

I05 - Confidence intervals

STAT 587 (Engineering) - Iowa State University

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Confidence intervals

Definition

The **coverage** of an interval estimator is the probability the interval will contain the true value of the parameter *when the data are considered to be random*. If an interval estimator has $100(1 - \alpha)\%$ coverage, then we call it a $100(1 - \alpha)\%$ **confidence interval** and $1 - \alpha$ is the **confidence level**.

The idea here is how often the interval will cover the truth when used on different data sets. In this interpretation, the interval itself is random because the endpoints are calculated based on random data.

Normal model

If $Y_i \stackrel{\text{ind}}{\sim} N(\mu, \sigma^2)$ and we assume the default prior $p(\mu, \sigma^2) \propto 1/\sigma^2$, then a $100(1 - a)\%$ credible interval for μ is given by

$$\bar{y} \pm t_{n-1, a/2} s / \sqrt{n}.$$

When the data are considered random

$$T_{n-1} = \frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}(0, 1)$$

thus the probability μ is within our credible interval is

$$\begin{aligned} & P\left(\bar{Y} - t_{n-1, a/2} S / \sqrt{n} < \mu < \bar{Y} + t_{n-1, a/2} S / \sqrt{n}\right) \\ &= P\left(-t_{n-1, a/2} < \frac{\bar{Y} - \mu}{S/\sqrt{n}} < t_{n-1, a/2}\right) \\ &= P\left(-t_{n-1, a/2} < T_{n-1} < t_{n-1, a/2}\right) \\ &= 1 - a. \end{aligned}$$

Thus, this $100(1 - a)\%$ credible interval is also a $100(1 - a)\%$ confidence interval.

Data example

Recall the corn yield example from I04 with 9 randomly selected fields in Iowa whose sample average yield is 186 and sample standard deviation is 22. Then a 95% confidence interval for the mean corn yield on Iowa farms is

$$186 \pm 2.31 \times 22/\sqrt{9} = (169, 202).$$

This **confidence interval** tells us nothing about the true mean yield of fields in Iowa. Instead, it tells us that if we use this **procedure** repeatedly on different data sets, then 95% of the time, the interval will contain the true parameter. But because this confidence interval happens to correspond to a **credible interval**, it does tell us what we should **believe** about the true mean yield.

Data example - R code

```
y %>% round(1)

[1] 153.5 205.7 178.8 170.2 224.8 184.1 169.9 201.3 181.6

n <- length(y)
ybar <- mean(y)
s <- sd(y)
a <- .05
t_crit <- qt(1-a/2, df = n-1)
L <- ybar - t_crit*s/sqrt(n)
U <- ybar + t_crit*s/sqrt(n)
c(L,U) %>% round

[1] 169 202
```

Sampling distribution

Definition

The **sampling distribution** of a statistic is the distribution of the statistic *when the data are considered random*.

Example

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$, then

$$T_{n-1} = \frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}(0, 1)$$

where \bar{Y} and S are the sample average and sample standard deviation over n observations.

Most of the time we don't know the sampling distribution of the statistic.

Approximate sampling distributions

If the estimator $\hat{\theta} = \hat{\theta}(Y)$ is based on an average or a sum, then the Central Limit Theorem tells us what its **approximate** sampling distribution is, i.e.

$$\hat{\theta} \sim N(E[\hat{\theta}], \text{Var}[\hat{\theta}]).$$

Example

If $Y_i \stackrel{\text{ind}}{\sim} N(\mu, \sigma^2)$, then

$$\hat{\mu} = \bar{Y} \sim N(E[\bar{Y}], \text{Var}[\bar{Y}])$$

where

$$\begin{aligned} E[\bar{Y}] &= \mu \\ \text{Var}[\bar{Y}] &= \sigma^2/n. \end{aligned}$$

Standard error

Definition

A **standard error** of an estimator is an *estimate* of the standard deviation of the sampling distribution of the estimator.

Example

If $Y_i \stackrel{\text{ind}}{\sim} N(\mu, \sigma^2)$ and $\hat{\mu} = \bar{Y}$, then

$$\text{Var}[\hat{\mu}] = \text{Var}[\bar{Y}] = \sigma^2/n \quad \text{and} \quad \sqrt{\text{Var}[\hat{\mu}]} = \sigma/\sqrt{n}.$$

We have the following *consistent* estimators

$$S^2 \xrightarrow{P} \sigma^2, \quad S \xrightarrow{P} \sigma, \quad \text{and} \quad S/\sqrt{n} \xrightarrow{P} \sigma/\sqrt{n}$$

and thus

$$SE[\hat{\mu}] = \sqrt{\widehat{\text{Var}[\hat{\mu}]}} = S/\sqrt{n}$$

where S is the sample standard deviation.

Approximate confidence intervals

If an estimator has an asymptotic normal distribution, then we can construct an **approximate** $100(1 - \alpha)\%$ confidence interval for $E[\hat{\theta}]$ using

$$\hat{\theta} \pm z_{\alpha/2} SE[\hat{\theta}].$$

where $SE(\hat{\theta}) = \sqrt{\widehat{Var}[\hat{\theta}]}$ is the **standard error** of the estimator.

