STAT 401A - Statistical Methods for Research Workers

Regression diagnostics

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All models are wrong!

George Box (Empirical Model-Building and Response Surfaces, 1987):

> *All models are wrong, but some are useful.*

`http://stats.stackexchange.com/questions/57407/what-is-the-meaning-of-all-models-are-wrong-but-some-are-useful`

> *"All models are wrong" that is, every model is wrong because it is a simplification of reality. Some models, especially in the "hard" sciences, are only a little wrong. They ignore things like friction or the gravitational effect of tiny bodies. Other models are a lot wrong - they ignore bigger things.*
>
> *"But some are useful" - simplifications of reality can be quite useful. They can help us explain, predict and understand the universe and all its various components.*
>
> *This isn't just true in statistics! Maps are a type of model; they are wrong. But good maps are very useful.*

Regression

The simpler linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

this can be rewritten as

$$Y_i = \beta_0 + \beta_1 X_i + e_i \quad e_i \stackrel{ind}{\sim} N(0, \sigma^2)$$

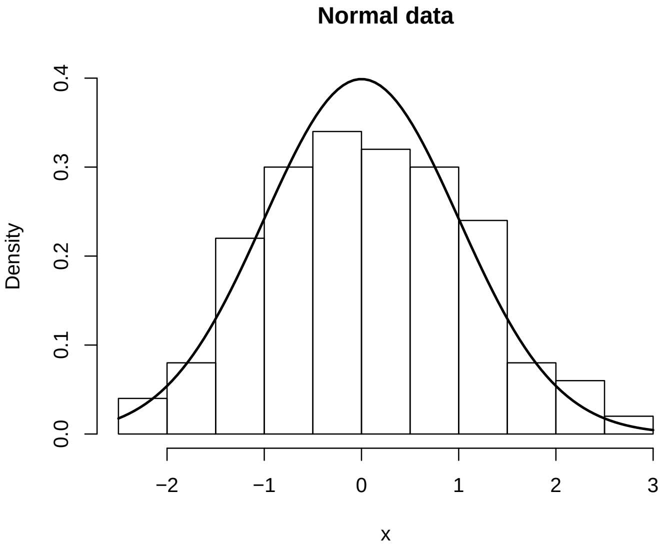
where we estimate the errors via the residuals

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i).$$

Key assumptions are:

- Normality of the errors
- Constant variance of the errors
- Independence of the errors
- Linearity between mean response and explanatory variable

Histograms with best fitting bell curves



Normal QQ-plot

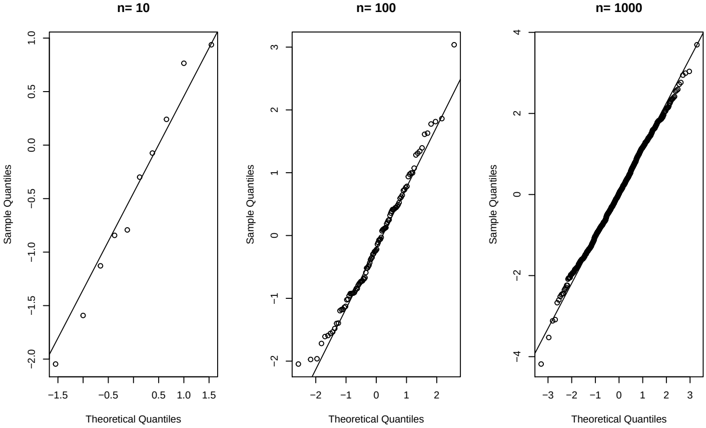
Definition

The quantile-quantile or qq-plot is an exploratory graphical device used to check the validity of a distributional assumption for a data set.

