STAT 401A - Statistical Methods for Research Workers

Simple linear regression

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Simple Linear Regression

Recall the one-way ANOVA model:

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

where Y_{ij} is the observation for individual i in group j.

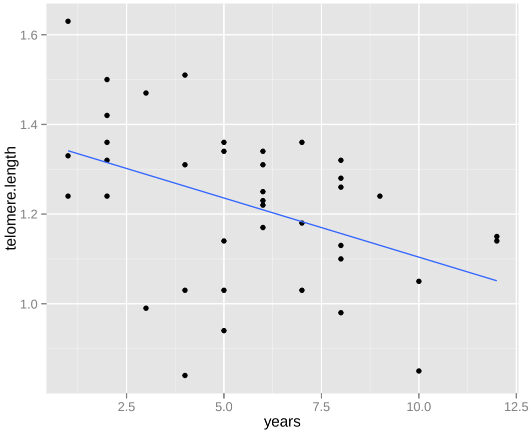
The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

Terminology (all of these are equivalent):

response	explanatory
outcome	covariate
dependent	independent
endogenous	exogenous


1.6
1.4
1.2
1.0
telomere.length
2.5
5.0
7.5
10.0
12.5
years

Interpretation

$$E[Y_i|X_i = x] = \beta_0 + \beta_1 x \qquad V[Y_i|X_i = x] = \sigma^2$$

- If $X_i = 0$, then $E[Y_i|X_i = 0] = \beta_0$.
  β_0 is the expected response when the explanatory variable is zero.

- If X_i increases from x to $x + 1$, then

  $$\begin{array}{rl} E[Y_i|X_i = x+1] & = \beta_0 + \beta_1 x + \beta_1 \\ -\ E[Y_i|X_i = x \qquad] & = \beta_0 + \beta_1 x \\ \hline & = \beta_1 \end{array}$$

  β_1 is the expected increase in the response for each unit increase in the explanatory variable.

- σ is the standard deviation of the response for a fixed value of the explanatory variable.

Remove the mean:

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

So the error is

$$e_i = Y_i - (\beta_0 + \beta_1 X_i)$$

which we approximate by the residual

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

The least squares, maximum likelihood, and Bayesian estimators are

$$\begin{array}{rl} \hat{\beta}_1 & = SXY/SXX \\ \hat{\beta}_0 & = \overline{Y} - \hat{\beta}_1 \overline{X} \\ \hat{\sigma}^2 & = SSE/(n-2) \qquad \text{df} = n-2 \end{array}$$

$$\begin{array}{rl} \overline{X} & = \frac{1}{n}\sum_{i=1}^n X_i \\ \overline{Y} & = \frac{1}{n}\sum_{i=1}^n Y_i \end{array}$$

$$\begin{array}{rl} SXY & = \sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) \\ SXX & = \sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^n (X_i - \overline{X})^2 \\ SSE & = \sum_{i=1}^n r_i^2 \end{array}$$

How certain are we about $\hat{\beta}_0$ and $\hat{\beta}_1$ being equal to β_0 and β_1?

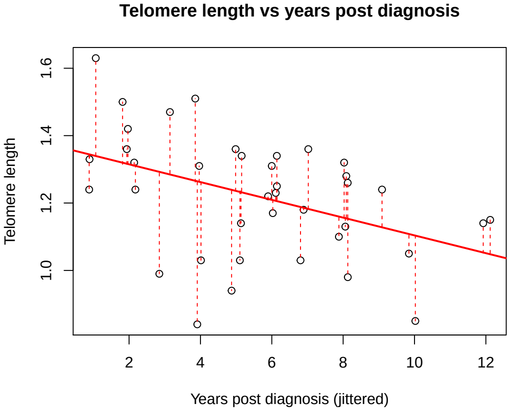
We quantify this uncertainty using their standard errors:

$$\begin{array}{rll} SE(\beta_0) & = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} & df = n-2 \\ SE(\beta_1) & = \hat{\sigma}\sqrt{\frac{1}{(n-1)s_X^2}} & df = n-2 \end{array}$$

$$\begin{array}{rl} s_X^2 & = SXX/(n-1) \\ s_Y^2 & = SYY/(n-1) \\ SYY & = \sum_{i=1}^n (Y_i - \overline{Y})^2 \end{array}$$

$$\begin{array}{rlll} r_{XY} & = \frac{SXY/(n-1)}{s_X s_Y} & & \text{correlation coefficient} \\ R^2 & = r_{XY}^2 & = \frac{SST-SSE}{SST} & \text{coefficient of determination} \\ SST & = SYY = \sum_{i=1}^n (Y_i - \overline{Y})^2 & & \end{array}$$

The coefficient of determination (R^2) is the percentage of the total response variation explained by the explanatory variable(s).


Telomere length vs years post diagnosis
Telomere length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
Years post diagnosis (jittered)

Pvalues and confidence interval

We can compute two-sided pvalues via

$$2P\left(t_{n-2} > \left|\frac{\hat{\beta}_0}{SE(\beta_0)}\right|\right) \qquad \text{and} \qquad 2P\left(t_{n-2} > \left|\frac{\hat{\beta}_1}{SE(\beta_1)}\right|\right)$$

These test the null hypothesis that the corresponding parameter is zero.

We can construct $100(1-\alpha)\%$ two-sided confidence intervals via

$$\hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\beta_0) \qquad \text{and} \qquad \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\beta_1)$$

These provide ranges of the parameters consistent with the data.

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years;
  RUN;
```

```
                              The REG Procedure
                                Model: MODEL1
                          Dependent Variable: length

                   Number of Observations Read          39
                   Number of Observations Used          39

                            Analysis of Variance

                                      Sum of           Mean
         Source               DF      Squares         Square    F Value    Pr > F

         Model                 1      0.22777        0.22777       8.42    0.0062
         Error                37      1.00033        0.02704
         Corrected Total      38      1.22810

                    Root MSE              0.16443    R-Square     0.1855
                    Dependent Mean        1.22026    Adj R-Sq     0.1634
                    Coeff Var            13.47473


                            Parameter Estimates
                       Parameter     Standard
    Variable     DF     Estimate        Error    t Value    Pr > |t|      95% Confidence Limits

    Intercept     1      1.36768      0.05721      23.91      <.0001      1.25176      1.48360
    years         1     -0.02637      0.00909      -2.90      0.0062     -0.04479     -0.00796
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
anova(m)
```

```
Analysis of Variance Table

Response: telomere.length
          Df Sum Sq Mean Sq F value Pr(>F)
years      1  0.228   0.228    8.42 0.0062 **
Residuals 37  1.000   0.027
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
summary(m)



Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
    Min      1Q  Median      3Q     Max 
-0.4222 -0.0854  0.0206  0.1074  0.2887 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.36768    0.05721    23.9   <2e-16 ***
years       -0.02637    0.00909    -2.9   0.0062 ** 
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.164 on 37 degrees of freedom
Multiple R-squared:  0.185, Adjusted R-squared:  0.163 
F-statistic: 8.42 on 1 and 37 DF,  p-value: 0.0062


confint(m)


               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

Summary

- The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

  where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

- Know how to use SAS/R to obtain $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$, R^2, pvalues, CIs, etc.

- Interpret SAS output
  - At a value of zero for the explanatory variable ($X_i = 0$), β_0 is the expected value for the response (Y_i).
  - For each unit increase in the explanatory variable value, β_1 is the expected increase in the response.
  - At a constant value of the explanatory variable, σ^2 is the variance of the responses.
  - The coefficient of determination (R^2) is the percentage of the total response variation explained by the explanatory variable(s).