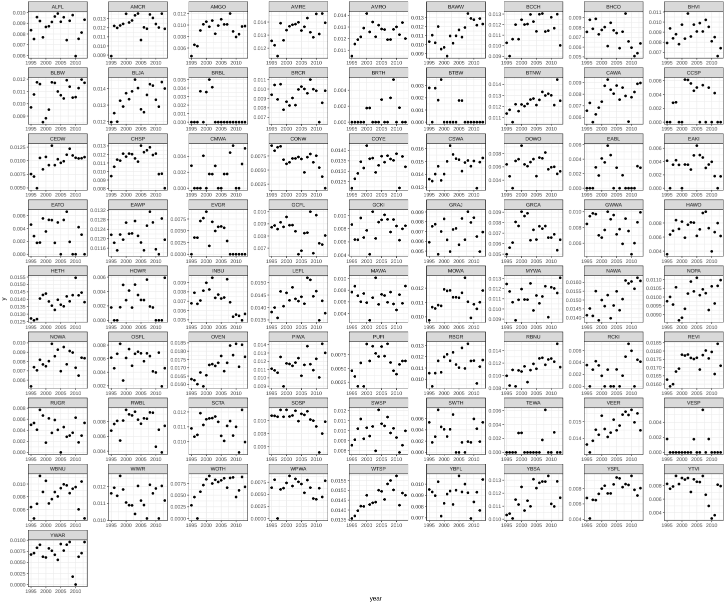
Hierarchical linear models (cont.)

Random intercept, random slope

Dr. Jarad Niemi

STAT 544 - Iowa State University

April 27, 2017


ALFL
AMCR
AMGO
AMRE
AMRO
BAWW
BCCH
BHCO
BHVI
BLBW
BLJA
BRBL
BRCR
BRTH
BTBW
BTNW
CAWA
CCSP
CEDW
CHSP
CMWA
CONW
COYE
CSWA
DOWO
EABL
EAKI
EATO
EAWP
EVGR
GCFL
GCKI
GRAJ
GRCA
GWWA
HAWO
HETH
HOWR
INBU
LEFL
MAWA
MOWA
MYWA
NAWA
NOPA
NOWA
OSFL
OVEN
PIWA
PUFI
RBGR
RBNU
RCKI
REVI
RUGR
RWBL
SCTA
SOSP
SWSP
SWTH
TEWA
VEER
VESP
WBNU
WIWR
WOTH
WPWA
WTSP
YBFL
YBSA
YSFL
YTVI
YWAR
y
year

