M4S1 - Central Limit Theorem

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STAT 226 - Iowa State University

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Outline

- Sampling distribution
- Central Limit Theorem
- Standard error

Sampling distribution

Definition

A summary statistic is a numerical value calculated from the sample.

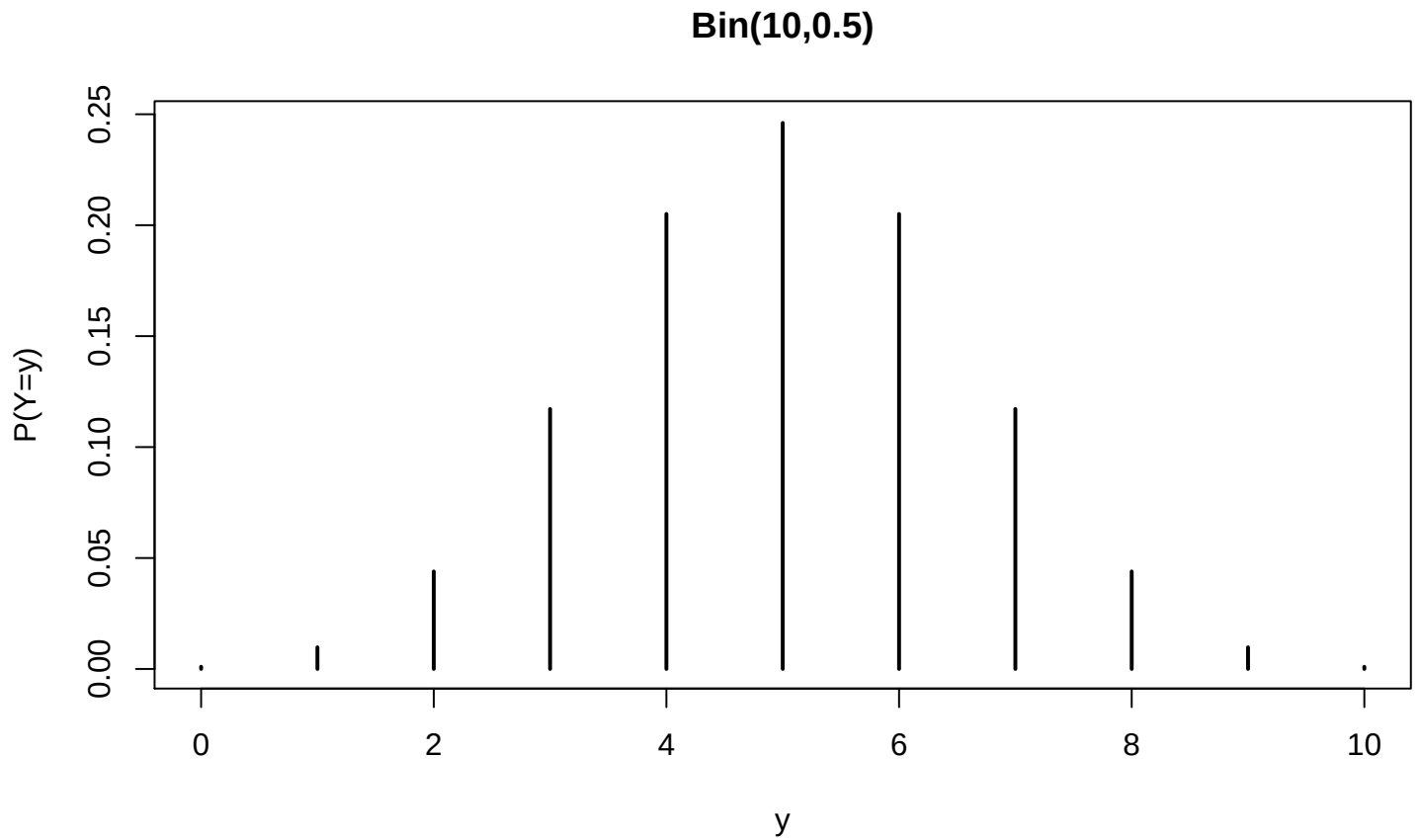
But this sample is only one of many possibilities. What could have happened if we had a different sample?

Definition

The sampling distribution of a statistic is the distribution of that statistic over different samples of a fixed size.

