Set R05 - Multiple Regression

STAT 401 (Engineering) - Iowa State University

April 3, 2017

Multiple regression

Recall the simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

The multiple regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}, \sigma^2)$$

where

- Y_i is the response for observation i and
- $X_{i,p}$ is the pth explanatory variable for observation i.

We may also write

$$Y_i \stackrel{ind}{\sim} N(\mu_i, \sigma^2) \quad \text{or} \quad Y_i = \mu_i + e_i, e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

where

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}.$$

Explanatory variables

There is a lot of flexibility in the mean

$$\mu_i = E[Y_i | X_{i,1}, \ldots, X_{i,p}] = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

as there are many possibilities for the explanatory variables $X_{i,1}, \ldots, X_{i,p}$:

- Higher order terms (X^2)
- Additional explanatory variables $(X_1 + X_2)$
- Dummy variables for categorical variables $(X_1 = \mathrm{I}())$
- Interactions $(X_1 X_2)$
    - Continuous-continuous
    - Continuous-categorical
    - Categorical-categorical

Interpretation

Model:
$$Y_i \overset{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}, \sigma^2)$$

The interpretation is

- β_0 is the expected value of the response Y_i when all explanatory variables are zero.
- β_p, $p \neq 0$ is the expected increase in the response for a one-unit increase in the pth explanatory variable when all other explanatory variables are held constant.
- R^2 is the proportion of the variability in the response explained by the model

Parameter estimation

Let

$$y = X\beta + \epsilon$$

where

- $y = (y_1, \ldots, y_n)^\top$
- X is $n \times p$ with ith row $X_i = (X_{i,1}, \ldots, X_{i,p})$
- $\beta = (\beta_1, \ldots, \beta_p)^\top$
- $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^\top$

Then we have

$$
\begin{aligned}
\hat{\beta} &= (X^\top X)^{-1} X^\top y \\
Var(\hat{\beta}) &= \sigma^2 (X^\top X)^{-1} \\
r &= y - X\hat{\beta} \\
\hat{\sigma}^2 &= \frac{1}{n-p} r^\top r
\end{aligned}
$$

Confidence/credible intervals and (two-sided) pvalues are constructed using

$$\hat{\beta}_j \pm t_{n-p, 1-a/2} SE(\hat{\beta}_j) \quad \text{and} \quad \text{pvalue} = 2P\left(T_{n-2} > \left| \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)} \right|\right)$$

where $SE(\hat{\beta}_j)$ is the jth diagonal element of $\hat{\sigma}^2 (X^\top X)^{-1}$.

Higher order terms (X^2)

Let

- Y_i be the distance for the ith run of the experiment and
- H_i be the height for the ith run of the experiment.

Simple linear regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i \qquad\qquad , \sigma^2)$$
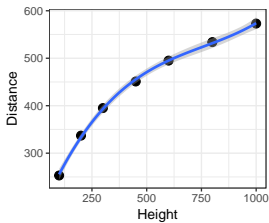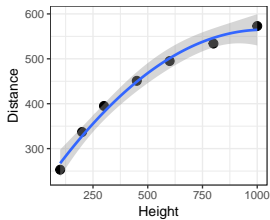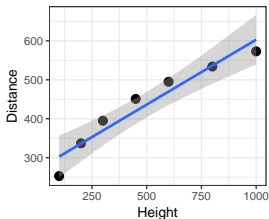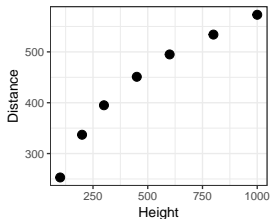
The quadratic multiple regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i + \beta_2 H_i^2 \qquad , \sigma^2)$$

The cubic multiple regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i + \beta_2 H_i^2 + \beta_3 H_i^3, \sigma^2)$$

