STAT 401A - Statistical Methods for Research Workers

Simple linear regression

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Simple Linear Regression

Recall the one-way ANOVA model:

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

where Y_{ij} is the observation for individual i in group j.

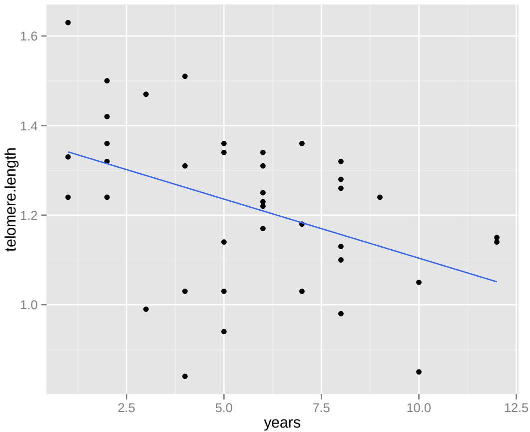
The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

Terminology (all of these are equivalent):

response	explanatory
outcome	covariate
dependent	independent
endogenous	exogenous



1.6
1.4
1.2
1.0
telomere.length
2.5
5.0
7.5
10.0
12.5
years

Telomere length

http://www.pnas.org/content/101/49/17312

People who are stressed over long periods tend to look haggard, and it is commonly thought that psychological stress leads to premature aging and the earlier onset of diseases of aging.

...

This design allowed us to examine the importance of perceived stress and measures of objective stress (caregiving status and chronicity of caregiving stress based on the number of years since a child's diagnosis).

...

Telomere length values were measured from DNA by a quantitative PCR assay that determines the relative ratio of telomere repeat copy number to single-copy gene copy number (T/S ratio) in experimental samples as compared with a reference DNA sample.

Interpretation

$$E[Y_i|X_i = x] = \beta_0 + \beta_1 x \qquad V[Y_i|X_i = x] = \sigma^2$$

- If $X_i = 0$, then $E[Y_i|X_i = 0] = \beta_0$.
  β_0 is the expected response when the explanatory variable is zero.

- If X_i increases from x to $x + 1$, then

$$\begin{array}{rl} E[Y_i|X_i = x+1] & = \beta_0 + \beta_1 x + \beta_1 \\ -\ E[Y_i|X_i = x] & = \beta_0 + \beta_1 x \\ \hline & = \beta_1 \end{array}$$

  β_1 is the expected increase in the response for each unit increase in the explanatory variable.

- σ is the standard deviation of the response for a fixed value of the explanatory variable.

Remove the mean:

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

So the error is

$$e_i = Y_i - (\beta_0 + \beta_1 X_i)$$

which we approximate by the residual

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

The least squares, maximum likelihood, and Bayesian estimators are

$$\begin{array}{rl} \hat{\beta}_1 & = SXY/SXX \\ \hat{\beta}_0 & = \overline{Y} - \hat{\beta}_1 \overline{X} \\ \hat{\sigma}^2 & = SSE/(n-2) \qquad \text{df} = n-2 \\ \\ \overline{X} & = \frac{1}{n}\sum_{i=1}^n X_i \\ \overline{Y} & = \frac{1}{n}\sum_{i=1}^n Y_i \\ \\ SXY & = \sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) \\ SXX & = \sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^n (X_i - \overline{X})^2 \\ SSE & = \sum_{i=1}^n r_i^2 \end{array}$$

How certain are we about $\hat{\beta}_0$ and $\hat{\beta}_1$ being equal to β_0 and β_1?

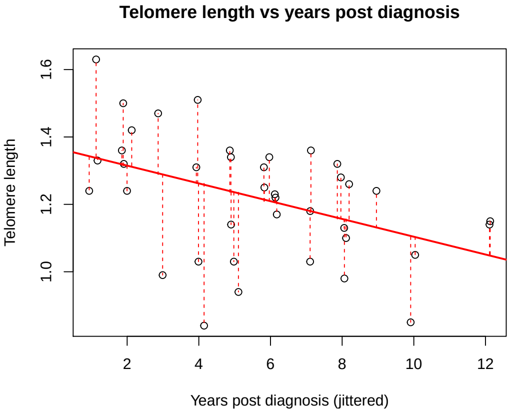
We quantify this uncertainty using their standard errors:

$$\begin{aligned} SE(\beta_0) &= \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} & df &= n-2 \\ SE(\beta_1) &= \hat{\sigma}\sqrt{\frac{1}{(n-1)s_X^2}} & df &= n-2 \end{aligned}$$

$$\begin{aligned} s_X^2 &= SXX/(n-1) \\ s_Y^2 &= SYY/(n-1) \\ SYY &= \sum_{i=1}^n (Y_i - \overline{Y})^2 \end{aligned}$$

$$\begin{aligned} r_{XY} &= \frac{SXY/(n-1)}{s_X s_Y} & & \text{correlation coefficient} \\ R^2 &= r_{XY}^2 = \frac{SST-SSE}{SST} & & \text{coefficient of determination} \\ SST &= SYY = \sum_{i=1}^n (Y_i - \overline{Y})^2 \end{aligned}$$

The coefficient of determination (R^2) is the proportion of the total response variation explained by the explanatory variable(s).