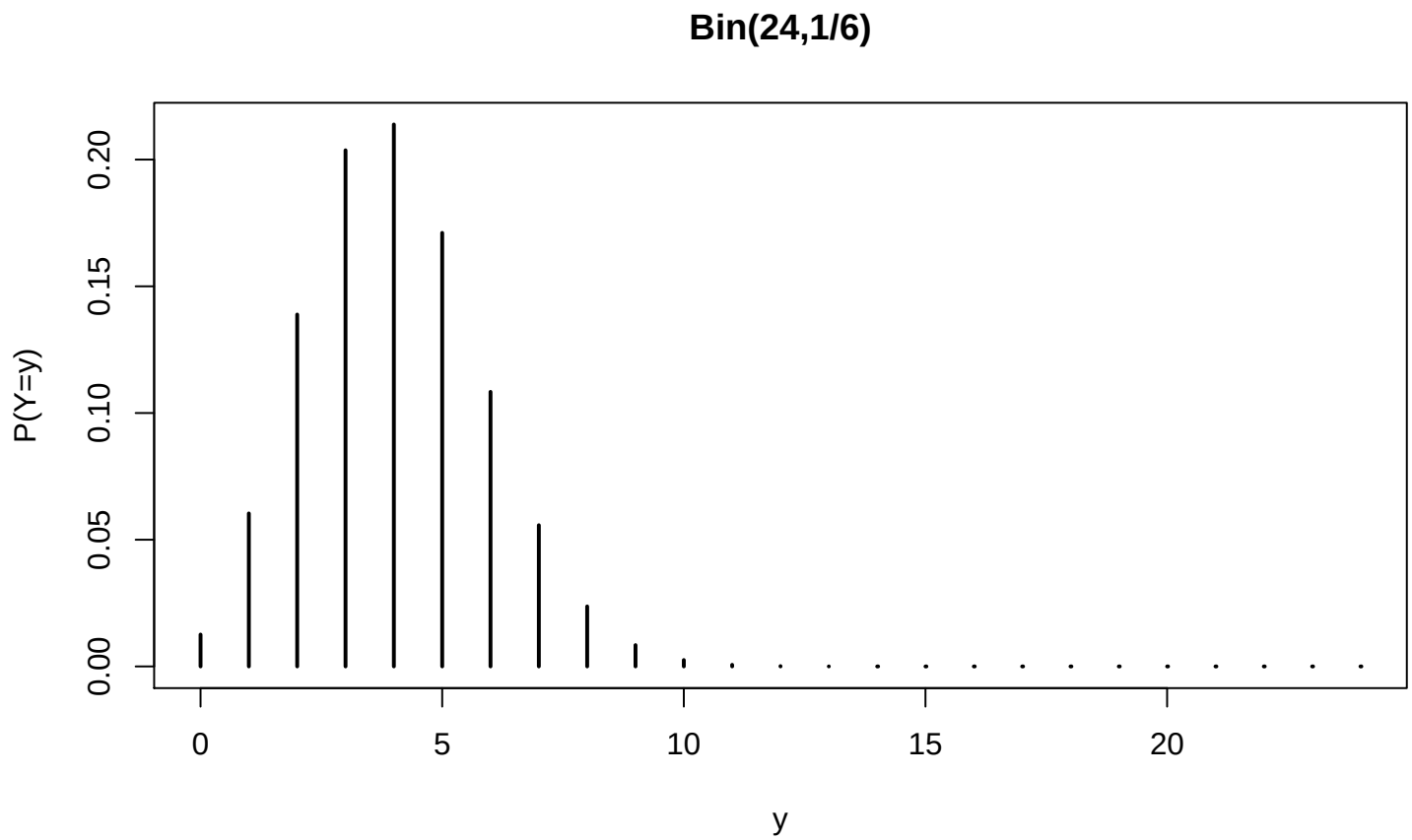
Flipping a coin

Suppose we repeatedly tossed a fair coin 10 times and recorded the number of heads. The sampling distribution is the binomial distribution with 10 attempts and probability of success 0.5.