Case1001

R code and output

```r
# Construct the variables by hand
case1001$Height2 = case1001$Height^2
case1001$Height3 = case1001$Height^3

m1 = lm(Distance~Height,                 case1001)
m2 = lm(Distance~Height+Height2,         case1001)
m3 = lm(Distance~Height+Height2+Height3, case1001)

coefficients(m1)

(Intercept)      Height
 269.712458    0.333337


coefficients(m2)

  (Intercept)         Height        Height2
 1.999128e+02   7.083225e-01  -3.436937e-04


coefficients(m3)

  (Intercept)         Height        Height2        Height3
 1.557755e+02   1.115298e+00  -1.244943e-03   5.477104e-07
```

R code and output

```
# Let R construct the variables for you
m = lm(Distance~poly(Height, 3, raw=TRUE), case1001)
summary(m)

Call:
lm(formula = Distance ~ poly(Height, 3, raw = TRUE), data = case1001)

Residuals:
      1        2        3        4        5        6        7
-2.40359  3.58091  1.89175 -4.46885 -0.08044  2.32159 -0.84138

Coefficients:
                              Estimate Std. Error t value Pr(>|t|)
(Intercept)                  1.558e+02  8.326e+00  18.710 0.000333 ***
poly(Height, 3, raw = TRUE)1 1.115e+00  6.567e-02  16.983 0.000445 ***
poly(Height, 3, raw = TRUE)2 -1.245e-03 1.384e-04  -8.994 0.002902 **
poly(Height, 3, raw = TRUE)3 5.477e-07  8.327e-08   6.577 0.007150 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.011 on 3 degrees of freedom
Multiple R-squared:  0.9994,  Adjusted R-squared:  0.9987
F-statistic:  1595 on 3 and 3 DF,  p-value: 2.662e-05
```

Longnose Dace Abundance

From http://udel.edu/~mcdonald/statmultreg.html:

> *I extracted some data from the Maryland Biological Stream Survey. ... The dependent variable is the number of Longnose Dace (Rhinichthys cataractae) per 75-meter section of [a] stream. The independent variables are the area (in acres) drained by the stream; the dissolved oxygen (in mg/liter); the maximum depth (in cm) of the 75-meter segment of stream; nitrate concentration (mg/liter); sulfate concentration (mg/liter); and the water temperature on the sampling date (in degrees C).*
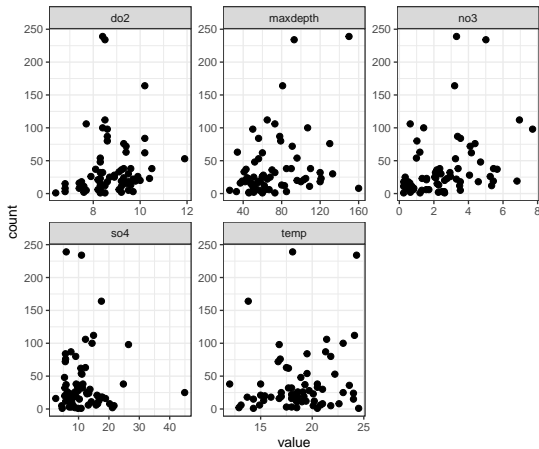
Consider the model

$$Y_i \overset{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}, \sigma^2)$$

where

- Y_i: count of Longnose Dace in stream i
- $X_{i,1}$: maximum depth (in cm) of stream i
- $X_{i,2}$: nitrate concentration (mg/liter) of stream i

Exploratory

R code and output

```
m <- lm(count~no3+maxdepth, longnosedace)
summary(m)


Call:
lm(formula = count ~ no3 + maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-55.060 -27.704  -8.679  11.794 165.310

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.5550    15.9586  -1.100  0.27544
no3           8.2847     2.9566   2.802  0.00671 **
maxdepth      0.4811     0.1811   2.656  0.00997 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.39 on 64 degrees of freedom
Multiple R-squared:  0.1936,  Adjusted R-squared:  0.1684
F-statistic: 7.682 on 2 and 64 DF,  p-value: 0.001022
```

Interpretation

- Intercept (β_0): The expected count of Longnose Dace when maximum depth and nitrate concentration are both zero is -18.
- Coefficient for maxdepth (β_1): Holding nitrate concentration constant, each cm increase in maximum depth is associated with an additional 0.48 Longnose Dace counted on average.
- Coefficient for no3 (β_2): Holding maximum depth constant, each mg/liter increase in nitrate concentration is associated with an addition 8.3 Longnose Dace counted on average.
- Coefficient of determination (R^2): The model explains 19% of the variability in the count of Longnose Dace.

