{diag}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda_i^G} \end{bmatrix} \right\}$$

$$\Delta = \mathrm{diag}\left\{ \begin{bmatrix} \frac{\lambda_i^F - \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2} & \\ & \frac{\lambda_i^F + \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2} \end{bmatrix} \right\} = \mathrm{diag}\{\delta_i\}$$

Directed graph (2^{nd} order system)

<u>Theorem</u>

If the graph Laplacian is diagonalizable, and the output matrix satisfies $C1_n = 0$, then the system performance is

$$\|G\|_{H_2}^2 = \mathrm{tr}\left((T^{-1}B)^* \hat{X} T^{-1} B\right)$$

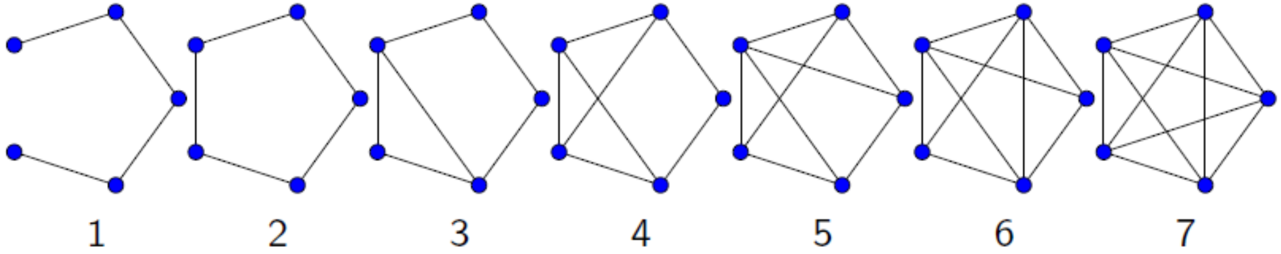
$$\left[\hat{X}\right]_{ls} = \frac{[T^* C^* C T]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}$$

where

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \ or \ s = 1 \\ \dfrac{[T^* C^* C T]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}, & otherwise. \end{cases}$$

The condition $C1_n = 0$ indicates that in the absence of absolute control the system performance is unbounded in any direction, except for the directions orthogonal to vector 1_n.

Advantages



Robust analysis of consensus algorithms

Two takeaways:

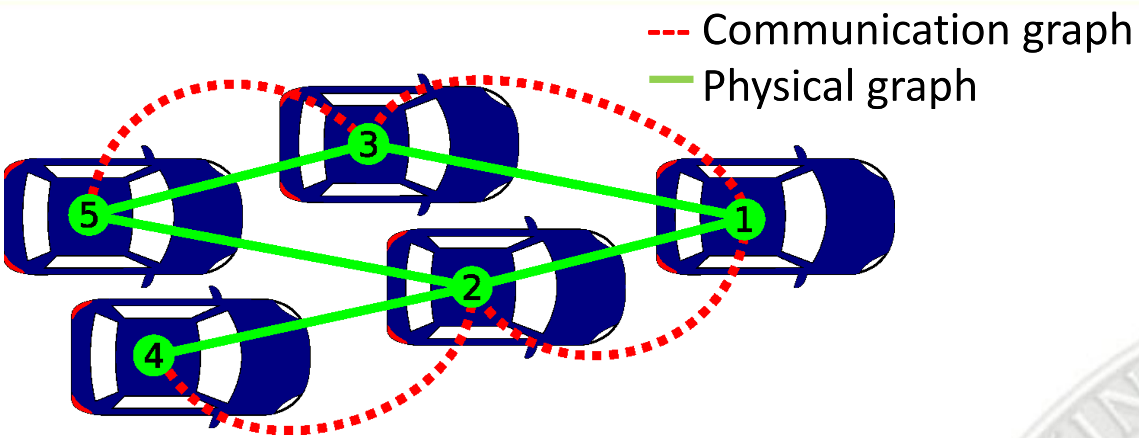
1. In this talk, the H2 norm metric is implemented to analyze the noisy (robust) performance of the consensus algorithms.
2. The performance is not restricted to the consensus performance.