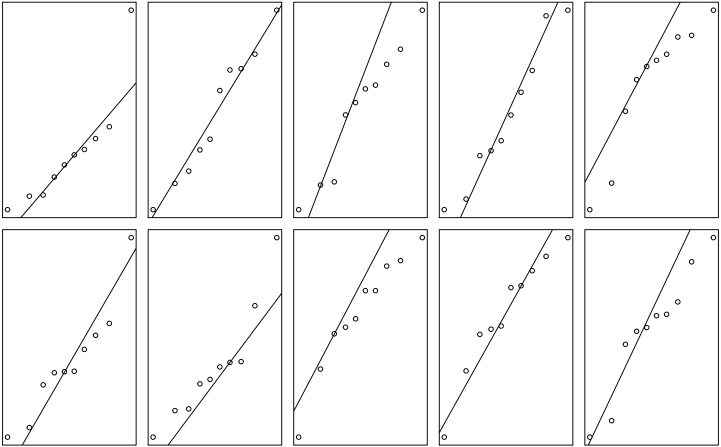
A normal qq-plot graphs the theoretical quantiles from a normal distribution versus the observed quantiles. With a line that indicates perfect normality.

Remark The bottom line is that, if the distribution assumption is satisfied, the points should fall roughly along the line. Systematic variation from this line indicates skewness,

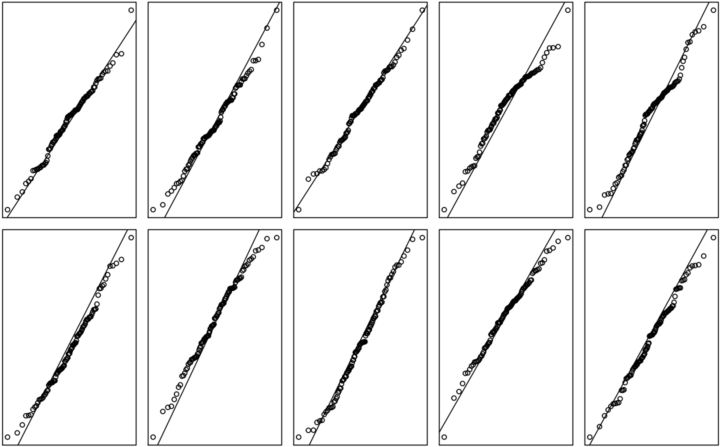
Normal



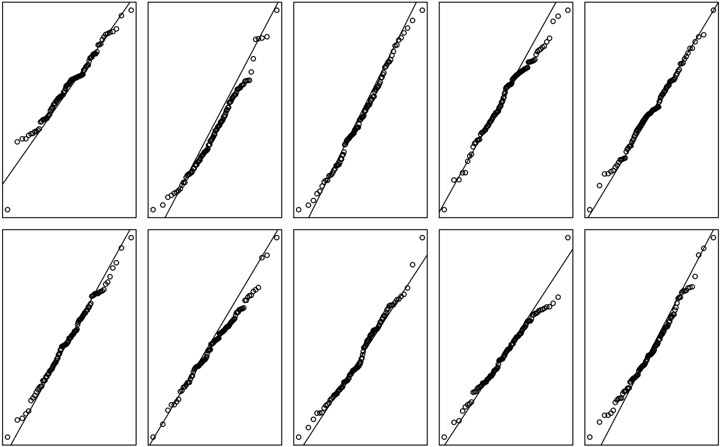
Normal (n=10)



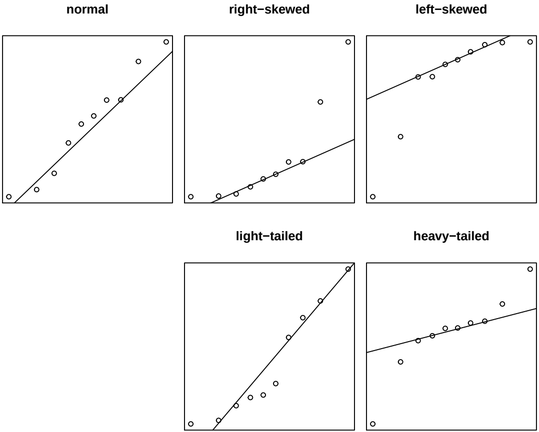
Normal(n=100)



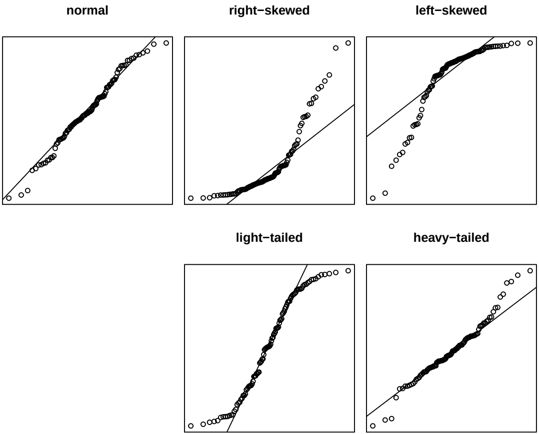
Normal (n=1000)



