# M3S2 - Normal Distribution

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STAT 226 - Iowa State University

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# Outline

- Continuous random variables
  - normal
  - Student's $t$ (later)
- Normal random variables
  - Expectation/mean
  - Variance/standard deviation
  - Standardizing (z-score)
  - Calculating probabilities (areas under the bell curve)
  - Empirical rule: 68%, 95%, 99.7%

# Normal

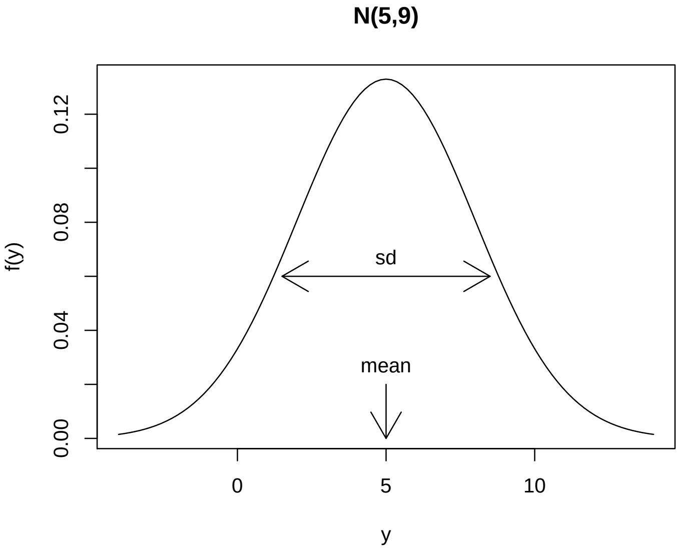
### Definition

A normal random variable with mean $\mu$ and standard deviation $\sigma$ has a probability distribution function

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)}$$

for $\sigma > 0$ where $e \approx 2.718$ is Euler's number. A normal random variable has mean $\mu$, i.e. $E[Y] = \mu$, and variance $Var[Y] = \sigma^2$ (and standard deviation $\sigma$). We write $Y \sim N(\mu, \sigma^2)$.

# Example normal pdf



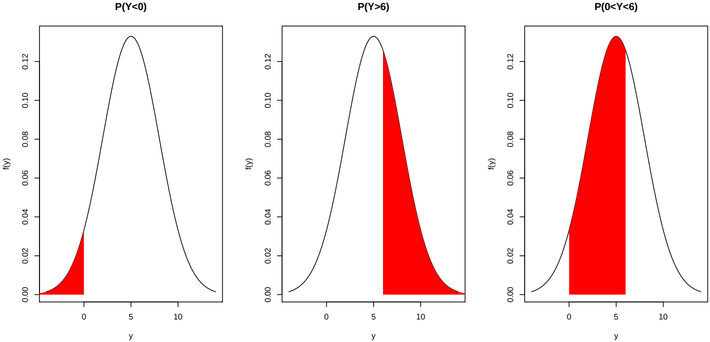
# Interpreting PDFs for continuous random variables

For continuous random variables, we calculate areas under the curve to evaluate probability statements. Suppose $Y \sim N(5, 9)$, then

- $P(Y < 0)$ is the area under the curve to the left of 0,
- $P(Y > 6)$ is the area under the curve to the right of 6, and
- $P(0 < Y < 6)$ is the area under the curve between 0 and 6

where the curve refers to the bell curve centered at 5 and with a standard deviation of 3 (variance of 9) because $Y \sim N(5, 9)$.