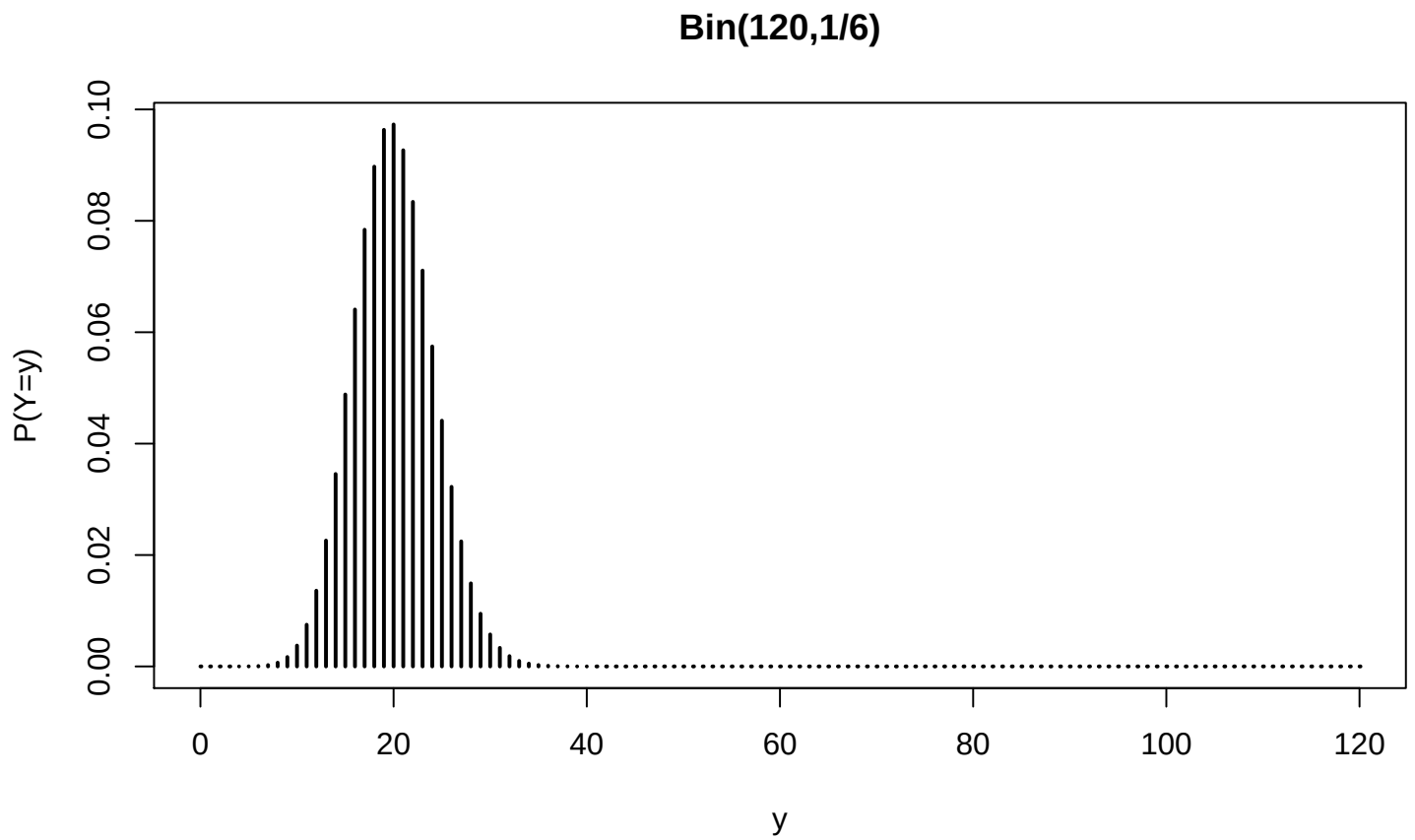
Rolling a die

Suppose we repeatedly rolled a fair 6-sided die 24 times and recorded the number of 1s. The sampling distribution is the binomial distribution with 24 attempts and probability of success 1/6.



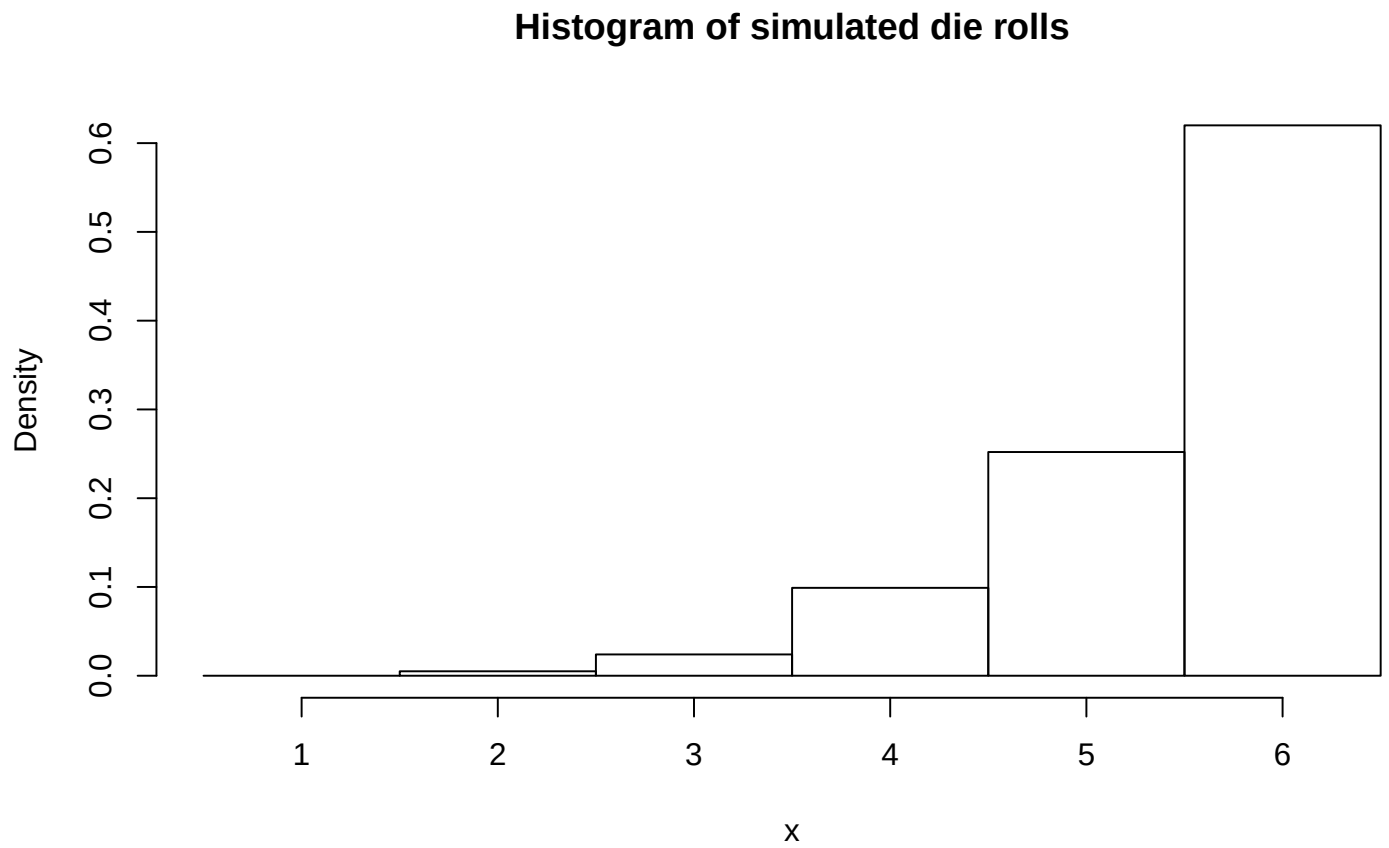
Rolling a die

Suppose we repeatedly rolled a fair 6-sided die 120 times and recorded the number of 1s. The sampling distribution is the binomial distribution with 120 attempts and probability of success 1/6.



