STAT 401A - Statistical Methods for Research Workers

Regression diagnostics

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All models are wrong!

George Box (Empirical Model-Building and Response Surfaces, 1987):

All models are wrong, but some are useful.

`http://stats.stackexchange.com/questions/57407/what-is-the-meaning-of-all-models-are-wrong-but-some-are-useful`

> *"All models are wrong" that is, every model is wrong because it is a simplification of reality. Some models, especially in the "hard" sciences, are only a little wrong. They ignore things like friction or the gravitational effect of tiny bodies. Other models are a lot wrong - they ignore bigger things.*
>
> *"But some are useful" - simplifications of reality can be quite useful. They can help us explain, predict and understand the universe and all its various components.*
>
> *This isn't just true in statistics! Maps are a type of model; they are wrong. But good maps are very useful.*

Regression

The simpler linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

this can be rewritten as

$$Y_i = \beta_0 + \beta_1 X_i + e_i \quad e_i \stackrel{ind}{\sim} N(0, \sigma^2)$$

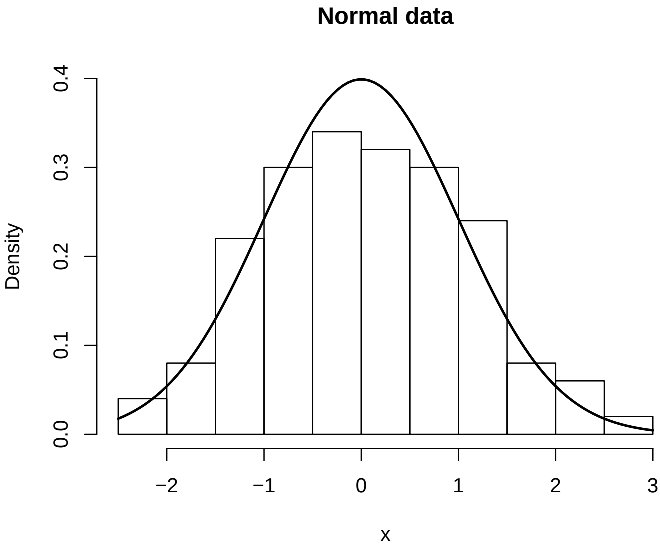
where we estimate the errors via the residuals

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i).$$

Key assumptions are:

- Normality of the errors
- Constant variance of the errors
- Independence of the errors
- Linearity between mean response and explanatory variable

Histograms with best fitting bell curves



Normal QQ-plot

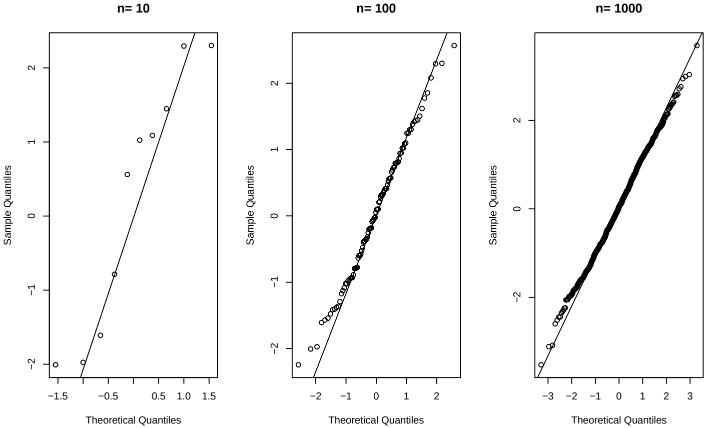
Definition

The quantile-quantile or qq-plot is an exploratory graphical device used to check the validity of a distributional assumption for a data set.

A normal qq-plot graphs the theoretical quantiles from a normal distribution versus the observed quantiles.