# Areas under the curve



# Standardizing

### Definition

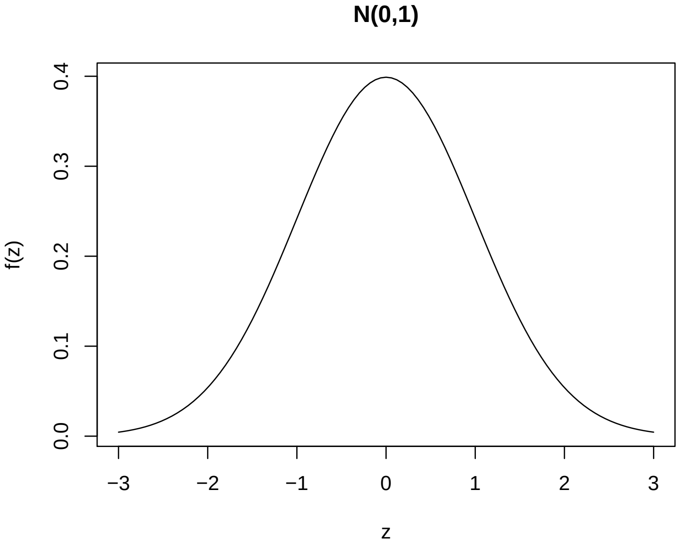
A standard normal random variable has mean $\mu = 0$ and standard deviation $\sigma = 1$. You can standardize any normal random variable by subtracting its mean and dividing by its standard deviation. If $Y \sim N(\mu, \sigma^2)$, then

$$Z = \frac{Y - \mu}{\sigma} \sim N(0, 1).$$

For an observed normal random variable $y$, a z-score is obtained by standardizing, i.e.

$$z = \frac{y - \mu}{\sigma}.$$

z-tables exist to calculate areas under the curve (probabilities) for standard normal random variables.


N(0,1)
f(z)
0.0
0.1
0.2
0.3
0.4
−3
−2
−1
0
1
2
3
z

# z-table

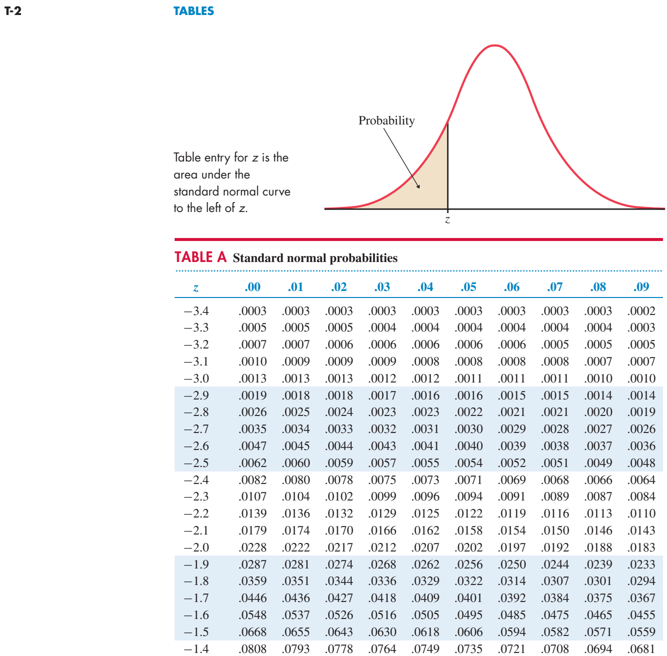


Table entry for *z* is the area under the standard normal curve to the left of *z*.

Probability
z

**TABLE A** **Standard normal probabilities**

| *z* | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −3.4 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |
| −3.3 | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| −3.2 | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| −3.1 | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| −3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| −2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| −2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| −2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| −2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| −2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| −2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| −2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| −2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| −2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| −2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| −1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| −1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| −1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| −1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| −1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| −1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |

# Calculating probabilities by standardizing

Using z-tables, we can calculate the probabilities for any normal random variable.

Suppose $Y \sim N(\mu, \sigma^2)$ and we want to calculate $P(Y < c)$, then

$$P\left(Y < c\right) = P\left(\frac{Y-\mu}{\sigma} < \frac{c-\mu}{\sigma}\right) = P\left(Z < \frac{c-\mu}{\sigma}\right).$$

Since $c$, $\mu$, and $\sigma$ are all known, $\frac{c-\mu}{\sigma}$ is just a number.