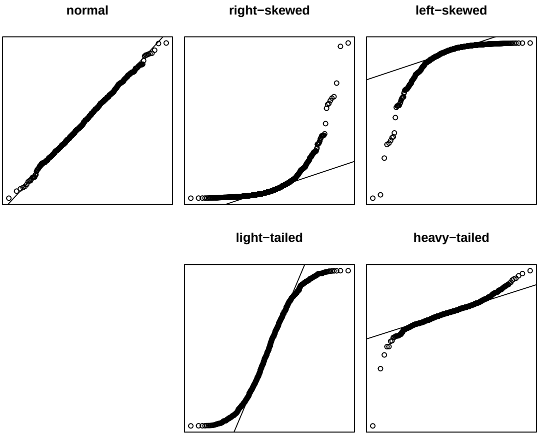
Not normal (n=10)



Not normal (n=100)



Not normal (n=1000)



Summary

For normal qq-plots with (standardized) residuals (y-axis) vs theoretical quantiles (x-axis), the following interpretations apply

- If the residuals fall roughly along the line, then normality is reasonable.
- If the residuals have a U pattern, then there is right-skewness.
- If the residuals have an upside down U pattern, then there is left-skewness.
- If the residuals have an S pattern, then there are light tails (we are not too concerned about this situation because our inferences will be conservative).
- If the residuals have a rotated Z pattern, then there are heavy tails.

Other patterns are certainly possible, but these are the most common.

Constant variance

Recall the model

$$Y_i = \beta_0 + \beta_1 X_i + e_i \quad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

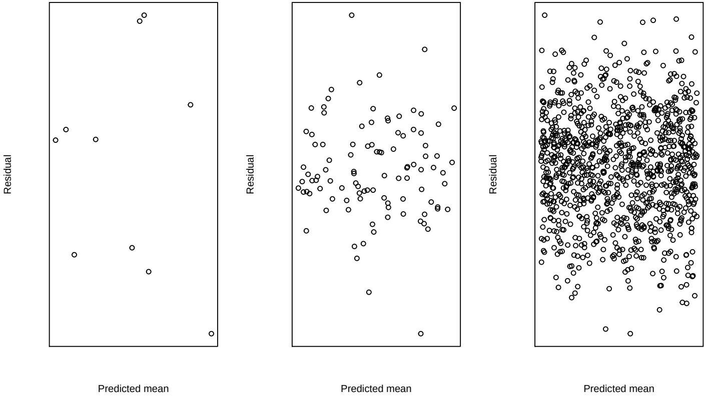
so the variance for the e_i is constant.

To assess this assumption, we look at plots of residuals vs anything and look for patterns that show different "spreads", e.g.

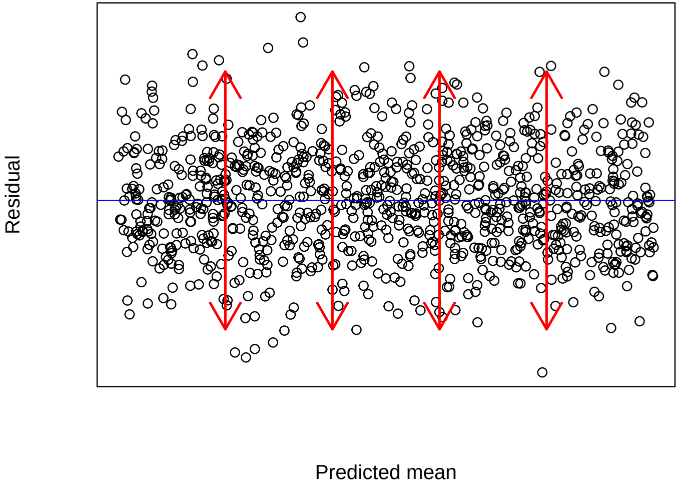
- funnels
- football shapes

The most common way this assumption is violated is by having increasing variance with increasing mean, thus we often look at a residuals vs predicted (fitted) mean plot.

