# R03 - Using logarithms in regression

STAT 401 (Engineering) - Iowa State University

March 23, 2018

# Parameter interpretation in regression

If

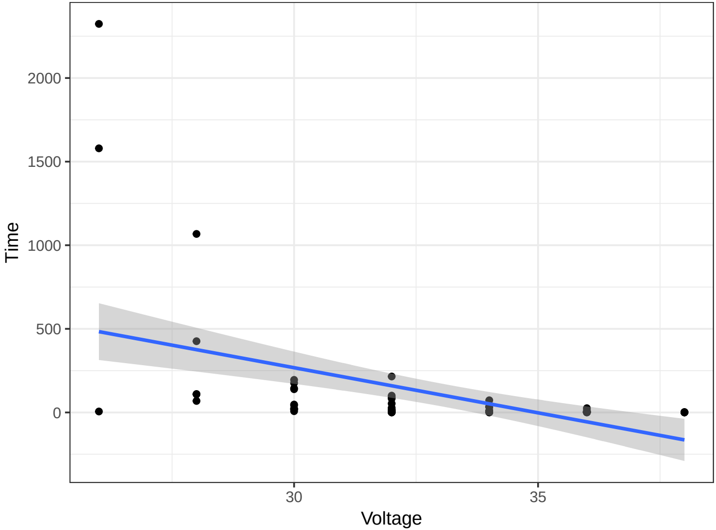
$$E[Y|X] = \beta_0 + \beta_1 X,$$

then

- $\beta_0$ is the expected response when $X$ is zero and
- $\beta_1$ is the expected increase in the response for a one unit increase in the explanatory variable.
- $d\beta_1$ is the expected increase in the response for a $d$ unit increase in the explanatory variable.

For the following discussion,

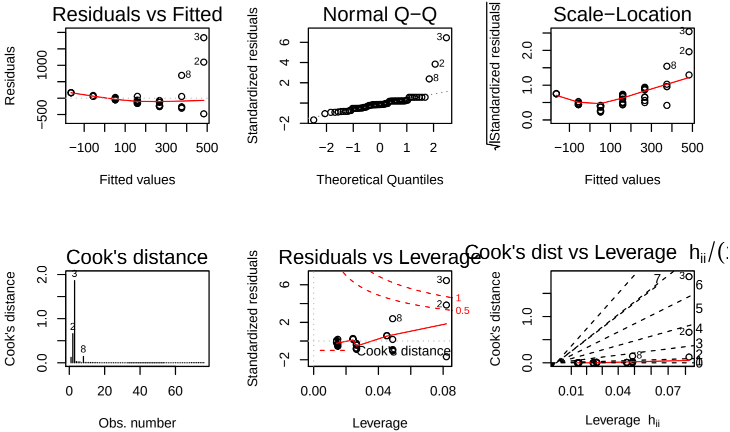
- $Y$ is always going to be the original response and
- $X$ is always going to be the original explanatory variable.


Time
2000
1500
1000
500
0
30
35
Voltage

# Run the regression and look at diagnostics

```
m <- lm(Time ~ Voltage, insulating)
```

```
opar = par(mfrow=c(2,3)); plot(m, 1:6, ask=FALSE); par(opar)
```



# Interpretations using logs

The most common transformation of either the response or explanatory variable(s) is to take logarithms because

- linearity will often then be approximately true,
- the variance will likely be approximately constant,
- influence of some observations may decrease, and
- there is a (relatively) convenient interpretation.

We will talk about interpretation of $\beta_0$ and $\beta_1$ when

- only the response is logged,
- only the explanatory variable is logged, and
- when both are logged.

# Example

Suppose

- $Y$ is corn yield (bushels/acre)
- $X$ is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 X$$

- $\beta_0$ is the expected corn yield (bushels/acre) when fertilizer level is zero and
- $\beta_1$ is the expected change in corn yield (bushels/acre) when fertilizer is increased by 1 lb/acre or
- $d\beta_1$ is the expected change in corn yield (bushels/acre) when fertilizer is increased by $d$ lb/acre.

# Response is logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 X$$

then

- $\beta_0$ is the expected $\log(Y)$ when $X$ is zero,
- $\beta_1$ is the expected change in $\log(Y)$ for a one unit increase in the explanatory variable, and
- $d\beta_1$ is the expected change in $\log(Y)$ for a $d$ unit increase in the explanatory variable.