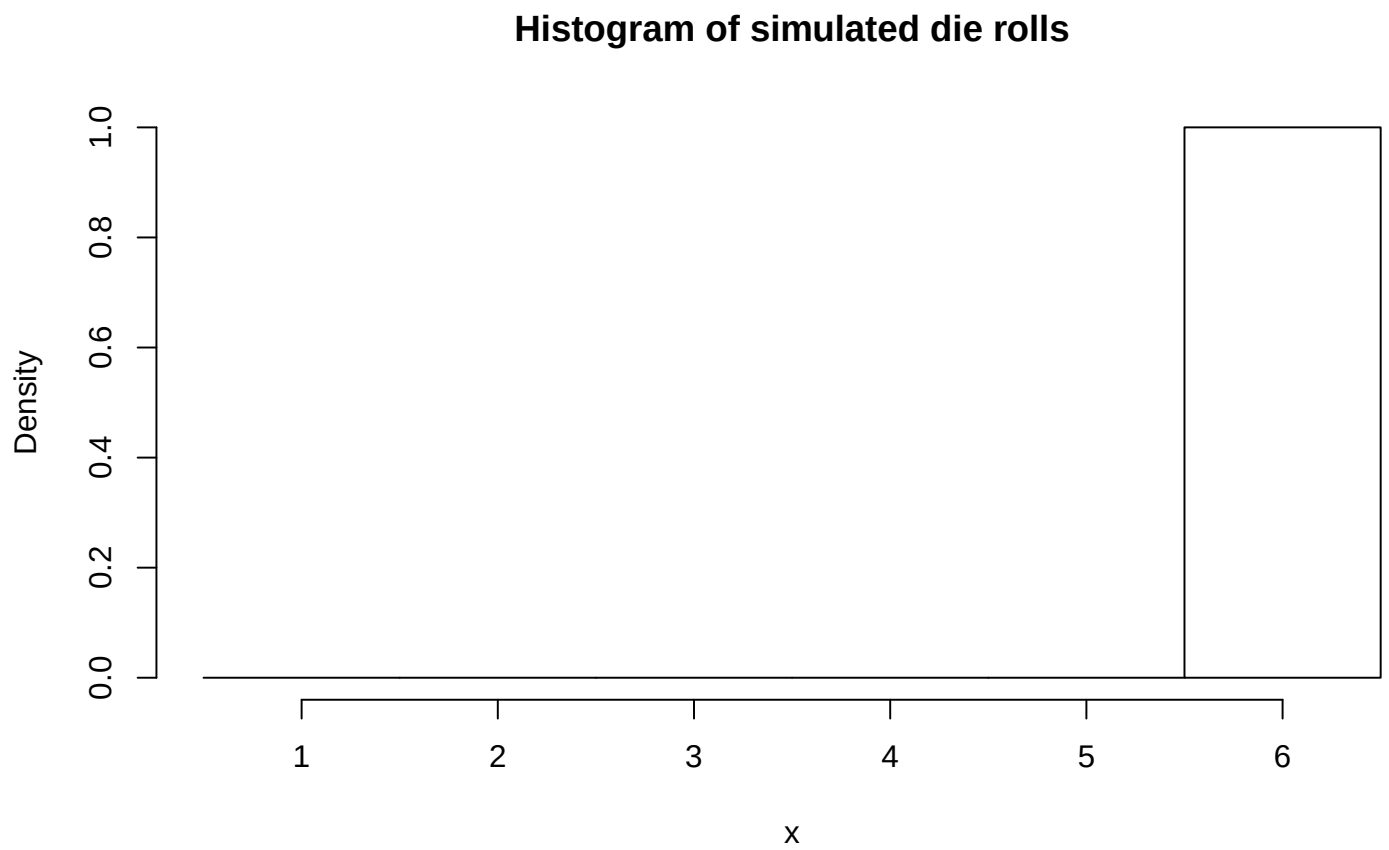
Rolling a die

Suppose we repeatedly rolled a fair 6-sided die 5 times and recorded the maximum. It's hard to analytically determine what happens, but we can use a computer to perform the experiment.