Telomere length vs years post diagnosis
Telomere length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
Years post diagnosis (jittered)

Pvalues and confidence interval

We can compute two-sided pvalues via

$$2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_0}{SE(\beta_0)}\right|\right) \qquad \text{and} \qquad 2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_1}{SE(\beta_1)}\right|\right)$$

These test the null hypothesis that the corresponding parameter is zero.

We can construct $100(1-\alpha)\%$ two-sided confidence intervals via

$$\hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\beta_0) \qquad \text{and} \qquad \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\beta_1)$$

These provide ranges of the parameters consistent with the data.

Calculations by hand

```
   n Xbar Ybar   s_X    s_Y    r_XY
1 39 5.59 1.22 2.935 0.1798 -0.4307
```

$$
\begin{aligned}
SXX &= (n-1)s_X^2 = (39-1)\times 2.935427^2 = 327.4358 \\
SYY &= (n-1)s_Y^2 = (39-1)\times 0.1797731^2 = 1.228098 \\
SXY &= (n-1)s_X s_Y r_{XY} = (39-1)\times 2.935427 \times 0.1797731 \times -0.4306534 = -8.635897 \\
\hat\beta_1 &= SXY/SXX = -8.635897/327.4358 = -0.02637432 \\
\hat\beta_0 &= \overline{Y} - \hat\beta_1 \overline{X} = 1.220256 - (-0.02637432)\times 5.589744 = 1.367682 \\
R^2 &= r_{XY}^2 = (-0.4306534)^2 = 0.1854624 \\
SSE &= SYY(1-R^2) = 1.228098(1-0.1854624) = 1.000332 \\
\hat\sigma^2 &= SSE/(n-2) = 1.000332/(39-2) = 0.027036 \\
\hat\sigma &= \sqrt{\hat\sigma^2} = \sqrt{0.027036} = 0.1644263 \\
SE(\hat\beta_0) &= \hat\sigma\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{39} + \frac{5.589744^2}{327.4358}} = 0.05721115 \\
SE(\hat\beta_1) &= \hat\sigma\sqrt{\frac{1}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{327.4358}} = 0.009086742 \\
p_{H_0:\beta_0=0} &= 2P\left(t_{n-2} < -\left|\frac{1.367682}{0.05721115}\right|\right) = 2P(t_{37} < -23.90586) < 0.0001 \\
p_{H_0:\beta_1=0} &= 2P\left(t_{n-2} < -\left|\frac{-0.02637432}{0.009086742}\right|\right) = 2P(t_{37} < -2.902506) < 0.0062 \\
CI_{95\%}\ \beta_0 &= \hat\beta_0 \pm t_{n-2}(1-\alpha/2)SE(\hat\beta_0) = 1.367682 \pm 2.026192 \times 0.05721115 = (1.251761, 1.483603) \\
CI_{95\%}\ \beta_1 &= \hat\beta_1 \pm t_{n-2}(1-\alpha/2)SE(\hat\beta_1) = -0.02637432 \pm 2.026192 \times 0.009086742 = (-0.044785804 - 0.007962836)
\end{aligned}
$$

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC CORR DATA=t;
  VAR length;
  WITH years;
  RUN;
```

```
                                        The CORR Procedure

                                    1 With Variables:    years
                                    1      Variables:    length


                                        Simple Statistics

   Variable            N          Mean       Std Dev           Sum       Minimum       Maximum

   years              39       5.58974       2.93543     218.00000       1.00000      12.00000
   length             39       1.22026       0.17977      47.59000       0.84000       1.63000


                              Pearson Correlation Coefficients, N = 39
                                     Prob > |r| under H0: Rho=0

                                                    length

                                        years     -0.43065
                                                    0.0062
```

```
PROC GLM DATA=t;
  MODEL length = years / SOLUTION CLPARM;
  RUN;
```

```
                          The GLM Procedure
                Number of Observations Read          39
                Number of Observations Used          39

