Set 09 - Bayesian statistics

STAT 401 (Engineering) - Iowa State University

February 7, 2017

Outline

- Bayesian parameter estimation
  - Condition on what is known
  - Describe uncertainty using probability
  - Binomial example

A Bayesian statistician

Let

- y be the data we will collect from an experiment,
- K be everything we know for certain about the world (aside from y), and
- θ be anything we don't know for certain.

My definition of a Bayesian statistician is an individual who makes decisions based on the probability distribution of those things we don't know conditional on what we know, i.e.

$$p(\theta|y, K).$$

Typically, the K is dropped from the notation.

Bayes' Rule

Bayes' Rule applied to a partition $P = \{A_1, A_2, \ldots\}$,

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{\infty} P(B|A_i)P(A_i)}$$

Bayes' Rule also applies to probability density (or mass) functions, e.g.

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

where the integral plays the role of the sum in the previous statement.

Parameter estimation

Let y be data from some model with unknown parameter θ. Then

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

and we use the following terminology

Terminology	Notation
Posterior	$p(\theta\|y)$
Prior	$p(\theta)$
Model	$p(y\|\theta)$
Prior predictive distribution (marginal likelihood)	$p(y)$

If θ is discrete (continuous),
  then $p(\theta)$ and $p(\theta|y)$ are probability mass (density) functions.
If y is discrete (continuous),
  then $p(y|\theta)$ and $p(y)$ are probability mass (density) functions.

Binomial model

Suppose $Y \sim Bin(n, \theta)$, then

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}.$$

A reasonable default prior is the uniform distribution on the interval $(0, 1)$

$$p(\theta) = \mathrm{I}(0 < \theta < 1).$$

Using Bayes Rule, you can find

$$\theta|y \sim Be(1+y, 1+n-y).$$

Beta distribution

Definition

The beta distribution defines a distribution for a probability, i.e. a number on the interval (0,1). The probability density function is

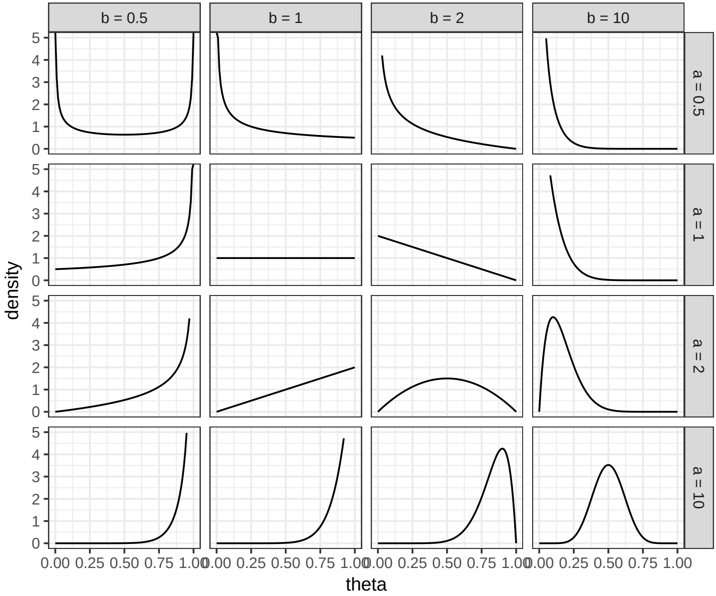
$$p(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{Beta(a,b)} \mathrm{I}(0 < \theta < 1)$$

where $a, b > 0$ and $Beta$ is the beta function, i.e.

$$Beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{and} \quad \Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

The beta distribution has the following properties:

- $E[\theta] = \frac{a}{a+b}$
- $Var[\theta] = \frac{ab}{(a+b)^2(a+b+1)}$.


b = 0.5
b = 1
b = 2
b = 10
a = 0.5
a = 1
a = 2
a = 10
density
0 1 2 3 4 5
0.00 0.25 0.50 0.75 1.00
theta