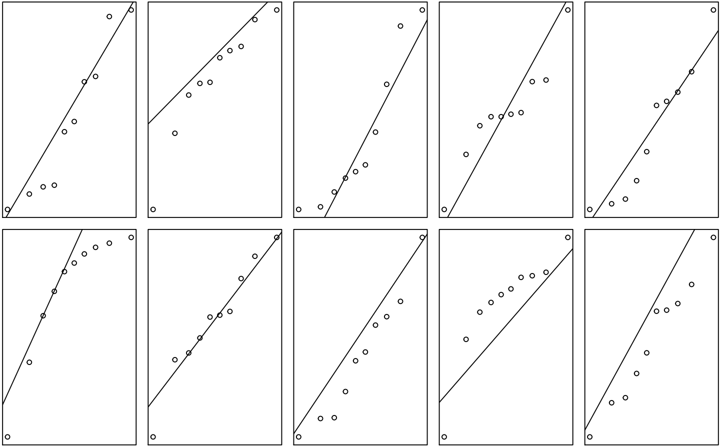
Remark The bottom line is that, if the distribution assumption is satisfied, the points should fall roughly along the y=x line.

Normal



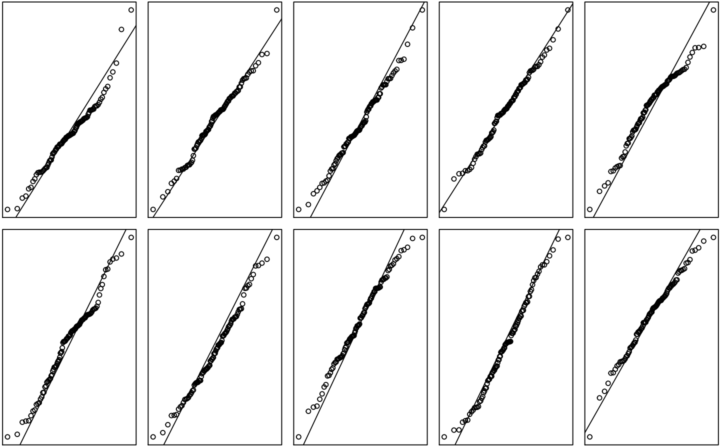
SAS swaps the x and y axes

Normal (n=10)



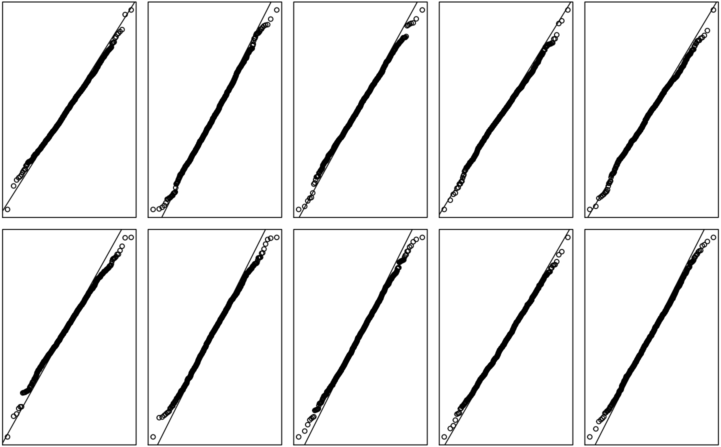
SAS swaps the x and y axes

Normal(n=100)



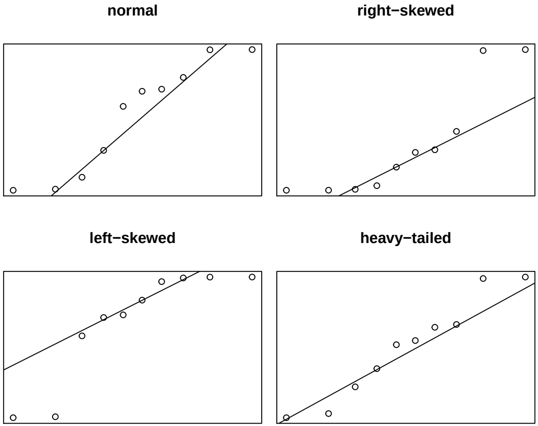
SAS swaps the x and y axes

Normal (n=1000)