Dependent Variable: length

                                        Sum of
     Source                 DF         Squares     Mean Square    F Value    Pr > F
     Model                   1      0.22776588      0.22776588       8.42    0.0062
     Error                  37      1.00033156      0.02703599
     Corrected Total        38      1.22809744

               R-Square     Coeff Var      Root MSE    length Mean
               0.185462      13.47473      0.164426       1.220256

     Source                 DF       Type I SS     Mean Square    F Value    Pr > F
     years                   1      0.22776588      0.22776588       8.42    0.0062

     Source                 DF     Type III SS     Mean Square    F Value    Pr > F
     years                   1      0.22776588      0.22776588       8.42    0.0062

                                    Standard
 Parameter         Estimate            Error    t Value    Pr > |t|      95% Confidence Limits
 Intercept      1.367682067       0.05721112      23.91      <.0001     1.251761335  1.483602799
 years         -0.026374315       0.00908674      -2.90      0.0062    -0.044785794 -0.007962836
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
with(Telomeres, cor(telomere.length,years))
```

```
[1] -0.4307
```

```
anova(m)
```

```
Analysis of Variance Table

Response: telomere.length
          Df Sum Sq Mean Sq F value Pr(>F)
years      1  0.228   0.228    8.42 0.0062 **
Residuals 37  1.000   0.027
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
summary(m)
```

```
Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
    Min      1Q  Median      3Q     Max
-0.4222 -0.0854  0.0206  0.1074  0.2887

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.36768    0.05721    23.9   <2e-16 ***
years       -0.02637    0.00909    -2.9   0.0062 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.164 on 37 degrees of freedom
Multiple R-squared:  0.185, Adjusted R-squared:  0.163
F-statistic: 8.42 on 1 and 37 DF,  p-value: 0.0062
```

```
confint(m)
```

```
               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

Conclusion

Telomere length at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% confidence interval of (1.25, 1.48). For each year increase since diagnosis, the length decreases by 0.026 with a 95% confidence interval of (0.008, 0.045). The proportional of variability in telomere length described by years since diagnosis is 18.5%.

http://www.pnas.org/content/101/49/17312

> *The zero-order correlation between chronicity of caregiving [years] and mean telomere length, r,is* -0.445 *(P <0.01). [*$R^2 = 0.198$ *was shown in the plot.]*

Remark I'm guessing our analysis and that reported in the paper don't match exactly due to a discrepancy in the data.

Summary

- The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

  where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

- Know how to use SAS/R to obtain $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$, R^2, pvalues, CIs, etc.

- Interpret SAS output
  - At a value of zero for the explanatory variable ($X_i = 0$), β_0 is the expected value for the response (Y_i).
  - For each unit increase in the explanatory variable value, β_1 is the expected increase in the response.
  - At a constant value of the explanatory variable, σ^2 is the variance of the responses.
  - The coefficient of determination (R^2) is the percentage of the total response variation explained by the explanatory variable(s).

What is $E[Y|X=x]$?

We know $\beta_0 = E[Y|X=0]$, but what about $X = x$?

$$E[Y|X=x] = \beta_0 + \beta_1 x$$

which we can estimate via

$$E[\widehat{Y|X=}\,x] = \hat{\beta}_0 + \hat{\beta}_1 x$$

but there is uncertainty in both β_0 and β_1. So the standard error of $E[Y|X=x]$ is

$$SE(E[Y|X=x]) = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(\overline{X}-x)^2}{(n-1)s_X^2}}$$

and a $100(1-\alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1-\alpha/2) SE(E[Y|X=x])$$

What do we predict about Y at $X = x$?

On the last slide, we calculated $E[Y|X = x]$ and it's uncertainty, but if we are trying to predict a new observation, we need to account for the sampling variablity σ^2. Thus a prediction about Y at a new $X = x$ is still

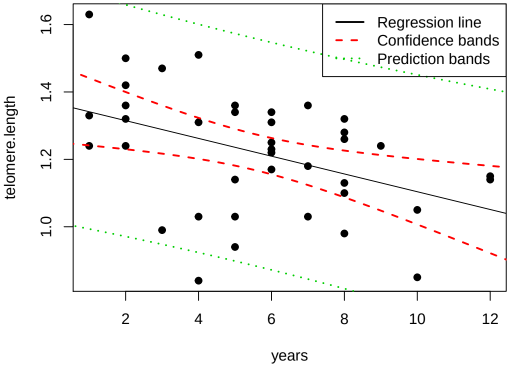
$$Pred(Y|X = x) = \hat{\beta}_0 + \hat{\beta}_1 x$$

but the uncertainty includes the variability due to σ^2. So the standard error of $Pred(Y|X = x)$ is

$$SE(Pred(Y|X = x])) = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(\overline{X} - x)^2}{(n-1)s_X^2}}$$

and a $100(1 - \alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1 - \alpha/2) SE(Pred(Y|X = x)).$$


