Higher Order Programming: Maps and Filters

Concepts of Programming Languages

Lecture 9

Warm Up with Options (Syntax)

» Recall option types

```
type 'a option = None | Some of 'a
```

» Formal Syntax for Constructing Options:

```
<expr> ::= None | Some ( <expr> )
```

» Formal Syntax for Destructing Options:

```
<expr> ::= match <expr> with
           | None -> <expr>
           | Some ( <var> ) -> <expr>
```

Typing Rules for Options

$$\overline{\Gamma \vdash \text{None} : \tau \text{ option}}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{Some}(e) : \tau \text{ option}}$$

» **None** is of type τ option in context Γ

» If e is of type τ in context Γ, then **Some(e)** is of type τ option in context Γ

Typing Rules for Options

$$\frac{}{\Gamma \vdash \text{None} : \tau \text{ option}} \text{ (none)}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{Some}(e) : \tau \text{ option}} \text{ (some)}$$

» None is of type τ option in context Γ

» If e is of type τ in context Γ, then Some(e) is of type τ option in context Γ

Typing Rule for Option Match

$$\frac{\Gamma \vdash e : \tau' \text{ option} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma, x : \tau' \vdash e_2 : \tau}{\Gamma \vdash \text{match } e \text{ with } | \text{None} \rightarrow e_1 | \text{Some}(x) \rightarrow e_2 : \tau}$$

If e is of type τ' option in context Γ, and
e_1 is of type τ in the same context Γ, and
e_2 is of type τ in context Γ extended with $x : \tau'$, then
match e with | None -> e_1 | Some(x) -> e_2
is of type τ in context Γ

Typing Rule for Option Match

$$\frac{\Gamma \vdash e : \tau' \text{ option} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma, x : \tau' \vdash e_2 : \tau}{\Gamma \vdash \text{match } e \text{ with } | \text{ None} \rightarrow e_1 | \text{ Some}(x) \rightarrow e_2 : \tau} \text{ (opt}-\text{match)}$$

If e is of type τ' option in context Γ, and
e_1 is of type τ in the same context Γ, and
e_2 is of type τ in context Γ extended with $x : \tau'$, then
match e with | None -> e_1 | Some(x) -> e_2
is of type τ in context Γ

Semantics Rules for Options

$$None \Downarrow None$$

$$\frac{e \Downarrow v}{Some(e) \Downarrow Some(v)}$$

» None evaluates to None

» If e evaluates to v,
  then Some(e) evaluates to Some(v)

Semantics Rules for Options

$$\frac{}{\text{None} \Downarrow \text{None}} \text{ (none-eval)}$$

$$\frac{e \Downarrow v}{\text{Some}(e) \Downarrow \text{Some}(v)} \text{ (some-eval)}$$

» None evaluates to None

» If e evaluates to v,
  then Some(e) evaluates to Some(v)

Semantic Rule 1 for Option Match

$$\frac{e \Downarrow None \qquad e_1 \Downarrow v}{\text{match } e \text{ with } | None \to e_1 | Some(x) \to e_2 \Downarrow v}$$

$$\frac{e \Downarrow Some(v) \qquad [v/x]e_2 = e_2' \qquad e_2' \Downarrow v'}{\text{match } e \text{ with } | None \to e_1 | Some(x) \to e_2 \Downarrow v'}$$

» If e evaluates to None, and e_1 evaluates to v, then match e with | None –> e_1 | Some(x) –> e_2 evaluates to v

» If e evaluates to Some(v), and e_2 with v substituted for x evaluates to v', then match e with | None –> e_1 | Some(x) –> e_2 evaluates to v'

Semantic Rule 1 for Option Match

$$\frac{e \Downarrow \textbf{None} \qquad e_1 \Downarrow v}{\textsf{match } e \textsf{ with } | \textbf{None} \to e_1 | \textbf{Some}(x) \to e_2 \Downarrow v} \text{ (opt−eval−none)}$$

$$\frac{e \Downarrow \textbf{Some}(v) \qquad e_2' = [v/x]e_2 \qquad e_2' \Downarrow v'}{\textsf{match } e \textsf{ with } | \textbf{None} \to e_1 | \textbf{Some}(x) \to e_2 \Downarrow v'} \text{ (opt−eval−some)}$$

» If e evaluates to None, and e_1 evaluates to v, then match e with | None –> e_1 | Some(x) –> e_2 evaluates to v

» If e evaluates to Some(v), and e_2 with v substituted for x evaluates to v', then match e with | None –> e_1 | Some(x) –> e_2 evaluates to v'

Outline

» Introduce the notion of **higher-order functions (HOFs)** as a way to write cleaner, more general code

» Examine two common HOFs: **map** and **filter**

Higher-Order Functions

Higher-Order Programming

In OCaml, functions are **first-class values**

They can be:

1. <u>returned</u> by another function

2. given names with <u>let-definitions</u>

3. <u>passed as arguments</u> to another function

Note. Types are *not* first-class values

Functions as Parameters

```
# let apply f x = f x;;
val apply : ('a -> 'b) -> 'a -> 'b = <fun>
# apply (fun x -> x+5) 10;;
- : int = 15
# let apply2 f x = f (f x);;
val apply : ('a -> 'a) -> 'a -> 'a = <fun>
# apply2 (fun x -> x+5) 10;;
- : int = 20
```

note the type

note the type

This allows us to create new functions which
are *parametrized* by old ones

First-Order Function Types

```
int -> string

t -> t

() -> bool

bool * bool -> bool
```

Second-Order Function Types

(**int -> string**) -> (**int -> string**)

t -> (**s -> t**)