[Ji et. al. 2019]

Example:
Collision potential analysis

Example: Vehicular platoons

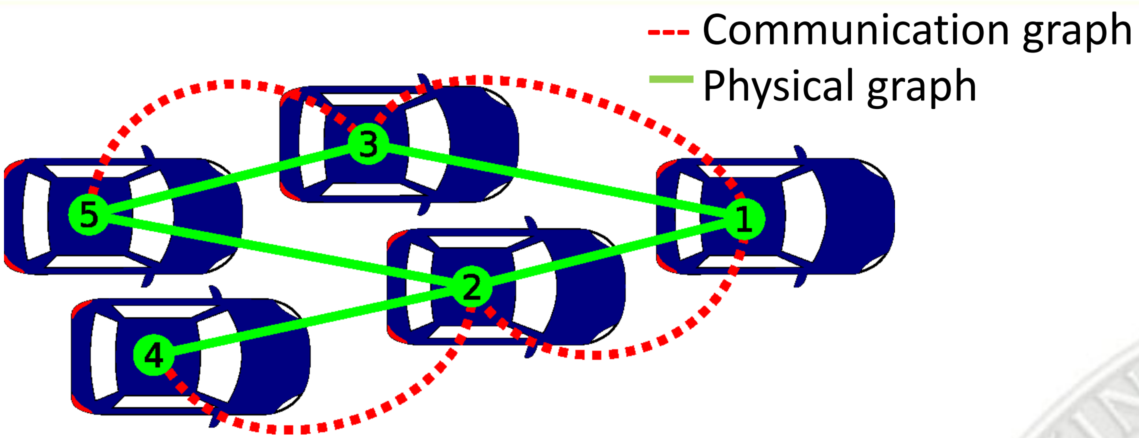
A vehicular platoon of n vehicles attempting to travel in formation



Dynamics of the i^{th} vehicle is given by $m\ddot{x}_i = u_i + w_i$

Example: Vehicular platoons

A vehicular platoon of n vehicles attempting to travel in formation



Dynamics of the i^{th} vehicle is given by $$m\ddot{x}_i = u_i + w_i$$

Consider two control laws for communication graph B

- Relative position and absolute velocity feedback (RPAV):

$$u_i = -\sum_{\{i,j\}\in E_B} b_{ij}\left(x_i - x_j\right) - \beta \dot{x}_i$$

- Relative position and relative velocity feedback (RPRV):

$$u_i = -\sum_{\{i,j\}\in E_B} b_{ij}\left(x_i - x_j\right) - \gamma \sum_{\{i,j\}\in E_B} b_{ij}\left(\dot{x}_i - \dot{x}_j\right)$$

Performance in terms of collision potential

- Consider the question of whether or not two vehicles collide given a particular disturbance $\mathbf{w} \in \mathrm{L}_2$
- Alternative interpretation of nodal performance

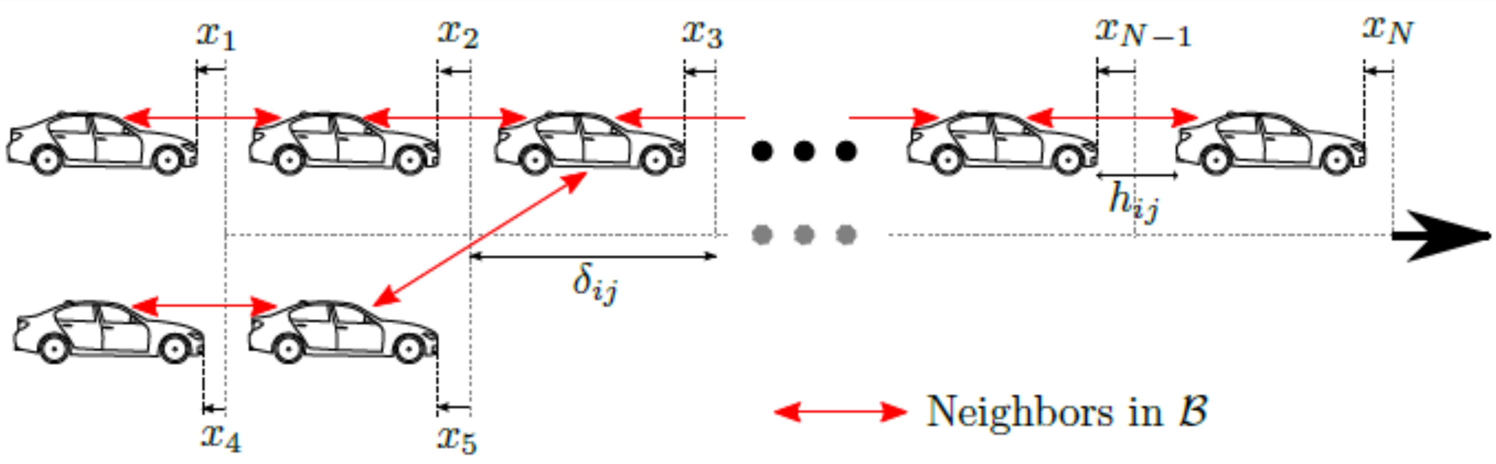
$$P_{ij} = \lim_{t \to \infty} E\left[\left(x_i - x_j\right)^2\right]$$

$$\sqrt{P_{ij}} = \|G\|_{\mathrm{H}_2} = \|G\|_{2,\infty}, \text{ the } \mathrm{L}_2 \text{ to } \mathrm{L}_\infty \text{ induced norm}$$

$$\|G\|_{2,\infty} = \sup_{\|\mathbf{w}\|_{\mathrm{L}_2}=1} |x_i - x_j|$$

[A. Megretski 2004, Grunberg & Gayme ACC 2017]