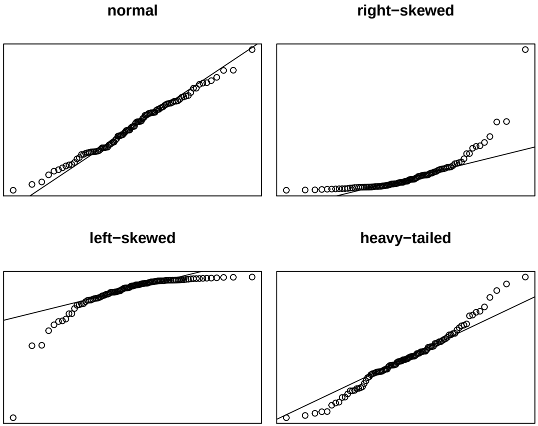
SAS swaps the x and y axes

Not normal (n=10)



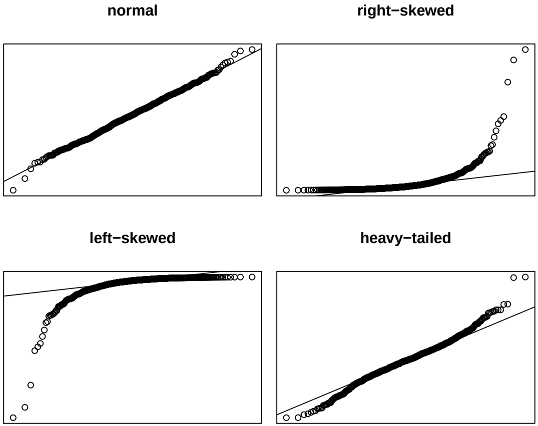
SAS swaps the x and y axes

Not normal (n=100)



SAS swaps the x and y axes

Not normal (n=1000)



SAS swaps the x and y axes

Constant variance

Recall the model

$$Y_i = \beta_0 + \beta_x X_i + e_i \quad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

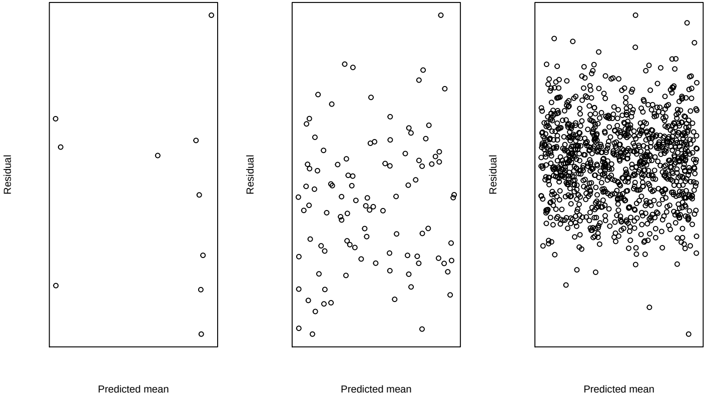
so the variance for the e_i is constant.

To assess this assumption, we look at plots of residuals vs anything and look for patterns that show different "spreads", e.g.

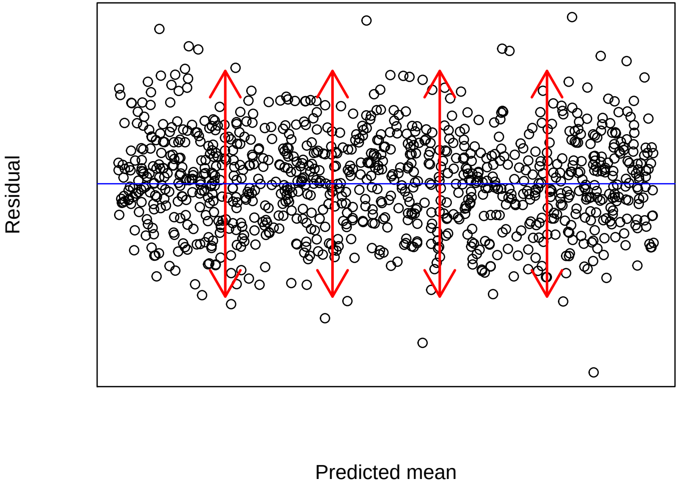
- funnels
- football shapes

The most common way this assumption is violated is by having increasing variance with increasing mean, thus we often look at a residuals vs predicted (fitted) mean plot.

