Student's t-distribution

STAT 587 (Engineering)
Iowa State University

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Student's t distribution

The random variable X has a Student's t distribution with degrees of freedom $\nu > 0$ if its probability density function is

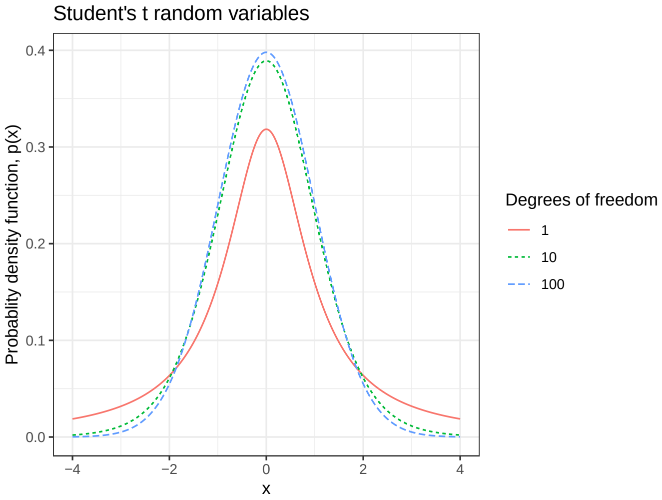
$$p(x|\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where $\Gamma(\alpha)$ is the gamma function,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

We write $X \sim t_\nu$.

Student's t probability density function



Student's t mean and variance

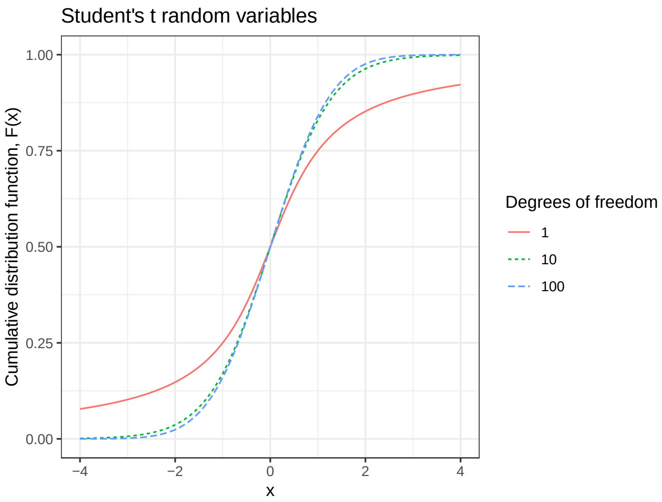
If $T \sim t_v$, then

$$E[X] = \int_{-\infty}^{\infty} x \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx = \cdots = 0, \quad \nu > 1$$

and

$$Var[X] = \int_{0}^{\infty} (x-0)^2 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx = \cdots = \frac{\nu}{\nu-2}, \quad \nu > 2.$$

Gamma cumulative distribution function - graphically



Location-scale t distribution

If $X \sim t_\nu$, then

$$Y = \mu + \sigma X \sim t_\nu(\mu, \sigma^2)$$

for parameters:

- degrees of freedom $\nu > 0$,
- location μ and
- scale $\sigma > 0$.

By properties of expectations and variances, we can find that

$$E[Y] = \mu, \qquad \nu > 1$$

and

$$Var[Y] = \frac{\nu}{\nu - 2}\sigma^2, \qquad \nu > 2.$$

Generalized Student's t probability density function

The random variable Y has a generalized Student's t distribution with

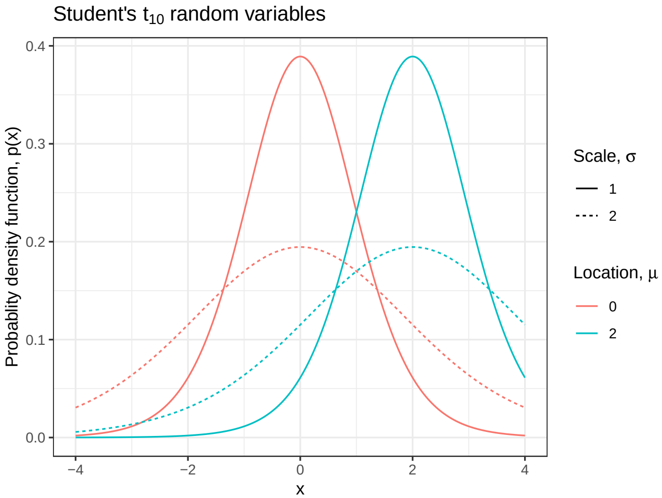
- degrees of freedom $\nu > 0$,
- location μ, and
- scale $\sigma > 0$

if its probability density function is

$$p(y) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}\sigma}\left(1 + \frac{1}{\nu}\left[\frac{y-\mu}{\sigma}\right]^2\right)^{-\frac{\nu+1}{2}}$$

We write $Y \sim t_\nu(\mu, \sigma^2)$.

Generalized Student's t probability density function



t with 1 degree of freedom

If $T \sim t_1(\mu, \sigma^2)$, then T has a Cauchy distribution and we write

$$T \sim Ca(\mu, \sigma^2).$$

If $T \sim t_1(0, 1)$, then T has a standard Cauchy distribution. A Cauchy random variable has no mean or variance.

As degrees of freedom increases

If $T_\nu \sim t_\nu(\mu, \sigma^2)$, then

$$\lim_{\nu \to \infty} T_\nu \stackrel{d}{=} X \sim N(\mu, \sigma^2)$$

t distribution arising from a normal sample

Let $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$. We calculate the sample mean

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

and the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2.$$

Then

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

Inverse-gamma scale mixture of a normal

If

$$X|\sigma^2 \sim N(\mu, \sigma^2/n) \qquad \text{and} \qquad \sigma^2 \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}s^2\right)$$

then

$$X \sim t_\nu(\mu, s^2/n)$$

which is obtained by

$$p_x(x) = \int p_{x|\sigma^2}(x|\sigma^2) p_{\sigma^2}(\sigma^2) d\sigma^2$$

where

- p_x is the marginal density for x
- $p_{x|\sigma^2}$ is the conditional density for x given σ^2, and
- p_{σ^2} is the marginal density for σ^2.

Summary

Student's t random variable:

- $T \sim t_\nu(\mu, \sigma^2)$, $\nu, \sigma > 0$
- $E[X] = \mu$, $\nu > 1$
- $Var[X] = \frac{\nu}{\nu-2}\sigma^2$, $\nu > 2$
- Relationships to other distributions