Introduction to Bayesian computation (cont.)

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Outline

Bayesian computation

- Adaptive rejection sampling
- Importance sampling

Adaptive rejection sampling

Definition

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$$f((1-t)x + t\,y) \geq (1-t)f(x) + t\,f(y)$$

for any $0 \leq t \leq 1$.

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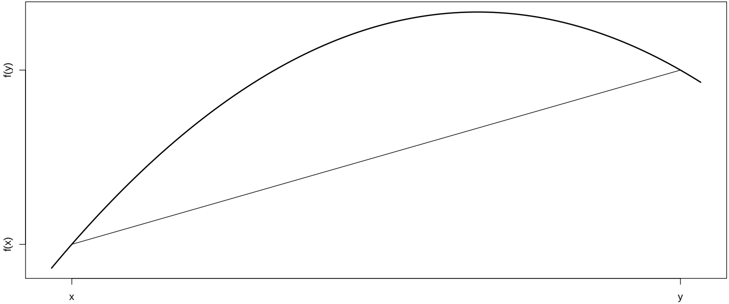

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For example, $X \sim N(0,1)$ has log-concave density since

$$\frac{d^2}{dx^2}\log e^{-x^2/2} = \frac{d^2}{dx^2} - x^2/2 = \frac{d}{dx} - x = -1.$$

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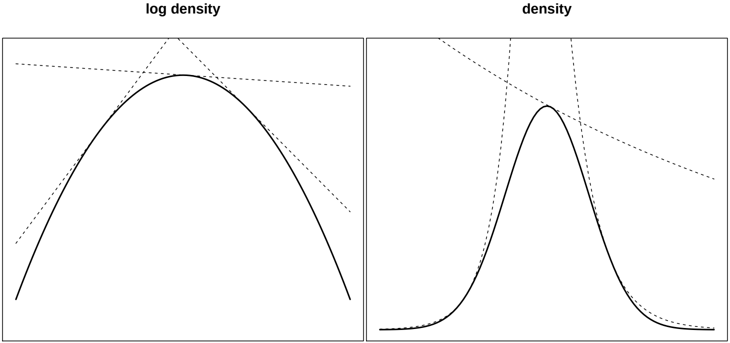

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Pseudo-algorithm for adaptive rejection sampling:

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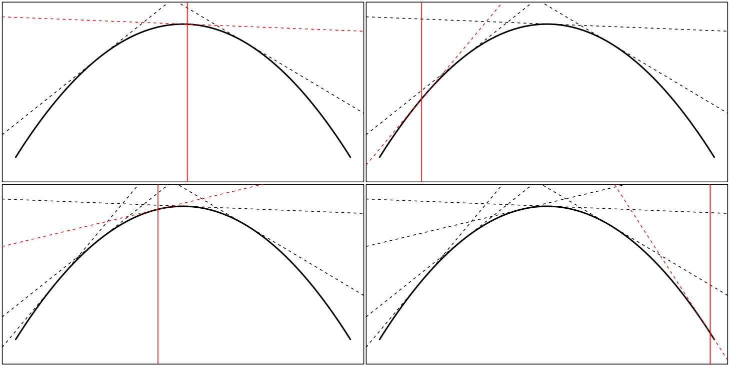
5. If rejected, add θ^* to Θ and return to 2.

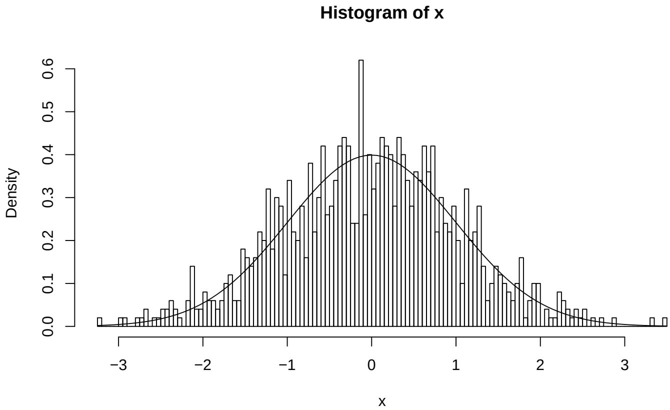
Updating the envelope

As values are proposed and rejected, the envelope gets updated:



Adaptive rejection sampling in R

```
library(ars)
x = ars(n=1000, function(x) -x^2/2, function(x) -x)
hist(x, prob=T, 100)
curve(dnorm, type='l', add=T)
```



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There is a vast literature on adaptive rejection sampling. To improve upon the basic idea presented here you can

- include a lower bound
- avoid calculating derivatives
- incorporate a Metropolis step to deal with non-log-concave densitis

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is known as the importance weight.

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is the normalized importance weight.

Example: Normal-Cauchy model

If $Y \sim N(\theta, 1)$ and $\theta \sim Ca(0, 1)$, then

$$p(\theta|y) \propto e^{-(y-\theta)^2/2} \frac{1}{(1+\theta^2)}$$

for all θ.

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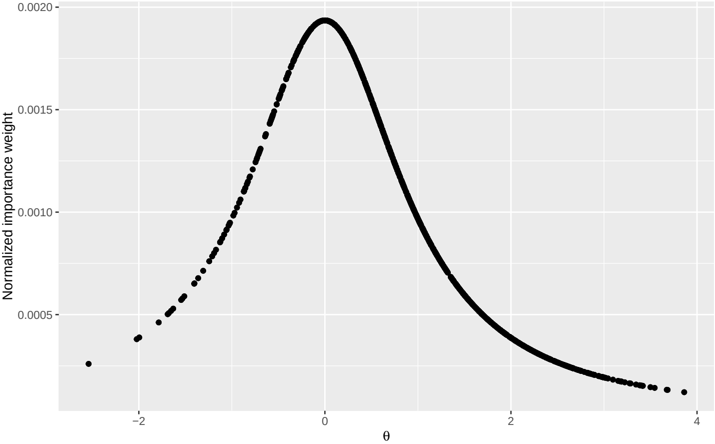
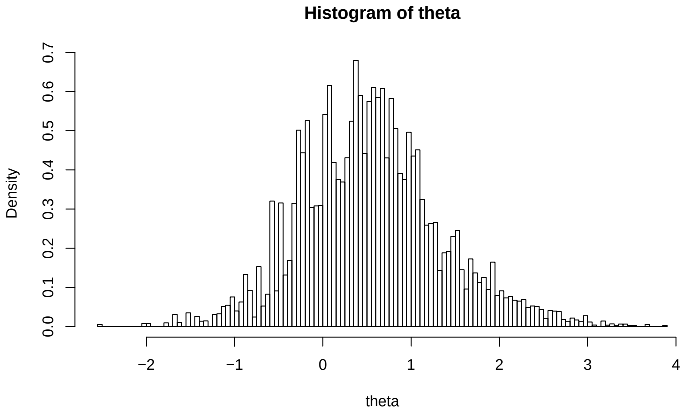
Normalized importance weights



```
library(weights)
sum(weight*theta/sum(weight)) # Estimate mean


[1] 0.5504221


wtd.hist(theta, 100, prob=TRUE, weight=weight)
```

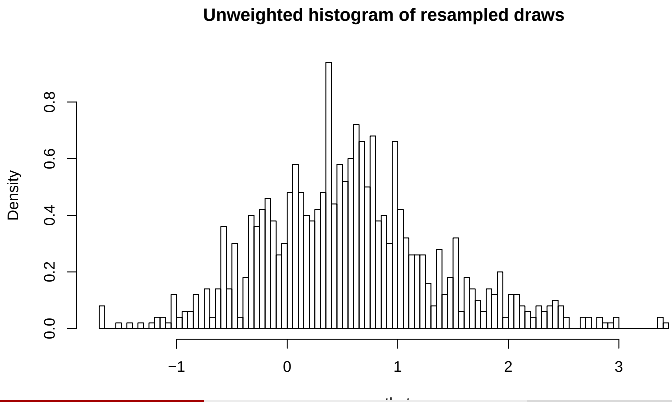


```
curve(q(x,y)/py(y), add=TRUE, col="red", lwd=2)
```