But since

$$E[\log(Y)|X] = \text{Median}[\log(Y)|X] = \log(\text{Median}[Y|X])$$

we have

$$\text{Median}[Y|X] = e^{\beta_0+\beta_1 X} = e^{\beta_0} e^{\beta_1 X}$$

then

- $e^{\beta_0}$ is the median of $Y$ when $X$ is zero,

# Response is logged

Suppose

- $Y$ is corn yield (bushels/acre)
- $X$ is fertilizer level in lbs/acre

If we assume

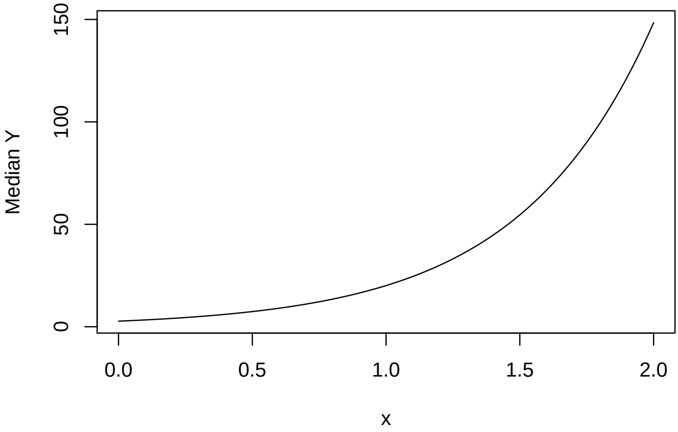
$$E[\log(Y)|X] = \beta_0 + \beta_1 X$$

then

$$\text{Median}[Y|X] = e^{\beta_0} e^{\beta_1 X}$$

- $e^{\beta_0}$ is the median corn yield (bushels/acre) when fertilizer level is 0,
- $e^{\beta_1}$ is the multiplicative effect in median corn yield (bushels/acre) when fertilizer is increased by 1 lb/acre, and
- $e^{d\beta_1}$ is the multiplicative effect in median corn yield (bushels/acre) when fertilizer is increased by $d$ lb/acre.

# Response is logged



# Explanatory variable is logged

If

$$E[Y|X] = \beta_0 + \beta_1 \log(X),$$

then

- $\beta_0$ is the expected response when $\log(X)$ is zero and
- $\beta_1$ is the expected change in the response for a one unit increase in $\log(X)$

Alternatively,

- $\beta_0$ is the expected response when $X$ is 1 and
- $\beta_1 \log(d)$ is the expected change in the response when $X$ increases multiplicatively by $d$, e.g.
    - $\beta_1 \log(2)$ is the expected change in the response for each doubling of $X$ or
    - $\beta_1 \log(10)$ is the expected change in the response for each ten-fold increase in $X$.

# Explanatory variable is logged

Suppose

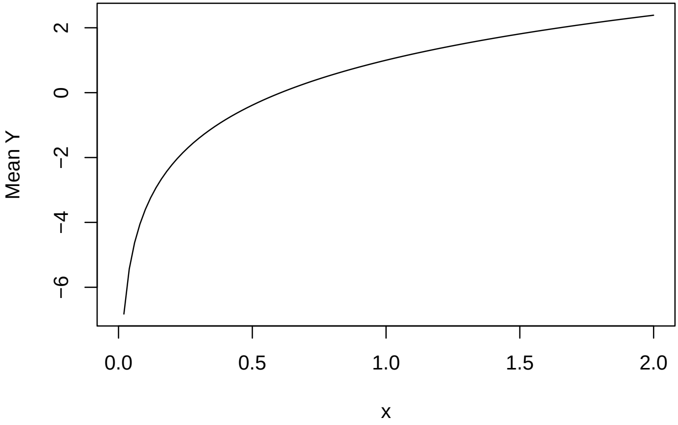
- $Y$ is corn yield (bushels/acre)
- $X$ is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 \log(X)$$

- $\beta_0$ is the expected corn yield (bushels/acre) when fertilizer level is 1 lb/acre and
- $\beta_1 \log(2)$ is the expected change in corn yield when fertilizer level is doubled.