Using a categorical variable as an explanatory variable.
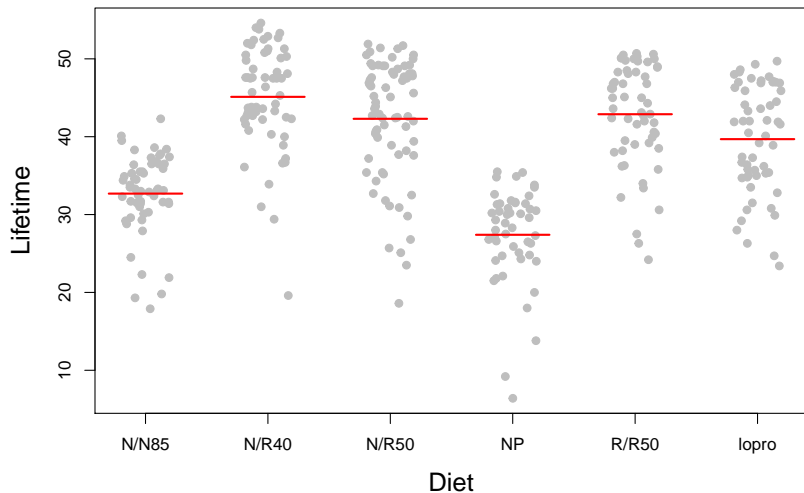
Regression with a categorical variable

- Choose one of the levels as the reference level, e.g. N/N85
- Construct dummy variables using indicator functions, i.e.

$$I(A) = \left\{ \begin{array}{ll} 1 & A \text{ is TRUE} \\ 0 & A \text{ is FALSE} \end{array} \right.$$

for the other levels, e.g.

$$X_{i,1} = I(\text{diet for observation } i \text{ is N/R40})$$
$$X_{i,2} = I(\text{diet for observation } i \text{ is N/R50})$$
$$X_{i,3} = I(\text{diet for observation } i \text{ is NP})$$
$$X_{i,4} = I(\text{diet for observation } i \text{ is R/R50})$$
$$X_{i,5} = I(\text{diet for observation } i \text{ is lopro})$$

- Estimate the parameters of a multiple regression model using these dummy variables.

R code and output

```
# by default, R uses the alphabetically first group as the reference level
case0501$Diet = relevel(case0501$Diet, ref='N/N85')

m = lm(Lifetime~Diet, case0501)
summary(m)


Call:
lm(formula = Lifetime ~ Diet, data = case0501)

Residuals:
    Min      1Q  Median      3Q     Max
-25.5167 -3.3857  0.8143  5.1833 10.0143

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  32.6912     0.8846  36.958  < 2e-16 ***
DietN/R40    12.4254     1.2352  10.059  < 2e-16 ***
DietN/R50     9.6060     1.1877   8.088 1.06e-14 ***
DietNP       -5.2892     1.3010  -4.065 5.95e-05 ***
DietR/R50    10.1945     1.2565   8.113 8.88e-15 ***
Dietlopro     6.9945     1.2565   5.567 5.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.678 on 343 degrees of freedom
Multiple R-squared:  0.4543,	Adjusted R-squared:  0.4463
F-statistic:  57.1 on 5 and 343 DF,  p-value: < 2.2e-16
```

Interpretation

- $\beta_0 = E[Y_i | \text{reference level}]$, i.e. expected response for the reference level

  Note: the only way $X_{i,1} = \cdots = X_{i,p} = 0$ is if all indicators are zero, i.e. at the reference level.

