P3 - Continuous distributions

STAT 401 (Engineering) - Iowa State University

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Cumulative distribution function

All properties of discrete random variables have direct counterparts for continuous random variables.

In particular,

Definition

The cumulative distribution function for a continuous random variable is

$$F_X(x) = P(X \leq x) = P(X < x)$$

since $P(X = x) = 0$ for any x.

we still have the properties

- $0 \leq F_X(x) \leq 1$ for all $x \in \mathbb{R}$
- F_X is monotone increasing, i.e. if $x_1 \leq x_2$ then $F_X(x_1) \leq F_X(x_2)$.
- $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

Probability density function

Definition

The probability density function (pdf) for a continuous random variable is

$$f_X(x) = \frac{d}{dx} F_X(x)$$

and

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

Thus, the probability density function has the following properties

- $f_X(x) \geq 0$ for all x and
- $\int_{-\infty}^{\infty} f(x)dx = 1.$

Example

Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$f_X(x)$ defines a valid pdf because $f_X(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 3x^2 dx = x^3|_0^1 = 1.$$

The cdf is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Expected value

Definition

Let X be a continuous random variable and h be some function. The expected value of a function of a continuous random variable is

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f_X(x)dx.$$

If $h(x) = x$, then

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx.$$

and we call this the expectation of X. We commonly use the symbol μ for this expectation.

Example (cont.)

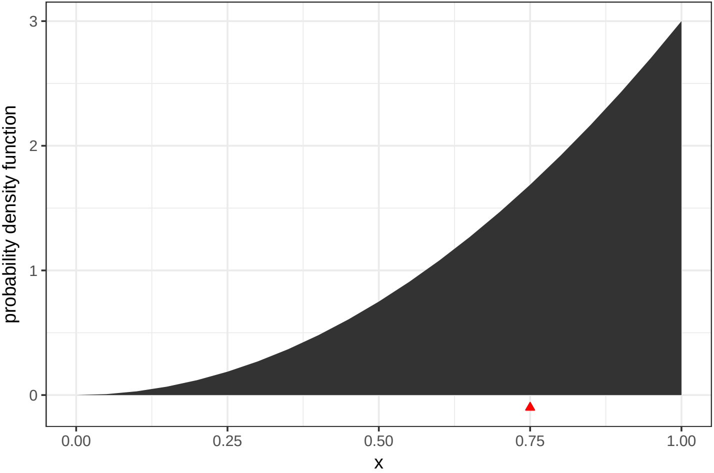
Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx \\ &= \int_0^1 3x^3 dx \\ &= 3\frac{x^4}{4}\Big|_0^1 = \frac{3}{4}. \end{aligned}$$

Example - Center of mass



Variance

Definition

The variance of a random variable is defined as the expected squared deviation from the mean. For continuous random variables, variance is

$$Var[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$$

where $\mu = E[X]$. The symbol σ^2 is commonly used for the variance.

Definition

The standard deviation is the positive square root of the variance

$$SD[X] = \sqrt{Var[X]}.$$

The symbol σ is commonly used for the standard deviation.

Example (cont.)

Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The variance is

$$\begin{aligned} Var[X] &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx \\ &= \int_0^1 \left(x - \frac{3}{4}\right)^2 3x^2 dx \\ &= \int_0^1 \left[x^2 - \frac{3}{2}x + \frac{9}{16}\right] 3x^2 dx \\ &= \int_0^1 3x^4 - \frac{9}{2}x^3 + \frac{27}{16}x^2 dx \\ &= \left[\frac{3}{5}x^5 - \frac{9}{8}x^4 + \frac{9}{16}x^3\right]\Big|_0^1 dx \\ &= \frac{3}{5} - \frac{9}{8} + \frac{9}{16} \\ &= \frac{3}{80} \end{aligned}$$

Comparison of discrete and continuous random variables

For simplicity here and later, we drop the subscript X.

	discrete	continuous
image	finite or countable	uncountable
pmf	$p(x) = P(X = x)$	
pdf		$p(x) = f(x) = F'(x)$
cdf	$F(x) = P(X \leq x) = \sum_{t \leq x} p(x)$	$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(t)dt$
expected value	$E[h(X)] = \sum_x h(x)p(x)$	$E[h(X)] = \int_x h(x)p(x)dx$
expectation	$\mu = E[X] = \sum_x x\,p(x)$	$\mu = E[X] = \int_x x\,p(x)dx$
variance	$Var[X] = E[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x)$	$Var[X] = E[(X-\mu)^2] = \int_x (x-\mu)^2 p(x)dx$

Note: we replace summations with integrals when using continuous as opposed to discrete random variables

Uniform

A uniform random variable on the interval (a, b) has equal probability for any value in that interval and we denote this $X \sim Unif(a, b)$. The pdf for a uniform random variable is

$$f(x) = \frac{1}{b-a}\mathrm{I}(a < x < b)$$

where $\mathrm{I}(A)$ is in indicator function that is 1 if A is true and 0 otherwise, i.e.

$$\mathrm{I}(A) = \begin{cases} 1 & A \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

The expectation is

$$E[X] = \int_a^b \frac{1}{b-a} x dx = \frac{b-a}{2}$$

and the variance is

$$Var[X] = \int_a^b \frac{1}{b-a} \left(x - \frac{b-a}{2}\right)^2 dx = \frac{1}{12}(b-a)^2.$$

Example (cont.)