# Response is logged



# Both response and explanatory variable are logged

If we assume

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X),$$

then

- $\beta_0$ is the expected $\log(Y)$ when $\log(X)$ is zero and
- $\beta_1$ is the expected change in $\log(Y)$ for a one unit increase in $\log(X)$.

But we also have

$$\text{Median}[Y|X] = e^{\beta_0+\beta_1 \log(X)} = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1},$$

and thus

- $e^{\beta_0}$ is the median of $Y$ when $X$ is 1 and
- $d^{\beta_1}$ is the multiplicative change in the median of the response when $X$ increases multiplicatively by $d$, e.g.
  - $2^{\beta_1}$ is the multiplicative effect on the median of the response for each doubling of $X$ or
  - $10^{\beta_1}$ is the multiplicative effect on the median of the response for each ten-fold increase in $X$.

# Both response and explanatory variables are logged

Suppose

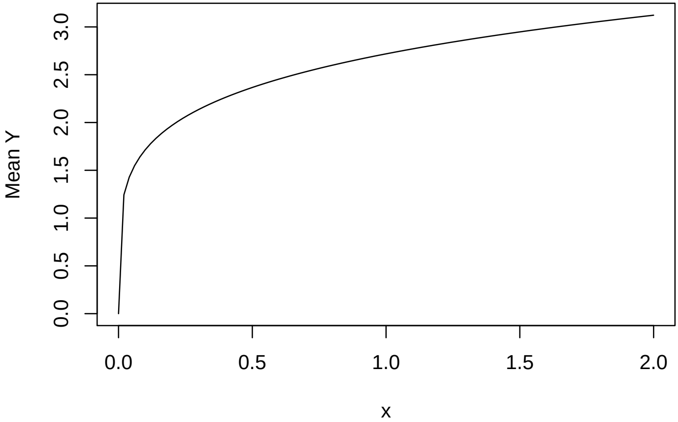
- $Y$ is corn yield (bushels/acre)
- $X$ is fertilizer level in lbs/acre

Then, if

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X) \quad \text{or} \quad \text{Median}[Y|X] = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1}$$

- $e^{\beta_0}$ is the median corn yield (bushels/acre) when fertilizer level is 1 lb/acre and
- $2^{\beta_1}$ is the multiplicative effect on median corn yield (bushels/acre) when fertilizer level doubles.

# Both response and explanatory variables are logged



# Summary of interpretations when using logarithms

- When using the log of the response,
  - $\beta_0$ will affect the median response
  - $\beta_1$ will affect the multiplicative change in the median response
- When using the log of the explanatory variable ($X$),
  - $\beta_0$ will affect the response when $X = 1$
  - $\beta_1$ will affect the change in the response when there is a multiplicative change in $X$

To construct confidence intervals for $f(\beta)$ (when $f()$ is monotonic, e.g. $f(x) = dx$ or $f(x) = \exp(x)$), find a confidence interval for $\beta$ and evaluate the function at the endpoints, i.e. if $(L, U)$ is a confidence interval for $\beta$, then $(f(L), f(U))$ is a confidence interval for $f(\beta)$.

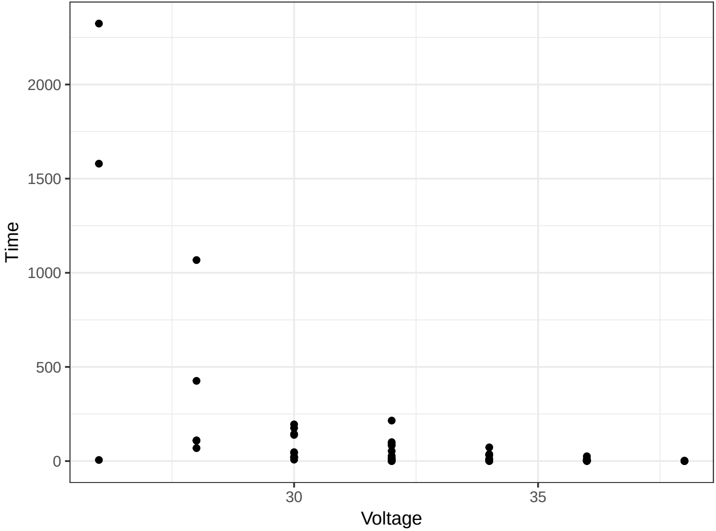
# Breakdown times

*In an industrial laboratory, under uniform conditions, batches of electrical insulating fluid were subjected to constant voltages (kV) until the insulating property of the fluids broke down. Seven different voltage levels were studied and the measured reponses were the times (minutes) until breakdown.*

```
summary(insulating)
```