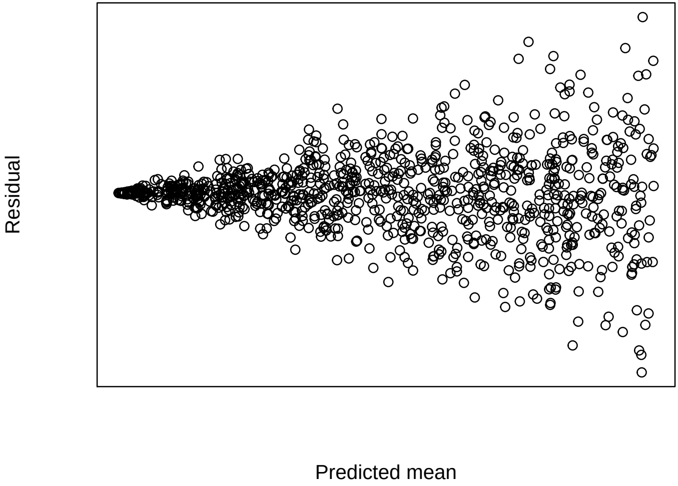
Constant variance



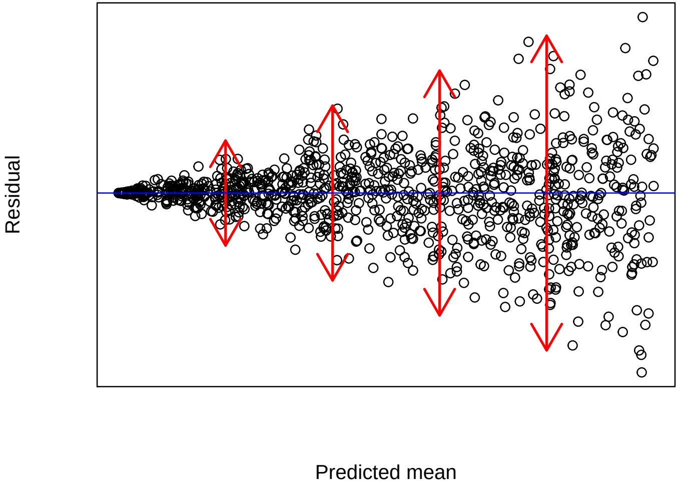
Constant variance



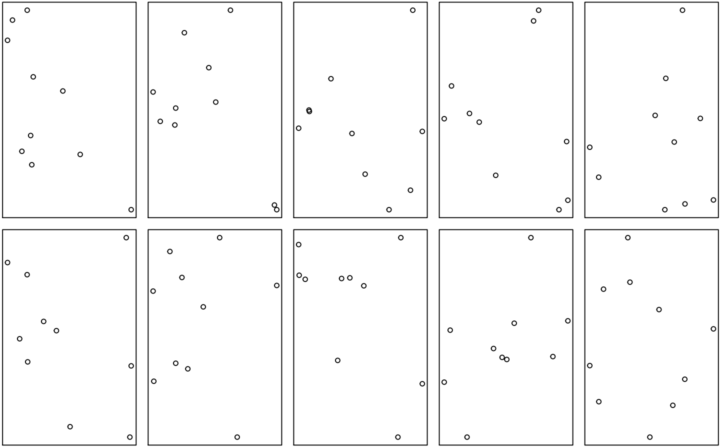
Extreme non-constant variance (funnel)



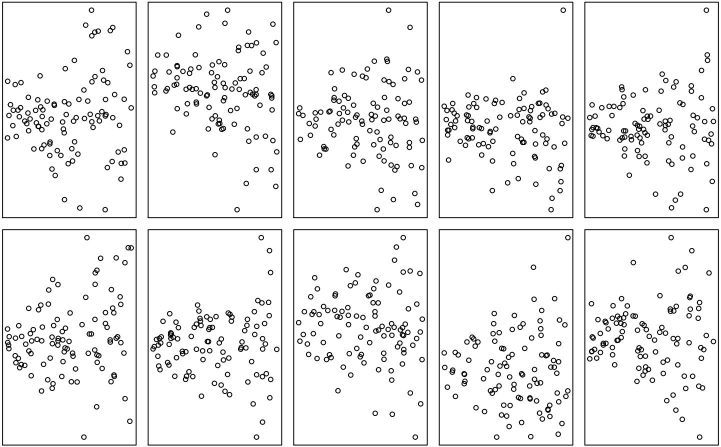
Extreme non-constant variance (funnel)