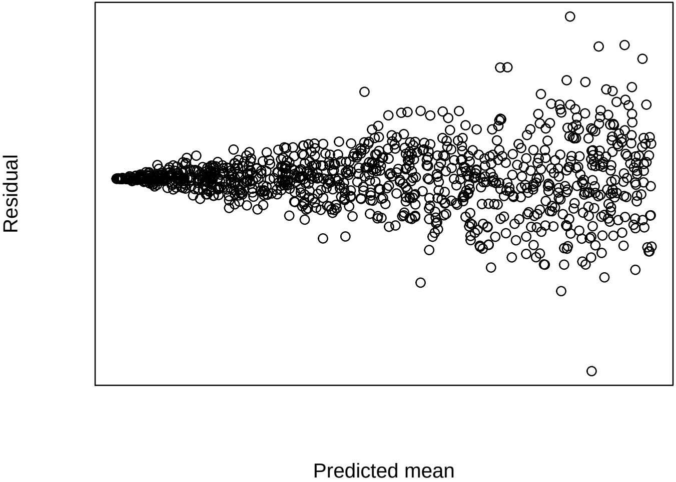
Constant variance



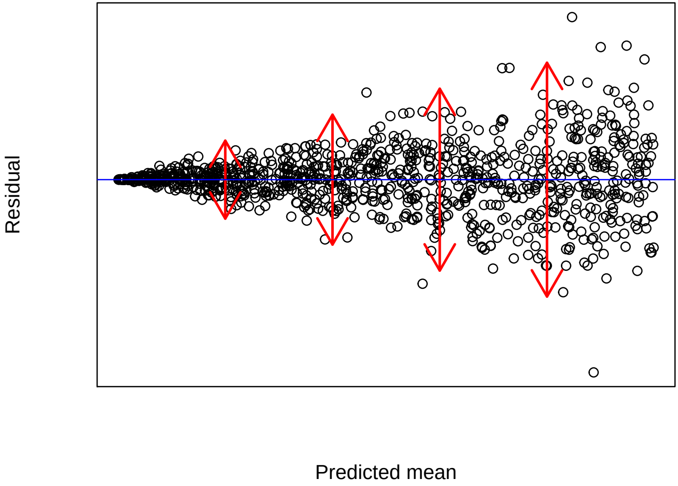
Constant variance



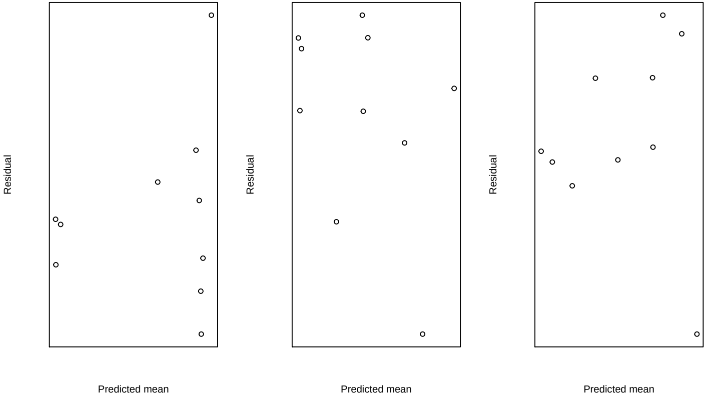
Extreme non-constant variance (funnel)