Independent regressions

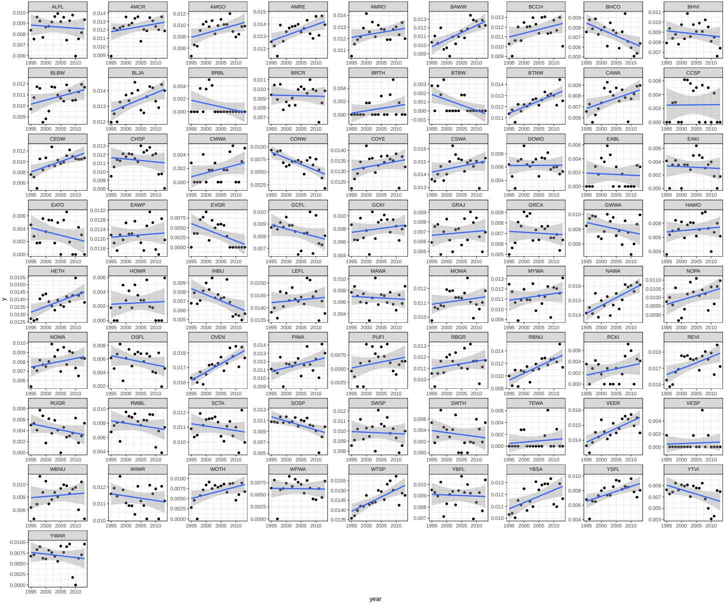
Initially, we could consider the model

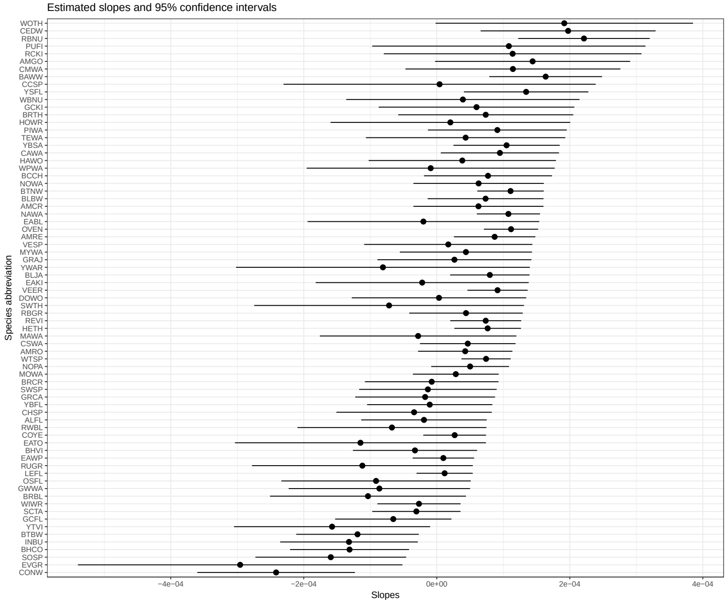
$$y_{st} \stackrel{ind}{\sim} N(\beta_{s,0} + x_{st}\beta_{s,1}, \sigma_s^2)$$

where

- y_{st} is the mean log count $(+1)$ for species s at time t
- x_{st} is the year (minus 2005) for species s at time t

This model treats each species completely independently.


ALFL
AMCR
AMGO
AMRE
AMRO
BAWW
BCCH
BHCO
BHVI
BLBW
BLJA
BRBL
BRCR
BRTH
BTBW
BTNW
CAWA
CCSP
CEDW
CHSP
CMWA
CONW
COYE
CSWA
DOWO
EABL
EAKI
EATO
EAWP
EVGR
GCFL
GCKI
GRAJ
GRCA
GWWA
HAWO
HETH
HOWR
INBU
LEFL
MAWA
MOWA
MYWA
NAWA
NOPA
NOWA
OSFL
OVEN
PIWA
PUFI
RBGR
RBNU
RCKI
REVI
RUGR
RWBL
SCTA
SOSP
SWSP
SWTH
TEWA
VEER
VESP
WBNU
WIWR
WOTH
WPWA
WTSP
YBFL
YBSA
YSFL
YTVI
YWAR
y
year


Estimated slopes and 95% confidence intervals
Species abbreviation
Slopes
WOTH
CEDW
RBNU
PUFI
RCKI
AMGO
CMWA
BAWW
CCSP
YSFL
WBNU
GCKI
BRTH
HOWR
PIWA
TEWA
YBSA
CAWA
HAWO
WPWA
BCCH
NOWA
BTNW
BLBW
AMCR
NAWA
EABL
OVEN
AMRE
VESP
MYWA
GRAJ
YWAR
BLJA
EAKI
VEER
DOWO
SWTH
RBGR
REVI
HETH
MAWA
CSWA
AMRO
WTSP
NOPA
MOWA
BRCR
SWSP
GRCA
YBFL
CHSP
ALFL
RWBL
COYE
EATO
BHVI
EAWP
RUGR
LEFL
OSFL
GWWA
BRBL
WIWR
SCTA
GCFL
YTVI
BTBW
INBU
BHCO
SOSP
EVGR
CONW
−4e−04
−2e−04
0e+00
2e−04
4e−04

Random intercept, random slope model

A reasonable assumption is to treat these species exchangeably and put a distribution on the intercept and slope.

