# STAT 401A - Statistical Methods for Research Workers

## Inference Using $t$-Distributions

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# Random variables

From: http://www.stats.gla.ac.uk/steps/glossary/probability_distributions.html

### Definition

A random variable is a function that associates a unique numerical value with every outcome of an experiment.

### Definition

A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4,... Discrete random variables are usually (but not necessarily) counts.

### Definition

A continuous random variable is one which takes an infinite number of possible values. Continuous random variables are usually measurements.

# Random variables

Examples:

- Discrete random variables
  - Coin toss: Heads (1) or Tails (0)
  - Die roll: 1, 2, 3, 4, 5, or 6
  - Number of Ovenbirds at a 10-minute point count
  - RNAseq feature count
- Continuous random variables
  - Pig average daily (weight) gain
  - Corn yield per acre

# Statistical notation

Let $Y$ be 1 if the coin toss is heads and 0 if tails, then

$$Y \sim Bin(n, p)$$

which means

> *$Y$ is a binomial random variable with $n$ trials and probability of success $p$*

For example, if $Y$ is the number of heads observed when tossing a fair coin ten times, then $Y \sim Bin(10, 0.5)$.

Later we will be constructing $100(1-\alpha)\%$ confidence intervals, these intervals are constructed such that if $n$ of them are constructed then $Y \sim Bin(n, 1-\alpha)$ will cover the true value.

# Statistical notation

Let $Y_i$ be the average daily (weight) gain in pounds for the $i$th pig, then

$$Y_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$$

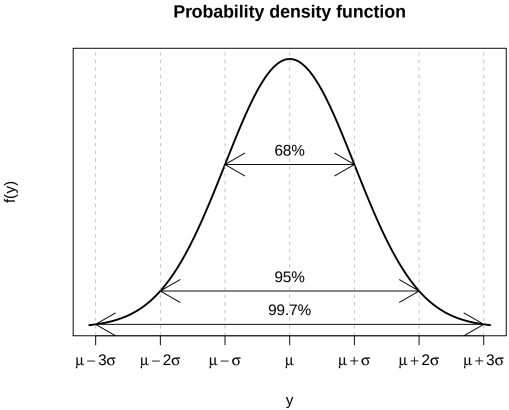
which means

> *$Y_i$ are independent and identically distributed normal (Gaussian) random variables with expected value $E[Y_i] = \mu$ and variance $V[Y_i] = \sigma^2$ (standard deviation $\sigma$).*

For example, if a litter of pigs is expected to gain 2 lbs/day with a standard deviation of 0.5 lbs/day and *the knowledge of how much one pig gained does not affect what we think about how much the others have gained*, then $Y_i \stackrel{iid}{\sim} N(2, 0.5^2)$.

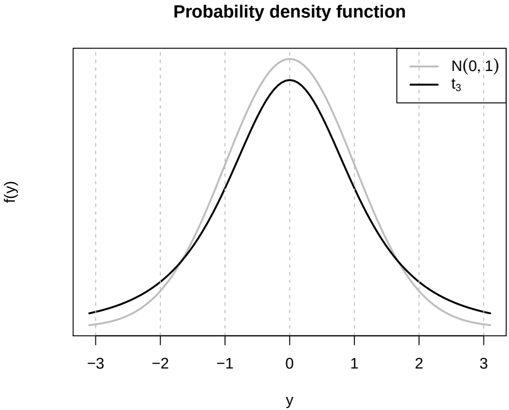
# Normal (Gaussian) distribution

A random variable $Y$ has a normal distribution, i.e. $Y \sim N(\mu, \sigma^2)$, with mean $\mu$ and variance $\sigma^2$ if draws from this distribution follow a bell curve centered at $\mu$ with spread determined by $\sigma^2$:



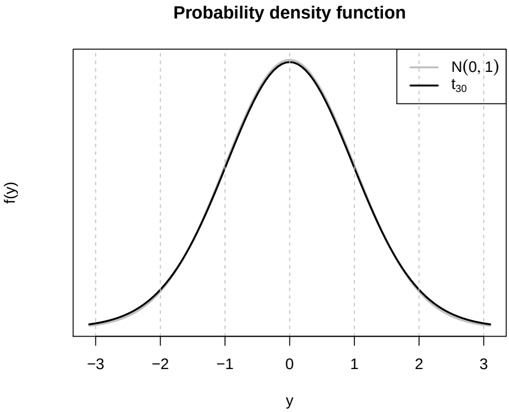
# *t*-distribution

A random variable $Y$ has a $t$-distribution, i.e. $Y \sim t_v$, with degrees of freedom $v$ if draws from this distribution follow a similar bell shaped pattern:



# $t$-distribution

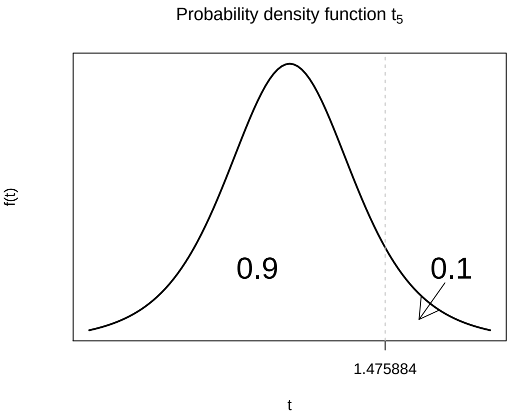
As $v \to \infty$, then $t_v \stackrel{d}{\to} N(0,1)$, i.e. as the degrees of freedom increase, a $t$ distribution gets closer and closer to a standard normal distribution, i.e. $N(0,1)$. If $v > 30$, the differences is negligible.



# $t$ critical value

## Definition

If $T \sim t_v$, a $t_v(1-\alpha/2)$ critical value is the value such that $P(T < t_v(1-\alpha/2)) = 1-\alpha/2$ (or $P(T > t_v(1-\alpha)) = \alpha/2$).



# Hypotheses

Three key features:

- a test statistic calculated from data
- a sampling distribution for the test statistic under the null hypothesis
- a region that is as or more extreme (one-sided vs two-sided hypotheses)

Calculate probability of being in the region:

### Definition

A pvalue is the probability of observing a test statistic as or more extreme than that observed, if the null hypothesis is true.

- If pvalue is less than or equal to $\alpha$, we reject the null hypothesis.
- If pvalue is greater than $\alpha$, we fail to reject the null hypothesis.

# Z-statistics

Let's assume, we have

- calculated a test statistic $z$ that
- has a $Z \sim N(0,1)$ sampling distribution if the null hypothesis is true.

We could easily replace $z$ with $t$ and have a sampling distribution that is $t_\nu$ for some $\nu$.

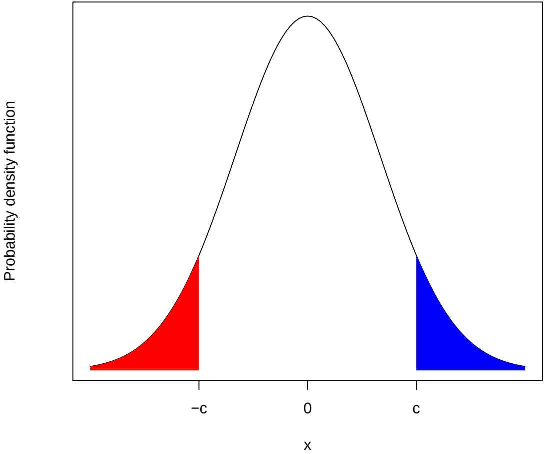
Now, we can have one of three types of hypotheses:

- Two-sided: $$\begin{aligned} P(|Z| > |z|) &= P(Z > |z|) + P(Z < -|z|) \\ &= 2P(Z < -|z|) \end{aligned}$$
- One-sided:
  - $P(Z > z) = P(Z < -z)$
  - $P(Z < z)$

$F(c) = P(Z < c)$ is the cumulative distribution function for the standard normal.