Pseudo-random number generators generate pseudo uniform values on (0,1). These values can be used in conjunction with the inverse of the cumulative distribution function to generate pseudo-random numbers from any distribution.

The inverse of the cumulative distribution function is

$$F_X^{-1}(u) = u^{1/3}.$$

A uniform random number on the interval (0,1) evaluted with the inverse cdf produces a random draw of X. So, in R

```
inverse_cdf = function(u) u^(1/3)
x = inverse_cdf(runif(1e6))
mean(x)
```

```
[1] 0.7502002
```

```
var(x); 3/80
```

```
[1] 0.03752111
[1] 0.0375
```

Normal distribution

The normal (or Gaussian) density is a "bell-shaped" curve. The density has two parameters: mean μ and variance σ^2 and is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{for } -\infty < x < \infty$$

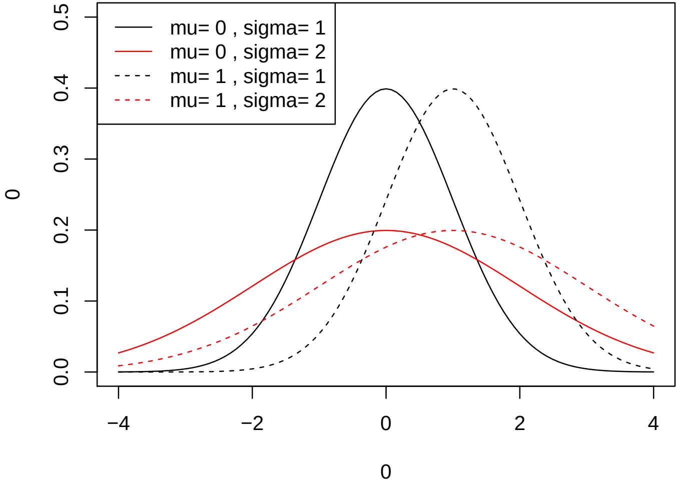
The expected value and variance of a normal distributed r.v. X are:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\, f(x) dx = \ldots &= \mu \\ Var[X] &= \int_{-\infty}^{\infty} (x-\mu)^2\, f(x) dx = \ldots &= \sigma^2. \end{aligned}$$

Thus, the parameters μ and σ^2 are actually the mean and the variance of the $N(\mu, \sigma^2)$ distribution.

There is no closed form cumulative distribution function for a normal random variable.

Example probability density functions



Properties of the normal distribution

Let $Z \sim N(0,1)$, i.e. a standard normal random variable. Then for constants m and s

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2).$$

alternatively

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

which is called standardizing.

Let $X_i \stackrel{ind}{\sim} N(\mu_i, \sigma_i^2)$. Then

$$Z_i = \frac{X_i - \mu_i}{\sigma_i} \stackrel{iid}{\sim} N(0,1) \quad \text{for all } i$$

and

$$Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

Calculating the standard normal cumulative distribution function

If $Z \sim N(0,1)$, what is $P(Z \leq 1.5)$? Although the cdf does not have a closed form, very good approximations exist and are available as tables or in software, e.g.

```
pnorm(1.5) # default is mean=0, sd=1
```

```
[1] 0.9331928
```

A standard normal random variable is often denoted Z, the standard normal cdf is often denoted $\Phi(z)$, and tables are called *standard normal tables* or *Z tables*. Sometimes these tables only have positive z values, but we can still compute $\Phi(z)$ for any z since the normal distribution is symmetric, i.e. $\Phi(-z) = 1 - \Phi(z)$. Finally, these tables usually only extend to $|z| < 4$, but that's okay since $P(Z < -4) = P(Z > 4) \approx 0.00003$.

Calculating any normal cumulative distribution function

If $X \sim N(15, 4)$ what is $P(X > 18)$?

$$\begin{aligned} P(X > 18) &= 1 - P(X \le 18) \\ &= 1 - P\left(\frac{X-15}{2} \le \frac{18-15}{2}\right) \\ &= 1 - P(Z \le 1.5) \\ &\approx 1 - 0.933 = 0.067 \end{aligned}$$

```
1-pnorm((18-15)/2)
```

```
[1] 0.0668072
```

```
1-pnorm(18, mean=15, sd=2)
```

```
[1] 0.0668072
```

Manufacturing

Suppose you are producing nails that must be within 5 and 6 centimeters in length. If the average length of nails the process produces is 5.3 cm and the standard deviation is 0.1 cm. What is the probability of producing a nail outside of the specification?

Let $X \sim N(\mu, \sigma^2)$ be the next nail produced with $\mu = 5.3$ cm and $\sigma = 0.1$ cm. We need to calculate

$$\begin{aligned} P(X < 5 \text{ or } X > 6) &= 1 - P(5 < X < 6) \\ &= 1 - [P(X < 6) - P(X < 5)] \qquad \text{or} \\ &= P(X < 5) + (1 - P(X < 6)). \end{aligned}$$

```
mu = 5.3
sigma = 0.1

1 - (pnorm(6, mean = mu, sd = sigma) - pnorm(5, mean = mu, sd = sigma))


[1] 0.001349898
```