Robustness in terms of collision potential



$h_{ij} > 0$: nominal spacing between vehicles i and j

- For the line graph above if vehicle j is in front of vehicle i, then these vehicles collide when

$$h_{ij} - x_i + x_j \leq 0$$

- Note: constructing the physical graph is nontrivial in some cases

Main result on collision potential

Theorem

Consider a vehicular network with collision graph C with either (1) relative position and absolute velocity feedback or (2) with both relative position and velocity feedback.

(i) If

$$\|\mathbf{w}\|_{\mathrm{L}_2} < \min_{\{i,j\}\in \mathrm{E}_C} \frac{h_{ij}}{\sqrt{P_{ij}}}$$

then there will be no collisions in the network

(ii) In addition

$$\forall \delta > 0 \text{ there exists } \mathbf{w} \in \mathrm{L}_2 \text{ with } \|\mathbf{w}\|_{\mathrm{L}_2} < (\delta + 1) \min_{\{i,j\}\in \mathrm{E}_C} \frac{h_{ij}}{\sqrt{P_{ij}}}$$

such that vehicles i and j collide

[Ji, Grunberg, Gayme 2018]

Symmetric vs asymmetric

Asymmetric communication has been shown to be beneficial

- Stability Margin
- Reducing the sensing requirements
    - The sensor measure the front and the back vehicle may have different weights.
    - Forward or backward sensor is malfunctioning.

How does asymmetric feedback affect collision potential?

Results: Second-order systems

<u>Theorem:</u> Given a second order system subjected to the relative position and absolute velocity control strategy, if the communication graph of the position is **strongly connected and diagonalizable**, and the output matrix satisfied $H\mathbf{1}_n = 0_n$, then we can represent the performance as

$$\|G_\alpha\|_{H_2}^2 = tr\left((P^{-1}T_{diag}^{-1}B)^*\hat{X}(P^{-1}T_{diag}^{-1}B)\right),$$

where

$$P = EFR.$$

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \ or \ s \leq 2, \\ \dfrac{-1}{\bar{\delta}_l + \delta_s}\bar{q}_l q_s, & otherwise. \end{cases}$$

$$q_l = \left[CT_{diag}P\right]_l,$$

Second-order systems (RPAV)

Performance analysis:

State matrix A is diagonalizable,

$$A = T_{diag} EFR\Delta R^{-1} F^{-1} E^T T_{diag}^{-1}, P = EFR$$

The performance can be representation as

$$\|G_\alpha\|_{H_2}^2 = tr(\hat{P}^* \int_0^\infty e^{\Delta^* t} P^* T_{diag}^* C^* \boxed{CT_{diag}P} e^{\Delta t} dt \; \hat{P})$$

Where

$$T_{diag} = \begin{bmatrix} T & \\ & T \end{bmatrix}$$

$$E = \begin{bmatrix} e_1 & e_{n+1} & e_2 & e_{n+2} & \cdots & e_i & e_{i+n} & \cdots & e_n & e_{2n} \end{bmatrix}$$

$$CT_{diag}P = CT_{diag}EFR$$

$$CT_{diag}E = \begin{bmatrix} t_{i1} - t_{j1} & 0 & t_{i2} - t_{j2} & \cdots & 0 & t_{in} - t_{jn} \end{bmatrix}$$

Notice that

- $t_{i1} = t_{ij} = 1$ (elements in the eigenvector associated with the zero eigenvalue of the Laplacian matrix)
- F and R are block diagonal matrix

Therefore

$$CT_{diag}P = CT_{diag}EFR = q = \begin{bmatrix} 0 & 0 & h_{2n-2} \end{bmatrix}$$