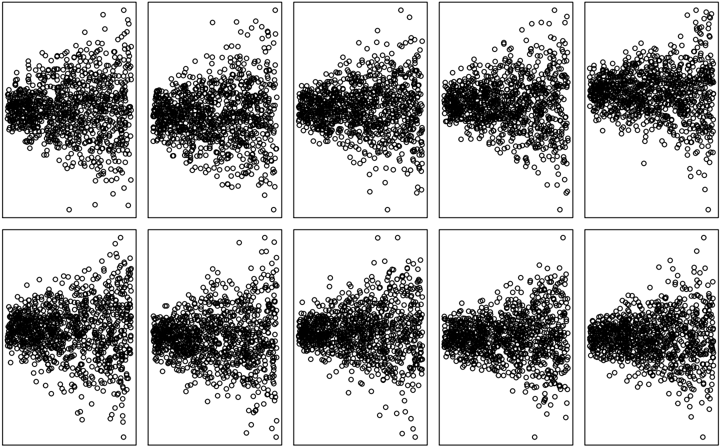
Non-constant variance (n=10, $\sigma_2/\sigma_1 = 4$)



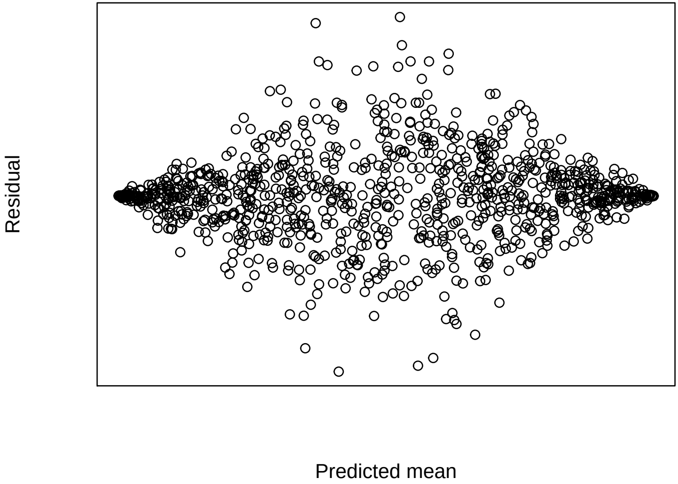
Non-constant variance (n=100, $\sigma_2/\sigma_1 = 4$)



Non-constant variance (n=1000, $\sigma_2/\sigma_1 = 4$)



Extreme non-constant variance (football)