- $\beta_p, p > 0$: expected change in the response moving from the reference level to the level associated with the pth dummy variable

  Note: the only way for $X_{i,p}$ to increase by one and all other indicators to stay constant is if initially $X_{i,1} = \cdots = X_{i,p} = 0$ and now $X_{i,p} = 1$

For example,

- The expected lifetime for mice on the N/N85 diet is 32.7 weeks.
- The expected increase in lifetime for mice on the N/R40 diet compared to the N/N85 diet is 12.4 weeks.
- The model explains 45% of the variability in mice lifetimes.

Using a categorical variable as an explanatory variable.
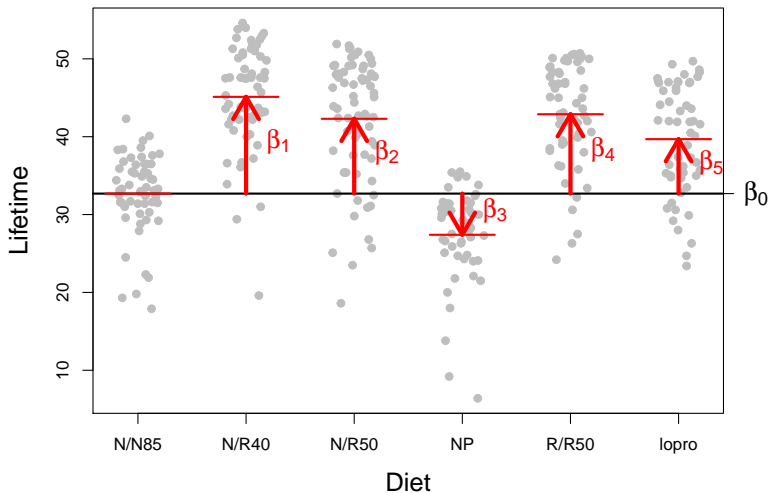
Interactions

Why an interaction?

Two explanatory variables are said to interact if the effect that one of them has on the mean response depends on the value of the other.

For example,

- Longnose dace: The effect of nitrate (no3) on longnose dace count depends on the maxdepth. (Continuous-continuous)
- Case1002: The effect of mass on energy depends on the species type. (Continuous-categorical)
- Yield: the effect of tillage method depends on the fertilizer brand (Categorical-categorical)

Continuous-continuous interaction

For observation i, let

- Y_i be the response
- $X_{i,1}$ be the first explanatory variable and
- $X_{i,2}$ be the second explanatory variable.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}.$$

The mean with the interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,1} X_{i,2}.$$

Intepretation - main effects only

Let $X_{i,1} = x_1$ and $X_{i,2} = x_2$, then we can rewrite the line (μ) as

$$\mu = (\beta_0 + \beta_2 x_2) + \beta_1 x_1$$

which indicates that the intercept of the line for x_1 depends on the value of x_2.

Similarly,

$$\mu = (\beta_0 + \beta_1 x_1) + \beta_2 x_2$$

which indicates that the intercept of the line for x_2 depends on the value of x_1.

Intepretation - with an interaction

Let $X_{i,1} = x_1$ and $X_{i,2} = x_2$, then we can rewrite the mean (μ) as

$$\mu = (\beta_0 + \beta_2 x_2) + (\beta_1 + \beta_3 x_2)x_1$$

which indicates that both the intercept and slope for x_1 depend on the value of x_2.
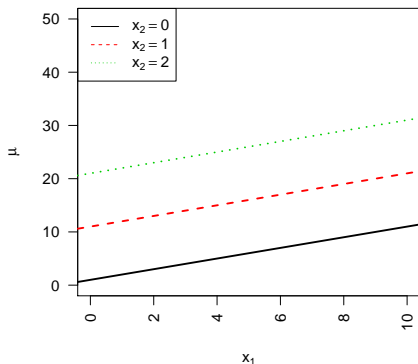
Similarly,