Results: Second-order systems

<u>Theorem:</u> Given a second order system subjected to the relative position and relative velocity control strategy, if the communication graph of the position and velocity are strongly connected and simultaneously diagonalizable, and the output matrix satisfied $H\mathbf{1}_n = 0_n$, then we can represent the performance as

$$\|G_\alpha\|_{H_2}^2 = tr\left((P^{-1}T_{diag}^{-1}B)^*\hat{X}(P^{-1}T_{diag}^{-1}B)\right),$$

where

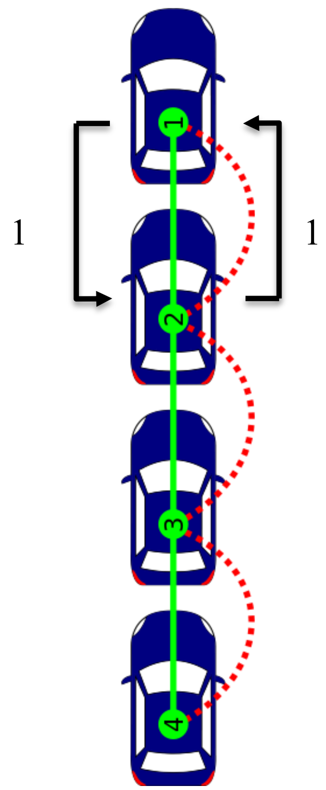
$$P = EFR.$$

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \ or \ s \le 2, \\ \dfrac{-1}{\bar{\delta}_l + \delta_s}\bar{q}_l q_s, & otherwise. \end{cases}$$

$$q_l = [CT_{diag}P]_l,$$

Numerical: Setup

- ▶ 50-Node Line Graph
- ▶ Control Strategies
  - ▶ Relative position and absolute velocity
  - ▶ Relative position and relative velocity
- ▶ **Symmetric feedback**

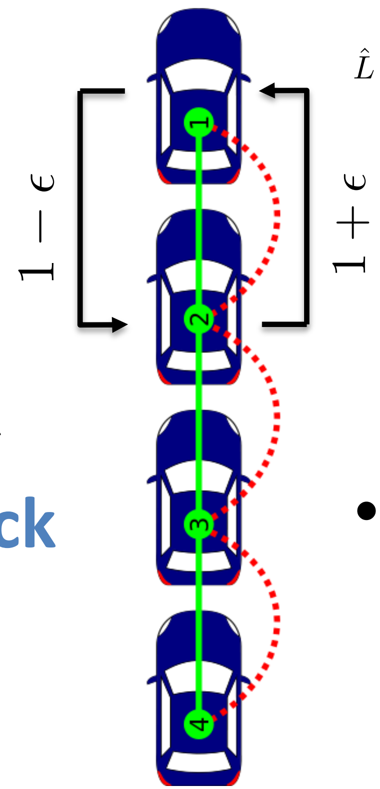


- Graph Laplacian

$$L = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Numerical: Setup

- 50-Node Line Graph
- Control Strategies
  - Relative position and absolute velocity
  - Relative position and relative velocity
- Symmetric feedback
- **Asymmetric feedback**



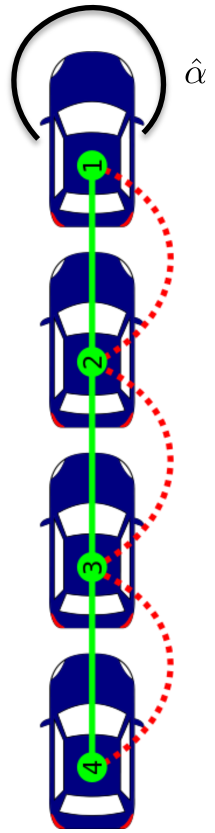
- Graph Laplacian

$$\hat{L} = \begin{bmatrix} 1-\epsilon+\hat{\alpha} & -1+\epsilon & & & \\ -1-\epsilon & 2 & -1+\epsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -1-\epsilon & 2 & -1+\epsilon \\ & & & -1-\epsilon & 1+\epsilon \end{bmatrix}$$

- Note:
  Feedback difference between the node in front and the node behind

Numerical: Setup

- 50-Node Line Graph
- Control Strategies
  - Relative position and absolute velocity
  - Relative position and relative velocity
- Symmetric feedback
- Asymmetric feedback
- **Leader**



