Set03 - Probability

STAT 401 (Engineering) - Iowa State University

January 17, 2017

Probability

Example

- Consider the event C: a successful connection to the internet from a laptop.
- From our experience with the wireless network and our internet service provider, we believe the probability we successfully connect is 90 %.
- We write: $P(C) = 0.9$
- To be able to work with probabilities, in particular, to be able to compute probabilities of events, a mathematical foundation is necessary.

Set comparison, operations, terminology

1. Review symbols $\in, \notin, \subset, \subseteq, \supset, \supseteq$.
   For example,
   - if a is a member of B, this is denoted $a \in B$ and
   - if every member of set A is also a member of set B, then A is said to be a subset of B, written $A \subseteq B$ (A is said to be contained in B).
2. Union ($\cup$): A union of events is an event consisting of all the outcomes in these events.
$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$
3. Intersection ($\cap$): An intersection of events is an event consisting of the common outcomes in these events.
$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}$$
4. Complement ($\bar{A}$): A complement of an event A is an event that occurs when event A does not happen.
$$\bar{A} = \{\omega \mid \omega \notin A\}$$

Set comparison, operations, terminology (cont.)

5. Set difference ($A \setminus B$): All elements in A that are not in B, i.e.

$$A \setminus B = \{\omega | \omega \in A \text{ and } \omega \notin B\}$$

6. Empty Set $\emptyset$ is a set having no elements, i.e. $\{\}$. The empty set is a subset of every set:

$$\emptyset \subseteq A$$

7. Disjoint sets: Sets A, B are disjoint if their intersection is empty:

$$A \cap B = \emptyset$$

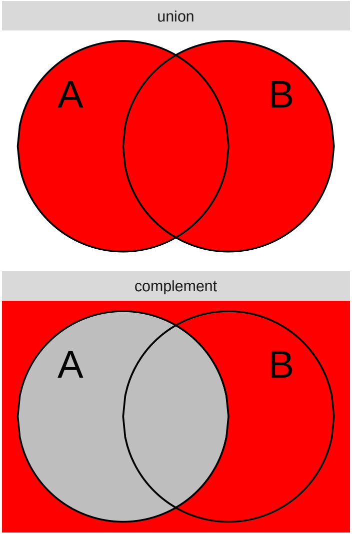
8. Mutually exclusive sets: Sets $A_1, A_2, \ldots$ are mutually exclusive if any two of these events are disjoint:

$$A_i \cap A_j = \emptyset \text{ for any } i \neq j$$

9. De Morgan's Laws:

$$(\overline{A \cup B}) = \bar{A} \cap \bar{B} \qquad \text{and} \qquad (\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

Venn diagrams



Kolmogorov's Axioms

To be able to work with probabilities properly - to compute with them - one must lay down a set of postulates.

A system of probabilities (a probability model) is an assignment of numbers $P(A)$ to events $A \subset \Omega$ such that

(i) $0 \leq P(A) \leq 1$ for all A

(ii) $P(\Omega) = 1$.

(iii) if $A_1, A_2, \ldots$ are (possibly, infinite many) mutually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$) then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots = \sum_i P(A_i).$$

Kolmogorov's Axioms (cont.)

These are the basic rules of operation of a probability model

- every valid model must obey these,
- any system that does, is a valid model.

Whether or not a particular model is appropriate for a specific application is another question.

Example: Draw a single card from a standard deck of playing cards:
$\Omega = \{red, black\}$ Two different, equally valid probability models are:

Model 1	Model 2
$P(\Omega) = 1$	$P(\Omega) = 1$
$P(red) = 0.5$	$P(red) = 0.3$
$P(black) = 0.5$	$P(black) = 0.7$

Mathematically, both schemes are equally valid. But, of course, our real world experience would favor to pick model 1 over model 2 as the 'correct' model.

Useful Consequences of Kolmogorov's Axioms

Let A, $B \subset \Omega$.