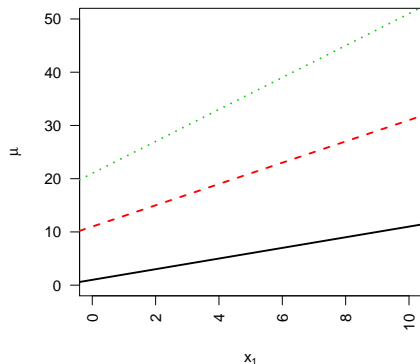$$\mu = (\beta_0 + \beta_1 x_1) + (\beta_2 + \beta_3 x_1)x_2$$

which indicates that both the intercept and slope for x_2 depend on the value of x_1.

Visualizing the models

R code and output - main effects only

```
mM = lm(count ~ no3+maxdepth, longnosedace)
summary(mM)


Call:
lm(formula = count ~ no3 + maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-55.060 -27.704  -8.679  11.794 165.310

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.5550    15.9586  -1.100  0.27544
no3           8.2847     2.9566   2.802  0.00671 **
maxdepth      0.4811     0.1811   2.656  0.00997 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.39 on 64 degrees of freedom
Multiple R-squared:  0.1936,	Adjusted R-squared:  0.1684
F-statistic: 7.682 on 2 and 64 DF,  p-value: 0.001022
```

R code and output - with an interaction

```
mI = lm(count ~ no3*maxdepth, longnosedace)
summary(mI)


Call:
lm(formula = count ~ no3 * maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-65.111 -21.399  -9.562   5.953 151.071

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  13.321043  23.455710   0.568   0.5721
no3          -4.646272   7.856932  -0.591   0.5564
maxdepth     -0.009338   0.329180  -0.028   0.9775
no3:maxdepth  0.201219   0.113576   1.772   0.0813 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 42.68 on 63 degrees of freedom
Multiple R-squared:  0.2319,	Adjusted R-squared:  0.1953
F-statistic: 6.339 on 3 and 63 DF,  p-value: 0.0007966
```
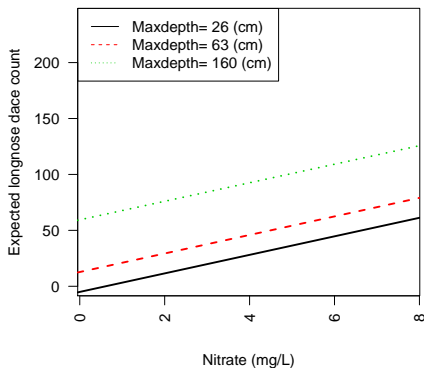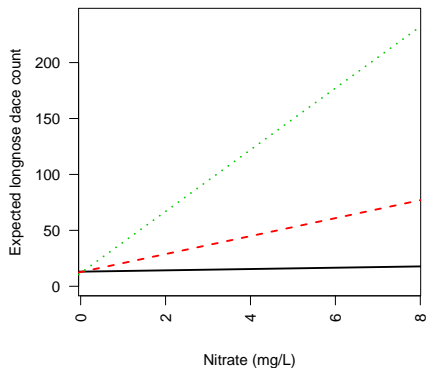
Visualizing the model

Continuous-categorical interaction

Let category A be the reference level. For observation i, let

- Y_i be the response
- $X_{i,1}$ be the continuous explanatory variable,
- B_i be a dummy variable for category B, and
- C_i be a dummy variable for category C.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i.$$

The mean with the interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i + \beta_4 X_{i,1} B_i + \beta_5 X_{i,1} C_i.$$

Think about this model as a different line for each level of the categorical explanatory variable.

Interpretation for the main effect model

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i.$$

For each category, the line is

Category	Line (μ)		
A	β_0	$+$	$\beta_1 X$
B	$(\beta_0 + \beta_2)$	$+$	$\beta_1 X$
C	$(\beta_0 + \beta_3)$	$+$	$\beta_1 X$

Each category has a different intercept, but a common slope.