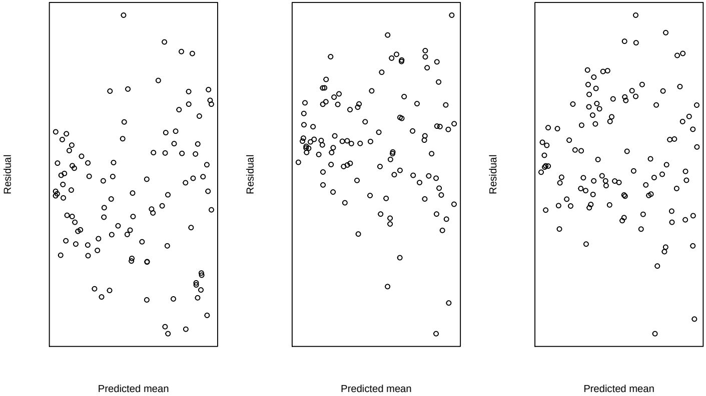
Extreme non-constant variance (funnel)



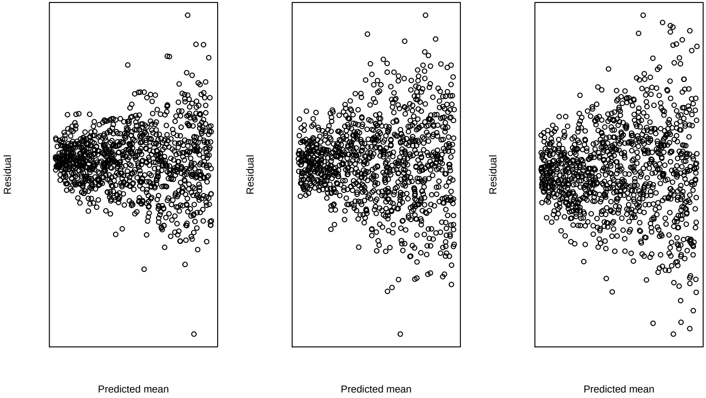
Non-constant variance (n=10, $\sigma_2/\sigma_1 = 4$)



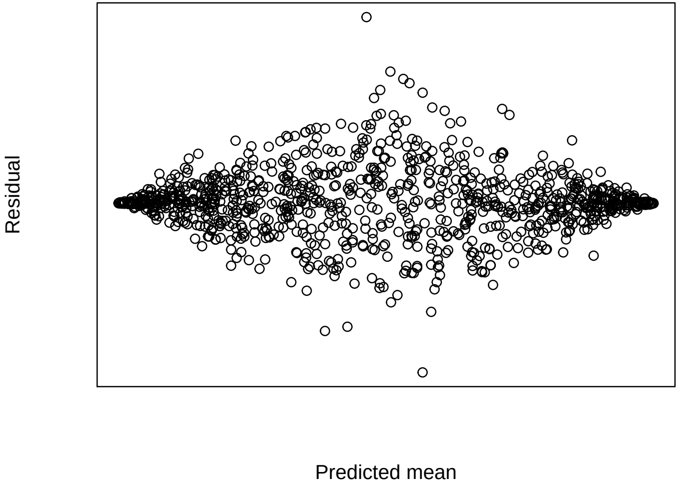
Non-constant variance (n=100, $\sigma_2/\sigma_1 = 4$)



Non-constant variance (n=1000, $\sigma_2/\sigma_1 = 4$)



Extreme non-constant variance (football)



Independence

Lack of independence includes

- Cluster effect
- Serial correlation
- Spatial association

Make plots of residuals vs relevant explanatory variables and look for patterns, e.g.

- Residuals vs groups
- Residuals vs time (or observation number)
- Residuals vs spatial variable

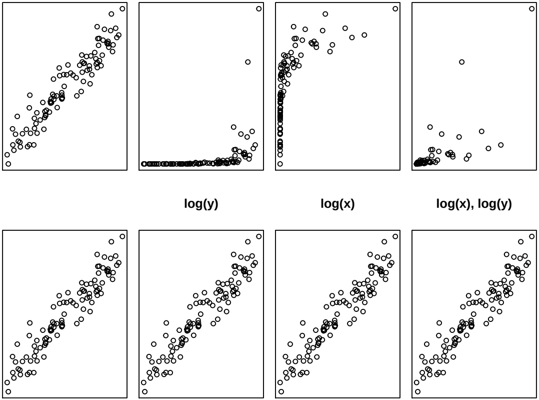
Summary

Often the best strategy is graphical exploration of the data, here are some relevant graphs:

- transformed response vs transformed explanatory
- transformed response vs transformed explanatory
- qqplot of residuals
- residual vs fitted value
- residual vs explanatory
- residual vs observation number
- residual vs any other variable

Linearity

Assess using scatterplots of (transformed) response vs (transformed) explanatory variable:



Testing Composite hypotheses

Comparing two models

- H_0 : (reduced)
- H_1 : (full)

Do the following

1. Calculate extra sum of squares.
2. Calculate extra degrees of freedom
3. Calculate

$$\text{F-statistic} = \frac{\text{Extra sum of squares / Extra degrees of freedom}}{\hat{\sigma}^2_{full}}$$

4. Compare this to an F-distribution with
   - numerator degrees of freedom = extra degrees of freedom
   - denominator degrees of freedom = degrees of freedom in estimating $\hat{\sigma}^2_{full}$

Lack-of-fit F-test

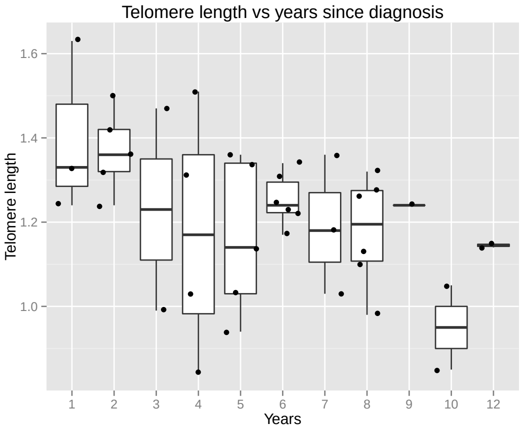
Let Y_{ij} be the i^{th} observation from the j^{th} group where the group is defined by those observations having the same explanatory variable value (X_j).

Two models:

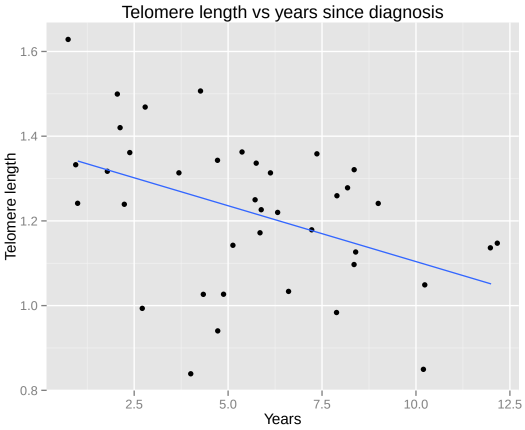
ANOVA:	$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$	(full)
Regression:	$Y_{ij} \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_j, \sigma^2)$	(reduced)

- Regression model is reduced:
  - ANOVA has J parameters for the mean
  - Regression has 2 parameters for the mean
  - Set $\mu_j = \beta_0 + \beta_1 X_j$.
- Small pvalues indicate a lack-of-fit, i.e. the reduced model is not adequate.
- Lack-of-fit F-test requires multiple observations at a few X_j values!

Telomere length



Telomere length



SAS code

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years / CLB LACKFIT;
  RUN;
```

```
                              The REG Procedure
                               Model: MODEL1
                         Dependent Variable: length

                    Number of Observations Read          39
                    Number of Observations Used          39


                             Analysis of Variance

                                        Sum of           Mean
        Source                  DF      Squares         Square    F Value    Pr > F

        Model                    1      0.22777        0.22777       8.42    0.0062
        Error                   37      1.00033        0.02704
          Lack of Fit            9      0.18223        0.02025       0.69    0.7093
          Pure Error            28      0.81810        0.02922
        Corrected Total         38      1.22810
```

Indicates no evidence for a lack of fit, i.e. regression seems adequate.