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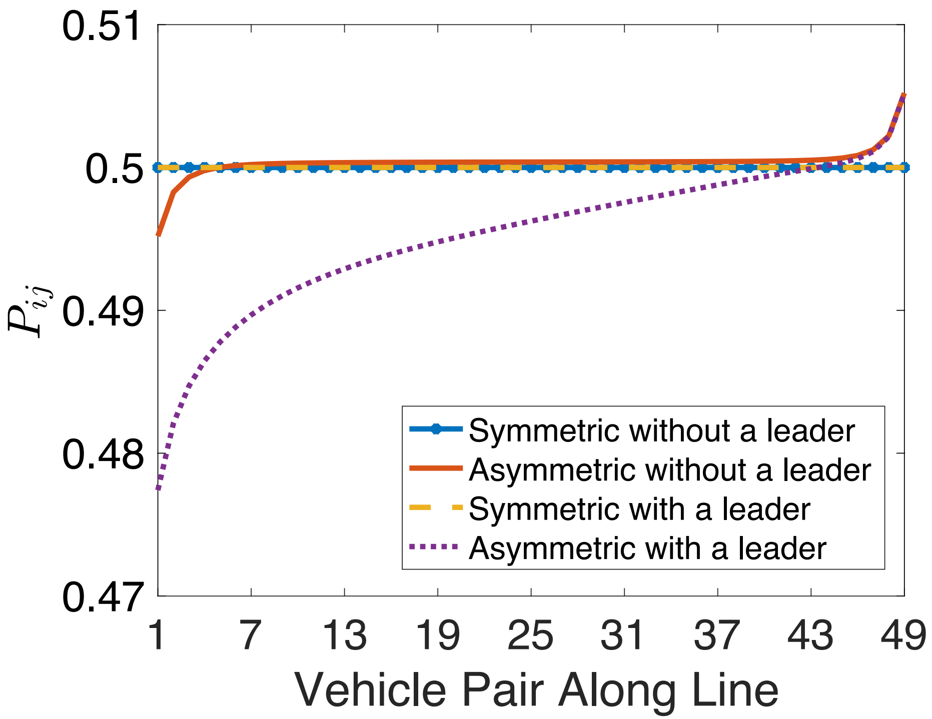
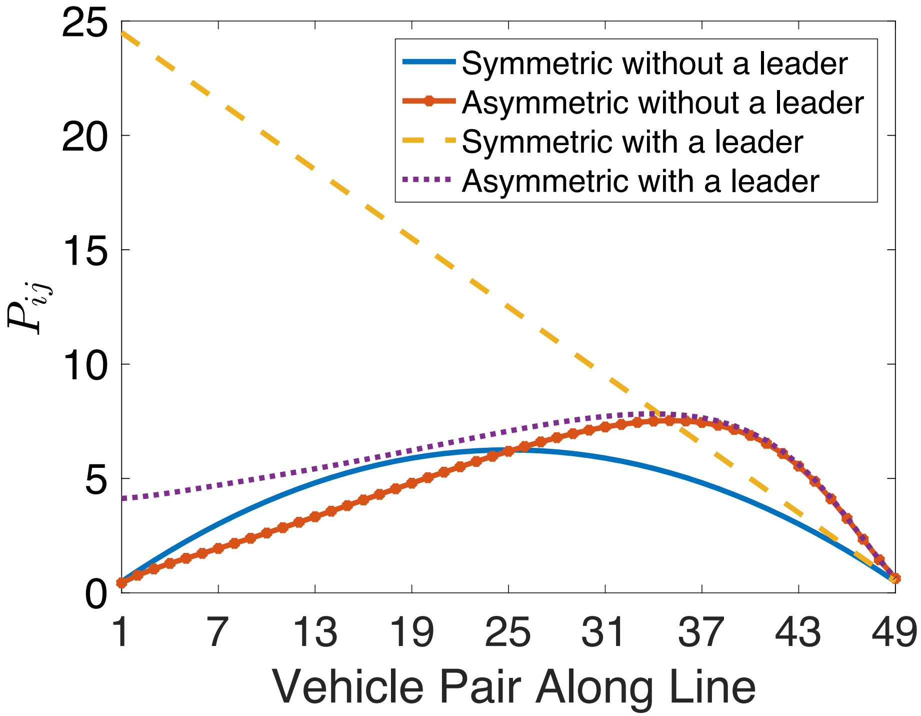
- (Metzler Matrix)

Numerical results



- Control Law: Relative position and absolute velocity
- Symmetric vs asymmetric
  - For both and without leader
  - Performance improved
    - First several nodes
  - Performance degraded
    - The nodes at the end of the platoon
  - Note: As we increasing the asymmetric, the performance of the whole platoon become worse and worse

Numerical results



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Augmented consensus algorithm

However?

- In the aforementioned part of this talk, we have analytically studied the relationship between the graph structure and the performance.

- Common approaches to improving consensus performance include:
  - Altering the network interconnection structure.
  - Introducing global state feedback.

- Drawbacks associated with these common approaches:
  (i) Changes to the consensus state attained (equilibrium value).
  (ii) Increases to the computational complexity.
  (iii) Cost or physical challenges associated with topological changes.