Interpretation for the model with an interaction

The model with an interaction is

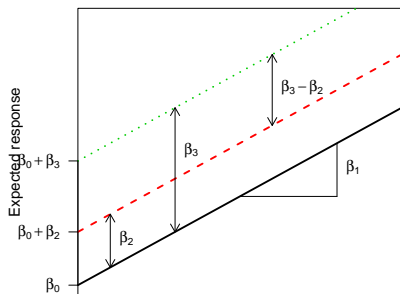$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i + \beta_4 X_{i,1} B_i + \beta_5 X_{i,1} C_i$$

For each category, the line is

Category	Line (μ)		
A	β_0	$+ \beta_1$	X
B	$(\beta_0 + \beta_2)$	$+(\beta_1 + \beta_4)X$	
C	$(\beta_0 + \beta_3)$	$+(\beta_1 + \beta_5)X$	

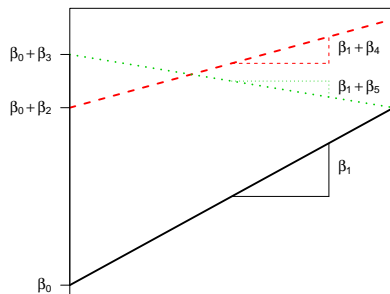Each category has its own intercept and its own slope.

Visualizing the models

Main effects only

Continuous explanatory variable

with an interaction

Continuous explanatory variable

R code and output - main effects only

```
case1002$Type = relevel(case1002$Type, ref='non-echolocating bats') # match SAS
summary(mM <- lm(log(Energy)~log(Mass)+Type, case1002))


Call:
lm(formula = log(Energy) ~ log(Mass) + Type, data = case1002)

Residuals:
    Min      1Q  Median      3Q     Max
-0.23224 -0.12199 -0.03637  0.12574  0.34457

Coefficients:
                           Estimate Std. Error t value Pr(>|t|)
(Intercept)                -1.57636    0.28724  -5.488 4.96e-05 ***
log(Mass)                   0.81496    0.04454  18.297 3.76e-12 ***
Typeecholocating bats       0.07866    0.20268   0.388    0.703
Typenon-echolocating birds  0.10226    0.11418   0.896    0.384
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.186 on 16 degrees of freedom
Multiple R-squared: 0.9815, Adjusted R-squared: 0.9781
F-statistic: 283.6 on 3 and 16 DF,  p-value: 4.464e-14
```

R code and output - with an interaction

```
summary(mI <- lm(log(Energy)~log(Mass)*Type, case1002))

Call:
lm(formula = log(Energy) ~ log(Mass) * Type, data = case1002)

Residuals:
     Min      1Q   Median      3Q      Max
-0.25152 -0.12643 -0.00954  0.08124  0.32840

Coefficients:
                                  Estimate Std. Error t value Pr(>|t|)
(Intercept)                        -0.2024     1.2613  -0.161   0.8748
log(Mass)                           0.5898     0.2061   2.861   0.0126 *
Typeecholocating bats              -1.2681     1.2854  -0.987   0.3406
Typenon-echolocating birds         -1.3784     1.2952  -1.064   0.3053
log(Mass):Typeecholocating bats     0.2149     0.2236   0.961   0.3529
log(Mass):Typenon-echolocating birds 0.2456    0.2134   1.151   0.2691
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1899 on 14 degrees of freedom
Multiple R-squared:  0.9832,     Adjusted R-squared:  0.9771
F-statistic: 163.4 on 5 and 14 DF,  p-value: 6.696e-12
```
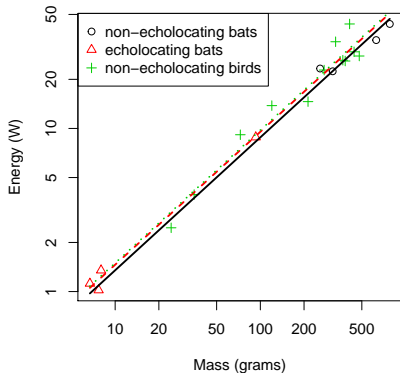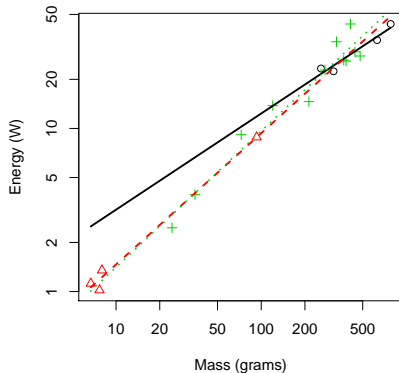
Visualizing the models