# Symmetric distributions

The standard normal and $t$ distributions are both symmetric around zero.




$P(Z > c) = P(Z < -c)$ blue area is equal to red area

# Paired t-test example

In a paired t-test, we might have

- a one-sided hypothesis that the difference is greater than zero
- a test statistic $t = 2.43$
- which has a $t$ distribution with 7 degrees of freedom when the null hypothesis is true.

So we need to calculate

$$P(t_7 > 2.43) = 1 - P(t_7 < 2.43)$$

where $t_7$ represents a t-distribution with 7 degrees of freedom.

# Using SAS or R

In SAS,

```
PROC IML;
  p = 1-Cdf('T', 2.43, 7);
  PRINT p;
  QUIT;
```

In R,

```
p = 1-pt(2.43,7)
```

Both obtain p=0.0227.

# Paired t-test example

In a paired t-test, we might have

- a one-sided hypothesis that the difference is greater than zero
- a test statistic $t = 2.43$
- which has a $t$ distribution with 7 degrees of freedom when the null hypothesis is true.

So we need to calculate

$$P(t_7 > 2.43) = 1 - P(t_7 < 2.43) = 0.02$$

So, we reject the null hypothesis that there is no difference, but how big is the difference?

# Confidence interval construction

Key steps in confidence interval construction:

1. Calculate statistic
2. Calculate standard error of the statistic
3. Find the appropriate critical value
4. Construct the interval
   - Two-sided interval:

$$\text{statistic} \pm \text{critical value} \times \text{standard error}$$

   - One-sided interval:

$$\text{statistic} - \text{critical value} \times \text{standard error}$$

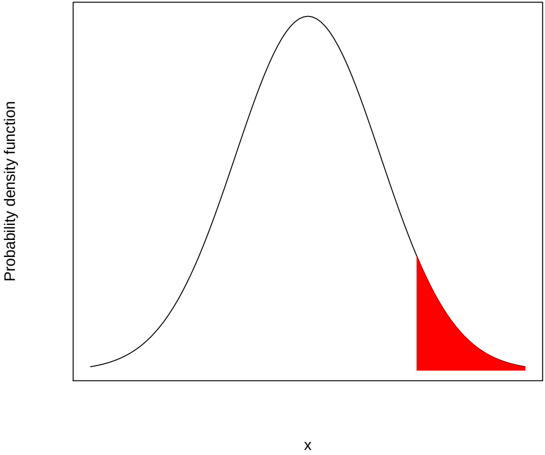
or

$$\text{statistic} + \text{critical value} \times \text{standard error}$$

# Critical values

A related quantity are critical values for confidence interval construction, e.g.

$$(\overline{Y}_2 - \overline{Y}_1) \pm t_\nu(1-\alpha/2)SE(\overline{Y}_2 - \overline{Y}_1).$$



# Using SAS or R

If $\alpha = 0.05$, then $1 - \alpha/2 = 0.975$.

In SAS,

```
PROC IML;
  q = QUANTILE('T', 0.975, 7);
  PRINT q;
  QUIT;
```

In R,

```
q = qt(0.975,7)
```

Both obtain q=2.364.

# Summary

Two main approaches to statistical inference:

- Statistical hypothesis (hypothesis test)
- Statistical question (confidence interval)

# Cedar-apple rust

Cedar-apple rust is a (non-fatal) disease that affects apple trees. Its most obvious symptom is rust-colored spots on apple leaves. Red cedar trees are the immediate source of the fungus that infects the apple trees. If you could remove all red cedar trees within a few miles of the orchard, you should eliminate the problem. In the first year of this experiment the number of affected leaves on 8 trees was counted; the following winter all red cedar trees within 100 yards of the orchard were removed and the following year the same trees were examined for affected leaves.