Independence

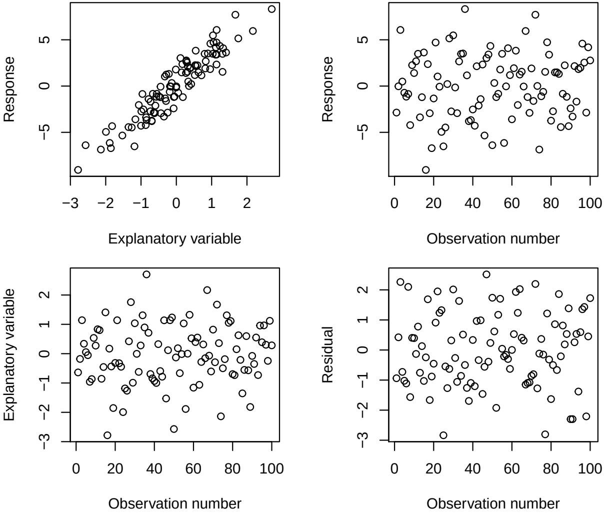
Lack of independence includes

- Cluster effect
- Serial correlation
- Spatial association

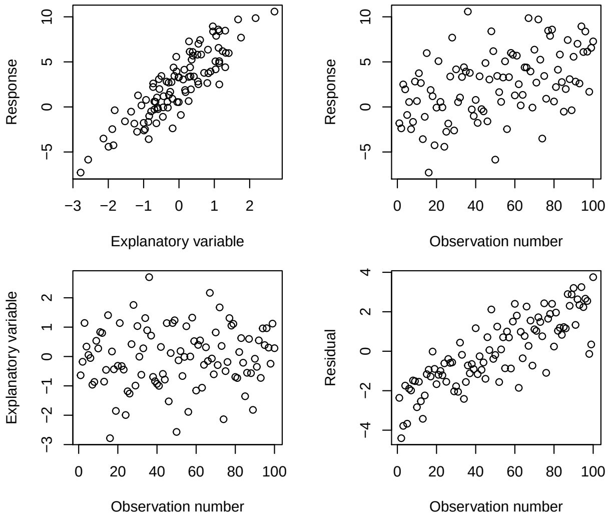
Make plots of residuals vs relevant explanatory variable(s) and look for patterns, e.g.

- Residuals vs groups
- Residuals vs time (or observation number)
- Residuals vs spatial variable

No evidence for lack of independence

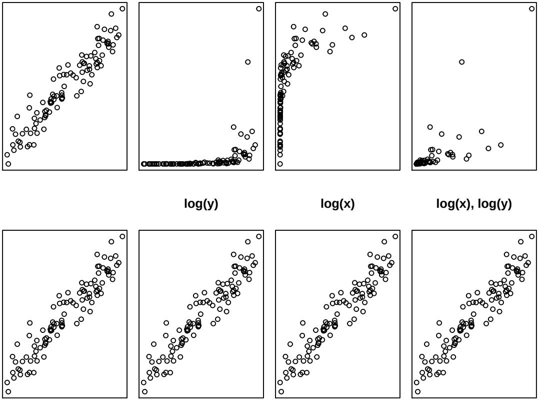


Evidence for lack of independence



Linearity

Assess using scatterplots of (transformed) response vs (transformed) explanatory variable:



Testing Composite hypotheses

Comparing two models

- H_0 : (reduced)
- H_1 : (full)

Do the following

1. Calculate extra sum of squares.
2. Calculate extra degrees of freedom
3. Calculate

$$\text{F-statistic} = \frac{\text{Extra sum of squares / Extra degrees of freedom}}{\hat{\sigma}^2_{full}}$$

4. Compare this to an F-distribution with
   - numerator degrees of freedom = extra degrees of freedom
   - denominator degrees of freedom = degrees of freedom in estimating $\hat{\sigma}^2_{full}$

Lack-of-fit F-test

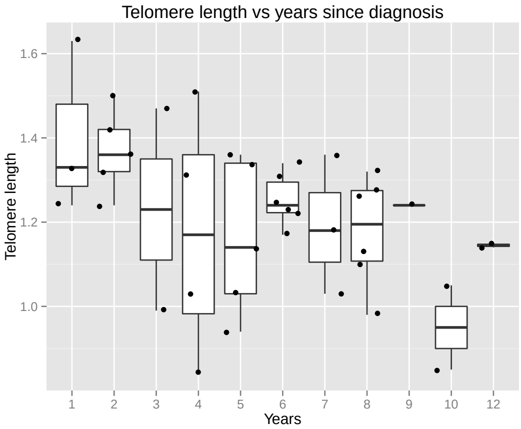
Let Y_{ij} be the i^{th} observation from the j^{th} group where the group is defined by those observations having the same explanatory variable value (X_j).

Two models:

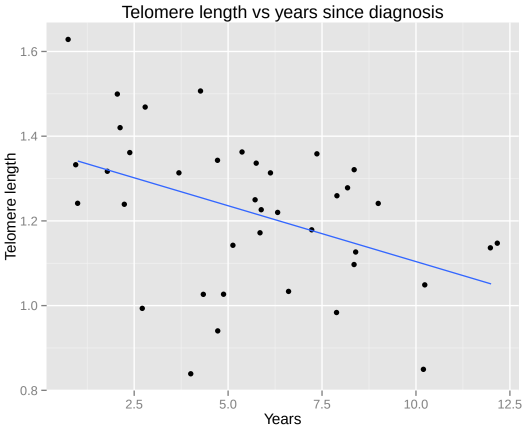
ANOVA:	$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$	(full)
Regression:	$Y_{ij} \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_j, \sigma^2)$	(reduced)

- Regression model is reduced:
  - ANOVA has J parameters for the mean
  - Regression has 2 parameters for the mean
  - Set $\mu_j = \beta_0 + \beta_1 X_j$.
- Small pvalues indicate a lack-of-fit, i.e. the reduced model is not adequate.
- Lack-of-fit F-test requires multiple observations at a few X_j values!