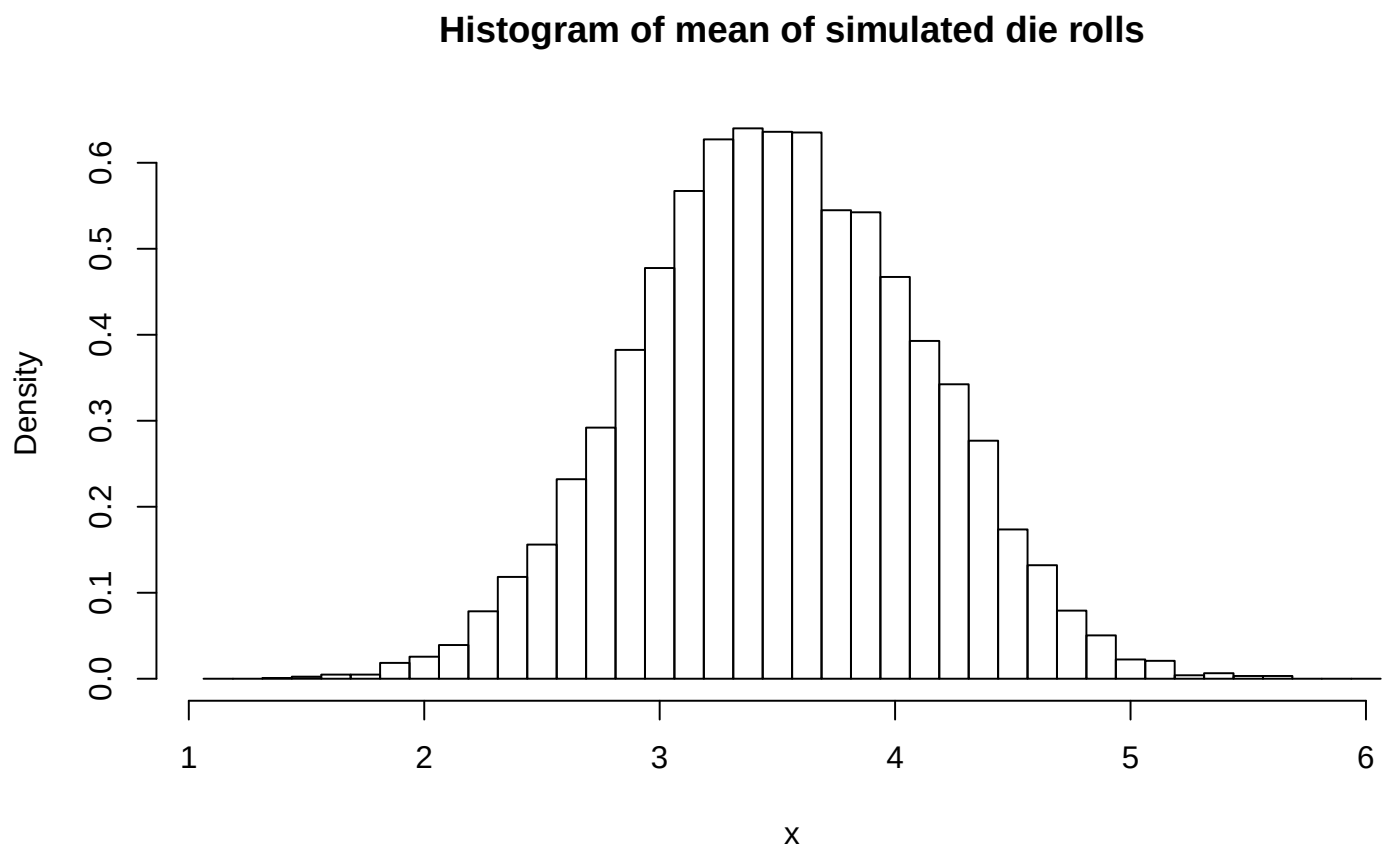
Rolling a die

Suppose we repeatedly rolled a fair 6-sided die 50 times and recorded the maximum. It's hard to analytically determine what happens, but we can use a computer to perform the experiment.



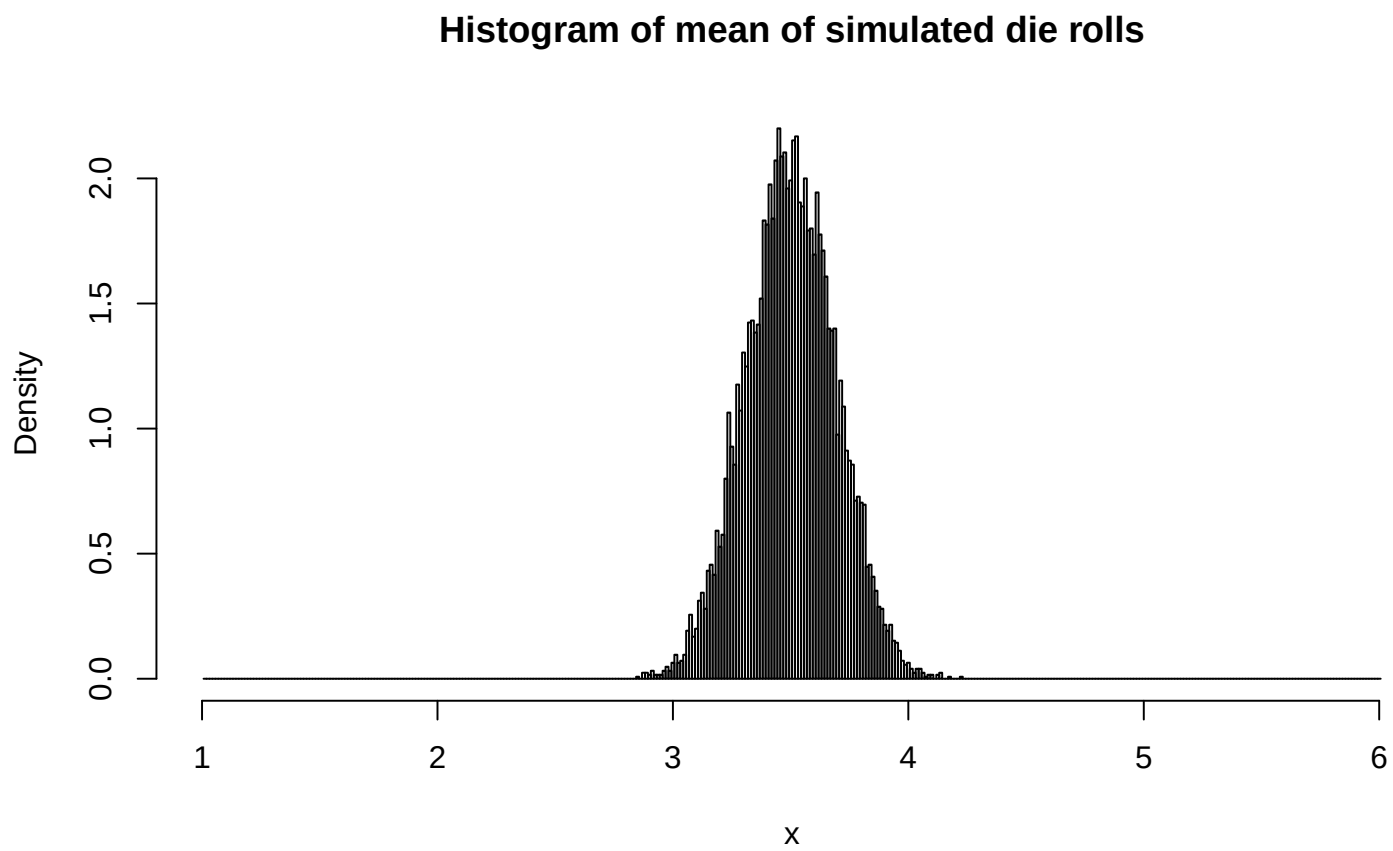
Sample mean

Suppose we repeatedly rolled a fair 6-sided die 8 times and recorded the mean. It's hard to analytically determine what happens, but we can use a computer to perform the experiment.