- Probability of the Complementary Event: $P(\overline{A}) = 1 - P(A)$
  Corollary: $P(\emptyset) = 0$
- Addition Rule of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If $A \subset B$, then $P(A) \leq P(B)$.
  Corollary: For any A, $P(A) \leq 1$.

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home. We connect successfully if and only if the wireless (WiFi) network works *and* the internet service provider (ISP) network works. Assume

$$\begin{aligned} P(\text{ WiFi up }) &= .9 \\ P(\text{ ISP up }) &= .6, \text{ and} \\ P(\text{ WiFi up and ISP up }) &= .55. \end{aligned}$$

1. What is the probability that the WiFi is up or the ISP is up?
2. What is the probability that both the WiFi and the ISP are down?
3. What is the probability that we fail to connect?

Solution

Let $A \equiv$ WiFi up; $B \equiv$ ISP up

1. What is the probability that the WiFi is up or the ISP is up?

$$P(\text{ WiFi up or ISP up}) = P(A \cup B) = 0.9 + 0.6 - 0.55 = 0.95$$

2. What is the probability that both the WiFi and the ISP are down?

$$\begin{aligned} P(\text{ WiFi down and ISP down}) &= P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) \\ &= 1 - .95 = .05 \end{aligned}$$

3. What is the probability that we fail to connect?

$$\begin{aligned} &P(\text{ WiFi down or ISP down}) \\ &\quad = P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B}) \\ &\quad = P(\bar{A} \cup \bar{B}) = (1 - .9) + (1 - .6) - .05 = .1 + .4 - .05 = .45 \end{aligned}$$

Conditional probability

Definition

The conditional probability of an event A given an event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) > 0$.

Intuitively, the fraction of outcomes in B that are also in A.

Corollary

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

Random CPUs

A box has 500 CPUs with a speed of 1.8 GHz and 500 with a speed of 2.0 GHz. The numbers of good (G) and defective (D) CPUs at the two different speeds are as shown below.

	1.8 GHz	2.0 GHz	Total
G	480	490	970
D	20	10	30
Total	500	500	1000

We select a CPU at random and observe its speed. What is the probability that the CPU is defective given that its speed is 1.8 GHz?
Let

- D be the event the CPU is defective and
- S be the event the CPU speed is 1.8 GHz.

Then

- $P(S) = 500/1000 = 0.5$
- $P(S \cap D) = 20/1000 = 0.02.$
- $P(D|S) = P(S \cap D)/P(S) = 0.02/0.5 = 0.04.$

Statistical independence

Definition

Events A and B are statistically independent if

$$P(A \cap B) = P(A) \times P(B)$$

or, equivalently,

$$P(A|B) = P(A).$$

Intuitively, the occurrence of one event does not affect the probability of the other.

Example

If I toss a fair coin and it comes up tails, does that affect the probability the next coin flip is a head?

WiFi example

In trying to connect my laptop to the internet, I need

- my WiFi network to be up (event A) and
- the ISP network to be up (event B).

Independently, assume the probability the WiFi network is up is 0.6 and the ISP network is up is 0.9. What is the probability we can connect to the internet?

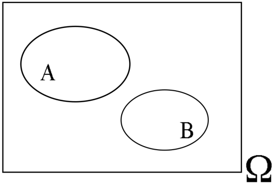
Since we have independence, we know

$$P(A \cap B) = P(A) \times P(B) = 0.6 \times 0.9 = 0.54.$$

Independence and disjoint

Warning: Independence and disjointedness are two very different concepts!