Telomere length



Telomere length



SAS code

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years / CLB LACKFIT;
  RUN;
```

```
                          The REG Procedure
                            Model: MODEL1
                    Dependent Variable: length

              Number of Observations Read          39
              Number of Observations Used          39


                        Analysis of Variance

                                   Sum of           Mean
Source                   DF       Squares         Square    F Value    Pr > F

Model                     1       0.22777        0.22777       8.42    0.0062
Error                    37       1.00033        0.02704
  Lack of Fit             9       0.18223        0.02025       0.69    0.7093
  Pure Error             28       0.81810        0.02922
Corrected Total          38       1.22810
```

Indicates no evidence for a lack of fit, i.e. regression seems adequate.

```
# Use as.factor to turn a continuous variable into a categorical variable
m_anova = lm(telomere.length ~ as.factor(years), Telomeres)
m_reg   = lm(telomere.length ~           years , Telomeres)
anova(m_reg, m_anova)


Analysis of Variance Table

Model 1: telomere.length ~ years
Model 2: telomere.length ~ as.factor(years)
  Res.Df   RSS Df Sum of Sq    F Pr(>F)
1     37 1.000
2     28 0.818  9     0.182 0.69   0.71
```

No evidence of a lack of fit.

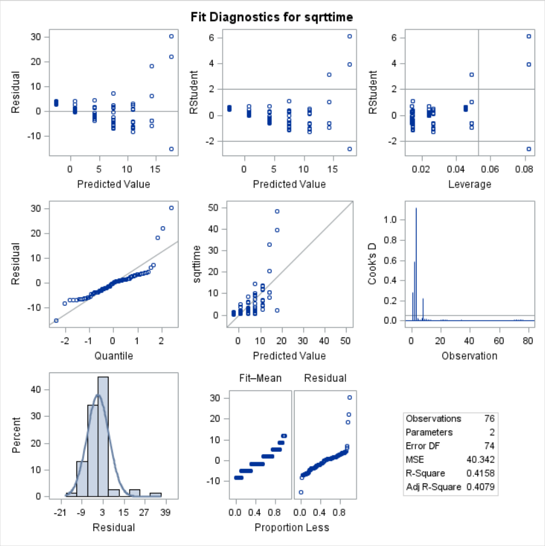
Lack-of-fit F-test summary

- Lack-of-fit F-test tests the assumption of linearity
- Needs multiple observations at various explanatory variable values
- Small pvalue indicates a lack-of-fit, i.e. means are not linear
  - Transform response, e.g. log
  - Transform explanatory variable
  - Add other explanatory variable(s)

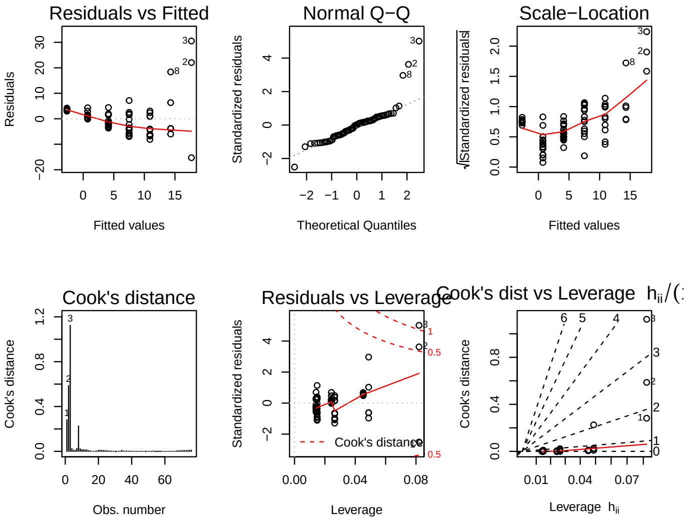
Summary of diagnostics

- Normality
  - Normal qq-plots should have points falling on the line
- Constant variance
  - Residuals vs predicted values should have random scatter on y-axis
  - Residuals vs explanatory variable(s) should have random scatter on y-axis
- Independence
  - Residuals vs anything should not have a pattern
- Linearity
  - Response vs explanatory variable should be linear
  - Lack-of-fit F-test should not be significant

Default diagnostics in SAS



Default diagnostics in R



Interpretations using logs

The most common transformation of either the response or explanatory variable(s) is to take logarithms because

- linearity will often then be approximately true,
- the variance will likely be approximately constant, and
- there is a (relatively) convenient interpretation.

We will talk about interpretation of β_0 and β_1 when

- only the response is logged,
- only the explanatory variable is logged, and
- when both are logged.

Neither response nor explanatory variable are logged

If

$$E[Y|X] = \beta_0 + \beta_1 X,$$

then

- β_0 is the expected response when X is zero and
- β_1 is the expected change in the response for a one unit increase in the explanatory variable.

For the following discussion,

- Y is always going to be the original response and
- X is always going to be the original explanatory variable.

Example

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 X$$

- β_0 is the expected corn yield per acre when fertilizer level is zero and