Sample mean

Suppose we repeatedly rolled a fair 6-sided die 80 times and recorded the mean. It's hard to analytically determine what happens, but we can use a computer to perform the experiment.



Central Limit Theorem

Theorem

Suppose you have a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ with mean $E[X_i] = \mu$ and variance $Var[X_i] = \sigma^2$. The Central Limit Theorem (CLT) says the sampling distribution of the sample mean converges to a normal distribution. Specifically

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \to N(0,1) \quad \text{as} \quad n \to \infty$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Thus, for large n, we can approximate the sample mean by a normal distribution, i.e.

$$\overline{X}_n \dot{\sim} N(\mu, \sigma^2/n)$$

where $\dot{\sim}$ means "approximately distributed." The standard deviation of the sampling distribution of a statistic is known as the standard error, i.e. $\sigma/\sqrt{n}$ is the standard error from the CLT.

Mean of the sample mean

Recall the following property:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

If we have $E[X_i] = \mu$ for all i, then

$$\begin{aligned} E[\overline{X}_n] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n}\sum_{i=1}^{n} \mu \\ &= \frac{1}{n} n \cdot \mu \\ &= \mu \end{aligned}$$

Variance of the sample mean

Recall the following property for independent random variables X and Y:

$$Var[aX + bY + c] = a^2 Var[X] + b^2 Var[Y]$$

If we have $Var[X_i] = \sigma^2$ for all i, then

$$\begin{aligned} Var[\overline{X}_n] &= Var\left[\frac{1}{n}\sum_{i=1}^n X_i\right] \\ &= \frac{1}{n^2} Var\left[\sum_{i=1}^n X_i\right] \\ &= \frac{1}{n^2}\sum_{i=1}^n Var[X_i] \\ &= \frac{1}{n^2}\sum_{i=1}^n \sigma^2 \\ &= \frac{1}{n^2} n \cdot \sigma^2 \\ &= \sigma^2/n \end{aligned}$$

Sampling distribution of sample mean

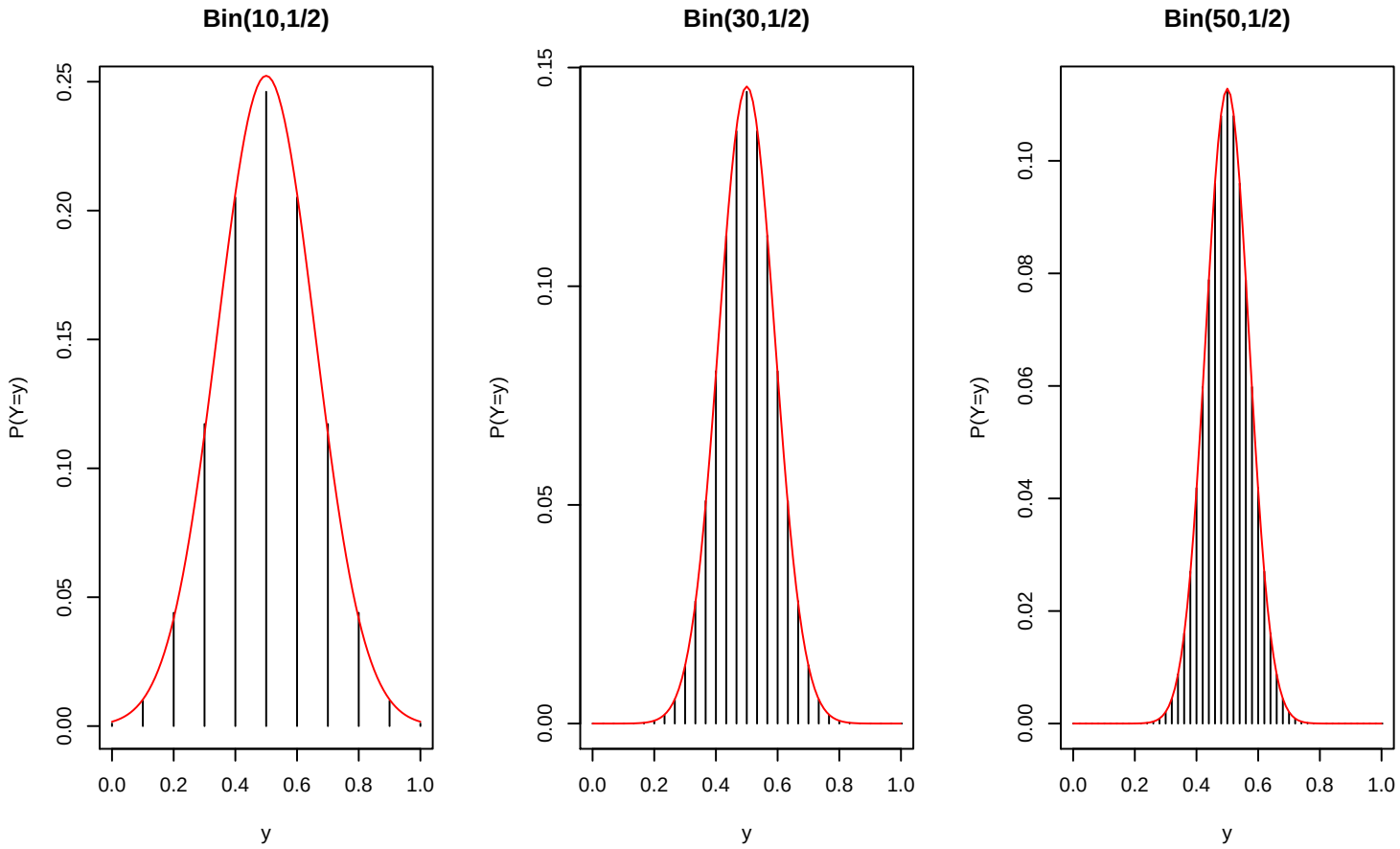
If $X_1, X_2, \ldots$ are a sequence of independent and identically distributed random variables with $E[X_i] = \mu$ and $Var[X_i] = \sigma^2$, then