Resampling

If an unweighted sample is desired, sample $\theta^{(s)}$ with replacement with probability equal to the normalized weights, $\tilde{w}\left(\theta^{(s)}\right)$.

```
# resampling
new_theta = sample(theta, replace=TRUE, prob=weight) # internally normalized
hist(new_theta, 100, prob=TRUE, main="Unweighted histogram of resampled draws"); curve(q(x,y)/py(y), add=TRUE,
```



Heavy-tailed proposals

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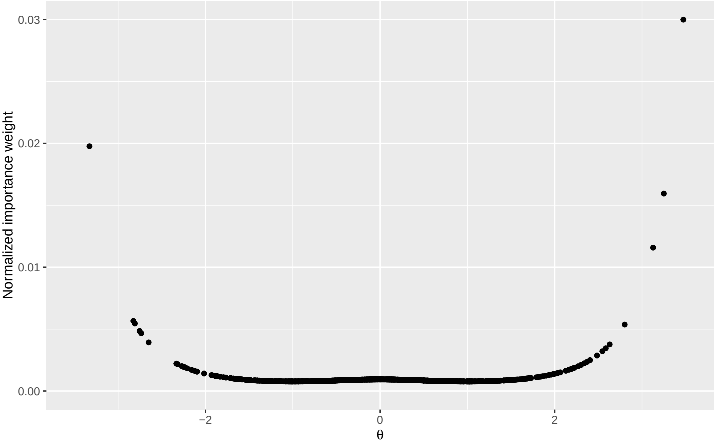
For example, suppose our target is a standard Cauchy and our proposal is a standard normal, the weights are

$$w\left(\theta^{(s)}\right) = \frac{p\left(\theta^{(s)}\middle| y\right)}{g(\theta^{(s)})} = \frac{\frac{1}{\pi(1+\theta^2)}}{\frac{1}{\sqrt{2\pi}}e^{-\theta^2/2}}$$

For $\theta^{(s)} \overset{iid}{\sim} N(0,1)$, the weights for the largest $|\theta^{(s)}|$ will dominate the others.

Importance weights for proposal with thin tails



Effective sample size

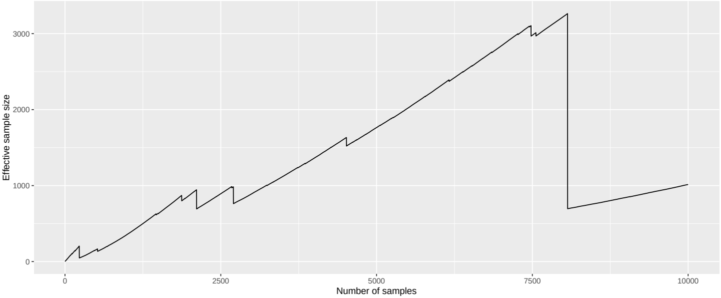
We can get a measure of how efficient the sample is by computing the effective sample size, i.e. how many independent unweighted draws do we effectively have:

$$S_{eff} = \frac{1}{\sum_{s=1}^{S} \left(\tilde{w}\left(\theta^{(s)}\right)\right)^2}$$

```
length(weight)

[1] 1000

1/sum(weight^2)

[1] 371.432
```

Effective sample size

```r
set.seed(5)
theta = rnorm(1e4)
lweight = dcauchy(theta,log=TRUE)-dnorm(theta,log=TRUE)
cumulative_ess = length(lweight)
for (i in 1:length(lweight)) {
  lw = lweight[1:i]
  w = exp(lw-max(lw))
  w = w/sum(w)
  cumulative_ess[i] = 1/sum(w^2)
}
qplot(x=1:length(cumulative_ess), y=cumulative_ess, geom="line") +
  labs(x="Number of samples", y="Effective sample size")
```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Number of samples