- β_1 is the expected change in corn yield per acre when fertilizer is increase by 1 lbs/acre.

Response is logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 X \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 X},$$

then

- β_0 is the expected $\log(Y)$ when X is zero and
- β_1 is the expected change in $\log(Y)$ for a one unit increase in the explanatory variable.

Alternatively,

- e^{β_0} is the median of Y when X is zero and
- e^{β_1} is the multiplicative effect on the median of Y for a one unit increase in the explanatory variable.

Response is logged

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[\log(Y)|X] = \beta_0 + \beta_1 X \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 X}$$

- e^{β_0} is the median corn yield per acre when fertilizer level is 0 and
- e^{β_1} is the multiplicative effect in median corn yield per acre when fertilizer is increase by 1 lbs/acre.

Explanatory variable is logged

If

$$E[Y|X] = \beta_0 + \beta_1 \log(X),$$

then

- β_0 is the expected response when $\log(X)$ is zero and
- β_1 is the expected change in the response for a one unit increase in $\log(X)$.

Alternatively,

- β_0 is the expected response when X is 1 and
- $\beta_1 \log(d)$ is the expected change in the response when X increase multiplicatively by d, e.g.
    - $\beta_1 \log(2)$ is the expected change in the response for each doubling of X or
    - $\beta_1 \log(10)$ is the expected change in the response for each ten-fold increase in X.

Explanatory variable is logged

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[Y|X] = \beta_0 + \beta_1 \log(X)$$

- β_0 is the expected corn yield per acre when fertilizer level is 1 lb/acre and
- $\beta_1 \log(2)$ is the expected change in corn yield when fertilizer level is doubled.

Both response and explanatory variable are logged

If

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X) \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1},$$

then

- β_0 is the expected $\log(Y)$ when $\log(X)$ is zero and
- β_1 is the expected change in $\log(Y)$ for a one unit increase in $\log(X)$.

Alternatively,

- e^{β_0} is the median of Y when X is 1 and
- d^{β_1} is the multiplicative change in the median of the response when X increase multiplicatively by d, e.g.
  - 2^{β_1} is the multiplicative effect on the median of the response for each doubling of X or
  - 10^{β_1} is the multiplicative effect on the median of the response for each ten-fold increase in X.

Both response and explanatory variable are logged

Suppose

- Y is corn yield per acre
- X is fertilizer level in lbs/acre

Then, if

$$E[\log(Y)|X] = \beta_0 + \beta_1 \log(X) \quad \text{or} \quad Median\{Y|X\} = e^{\beta_0} e^{\beta_1 \log(X)} = e^{\beta_0} X^{\beta_1}$$

- e^{β_0} is the median corn yield per acre when fertilizer level is 1 lb/acre and
- 2^{β_1} is the multiplicative effect on median corn yield per acre when fertilizer level doubles.

Summary of interpretations when using logarithms

- When using the log of the response,
  - β_0 will affect the median response
  - β_1 will affect the multiplicative change in the median response
- When using the log of the explanatory variable (X),
  - β_0 will affect the response when $X = 1$
  - β_1 will affect the change in the response when there is a multiplicative change in X