Disjointedness:

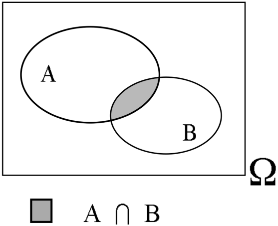


A, B are disjoint

If A and B are disjoint, their intersection is empty, has therefore probability 0:

$$P(A \cap B) = P(\emptyset) = 0.$$

Independence:

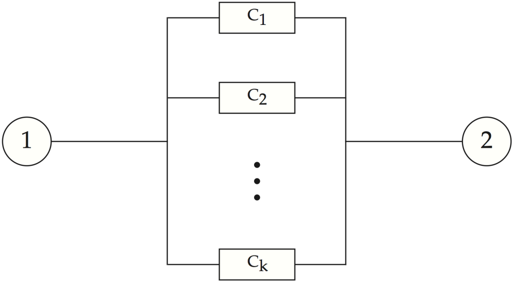


If A and B are independent events, the probability of their intersection can be computed as the product of their individual probabilities:
$P(A \cap B) = P(A) \cdot P(B)$
The probability for the intersection is zero only if A or B is empty.

Parallel system

Definition

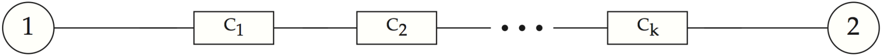
A parallel system consists of K components $c_1, \ldots, c_K$ arranged in such a way that the system works if **at least one** of the K components functions properly.



Serial system

Definition

A serial system consists of K components $c_1, \ldots, c_K$ arranged in such a way that the system works if and only if **all** of the components function properly.



Reliability

Definition

Reliability of a system is the probability the system works.

Example

The reliability of the WiFi-ISP network (assuming independence) is 0.54

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

$$
\begin{aligned}
P(\text{ system works }) &= P(\text{ at least one component works }) \\
&= 1 - P(\text{ all components fail }) \\
&= 1 - P(c_1 \text{ fails and } c_2 \text{ fails } \ldots \text{ and } c_k \text{ fails }) \\
&= 1 - \textstyle\prod_{k=1}^{K} P(c_k \text{ fails}) \\
&= 1 - \textstyle\prod_{k=1}^{K} (1 - p_k).
\end{aligned}
$$

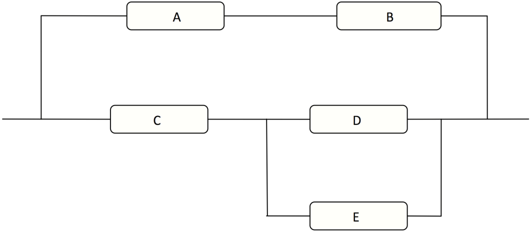
Reliability of serial systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a serial system. Assume the K components operate independently and $P(c_k \text{ works }) = p_k$. What is the reliability of the system?

$$\begin{aligned} P(\text{ system works }) &= P(\text{ all components work}) \\ &= \prod_{k=1}^{K} P(c_k \text{ works}) \\ &= \prod_{k=1}^{K} p_k. \end{aligned}$$

Reliability example

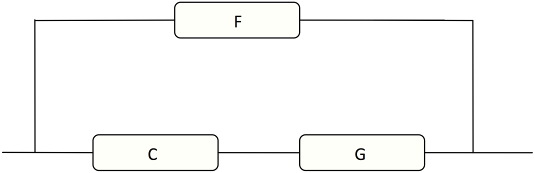
Each component in the system shown below is opearable with probability 0.92 independently of other components. Calculate the reliability.



1. Serial components A B can be replaced by a component F that operates with probability $P(A \cap B) = (0.92)^2 = 0.8464$.
2. Parallel components D and E can be replaced by component G that operates with probability $P(D \cup E) = 1 - (1 - 0.92)^2 = 0.9936$.

Reliability example (cont.)

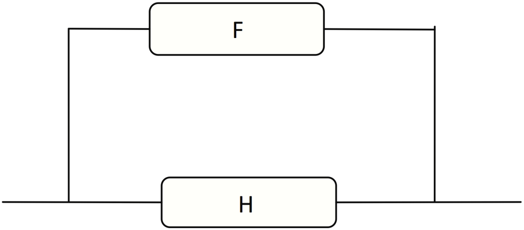
Updated circuit:



3. Serial components C and G connected can be replaced by a component H that operates with probability $P(C \cap G) = (0.92)(0.9936) = 0.9141$.