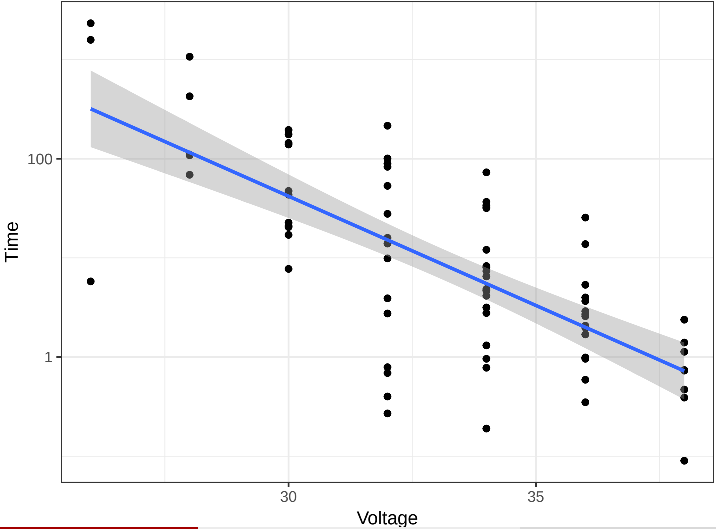
```
      Time              Voltage          Group   
 Min.   :   0.090   Min.   :26.00   Group1: 3  
 1st Qu.:   1.617   1st Qu.:31.50   Group2: 5  
 Median :   6.925   Median :34.00   Group3:11  
 Mean   :  98.558   Mean   :33.13   Group4:15  
 3rd Qu.:  38.383   3rd Qu.:36.00   Group5:19  
 Max.   :2323.700   Max.   :38.00   Group6:15  
                                    Group7: 8  
```

```
g <- ggplot(insulating, aes(Voltage, Time)) + geom_point() + theme_bw(); g
```



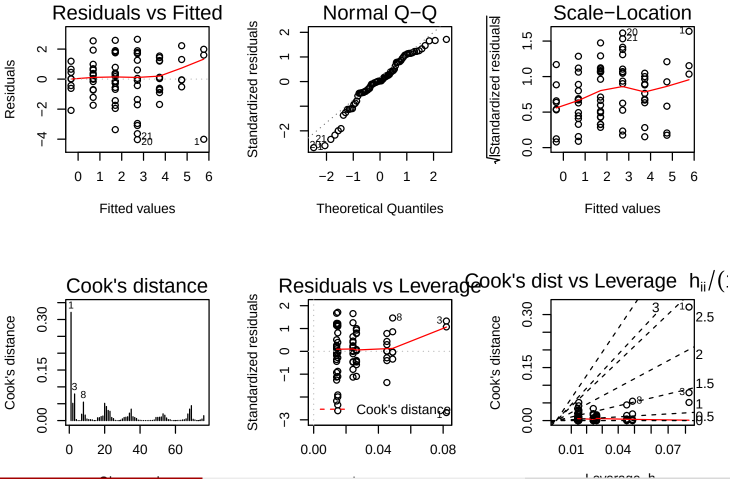
# Take log of time

```
g + stat_smooth(method="lm") + scale_y_log10()
```



# Take log of time

```
m <- lm(log(Time) ~ Voltage, insulating)
opar = par(mfrow=c(2,3)); plot(m, 1:6, ask=FALSE); par(opar)
```



# Summary

```
m$coefficients %>% exp
```

```
 (Intercept)      Voltage
1.707069e+08 6.020800e-01
```

```
confint(m) %>% exp()
```

```
                   2.5 %       97.5 %
(Intercept) 3.796778e+06 7.675154e+09
Voltage     5.370152e-01 6.750281e-01
```

```
lm(log(Time) ~ Voltage, insulating, subset= Time != 5.79) %>% confint() %>% exp() # remove first observation
```

```
                   2.5 %       97.5 %
(Intercept) 1.658205e+07 3.219178e+10
Voltage     5.153150e-01 6.465834e-01
```

Summary:
Each 1 kV increase in voltage caused a multiplicative effect of 0.6 (0.5,0.7) on median breakdown time.