# Hypotheses

Three key features:

- a test statistic calculated from data
- a sampling distribution for the test statistic under the null hypothesis
- a region that is as or more extreme (one-sided vs two-sided hypotheses)

Calculate probability of being in the region:

**Definition**

A pvalue is the probability of observing a test statistic as or more extreme than that observed, if the null hypothesis is true.

- If pvalue is less than or equal to $\alpha$, we reject the null hypothesis.
- If pvalue is greater than $\alpha$, we fail to reject the null hypothesis.

# Hypothesis framework

Let's assume, we have

- $Y_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$ for $i = 1, \ldots, n$ and have
- calculated a test statistic $t$, and
- if the null hypothesis is true, $t$ has a $t_\nu$ sampling distribution.

Now, we can have one of three types of hypotheses:

- Two-sided ($H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$):

$$\text{pvalue} = P(|t_\nu| > |t|) = P(t_\nu > |t|) + P(t_\nu < -|t|) = 2P(t_\nu < -|t|)$$

- One-sided ($H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$):

$$\text{pvalue} = P(t_\nu > t) = P(t_\nu < -t)$$

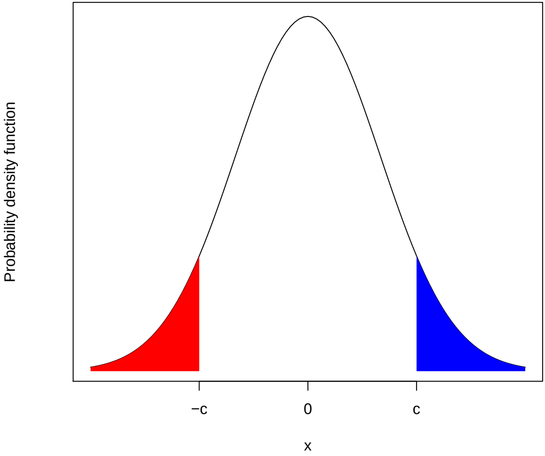
- One-sided ($H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$):

$$\text{pvalue} = P(t_\nu < t)$$

$F(c) = P(t_\nu < c)$ is the cumulative distribution function for a $t$ distribution with $\nu$ degrees of freedom.

# Symmetric distributions

The standard normal and $t$ distributions are both symmetric around zero.




$$P(T_\nu > c) = P(t_\nu < -c) \quad \text{blue area is equal to red area}$$

# Paired t-test example

In the paired t-test example, we had a test statistic $t = 2.43$ with a $t_7$ sampling distribution if the null hypothesis is true.

Consider the following hypotheses ($\mu$ is the expected difference):

- Two-sided ($H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$):

$$\text{pvalue} = 2P(t_7 < -2.43) = 0.0454$$

- One-sided ($H_0 : \mu \leq 0$ vs $H_1 : \mu > 0$):

$$\text{pvalue} = \quad P(t_7 < -2.43) = 0.0227$$

- One-sided ($H_0 : \mu \geq 0$ vs $H_1 : \mu < 0$):

$$\text{pvalue} = \quad P(t_7 < 2.43) = 0.9773$$

# Two-sample t-test example

In a two-sample t-test, we might have a test statistic $t = -2$ with a $t_{30}$ sampling distribution if the null hypothesis is true.

Consider the following hypotheses ($\mu_1 - \mu_2$ is the expected difference):

- Two-sided ($H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 - \mu_2 \neq 0$):
$$\text{pvalue} = 2P(t_{30} < -2) = 0.0546$$
- One-sided ($H_0 : \mu_1 - \mu_2 \leq 0$ vs $H_1 : \mu_1 - \mu_2 > 0$):
$$\text{pvalue} = \quad P(t_{30} < 2) = 0.9727$$
- One-sided ($H_0 : \mu_1 - \mu_2 \geq 0$ vs $H_1 : \mu_1 - \mu_2 < 0$):
$$\text{pvalue} = \quad P(t_{30} < -2) = 0.0273$$

# Confidence interval construction

Key steps in confidence interval construction:

1. Calculate point estimate
2. Calculate standard error of the statistic
3. Set error level $\alpha$
4. Find the appropriate critical value
5. Construct the $100(1-\alpha)\%$ confidence interval
   - Two-sided ($H_0: \mu = \mu_0$ vs $H_1: \mu \ne \mu_0$): $(L, U)$

     $$(L, U) = \text{estimate} \pm \text{critical value}(1-\alpha/2) \times \text{standard error}$$

   - One-sided ($H_0: \mu \le \mu_0$ vs $H_1: \mu > \mu_0$): $(L, \infty)$

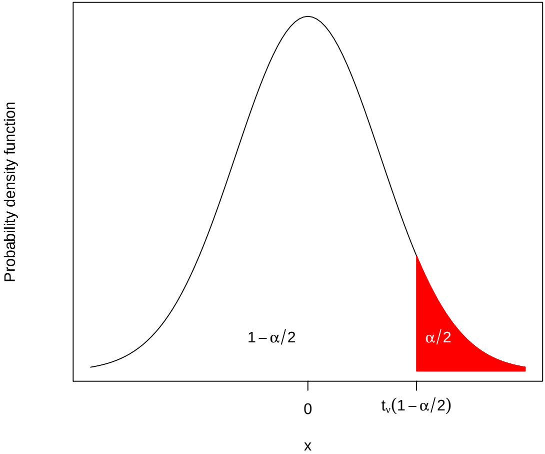
     $$L = \text{estimate} - \text{critical value}(1-\alpha) \times \text{standard error}$$

   - One-sided ($H_0: \mu \ge \mu_0$ vs $H_1: \mu < \mu_0$): $(-\infty, U)$

     $$U = \text{estimate} + \text{critical value}(1-\alpha) \times \text{standard error}$$

# Critical values

A related quantity are critical values, e.g. $t_\nu(1-\alpha/2)$.




Let $c = t_\nu(1-\alpha/2)$, then we need $P(t_\nu < c) = 1-\alpha/2$, i.e. the inverse of the cumulative distribution function.

# Paired t-test example

In paired t-test example, we had an estimate $\hat{\mu} = 10.5$ and a standard error of 4.3136 with 7 degrees of freedom.

The 95%, i.e. $\alpha = 0.05$, confidence intervals for $\mu$ are

- Two-sided ($t_7(.975) = 2.364624$)

$$10.5 \pm 2.364624 \cdot 4.3136 = (0.30, 20.7)$$

- One-sided (positive) ($t_7(.95) = 1.894579$)

$$(10.5 - 1.894579 \cdot 4.3136, \infty) = (2.33, \infty)$$

- One-sided (negative) ($t_7(.95) = 1.894579$)

$$(-\infty, 10.5 + 1.894579 \cdot 4.3136) = (-\infty, 18.7)$$

# Two-sample t-test example

In the two-sample t-test example, we had an estimate $\mu_1 \hat{-} \mu_2 = 10.33643$ and a pooled standard error of 0.8190 with 326 degrees of freedom.

The 90%, i.e. $\alpha = 0.10$, confidence intervals for $\mu$ are

- Two-sided ($t_{326}(.95) = 1.649541$)

$$10.33643 \pm 1.649541 \cdot 0.8190 = (9.0, 11.7)$$

- One-sided (positive) ($t_{326}(.90) = 1.285149$)

$$(10.33643 - 1.285149 \cdot 0.8190, \infty) = (9.3, \infty)$$

- One-sided (negative) ($t_{326}(.90) = 1.285149$)

$$(-\infty, 10.33643 + 1.285149 \cdot 0.8190) = (-\infty, 11.4)$$

# Using SAS or R

If $\alpha = 0.05$, then $1 - \alpha/2 = 0.975$.

In SAS,

```
PROC IML;
  q = QUANTILE('T', 0.975, 7);
  PRINT q;
  QUIT;
```

In R,

```
q = qt(0.975,7)
```

Both obtain q=2.364.

# Summary

Two main approaches to statistical inference:

- Statistical hypothesis (hypothesis test)
- Statistical question (confidence interval)