Reliability example (cont.)

Updated circuit:



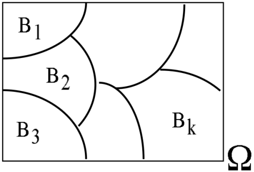
4. Parallel componenents F and H are in parallel, so the reliability of the system is $P(F \cup H) = 1 - (1 - 0.8424)(1 - 0.9141) = 0.9868$.

Partition

Definition

A collection of events $B_1, \ldots B_K$ is called a partition (or cover) of Ω if

- the events are mutually exclusive (i.e., $B_i \cap B_j = \emptyset$ for $i \neq j$), and
- the union of the events is Ω (i.e., $\bigcup_{k=1}^{K} B_k = \Omega$).



The branches of our tree (B_1, B_2, B_3) formed a partition.

Law of Total Probability

Theorem (Law of Total Probability)

If the collection of events $B_1, \ldots, B_K$ is a partition of Ω, and A is an event, then

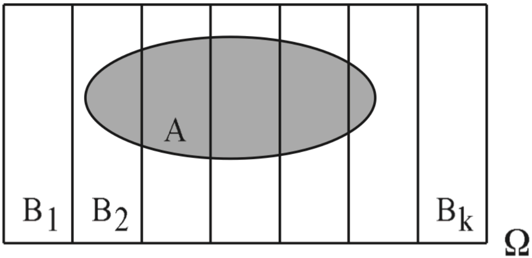
$$P(A) = \sum_{k=1}^{K} P(A|B_k)P(B_k).$$

Proof.

$$\begin{aligned} P(A) &= P\left(\bigcup_{k=1}^{K} A \cap B_k\right) && B_1, \ldots, B_K \text{ is a partition} \\ &= \sum_{k=1}^{K} P(A \cap B_k) && A \cap B_1, \ldots, A \cap B_K \text{ are disjoint} \\ &= \sum_{k=1}^{K} P(A|B_k)P(B_k) && \text{definition of conditional probability} \end{aligned}$$

□

Law of Total Probability graphic



Bayes' rule

Theorem (Bayes' Rule)

If $B_1, \ldots, B_K$ is a partition of Ω, and A is an event, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^{K} P(A|B_k)P(B_k)}.$$

Proof.

$$\begin{aligned} P(B_k|A) &= \tfrac{P(A\cap B_k)}{P(A)} && \text{by definition of conditional probability} \\ &= \tfrac{P(A|B_k)P(B_k)}{P(A)} && \text{by definition of conditional probability} \\ &= \tfrac{P(A|B_k)P(B_k)}{\sum_{k=1}^{K} P(A|B_k)P(B_k)} && \text{by Law of Total Probability} \end{aligned}$$

$\square$

CPU testing

Example

A given lot of CPUs contains 2% defective CPUs. Each CPU is tested before delivery. However, the tester is not wholly reliable:

$$\begin{aligned} P(\text{ tester says CPU is good } \mid \text{ CPU is good }) &= 0.95 \\ P(\text{ tester says CPU is defective } \mid \text{ CPU is defective }) &= 0.94 \end{aligned}$$

If the test device says the CPU is defective, what is the probability that the CPU actually is defective?

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

- $0.02 = P(C_d) \quad = 1 - P(C_g)$
- $0.95 = P(T_g|C_g) = 1 - P(T_d|C_g)$
- $0.94 = P(T_d|C_d) = 1 - P(T_g|C_d)$

Using Bayes' Rule, we have

$$\begin{aligned} P(C_d|T_d) &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+P(T_d|C_g)P(C_g)} \\ &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+[1-P(T_g|C_g)][1-P(C_d)]} \\ &= \frac{0.94\times0.02}{0.94\times0.02+[1-0.95]\times[1-0.02]} \\ &= 0.28 \end{aligned}$$