Augmented consensus algorithm

- **Goal:** Develop an easy-to-implement consensus algorithm that improves consensus performance without altering the equilibrium value (the consensus state).

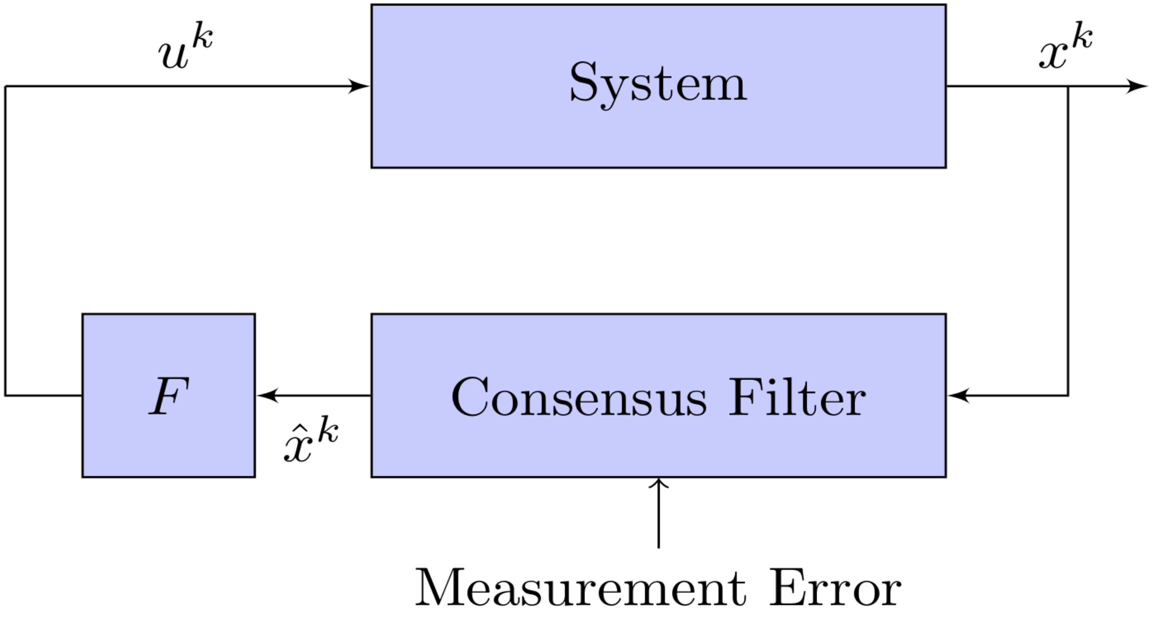
- **Augmented Consensus Algorithm (ACA)**

  1. the ACA **attains the same equilibrium value** (**Nominal Consensus**) as the TCA without altering the network structure.
  2. Better consensus performance and identify a particular class of problems
     - **improves the consensus performance and eliminates the build up of measurement error known to degrade performance in the TCA.**

Augmented consensus algorithm

Consider a networked system with underlying graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{W})$. The system dynamics at step k is given by,

$$x^k = x^{k-1} + u^{k-1}$$



The Consensus Filter is integrated into the feedback loop of the system and generates the state estimation $\hat{x}^k$. The new control input is $u^k = F\hat{x}^k$. In TCA, there is no consensus filter, i.e., $u^k = Fx^k$.

Augmented consensus algorithm

Consider a networked system with underlying graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{W})$.
The system dynamics at step k is given by,

$$x^k = x^{k-1} + u^{k-1}$$

The **consensus filter** is composed of three parts:

$$\begin{aligned} &\text{Measurement:} && z^k = Hx^k + v^k \\ &\text{Prediction:} && \hat{x}^{k|k-1} = \hat{x}^{k-1} + F\hat{x}^{k-1} \\ &\text{Correction:} && \hat{x}^k = \text{argmin } J(\bullet)\big|_{(\hat{x}^{k|k-1}, z^k)} \end{aligned}$$

v^k: measurement error at step k z^k: current measurement at step k
$\hat{x}^{k|k-1}$: state prediction based on state estimation of previous step $\hat{x}^{k-1}$
$\hat{x}^k$: state estimation at step k
F : is the negative Laplacian Matrix with a simple zero eigenvalue.

