P3 - Continuous distributions

STAT 587 (Engineering)
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Cumulative distribution function

All properties of discrete random variables have direct counterparts for continuous random variables with the caveat that $P(X = x) = 0$ for all x.

The cumulative distribution function for a continuous random variable is

$$F_X(x) = P(X \leq x) = P(X < x)$$

since $P(X = x) = 0$ for any x.

The cdf still has the properties

- $0 \leq F_X(x) \leq 1$ for all $x \in \mathbb{R}$
- F_X is monotone increasing,
  i.e. if $x_1 \leq x_2$ then $F_X(x_1) \leq F_X(x_2)$.
- $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

Probability density function

The probability density function (pdf) for a continuous random variable is

$$f_X(x) = \frac{d}{dx} F_X(x)$$

and

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

Thus, the pdf has the following properties

- $f_X(x) \geq 0$ for all x and
- $\int_{-\infty}^{\infty} f(x)dx = 1.$

Example

Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$f_X(x)$ defines a valid pdf because $f_X(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 3x^2 dx = x^3|_0^1 = 1.$$

The cdf is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}.$$

Expected value

Let X be a continuous random variable and h be some function. The expected value of a function of a continuous random variable is

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f_X(x) dx.$$

If $h(x) = x$, then

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx.$$

and we call this the expectation of X. We commonly use the symbol μ for this expectation.

Example (cont.)

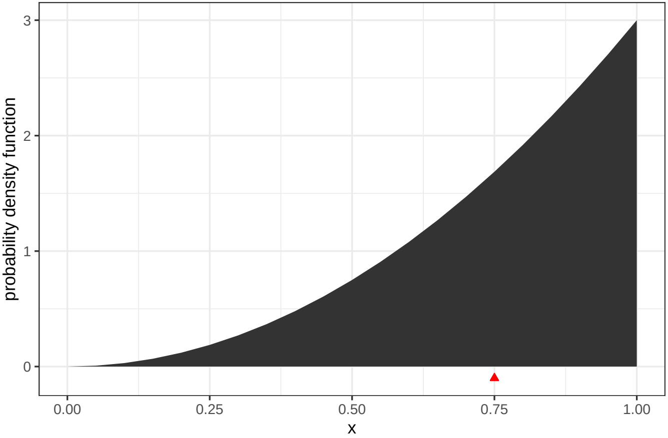
Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx \\ &= \int_0^1 3x^3 dx \\ &= 3\frac{x^4}{4}\Big|_0^1 = \frac{3}{4}. \end{aligned}$$

Example - Center of mass



Variance

The variance of a random variable is defined as the expected squared deviation from the mean. For continuous random variables, variance is

$$Var[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$$

where $\mu = E[X]$. The symbol σ^2 is commonly used for the variance.

The standard deviation is the positive square root of the variance

$$SD[X] = \sqrt{Var[X]}.$$

The symbol σ is commonly used for the standard deviation.

Example (cont.)

Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The variance is

$$\begin{aligned} Var[X] &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx \\ &= \int_0^1 \left(x - \tfrac{3}{4}\right)^2 3x^2 dx \\ &= \int_0^1 \left[x^2 - \tfrac{3}{2}x + \tfrac{9}{16}\right] 3x^2 dx \\ &= \int_0^1 3x^4 - \tfrac{9}{2}x^3 + \tfrac{27}{16}x^2 dx \\ &= \left[\tfrac{3}{5}x^5 - \tfrac{9}{8}x^4 + \tfrac{9}{16}x^3\right] |_0^1 dx \\ &= \tfrac{3}{5} - \tfrac{9}{8} + \tfrac{9}{16} \\ &= \tfrac{3}{80}. \end{aligned}$$

Comparison of discrete and continuous random variables

For simplicity here and later, we drop the subscript X.

	discrete	continuous
image	finite or countable	uncountable
pmf	$p(x) = P(X = x)$	
pdf		$p(x) = f(x) = F'(x)$
cdf	$\begin{aligned} F(x) &= P(X \leq x) \\ &= \sum_{t \leq x} p(x) \end{aligned}$	$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} p(t)dt \end{aligned}$
expected value	$E[h(X)] = \sum_x h(x)p(x)$	$E[h(X)] = \int_x h(x)p(x)dx$
expectation	$\mu = E[X] = \sum_x x\, p(x)$	$\mu = E[X] = \int_x x\, p(x)dx$
variance	$\begin{aligned} Var[X] &= E[(X-\mu)^2] \\ &= \sum_x (x-\mu)^2 p(x) \end{aligned}$	$\begin{aligned} Var[X] &= E[(X-\mu)^2] \\ &= \int_x (x-\mu)^2 p(x)dx \end{aligned}$

Note: we replace summations with integrals when using continuous as opposed to discrete random variables

Uniform

A uniform random variable on the interval (a, b) has equal probability for any value in that interval and we denote this $X \sim Unif(a, b)$. The pdf for a uniform random variable is

$$f(x) = \frac{1}{b-a}\mathrm{I}(a < x < b)$$

where $\mathrm{I}(A)$ is in indicator function that is 1 if A is true and 0 otherwise, i.e.

$$\mathrm{I}(A) = \begin{cases} 1 & A \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

The expectation is

$$E[X] = \int_a^b \frac{1}{b-a}\, x\, dx = \frac{a+b}{2}$$

and the variance is

$$Var[X] = \int_a^b \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{12}(b-a)^2.$$

Example (cont.)

Pseudo-random number generators generate pseudo uniform values on (0,1). These values can be used in conjunction with the inverse of the cumulative distribution function to generate pseudo-random numbers from any distribution.