$$E[\overline{X}_n] = \mu \qquad Var[\overline{X}_n] = \sigma^2/n$$

for any n. The CLT says that, as n gets large, the sampling distribution of the sample mean converges to a normal distribution.

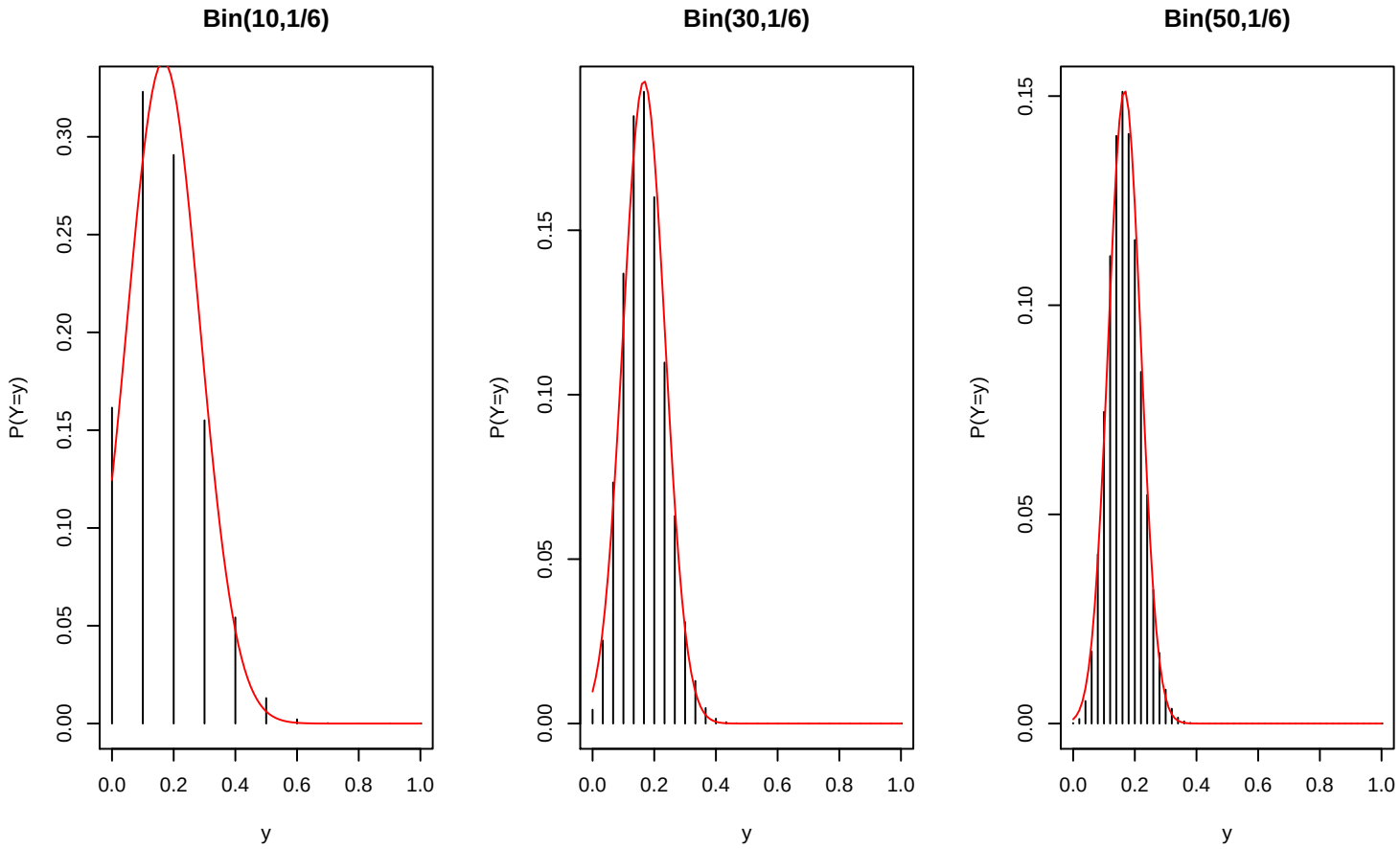
Coin flipping

Sampling distribution for the proportion of heads on an unbiased coin flip.