In addition, we have the following rules

$$\begin{array}{lrll} P(Y > c) & = 1 - P(Y \le & c) & \text{probabilities sum to 1} \\ P(Y \le c) & = \qquad P(Y < & c) & \text{continuous random variable} \\ P(Z < c) & = \qquad P(Z > & -c) & \text{symmetric around 0} \end{array}$$

# Example z-table use

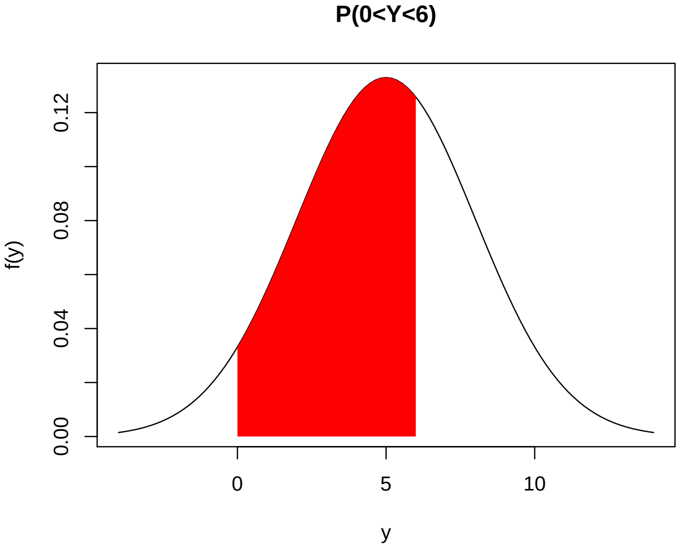
Suppose $Y \sim N(5,9)$, then

$$\begin{aligned}
P(Y<0) &= P\left(\frac{Y-5}{3} < \frac{0-5}{3}\right) && \text{standardize} \\
&\approx P\left(Z < -1.67\right) && \text{calculation} \\
&= 0.0475 && \text{z-table lookup}
\end{aligned}$$

$$\begin{aligned}
P(Y>6) &= P\left(\frac{Y-5}{3} > \frac{6-5}{3}\right) && \text{standardize} \\
&\approx P\left(Z > 0.33\right) && \text{calculation} \\
&= P\left(Z < -0.33\right) && \text{symmetric around 0} \\
&= 0.3707 && \text{z-table lookup}
\end{aligned}$$

$$\begin{aligned}
P(0<Y<6) &= P(Y<6) - P(Y<0) && \\
&= [1-P(Y>6)] - P(Y<0) && \text{probabilities sum to 1} \\
&= [1-0.3707] - 0.0475 && \text{previous slides} \\
&= 0.5818 &&
\end{aligned}$$

# Differences of probabilities



# Inventory management

Suppose that based on past history Wheatsfield Coop knows that during any given month, the amount of wheat flour that is purchased follows a normal distribution with mean 20 lbs and standard deviation 4 lbs. Currently, Wheatsfield has 25 lbs of wheat flour in stock for this month. What is the probability Wheatsfield runs out of wheat flour this month?

Let $Y$ be the amount of wheat flour purchased this month and assume $Y \sim N(20, 4^2)$. Then

$$\begin{aligned} P(Y > 22) &= P\left(\tfrac{Y-20}{3} > \tfrac{25-20}{4}\right) \\ &= P(Z > 1.25) \\ &= P(Z < -1.25) \\ &= 0.1056 \end{aligned}$$

There is approximately an 11% probability Wheatsfield will run out of wheat flour this month.