The inverse of the cdf $F_X(x) = x^3$ is

$$F_X^{-1}(u) = u^{1/3}.$$

A uniform random number on the interval (0,1) generated using the inverse cdf produces a random draw of X.

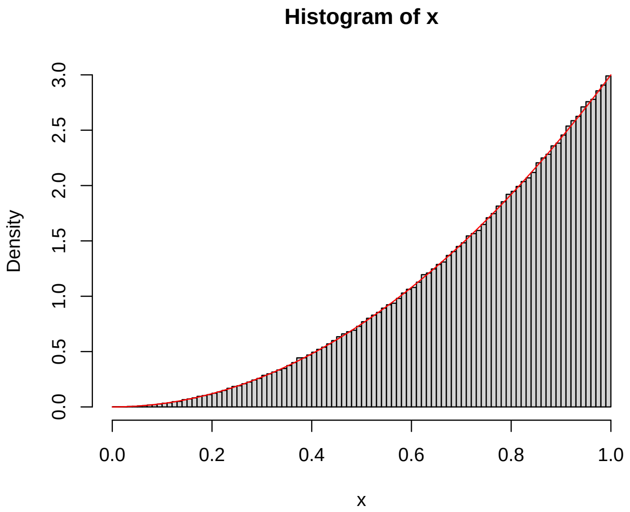
So, in R

```
inverse_cdf = function(u) u^(1/3)
x = inverse_cdf(runif(1e6))
mean(x)

[1] 0.7502002

var(x); 3/80

[1] 0.03752111
[1] 0.0375
```


Histogram of x
Density
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.2
0.4
0.6
0.8
1.0
x

Normal distribution

The normal (or Gaussian) density is a "bell-shaped" curve. The density has two parameters: mean μ and variance σ^2 and is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{for } -\infty < x < \infty$$

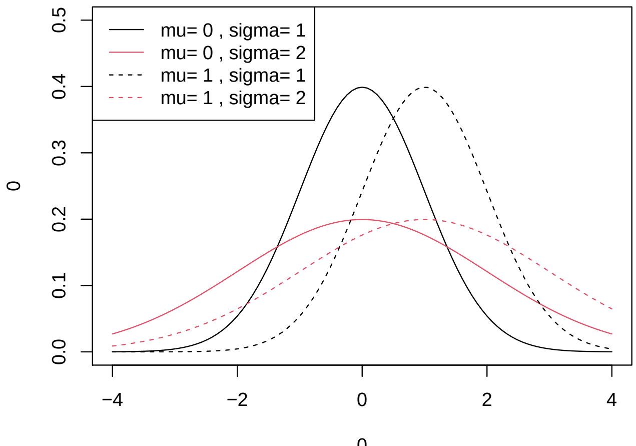
The expected value and variance of a normal distributed r.v. X are:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\, f(x) dx = \dots & &= \mu \\ Var[X] &= \int_{-\infty}^{\infty} (x-\mu)^2\, f(x) dx = \dots & &= \sigma^2. \end{aligned}$$

Thus, the parameters μ and σ^2 are actually the mean and the variance of the $N(\mu, \sigma^2)$ distribution.

There is no closed form cumulative distribution function for a normal random variable.

Example normal probability density functions



Properties of normal random variables

Let $Z \sim N(0,1)$, i.e. a standard normal random variable. (If you see Z without explanation, it is a standard normal random variable.) Then for constants μ and σ

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2)$$

and

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1).$$

which is called standardizing.

Let $X_i \stackrel{ind}{\sim} N(\mu_i, \sigma_i^2)$. Then

$$Z_i = \frac{X_i - \mu_i}{\sigma_i} \stackrel{iid}{\sim} N(0,1) \quad \text{for all } i$$

and

$$Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

Calculating the standard normal cdf

If $Z \sim N(0,1)$, what is $P(Z \leq 1.5)$? Although the cdf does not have a closed form, very good approximations exist and are available as tables or in software, e.g.

```
pnorm(1.5) # default is mean=0, sd=1
```

```
[1] 0.9331928
```

If $Z \sim N(0,1)$, then

- $P(Z \leq z) = \Phi(z)$
- $\Phi(z) = 1 - \Phi(-z)$ since a normal pdf is symmetric around its mean.

Calculating any normal cumulative distribution function

If $X \sim N(15, 4)$ what is $P(X > 18)$?

$$
\begin{aligned}
P(X > 18) \quad &= 1 - P(X \le 18) \\
&= 1 - P\left(\frac{X-15}{2} \le \frac{18-15}{2}\right) \\
&= 1 - P(Z \le 1.5) \\
&\approx 1 - 0.933 = 0.067
\end{aligned}
$$

```
1-pnorm((18-15)/2)          # by standardizing

[1] 0.0668072

1-pnorm(18, mean = 15, sd = 2) # using the mean and sd arguments

[1] 0.0668072
```

Manufacturing

Suppose you are producing nails that must be within 5 and 6 centimeters in length. If the average length of nails the process produces is 5.3 cm and the standard deviation is 0.1 cm. What is the probability the next nail produced is outside of the specification?

Let $X \sim N(\mu, \sigma^2)$ be the next nail produced with $\mu = 5.3$ cm and $\sigma = 0.1$ cm. We need to calculate

$$P(X < 5 \text{ or } X > 6) = 1 - P(5 < X < 6).$$

```r
mu = 5.3
sigma = 0.1

1-diff(pnorm(c(5,6), mean = mu, sd = sigma))
```

```
[1] 0.001349898
```