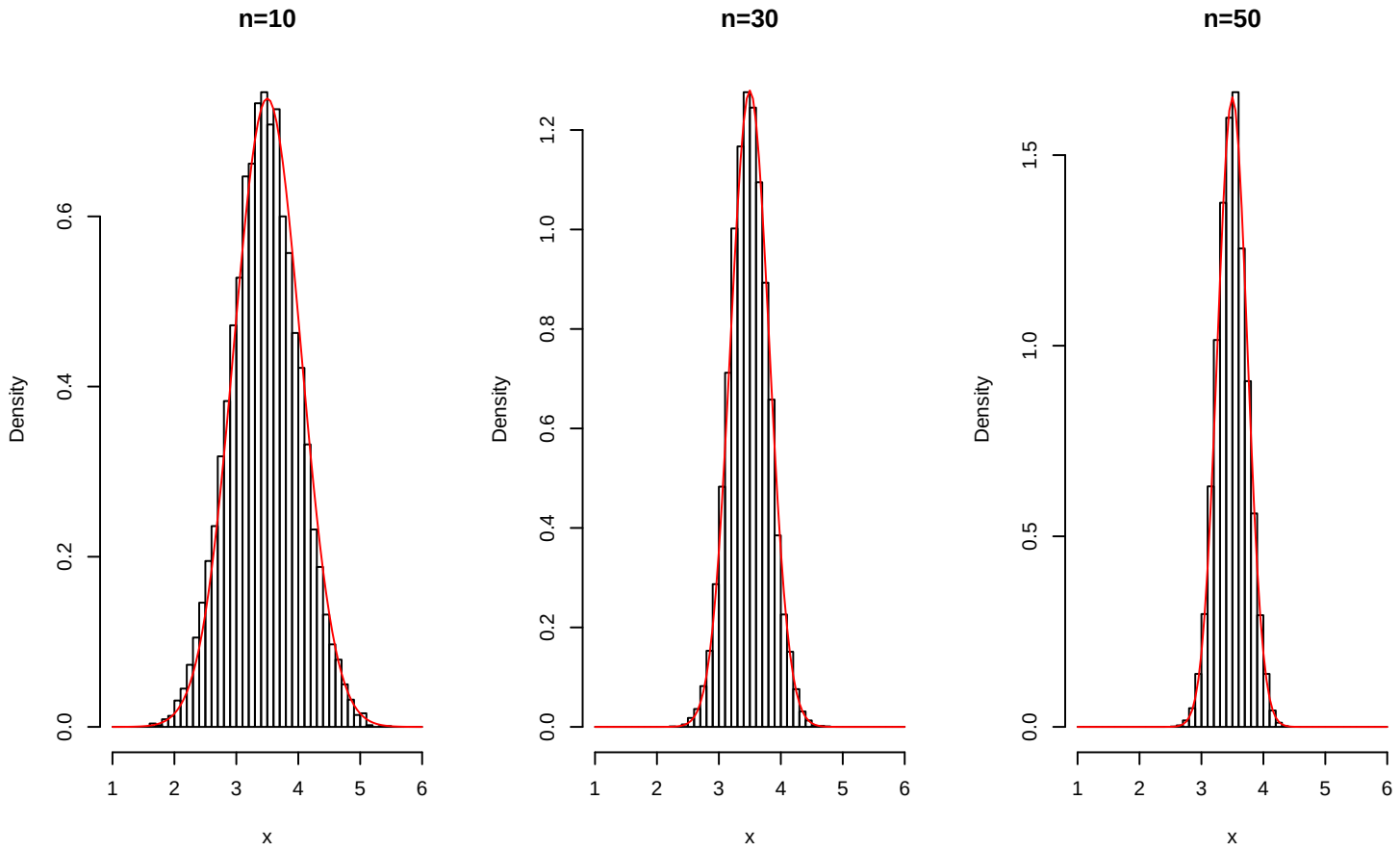
Die rolling

Sampling distribution for the proportion of 1s on an unbiased 6-sided die roll.



Die rolling

Sampling distribution for the sample mean of an unbiased 6-sided die roll.



Welfare

A certain group of welfare recipients receives SNAP benefits of \$110 per week with a standard deviation of \$20. A random sample of 30 people is taken and sample mean is calculated.

- What is the expected value of the sample mean?

  Let X_i be the SNAP benefit for individual i. We know $E[X_i] = \$110$ and $Var[X_i] = \$20^2$. Thus, $E[\overline{X}_{30}] = \$110$.

- What is the the standard error of the sample mean?

  The standard error is $\sigma/\sqrt{n} = \$20/\sqrt{30} \approx \$3.65$.

- What is the approximate probability the sample mean will be greater than \$120?

  We know $\overline{X}_{30} \dot{\sim} N(\$110, \$3.65^2)$.

$$\begin{aligned} P(\overline{X}_{30} > \$120) &= P\left(\frac{\overline{X}_{30} - \$110}{\$3.65} > \frac{\$120 - \$110}{\$3.65}\right) \\ &\approx P(Z > 2.74) \\ &= 1 - P(Z < 2.74) \\ &= 1 - 0.9969 = 0.0031 \end{aligned}$$

Process to use CLT

Given a scientific question, do the following

1. Identify the random variables $X_1, X_2, \ldots$.
2. Verify these are independent and identically distributed.
3. Determine the expectation/mean and variance (or standard deviation) of the X_i.
4. Determine the sample size. Is the sample size large enough for the CLT to apply?
5. If yes, determine the approximate sampling distribution for the sample mean.
6. Write the scientific question in mathematical/probabilistic notation.
7. Calculate your answer.