Decision theory

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Bayesian decision theory

Suppose we have an unknown quantity θ which we believe follows a probability distribution $p(\theta)$ and a decision (or action) δ. For each decision, we have a loss function $L(\theta, \delta)$ that describes how much we lose if θ is the truth. The expected loss is taken with respect to $\theta \sim p(\theta)$, i.e.

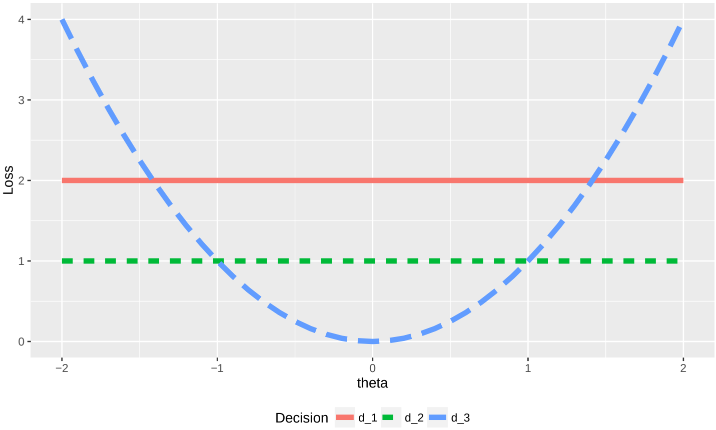
$$E_\theta[L(\theta, \delta)] = \int L(\theta, \delta) p(\theta) d\theta = f(\delta).$$

The optimal Bayesian decision is to choose δ that minimizes the expected loss, i.e.

$$\delta_{opt} = \text{argmin}_\delta E[L(\theta, \delta)] = \text{argmin}_\delta f(\delta).$$

Economists typically maximize expected utility where utility is the negative of loss, i.e. $U(\theta, \delta) = -L(\theta, \delta)$. If we have data, just replace the prior $p(\theta)$ with the posterior $p(\theta|y)$.

Depicting loss/utility functions



Which hand?

The setup:

- Randomly put a quarter in one of two hands with probability p.
- Let $\theta \in \{0, 1\}$ indicate that the quarter is in the right hand.
- You get to choose whether the quarter is in the right hand or not.
- If you guess the quarter is in the right hand and it is, you get to keep the quarter. Otherwise, you don't get anything.

We have $\theta \sim Ber(p)$ and two actions

- a_0: say the quarter is not in the right hand and
- a_1: say the quarter is in the right hand.

Thus, the utility is

$$U(\theta, a_i) = \begin{cases} \$0.25\theta & \text{if } a_1 \\ 0 & \text{if } a_0 \end{cases}$$

and the expected utility is

$$E[U(\theta, a_i)] = \begin{cases} \$0.25p & \text{if } a_1 \\ 0 & \text{if } a_0 \end{cases}$$

So, we maximize expected utility by taking a_1 if $p > 0$.

How many quarters in the jar?

Suppose a jar is filled up to a pre-specified line. Let θ be the number of quarters in the jar. Provide a probability distribution for your uncertainty in θ. Suppose you choose

$$\theta \sim N(\mu, \sigma^2)$$

Since $\theta \in \mathbb{N}^+$, we can provide a formal prior by letting

$$P(\theta = q) \propto N(q; \mu, \sigma^2)\mathrm{I}(0 < q \le U)$$

for some upper bound U.

Guessing how many quarters are in the jar.

Now you are asked to guess how many quarters are in the jar. What should you guess?

Let q be the guess that the number of quarters is q, then our utility is

$$U(\theta, q) = q\mathrm{I}(\theta = q)$$

and our expected utility is

$$E_\theta[U(\theta, q)] = qP(\theta = q) \propto qN(q; \mu, \sigma^2)\mathrm{I}(0 \le q \le U).$$

Deriving the optimal decision

Here are three approaches for deriving the optimal decision:

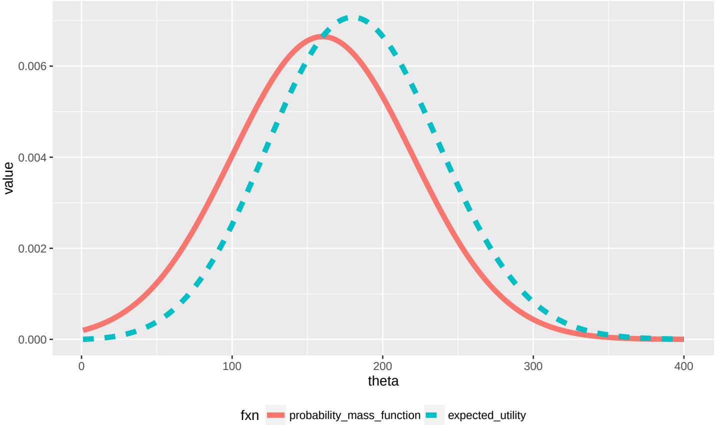
$$\text{argmax}_q f(q), \quad f(q) = qN(q; \mu, \sigma^2)\mathrm{I}(0 \le q \le U)$$

1. Evaluate $f(q)$ for $q \in \{1, 2, \ldots, U\}$ and find which one is the maximum.
2. Treat q as continuous and use a numerical optimization routine.
3. Take the derivative of $f(q)$, set it equal to zero, and solve for q.

In all cases, you are better off taking the log $f(q)$ which is monotonic and therefore will still provide the same maximum as $f(q)$.

Visualizing the expected log utility

```
# p(theta) \propto N(theta;mu,sigma^2)I(1<= theta <= 400)
mu=160; sigma=60; U=400
```



Computational approaches

```r
log_f = Vectorize(function(q, mu, sigma, U) {
  if (q<0 | q>U) return(-Inf)
  return(log(q) + dnorm(q, mu, sigma, log=TRUE))
})

# Evaluate all options
log_expected_utility = log_f(1:U, mu=mu, sigma=sigma, U=U)
which.max(log_expected_utility) # since we are using integers 1:U
```

```
[1] 180
```

```r
# Numerical optimization
optimize(function(x) log_f(x, mu=mu, sigma=sigma, U=U), c(1,U), maximum=TRUE)
```

```
$maximum
[1] 180

$objective
[1] 0.1241182
```

Derivation

The function to maximize is

$$\log f(q) = \log(q) - (q-\mu)^2/2\sigma^2.$$

The derivative is

$$\frac{d}{dq}\log f(q) = \frac{1}{q} - (q-\mu)/\sigma^2.$$

Setting this equal to zero and multiplying by $-q\sigma^2$ results in

$$q^2 - \mu q - \sigma^2 = 0.$$

This is a quadratic with roots at

$$\frac{\mu \pm \sqrt{\mu^2 + 4\sigma^2}}{2}.$$

Since q must be positive, the answer is

```
(mu+sqrt(mu^2+4*sigma^2))/2
```

```
[1] 180
```