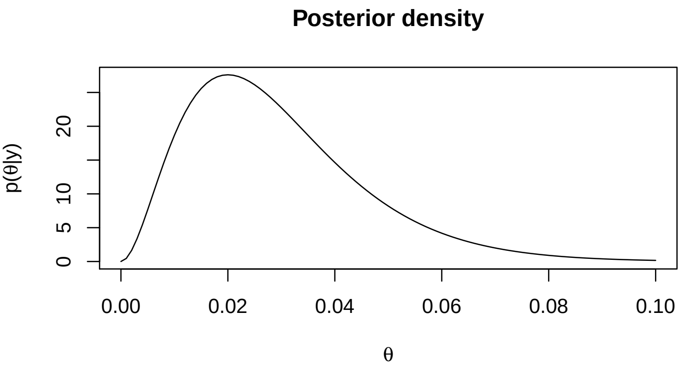
Beta posterior

Suppose we have made 100 sensors according to a particular protocol and 2 have a sensitivity below a pre-determined threshold. Let Y be the number below the threshold. Assume $Y \sim Bin(n, \theta)$ with $n = 100$ and $\theta \sim Unif(0,1) \stackrel{d}{=} Be(1,1)$, then

$$\theta|y \sim Be(1+y, 1+n-y) \stackrel{d}{=} Be(3,99).$$

Posterior density

```r
n <- 100
y <- 2
curve(dbeta(x, 1+y, 1+n-y), 0, 0.1,
      main = "Posterior density",
      xlab = expression(theta),
      ylab = expression(paste("p(",theta,"|y)")))
```



Posterior expectation

Often times it is inconvenient to provide a full posterior and so we often summarize using a point estimate from the posterior. For a point estimate, we can use the posterior expectation:

$$\hat{\theta}_{Bayes} = E[\theta|y] = \frac{1+y}{(1+y)+(1+n-y)} = \frac{1+y}{1+n}$$

```
(1+y)/(2+n)
```

```
[1] 0.02941176
```

Not that this is close, but not exactly equal to $\hat{\theta}_{MLE} = y/n$. Since the MLE is unbiased, this posterior expectation will generally be biased but it is still consistent since $\hat{\theta}_{Bayes} \to \hat{\theta}_{MLE}$.

Credible intervals

Definition

A $100(1-\alpha)\%$ credible interval is any interval (L, U) such that

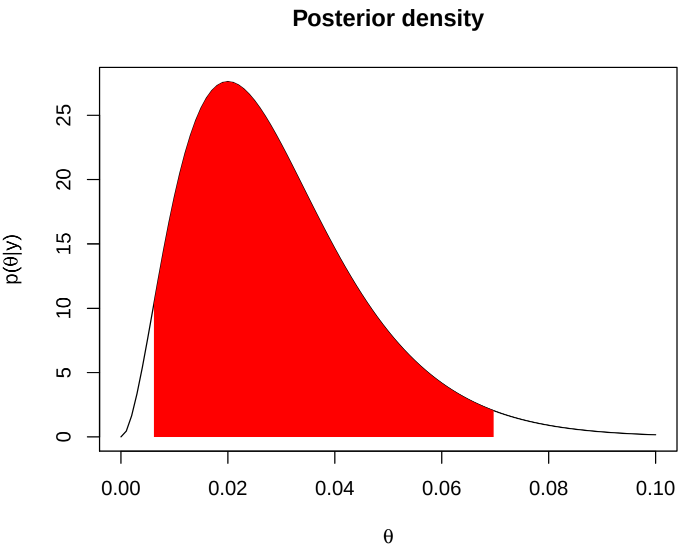
$$1-\alpha = \int_L^U p(\theta|y)d\theta.$$

An equal-tail $100(1-\alpha)\%$ credible interval is the interval $L, U)$ such that

$$\alpha/2 = \int_{-\infty}^L p(\theta|y)d\theta = \int_U^\infty p(\theta|y)d\theta.$$

```
[1] 0.006 0.070
```

Equal-tail 95% credible interval



Summary

Bayesian parameter estimation involves

1. Specifying a model $p(y|\theta)$ for your data.
2. Specifying a prior $p(\theta)$ for the parameter.
3. Deriving the posterior

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto p(y|\theta)p(\theta).$$

4. Calculating quantities of interest, e.g.
   - Posterior expectation, $E[\theta|y]$
   - Credible interval