# Empirical rule

### Definition

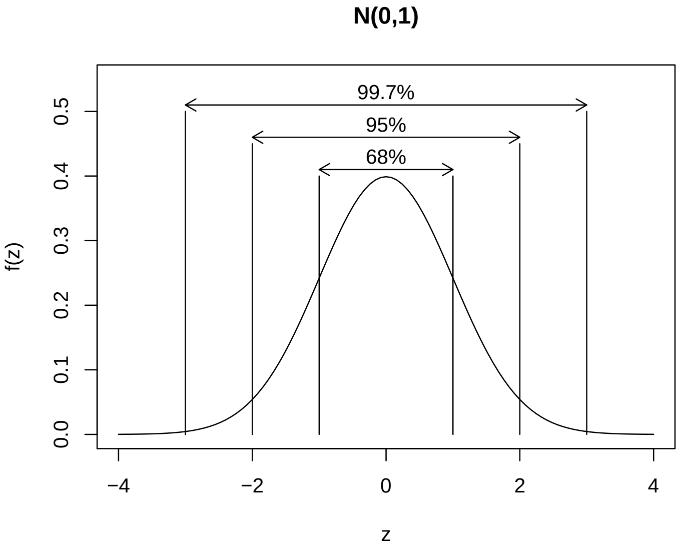
The empirical rule states that for a normal distribution, on average,

- 68% of observations will fall within 1 standard deviation of the mean,
- 95% of observations will fall within 2 standard deviations of the mean, and
- 99.7% of observations will fall within 3 standard deviations of the mean.

For a standard normal, i.e. $Z \sim N(0,1)$,

$$\begin{aligned}
P(-1 < Z < 1) &= P(Z < 1) - P(Z < -1) \\
&= [1 - P(Z < -1)] - P(Z < -1) \\
&= 1 - 2 \cdot P(Z < -1) = 1 - 2 \cdot 0.1587 \quad \approx 0.68 \\
P(-2 < Z < 2) &= 1 - 2 \cdot P(Z < -2) = 1 - 2 \cdot 0.0228 \quad \approx 0.95 \\
P(-3 < Z < 3) &= 1 - 2 \cdot P(Z < -3) = 1 - 2 \cdot 0.0013 \quad \approx 0.997
\end{aligned}$$

# Empirical rule - graphically



# Empirical rule

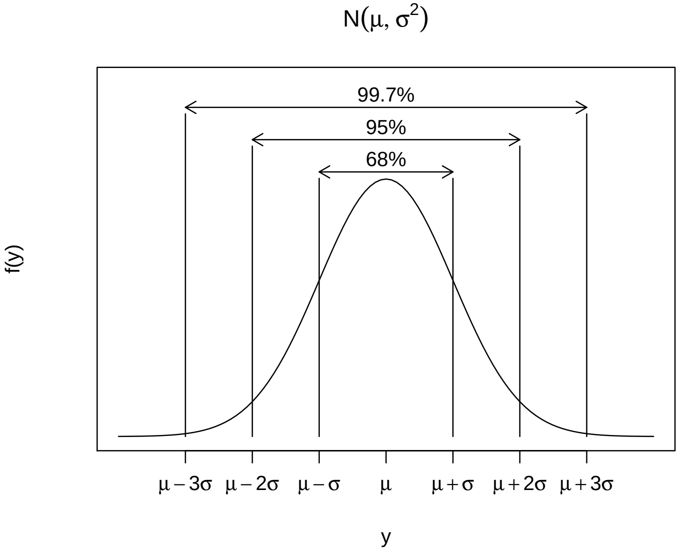
Let $Y \sim N(\mu, \sigma^2)$, then the probability $Y$ is within $c$ standard deviations of the mean is

$$P(\mu - c \cdot \sigma < Y < \mu + c \cdot \sigma) = P\left(-c < \frac{Y - \mu}{\sigma}\right) = P(-c < Z < c).$$

Thus

- 68% of observations will fall within 1 standard deviation of the mean,
- 95% of observations will fall within 2 standard deviations of the mean, and
- 99.7% of observations will fall within 3 standard deviations of the mean.

# Empirical rule - graphically


$N(\mu, \sigma^2)$
f(y)
$\mu - 3\sigma$
$\mu - 2\sigma$
$\mu - \sigma$
$\mu$
$\mu + \sigma$
$\mu + 2\sigma$
$\mu + 3\sigma$
y