Augmented consensus algorithm

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&\text{Correction:} \quad && \hat{x}^k = \operatorname{argmin} \, J(\bullet)\big|_{(\hat{x}^{k|k-1}, z^k)}
\end{aligned}$$

The **correction step** is obtained as the solution of the problem:

$$\begin{aligned}
\min \; J(\hat{x}^k) = &\frac{1}{2}(\hat{x}^k - \hat{x}^{k|k-1})^T(\hat{x}^k - \hat{x}^{k|k-1}) \\
&+ \frac{1}{2}(z^k - H\hat{x}^k)^T R^{-1}(z^k - H\hat{x}^k)
\end{aligned}$$

where $R = R^T$ is a weighting matrix

Augmented consensus algorithm

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\end{aligned}$$

Remark 1:When $H = I$ the measurements correspond to the states.
Remark 2: This optimization problem can be adapted to other system errors.

Attaining nominal consensus

The system dynamics is

$$\begin{bmatrix} x^k \\ \hat{x}^k \end{bmatrix} = \hat{A} \begin{bmatrix} x^{k-1} \\ \hat{x}^{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ H^T R^{-1} \end{bmatrix} v^k, \quad \hat{A} = \begin{bmatrix} I & F \\ Q & QF + (I+Q)^{-1} A \end{bmatrix}$$

with $Q = H^T R H$ and $A = I + F$.

Theorem:
If F and Q are simultaneously diagonalizable and the eigenvalues, λ_i and θ_i, of F and Q that correspond to the same eigenvector satisfy,

$$\rho\left(\begin{bmatrix} 1 & \lambda_i \\ \theta_i & \lambda_i \theta_i + \frac{1+\lambda_i}{1+\theta_i} \end{bmatrix}\right) < 1, \ \forall i \neq 1,$$

then the expected value of the system states reach nominal consensus.

Better performance

Consider two networks, $\mathcal{G}_{TCA}$ and $\mathcal{G}_{ACA}$, that are updated according to the TCA and ACA, respectively.

Steady-state variance of the deviation from consensus

- The steady-state variance of $\mathcal{G}_{ACA}$ is lower than $\mathcal{G}_{TCA}$ if,
- $\|\mathcal{G}_{ACA}\|_{\mathcal{H}_2} < \|\mathcal{G}_{TCA}\|_{\mathcal{H}_2}$, with $y^k = \left(I - \frac{1}{n} 1_n 1_n^T\right) x^k$.

State deviation at a particular time step, *p*

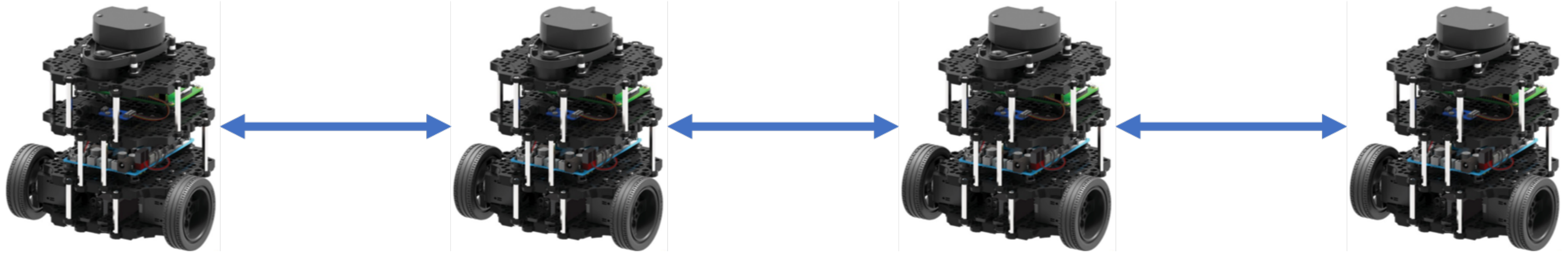
- The state deviation at step *p* is lower of $\mathcal{G}_{ACA}$ is lower than $\mathcal{G}_{TCA}$, if
$$\left\| \begin{bmatrix} I & 0 \end{bmatrix} \sum_{i=1}^{p} \hat{A}^{p-i} \begin{bmatrix} 0_{n\times n} \\ H^T R^{-1} \end{bmatrix} v^i \right\| < \left\| \sum_{i=1}^{p} A^{p-i} F v^i \right\| .$$

No additional error buildup

If the measurement error at step h satisfies, $H^T R^{-1} v^h \in span\{1_n\}$, then for any step $k > h$, there is no additional buildup of measurement error.

Remark: In the TCA, the measurement error can only be eliminated if and only if $v^h \in span\{1_n\}$. The ACA can **bypass** this strong condition by tuning the algorithm parameters R and H.