- Statistical hypothesis:
  - $H_0$: Removing red cedar trees increases or maintains the same mean number of rusty leaves.
  - $H_1$: Removing red cedar trees decreases the mean number of rusty leaves.
- Statistical question:

  What is the expected reduction of rusty leaves in our sample between year 1 and year 2 (perhaps due to removal of red cedar trees)?

# Data

Here are the data

```
library(plyr)
y1 = c(38,10,84,36,50,35,73,48)
y2 = c(32,16,57,28,55,12,61,29)
leaves  = data.frame(year1=y1, year2=y2, diff=y1-y2)
leaves


  year1 year2 diff
1    38    32    6
2    10    16   -6
3    84    57   27
4    36    28    8
5    50    55   -5
6    35    12   23
7    73    61   12
8    48    29   19


summarize(leaves, n=length(diff), mean=mean(diff), sd=sd(diff))


  n mean   sd
1 8 10.5 12.2
```

Is this a statistically significant difference?

# Assumptions

Let

- $Y_{1j}$ be the number of rusty leaves on tree $j$ in year 1
- $Y_{2j}$ be the number of rusty leaves on tree $j$ in year 2

Assume

$$D_j = Y_{1j} - Y_{2j} \stackrel{iid}{\sim} N(\mu, \sigma^2)$$

Then the statistical hypothesis test is

$H_0$: $\mu = 0$ ($\mu \le 0$)

$H_1$: $\mu > 0$

while the statistical question is 'what is $\mu$?'

# Paired t-test pvalue

Test statistic

$$t = \frac{\overline{D} - \mu}{SE(\overline{D})}$$

where $SE(\overline{D}) = s/\sqrt{n}$ with

- $n$ being the number of observations (differences),
- $s$ being the sample standard deviation of the differences, and
- $\overline{D}$ being the average difference.

If $H_0$ is true, then $\mu = 0$ and $t \sim t_{n-1}$. The pvalue is $P(t_{n-1} > t)$ since this is a one-sided test. By symmetry, $P(t_{n-1} > t) = P(t_{n-1} < -t)$.

For these data,

$$\overline{D} = 10.5, \mathrm{SE}(\overline{D}) = 4.31, t_7 = 2.43, and \mathrm{p} = 0.02$$

# Confidence interval for $\mu$

The 100(1-$\alpha$)% confidence interval has lower endpoint

$$\overline{D} - t_{n-1}(1-\alpha)SE(\overline{D})$$

and upper endpoint at infinity

For these data at 95% confidence, $t_7(0.9) = 1.89$ and thus the lower endpoint is

$$10.5 - 1.89 \cdot 4.31 = 2.33$$

So we are 95% confident that the true difference in the number of rusty leaves is greater than 2.33.

# SAS code for paired t-test

```
DATA leaves;
  INPUT tree year1 year2;
  DATALINES;
1 38 32
2 10 16
3 84 57
4 36 28
5 50 55
6 35 12
7 73 61
8 48 29
;

PROC TTEST DATA=leaves SIDES=U;
    PAIRED year1*year2;
    RUN;
```

# SAS output for paired t-test

```
                    The TTEST Procedure

                Difference:  year1 - year2

 N        Mean     Std Dev     Std Err     Minimum     Maximum

 8     10.5000     12.2007      4.3136     -6.0000     27.0000

    Mean       95% CL Mean        Std Dev      95% CL Std Dev

 10.5000      2.3275   Infty      12.2007      8.0668  24.8317

                     df     t Value    Pr > t

                      7       2.43     0.0226
```

# R output for paired t-test

```
t.test(leaves$year1, leaves$year2, paired=TRUE, alternative="greater")



Paired t-test

data:  leaves$year1 and leaves$year2
t = 2.434, df = 7, p-value = 0.02257
alternative hypothesis: true difference in means is greater than 0
95 percent confidence interval:
 2.328   Inf
sample estimates:
mean of the differences
                   10.5
```

# Statistical Conclusion

Removal of red cedar trees within 100 yards is associated with a significant reduction in rusty apple leaves (paired t-test $t_7$=2.43, p=0.023). The mean reduction in rust color leaves is 10.5 [95% CI (2.33,$\infty$)].