Categorical-categorical

Let category A and type 0 be the reference level. For observation i, let

- Y_i be the response,
- 1_i be a dummy variable for type 1,
- B_i be a dummy variable for category B, and
- C_i be a dummy variable for category C.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i.$$

The mean with an interaction is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i + \beta_4 1_i B_i + \beta_5 1_i C_i.$$

Interpretation for the main effects model

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i.$$

- β_0 is the expected response for category A and type 0
- β_1 is the change in response for moving from type 0 to type 1
- β_2 is the change in response for moving from category A to category B
- β_3 is the change in response for moving from category A to category C

The means are then

Type	A	B	C
0	β_0	$\beta_0 + \beta_2$	$\beta_0 + \beta_3$
1	$\beta_0 + \beta_1$	$\beta_0 + \beta_1 + \beta_2$	$\beta_0 + \beta_1 + \beta_3$

Category

Interpretation for the model with an interaction

The mean with an interaction is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i + \beta_4 1_i B_i + \beta_5 1_i C_i.$$
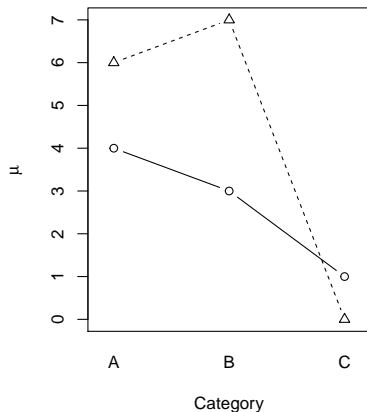
- β_0 is the expected response for category A and type 0
- β_1 is the change in response for moving from type 0 to type 1 for category A
- β_2 is the change in response for moving from category A to category B for type 0
- β_3 is the change in response for moving from category A to category C for type 0
- β_4 is the difference in change in response for moving from category A to category B for type 1 compared to type 0
- β_5 is the difference in change in response for moving from category A to category C for type 1 compared to type 0

The means are then

| | Category | | |
Type	A	B	C
0	β_0	$\beta_0 \quad + \beta_2$	$\beta_0 \quad + \beta_3$
1	$\beta_0 + \beta_1$	$\beta_0 + \beta_1 + \beta_2 + \beta_4$	$\beta_0 + \beta_1 + \beta_3 + \beta_5$

This is equivalent to a cell-means model where each combination has its own mean.