This comes from the fact that if $\hat{\theta} \sim N(E[\hat{\theta}], SE[\hat{\theta}])$, then

$$\begin{aligned} &P\left(\hat{\theta} - z_{\alpha/2} SE(\hat{\theta}) < E[\hat{\theta}] < \hat{\theta} + z_{\alpha/2} SE(\hat{\theta})\right) \\ &= P\left(-z_{\alpha/2} < \frac{\hat{\theta} - E[\hat{\theta}]}{SE(\hat{\theta})} < z_{\alpha/2}\right) \\ &= P\left(-z_{\alpha/2} < \frac{\hat{\theta} - E[\hat{\theta}]}{\sqrt{\widehat{Var}[\hat{\theta}]}} < z_{\alpha/2}\right) \\ &\approx P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) \\ &= 1 - \alpha. \end{aligned}$$

Normal example

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ and we have the estimator $\hat{\mu} = \bar{Y}$, then

$$\begin{aligned} E[\hat{\mu}] &= \mu \\ SE[\hat{\mu}] &= S/\sqrt{n} \end{aligned}$$

Thus an **approximate** $100(1 - \alpha)\%$ confidence interval for $\mu = E[\hat{\mu}]$ is

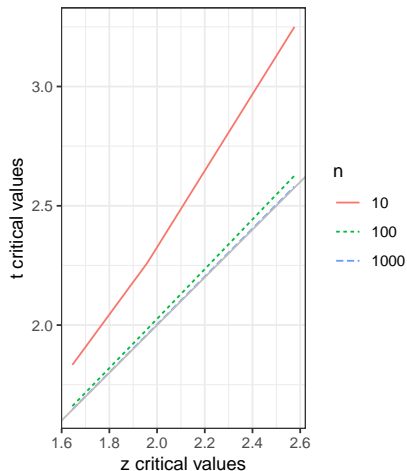
$$\hat{\mu} \pm z_{\alpha/2} SE[\hat{\mu}] = \bar{Y} \pm z_{\alpha/2} S/\sqrt{n}.$$

Note that this is almost identical to the **exact** $100(1 - \alpha)\%$ confidence interval for μ ,

$$\bar{Y} \pm t_{n-1, \alpha/2} S/\sqrt{n}$$

and when n is large $z_{\alpha/2} \approx t_{n-1, \alpha/2}$.

T critical values vs Z critical values



Binomial example

Suppose $Y \sim \text{Bin}(n, \theta)$ and we are interested in an confidence interval for θ . An unbiased estimator for θ is

$$\hat{\theta} = \frac{Y}{n}$$

since $E[\hat{\theta}] = \theta$. The variance of this estimator *when the data are considered random* is

$$\text{Var}[\hat{\theta}] = \frac{\theta(1 - \theta)}{n}.$$

A standard error for this estimator is

$$SE[\hat{\theta}] = \sqrt{\frac{\hat{\theta}(1 - \hat{\theta})}{n}}$$

since $\hat{\theta} = Y/n$ is a consistent estimator for θ

Approximate confidence interval for binomial proportion

If $Y \sim \text{Bin}(n, \theta)$, then an **approximate** $100(1 - \alpha)\%$ confidence interval for θ is

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\frac{\hat{\theta}(1 - \hat{\theta})}{n}}.$$

where $\hat{\theta} = Y/n$.

Example

The Gallup poll is a poll of approximately 1,500 (randomly selected?) U.S. adults. In a poll dated 2017/02/19, 32.1% of respondents indicated that they were engaged at work. Thus an approximate 95% confidence interval for the proportion of all U.S. adults is

$$0.321 \pm 1.96 \times \sqrt{\frac{.321(1 - .321)}{1500}} = (0.30, 0.34).$$

But this confidence interval actually says nothing about where the true proportion is, nor where we should believe it is. Instead it says something about how many of our intervals would cover the truth if we were to repeat this procedure over and over.

Confidence interval summary

Model	Estimator	Confidence Interval	Type
$Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$	$\hat{\mu} = \bar{y}$	$\hat{\mu} \pm t_{n-1, a/2} s / \sqrt{n}$	exact
$Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$	$\hat{\mu} = \bar{y}$	$\hat{\mu} \pm z_{a/2} s / \sqrt{n}$	approximate
$Y_i \stackrel{ind}{\sim} Ber(\theta)$	$\hat{\theta} = \bar{y}$	$\hat{\theta} \pm z_{a/2} \sqrt{\hat{\theta}(1 - \hat{\theta})/n}$	approximate
$Y \sim Bin(n, \theta)$	$\hat{\theta} = y/n$	$\hat{\theta} \pm z_{a/2} \sqrt{\hat{\theta}(1 - \hat{\theta})/n}$	approximate

Bayesian credible intervals based on the priors we have discussed are generally approximate confidence intervals.

Approximate here means that the coverage of the interval procedure will get closer and closer to the desired probability, i.e. $100(1 - \alpha)\%$, as the sample size gets larger.