# Do Japanese cars get better mileage than American cars?

- Statistical hypothesis:

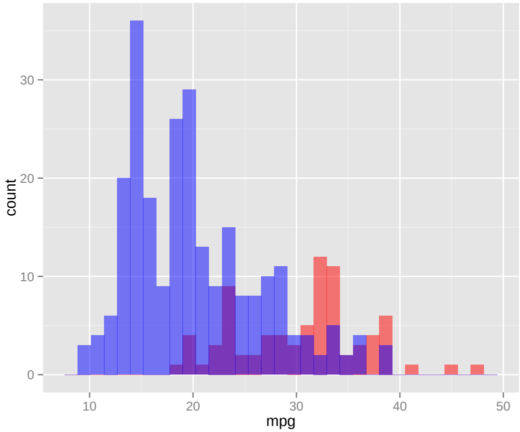
  $H_0$: Mean mpg of Japanese cars is the same as mean mpg of American cars.
  $H_1$: Mean mpg of Japanese cars is different than mean mpg of American cars.

- Statistical question:

  What is the difference in mean mpg between Japanese and American cars?

- Data collection:
  - Collect a random sample of Japanese/American cars

```
mpg = read.csv("mpg.csv")
library(ggplot2)
ggplot(mpg, aes(x=mpg))+
  geom_histogram(data=subset(mpg,country=="Japan"), fill="red", alpha=0.5)+
  geom_histogram(data=subset(mpg,country=="US"), fill="blue", alpha=0.5)
```



# Assumptions

Let

- $Y_{1j}$ represent the $j$th Japanese car
- $Y_{2j}$ represent the $j$th American car

Assume

$$Y_{1j} \stackrel{iid}{\sim} N(\mu_1, \sigma^2) \qquad Y_{2j} \stackrel{iid}{\sim} N(\mu_2, \sigma^2)$$

Restate the hypotheses using this notation

$H_0$: $\mu_1 = \mu_2$

$H_1$: $\mu_1 \neq \mu_2$

Alternatively

$H_0$: $\mu_1 - \mu_2 = 0$

$H_1$: $\mu_1 - \mu_2 \neq 0$

# Test statistic

The test statistic we use here is

$$\frac{\overline{Y}_1 - \overline{Y}_2 - (\mu_1 - \mu_2)}{SE(\overline{Y}_1 - \overline{Y}_2)}$$

where

- $\overline{Y}_1$ is the sample average mpg of the Japanese cars
- $\overline{Y}_2$ is the sample average mpg of the American cars

and

$$SE(\overline{Y}_1 - \overline{Y}_2) = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \qquad s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{(n_1+n_2-2)}}$$

where

- $s_1$ is the sample standard deviation of the mpg of the Japanese cars
- $s_2$ is the sample standard deviation of the mpg of the American cars

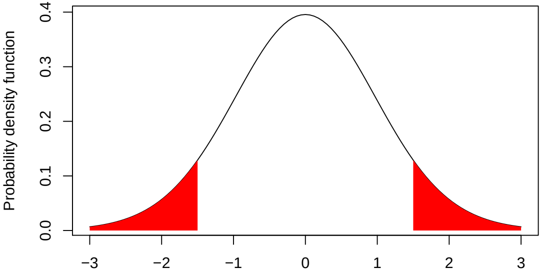
# Pvalue

If $H_0$ is true, then $\mu_1 = \mu_2$ and the test statistic

$$t = \frac{\overline{Y}_1 - \overline{Y}_2 - (\mu_1 - \mu_2)}{SE(\overline{Y}_1 - \overline{Y}_2)} \sim t_{n_1+n_2-2}$$

where $t_\nu$ is a t-distribution with $\nu$ degrees of freedom.