Visualizing the models
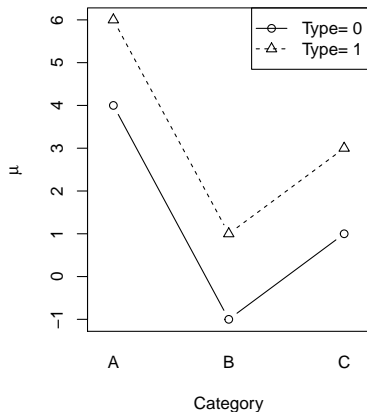
R code and output - main effects only

```
# Set the reference levels
case1301$Block = relevel(case1301$Block, ref='B1')
case1301$Treat = relevel(case1301$Treat, ref='L' )
summary(mM <- lm(Cover~Block+Treat, case1301, subset=Block %in% c("B1","B2") & Treat %in% c("L","Lf","LfF")))


Call:
lm(formula = Cover ~ Block + Treat, data = case1301, subset = Block %in%
    c("B1", "B2") & Treat %in% c("L", "Lf", "LfF"))

Residuals:
    Min      1Q  Median      3Q     Max
-2.3333 -0.6667  0.0000  0.7917  1.8333

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   4.6667     0.7683   6.074 0.000298 ***
BlockB2       2.1667     0.7683   2.820 0.022491 *
TreatLf      -1.5000     0.9410  -1.594 0.149578
TreatLfF     -3.0000     0.9410  -3.188 0.012838 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.331 on 8 degrees of freedom
Multiple R-squared:  0.6937,    Adjusted R-squared:  0.5788
F-statistic: 6.039 on 3 and 8 DF,  p-value: 0.01881
```

R code and output - with an interaction

```
summary(mI <- lm(Cover~Block*Treat, case1301, subset=Block %in% c("B1","B2") & Treat %in% c("L","Lf","LfF")))

Call:
lm(formula = Cover ~ Block * Treat, data = case1301, subset = Block %in%
    c("B1", "B2") & Treat %in% c("L", "Lf", "LfF"))

Residuals:
    Min    1Q  Median      3Q     Max
 -1.500 -0.625   0.000   0.625   1.500

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       4.0000     0.8898   4.496  0.00412 **
BlockB2           3.5000     1.2583   2.782  0.03193 *
TreatLf           0.0000     1.2583   0.000  1.00000
TreatLfF         -2.5000     1.2583  -1.987  0.09413 .
BlockB2:TreatLf  -3.0000     1.7795  -1.686  0.14280
BlockB2:TreatLfF -1.0000     1.7795  -0.562  0.59450
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.258 on 6 degrees of freedom
Multiple R-squared:  0.7946,	Adjusted R-squared:  0.6234
F-statistic: 4.642 on 5 and 6 DF,  p-value: 0.04429
```
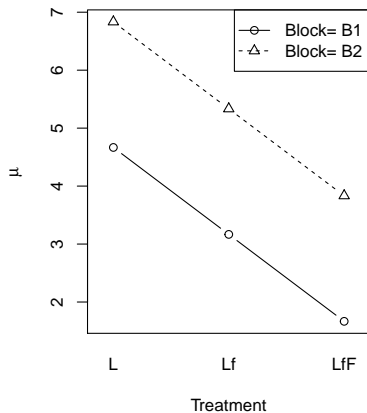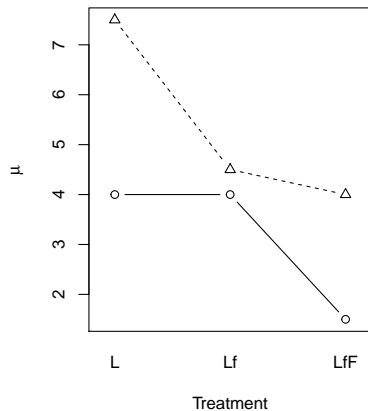
Visualizing the models

Main effect only

with interaction

When to include interaction terms

From The Statistical Sleuth (3rd ed) page 250:

- when a question of interest pertains to an interaction
- when good reason exists to suspect an interaction or
- when interactions are proposed as a more general model for the purpose of examining the goodness of fit of a model without interaction.

Multiple regression explanatory variables

The possibilities for explanatory variables are

- Higher order terms (X^2)
- Additional explanatory variables (X_1 and X_2)
- Dummy variables for categorical variables ($X_1 = \mathrm{I}()$)
- Interactions $(X_1 X_2)$
    - Continuous-continuous
    - Continuous-categorical
    - Categorical-categorical

We can also combine these explanatory variables, e.g.

- including higher order terms for continuous variables along with dummy variables for categorical variables and
- including higher order interactions $(X_1 X_2 X_3)$.