(**() -> bool**) -> bool

bool -> **bool -> bool**

Third-Order Functions

(int -> string) -> **(int -> string) -> (int -> string)**

(**t -> (s -> t)**) -> t

((**() -> bool) -> bool**) -> bool

(**bool -> bool -> bool**) * bool -> bool

And so on...

```
1st: int
2nd: int -> int
3rd: (int -> int) -> int
4th: ((int -> int) -> int) -> int
5th: (((int -> int) -> int) -> int) -> int
6th: ((((int -> int) -> int) -> int) -> int) -> int
7th: (((((int -> int) -> int) -> int) -> int) -> int) -> int
8th: ((((((int -> int) -> int) -> int) -> int) -> int) -> int) -> int
⋮
```

The **higher-order** part comes from the fact that
we can do this *ad infinitum*

(In practice, we rarely use higher than third-order or fourth-order functions)

The Abstraction Principle

Motivation

» Observe the main computation happening in a function; see what operations are needed

» Can those operations become parameters? Yes! Because operations are like functions, and HOFs allow functions as parameters

» We will see two examples: **map** and **filter**

Observe the Pattern!

```
let rec fact n =
    if n = 0 then 0
    else n * fact (n – 1)

let rec sum n =
    if n = 0 then 0
    else n + sum (n – 1)
```

Can we abstract the core functionality?

demo
(accumulate)

Accumulation Pattern

```
let rec accum f n start =
    let rec go n =
        match n with
        | 0 -> start
        | n -> f n (go (n - 1))
    in go n
```

In order to generalize this function, we need
to be able to take the *operation as a parameter*

Now we have a single function which we can
reuse elsewhere

Practice Problem

Implement the function

val sum_squares : int -> int

so that **sum_squares n** *is the sum* **1^2 + 2^2 + … + n^2**

val sum_cubes : int -> int

so that **sum_cubes n** *is the sum* **1^3 + 2^3 + … + n^3**

Write a single function that can be used to implement both

⭐

Map

Observe the Pattern!

» Each element is being modified

» Modification function is the same for all elements

» **map** is used to apply a function to every element in a list (or other structure)

```
int list -> int list
let rec inc l =
  match l with
  | [] -> []
  | h::t -> (h+1)::(inc t)

int list -> int list
let rec dec l =
  match l with
  | [] -> []
  | h::t -> (h-1)::(dec t)

int list -> int list
let rec double l =
  match l with
  | [] -> []
  | h::t -> (2*h)::(double t)
```

Definition of Map

```
let rec map f l =
  match l with
  | [] -> []
  | x :: xs -> f x :: map f xs
```

Is this tail recursive?

» *If the list is empty there is nothing to do*

» *If the list is nonempty, we apply f to its first element, and recurse*

Tail-Recursive Map

```
let rec map_tr f l =
  let rec go l acc =
    match l with
    | [] -> List.rev acc
    | x :: xs -> go xs (f x :: acc)
  in go l []
```

For a tail-recursive version we can build the list in reverse in acc and then *reverse it at the end*

This may seem inefficient, but it's just a *constant factor* slower

They are so easy now!

```
int list -> int list
let inc l = map (fun x -> x + 1) l

int list -> int list
let dec l = map (fun x -> x - 1) l

int list -> int list
let double l = map (fun x -> 2 * x) l
```

```
int list -> int list
let rec inc l =
    match l with
    | [] -> []
    | h::t -> (h+1)::(inc t)

int list -> int list
let rec dec l =
    match l with
    | [] -> []
    | h::t -> (h-1)::(dec t)

int list -> int list
let rec double l =
    match l with
    | [] -> []
    | h::t -> (2*h)::(double t)
```

demo
(int/float lists)

Practice Problem

Implement then function

val pointwise_max : ('a -> 'b) -> ('a -> 'b)
-> 'a list -> 'b list

*so that **pointwise_max f g l** is **l** but with **f** or **g** applied to each element, whichever gives the larger value*

★

Filter

Observe the Pattern!

```
int list -> int list
let rec evens l =
  match l with
  | [] -> []
  | x::xs -> (if x mod 2 = 0 then [x] else []) @ evens xs
```

```
int list -> int list
let rec pos l =
  match l with
  | [] -> []
  | x::xs -> (if x >= 0 then [x] else []) @ pos xs
```

filter is used to do grab all elements in a list which *satisfy a given property*

Predicates

> **Definition:** A **Boolean predicate** on `'a` is a function of type `'a -> bool`

A predicate is a function which defines a *property*

Examples:

```
let even n = n mod 2 = 0
let even_length l = even (List.length l)
```

Definition of Filter

```
let rec filter p l =
  match l with
  | [] -> []
  | x :: xs ->
    (if p x then [x] else []) @ filter p xs
    Is this tail recursive?
```

» *If the list is empty there is nothing to do*

» *If the first element satisfies our predicate we keep it and recurse*

» *Otherwise, we drop it and recurse*

Tail-Recursive Definition of Filter

```
let filter_tr p =
  let rec go acc l =
    match l with
    | [] -> List.rev acc
    | x :: xs -> go (((if p x then [x] else []) @ acc) xs
  in go []
```

As with map, we have to reverse the output
before returning it

This is so easy now!

```
int list -> int list
let evens l = filter (fun x -> x mod 2 = 0) l


int list -> int list
let pos l = filter (fun x -> x >= 0) l
```

filter just needs a filtering function as parameter

Understanding Check

```
let h p q = filter (fun i -> p i && q i)
```

What does the above function do?

demo
(filter function on functions)

Summary

Higher-order function allow for better **abstraction** because we can **parameterize** functions by other functions

List.map and **List.filter** are very common patterns which can be used to write clean and simple code