Pvalue is $P(|t_{n_1+n_2-2}| > |t|) = P(t_{n_1+n_2-2} > |t|) + P(t_{n_1+n_2-2} < -|t|)$ or as a picture



# Hand calculation

To calculate the quantity by hand, we need 6 numbers:

```
library(plyr)
ddply(mpg, .(country), summarize, n=length(mpg), mean=mean(mpg), sd=sd(mpg))


  country   n  mean    sd
1   Japan  79 30.48 6.108
2      US 249 20.14 6.415
```

Calculate

$$\begin{aligned} s_p &= \sqrt{\frac{(79-1)\cdot 6.11^2+(249-1)\cdot 6.41^2}{79+249-2}} &= 6.34 \\ SE(\overline{Y}_1 - \overline{Y}_2) &= 6.34\sqrt{\frac{1}{79}+\frac{1}{249}} &= 0.82 \\ t &= \frac{30.5-20.1}{0.82} &= 12.6 \end{aligned}$$

Finally, we are interested in finding $P(|t_{326}| > |12.6|) = 2P(t_{326} < -|12.6|) < 0.0001$ which is found using a table or software.

# Confidence interval

Alternatively, we can construct a 100(1-$\alpha$)% confidence interval. The formula is

$$\overline{Y}_1 - \overline{Y}_2 \pm t_{n_1+n_2-2}(1-\alpha/2)SE(\overline{Y}_1 - \overline{Y}_2)$$

where $\pm$ indicates plus and minus and $t_\nu(1-\alpha/2)$ is the value such that $P(t_\nu < t_\nu(1-\alpha/2)) = 1-\alpha/2$. If $\alpha = 0.05$ and $\nu = 326$, then $t_\nu(1-\alpha/2) = 1.97$.

The 95% confidence interval is

$$30.5 - 20.1 \pm 1.97 \cdot 0.82 = (8.73, 11.9)$$

We are 95% confident that, on average, Japanese cars get between 8.73 and 11.9 more mpg than American cars.

# SAS code for two-sample t-test

```
DATA mpg;
    INFILE 'mpg.csv' DELIMITER=',' FIRSTOBS=2;
    INPUT mpg country $;

PROC TTEST DATA=mpg;
    CLASS country;
    VAR mpg;
    RUN;
```

```
                              The TTEST Procedure

                                Variable:  mpg

      country           N        Mean     Std Dev     Std Err     Minimum     Maximum

      Japan            79     30.4810      6.1077      0.6872     18.0000     47.0000
      US              249     20.1446      6.4147      0.4065      9.0000     39.0000
      Diff (1-2)              10.3364      6.3426      0.8190

 country       Method               Mean       95% CL Mean        Std Dev      95% CL Std Dev

 Japan                           30.4810     29.1130  31.8491      6.1077      5.2814   7.2429
 US                              20.1446     19.3439  20.9452      6.4147      5.8964   7.0336
 Diff (1-2)    Pooled            10.3364      8.7252  11.9477      6.3426      5.8909   6.8699
 Diff (1-2)    Satterthwaite     10.3364      8.7576  11.9152

                 Method           Variances        df    t Value    Pr > |t|

                 Pooled           Equal           326      12.62      <.0001
                 Satterthwaite    Unequal      136.87      12.95      <.0001

                             Equality of Variances

                    Method      Num df    Den df    F Value    Pr > F

                    Folded F       248        78       1.10    0.6194
```

# R code/output for two-sample t-test

```
t.test(mpg~country, data=mpg, var.equal=TRUE)



Two Sample t-test

data:  mpg by country
t = 12.62, df = 326, p-value < 2.2e-16
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  8.725 11.948
sample estimates:
mean in group Japan    mean in group US
              30.48               20.14
```

# Conclusion

Mean miles per gallon of Japanese cars is significantly different than mean miles per gallon of American cars (two-sample t-test t=12.62, $p < 0.0001$). Japanese cars get an average of 10.3 [95% CI (8.7,11.9)] more miles per gallon than American cars.