Numerical analysis



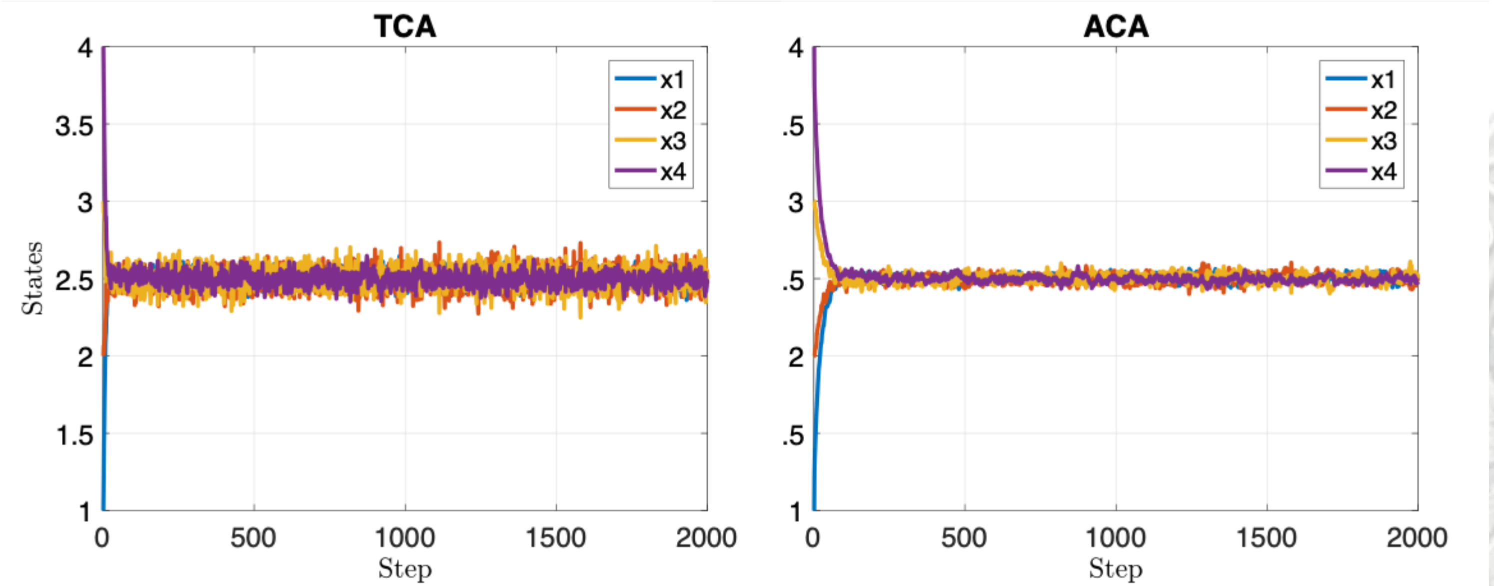


Figure 1. The average variance of the state deviation for TCA is 0.02, whereas the ACA reduces the average variance to 0.005.

Conclusions

- Proposed a new performance analyzing framework that can be used to evaluate input-output system performance
  - Illustrated our framework with a vehicle platoon example
- Proposed a filter-based algorithm that improve the consensus performance of dynamical systems
  - Three advantages: Attaining nominal consensus; Better performance; No additional error buildup

Publications

Augmented Consensus Algorithm for Discrete-time Dynamical Systems

Chengda Ji * Yue Shen ** Marin Kobilarov *
Dennice F. Gayme *

Evaluating Robustness of Consensus Algorithms Under Measurement Error over Digraphs

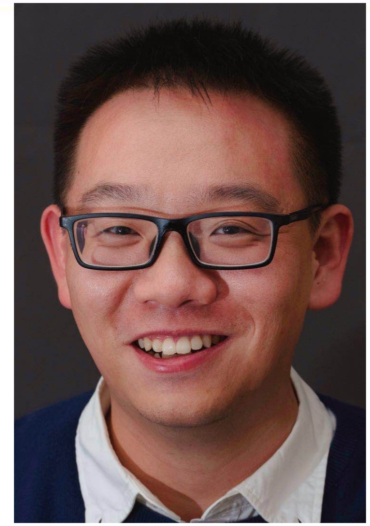
Chengda Ji, Enrique Mallada and Dennice F. Gayme

Collision Potential Analysis in First and Second Order Integrator Networks Over Strongly Connected Digraphs

Chengda Ji and Dennice F. Gayme

Acknowledgements

Yue Shen
Hopkins
Advisor: Prof. Enrique Mallada



Yi Guo
UT Dallas
Advisor: Prof. Tyler Summers

Funding:

National Science Foundation: *ECCS 1230788 & CNS 1544771*

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NSF

Acknowledgements

Thank you for your attention!

If you have any **questions/ comments / suggestions** regards this work, feel free to reach me at:

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