```
# Use as.factor to turn a continuous variable into a categorical variable
m_anova = lm(telomere.length ~ as.factor(years), Telomeres)
m_reg   = lm(telomere.length ~           years , Telomeres)
anova(m_reg, m_anova)


Analysis of Variance Table

Model 1: telomere.length ~ years
Model 2: telomere.length ~ as.factor(years)
  Res.Df   RSS Df Sum of Sq    F Pr(>F)
1     37 1.000
2     28 0.818  9     0.182 0.69   0.71
```

No evidence of a lack of fit.

Lack-of-fit F-test summary

- Lack-of-fit F-test tests the assumption of linearity
- Needs multiple observations at various explanatory variable values
- Small pvalue indicates a lack-of-fit, i.e. means are not linear
  - Transform response, e.g. log
  - Transform explanatory variable
  - Add other explanatory variables

Interpretations using logs

The most common transformation of either the response or explanatory is to take logarithms because

- linearity will often then be approximately true,
- the variance will likely be approximately constant, and
- there is a (relatively) convenient interpretation.

We will talk about interpretation of β_0 and β_1 when

- only the response is logged,
- only the explanatory variable is logged, and
- when both are logged.

Neither response nor explanatory variable are logged

If

$$E[Y|X] = \beta_0 + \beta_1 X$$

, then

- β_0 is the expected response when X is zero and
- β_1 is the expected change in the response for a one unit increase in the explanatory variable.

For the following discussion,

- Y is always going to be the original response and
- X is always going to be the original explanatory variable.

Example

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 X$$

- β_0 is the expected corn yield per acre when fertilizer level is zero and
- β_1 is the expected change in corn yield per acre when fertilizer is increase by 1 lbs/acre.

Response is logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 X \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 X}$$

, then

- β_0 is the expected $\log(Y)$ when X is zero and
- β_1 is the expected change in $\log(Y)$ for a one unit increase in the explanatory variable.

Alternatively,

- $\exp(\beta_0)$ is the median of Y when X is zero and
- $\exp(\beta_1)$ is the multiplicative effect on the median of Y for a one unit increase in the explanatory variable.

Response is logged

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[\log(Y)|X] = \beta_0 + \beta_1 X \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 X}$$

- $\exp(\beta_0)$ is the median corn yield per acre when fertilizer level is zero and
- $\exp(\beta_1)$ is the multiplicative effect in median corn yield per acre when fertilizer is increase by 1 lbs/acre.

Explanatory variable is logged

If

$$E[Y|X] = \beta_0 + \beta_1 \log(X)$$

, then

- β_0 is the expected response when $\log(X)$ is zero and
- β_1 is the expected change in the response for a one unit increase in $\log(X)$.

Alternatively,

- β_0 is the expected response when X is 1 and
- $\beta_1 \log(d)$ is the expected change in the response when X increase multiplicatively by d, e.g.
  - $\beta_1 \log(2)$ is the expected change in the response for each doubling of X or
  - $\beta_1 \log(10)$ is the expected change in the response for each ten-fold increase in X.

Explanatory variable is logged

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 \log(X)$$

- β_0 is the median corn yield per acre when fertilizer level is 1 lb/acre and
- $\beta_1 \log(2)$ is the expected change in corn yield when fertilizer level is doubled.

Both response and explanatory variables are logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X) \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1}$$

, then

- β_0 is the expected $\log(Y)$ when $\log(X)$ is zero and
- β_1 is the expected change in $\log(Y)$ for a one unit increase in $\log(X)$.

Alternatively,

- $\exp(\beta_0)$ is the median of Y when X is 1 and
- d^{β_1} is the multiplicative change in the median of the response when X increase multiplicatively by d, e.g.
    - 2^{β_1} is the multiplicative effect on the median of the response for each doubling of X or
    - 10^{β_1} is the multiplicative effect on the median of the response for each ten-fold increase in X.

Both response and explanatory variables are logged

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X) \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1}$$

- $\exp(\beta_0)$ is the median corn yield per acre when fertilizer level is 1 lb/acre and
- 2^{β_1} is the multiplicative effect on median corn yield per acre when fertilizer level doubles.