Regression line
Confidence bands
Prediction bands
telomere.length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
years

Testing Composite hypotheses

Comparing two models

- H_0 : (reduced)
- H_1 : (full)

Do the following

1. Calculate extra sum of squares.
2. Calculate extra degrees of freedom
3. Calculate

$$\text{F-statistic} = \frac{\text{Extra sum of squares / Extra degrees of freedom}}{\hat{\sigma}^2_{full}}$$

4. Compare this to an F-distribution with
   - numerator degrees of freedom = extra degrees of freedom
   - denominator degrees of freedom = degrees of freedom in estimating $\hat{\sigma}^2_{full}$

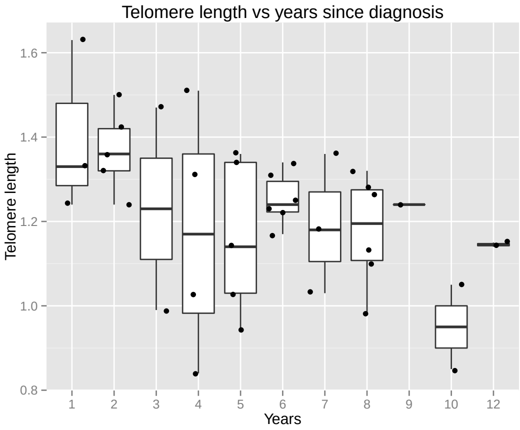
Simple Linear Regression

Two models:

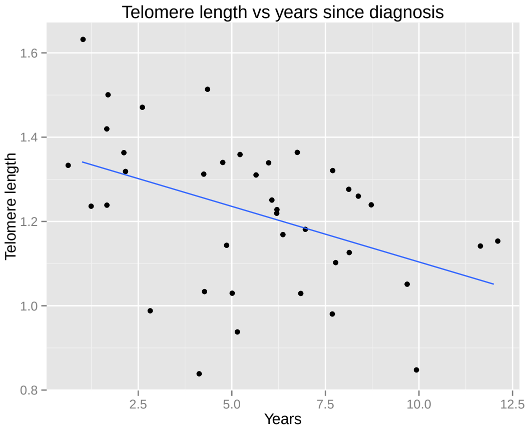
ANOVA:	$Y_{ij} \stackrel{ind}{\sim} N(\mu_i, \sigma^2)$	(full)
Regression:	$Y_{ij} \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$	(reduced)

- Regression model is reduced:
  - ANOVA has J parameters for the mean
  - Regression has 2 parameters for the mean
  - $\mu_i = \beta_0 + \beta_1 X_i$
- Small pvalues indicate a lack-of-fit, i.e. the reduced model is not adequate.
- Lack-of-fit F-test requires multiple observations at a few X_i values.

Telomere length



Telomere length



SAS code

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years / CLB LACKFIT;
  RUN;
```

```
                              The REG Procedure
                                Model: MODEL1
                         Dependent Variable: length

                    Number of Observations Read          39
                    Number of Observations Used          39


                             Analysis of Variance

                                   Sum of           Mean
Source                   DF        Squares         Square    F Value    Pr > F

Model                     1        0.22777        0.22777       8.42    0.0062
Error                    37        1.00033        0.02704
  Lack of Fit             9        0.18223        0.02025       0.69    0.7093
  Pure Error             28        0.81810        0.02922
Corrected Total          38        1.22810
```

Indicates no evidence for a lack of fit, i.e. regression seems adequate.

```
m_anova = lm(telomere.length ~ as.factor(years), Telomeres)
m_reg   = lm(telomere.length ~           years , Telomeres)
anova(m_reg, m_anova)


Analysis of Variance Table

Model 1: telomere.length ~ years
Model 2: telomere.length ~ as.factor(years)
  Res.Df   RSS Df Sum of Sq    F Pr(>F)
1     37 1.000
2     28 0.818  9     0.182 0.69   0.71
```

No evidence of a lack of fit.

Summary

- Lack-of-fit F-test tests the assumption of linearity
- Needs multiple observations at various explanatory variable values
- Small pvalue indicates a lack-of-fit, i.e. means are not linear
    - Transform response, e.g. log
    - Transform explanatory variable
    - Add other explanatory variables