Then a random intercept, random slope model is

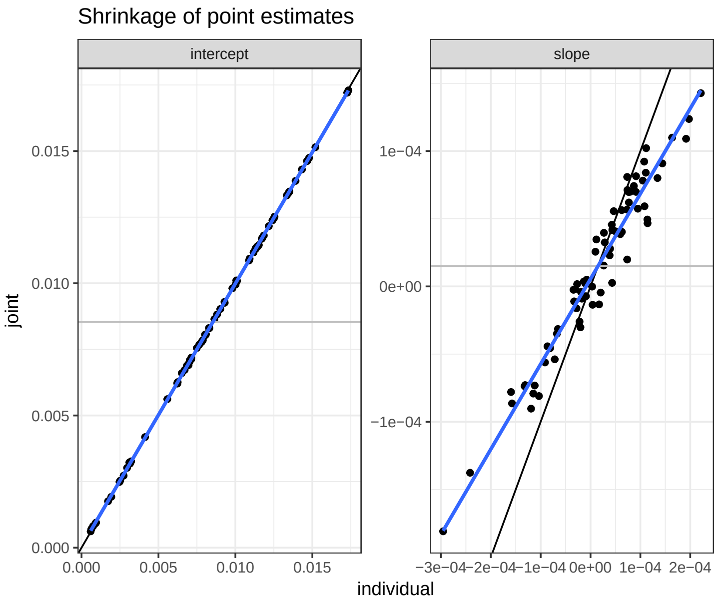
$$\begin{aligned} y_{st} &\stackrel{ind}{\sim} N(\beta_{s,0} + x_{st}\beta_{s,1}, \sigma^2) \\ \beta_s &\stackrel{ind}{\sim} N(\mu_\beta, \Sigma_\beta) \end{aligned}$$

where $\beta_s = (\beta_{s,0}, \beta_{s,1})'$ and σ^2, μ_β, and Σ_β are parameters to be estimated.

Notice that there is now a common variance for all species.

Random intercept and random slope model in R

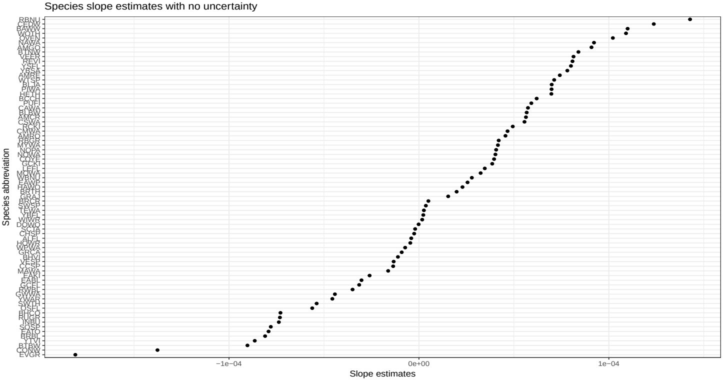
```
m2 = lmer(y~I(year-2005) + (I(year-2005)|abbrev), d)
summary(m2)


Linear mixed model fit by REML ['lmerMod']
Formula: y ~ I(year - 2005) + (I(year - 2005) | abbrev)
   Data: d

REML criterion at convergence: -13786.6

Scaled residuals:
    Min      1Q  Median      3Q     Max
-4.6888 -0.5157  0.0381  0.5437  3.6365

Random effects:
 Groups   Name           Variance  Std.Dev.  Corr
 abbrev   (Intercept)    1.799e-05 4.242e-03
          I(year - 2005) 5.998e-09 7.745e-05 0.49
 Residual                2.015e-06 1.419e-03
Number of obs: 1387, groups:  abbrev, 73

Fixed effects:
                Estimate Std. Error t value
(Intercept)    8.543e-03  4.980e-04  17.156
I(year - 2005) 1.502e-05  1.143e-05   1.314

Correlation of Fixed Effects:
            (Intr)
I(yer-2005) 0.393
```



Shrinkage of point estimates
intercept
slope
joint
individual
0.000
0.005
0.010
0.015
1e−04
0e+00
−1e−04
−3e−04
−2e−04
−1e−04
0e+00
1e−04
2e−04

Which species have significant decline?

The quantities of interest here are $\beta_{s,1}$ and whether these quantities are negative, i.e. indicating an average decease in counts over time. But how can we calculate pvalues or confidence intervals for the random effects themselves?



Bayesian random intercept, random slope model

The model

$$\begin{aligned} y_{st} &\stackrel{ind}{\sim} N(\beta_{s,0} + x_{st}\beta_{s,1}, \sigma^2) \\ \beta_s &\stackrel{ind}{\sim} N(\mu_\beta, \Sigma_\beta) \end{aligned}$$

and a prior

$$p(\sigma, \mu_\beta, \Sigma_\beta) \propto p(\sigma)p(\mu_\beta)p(\Sigma_\beta)$$

and

- $\sigma \sim Ca^+(0,1)$,
- $p(\mu_\beta) \propto 1$, and
- $\Sigma_\beta \sim$?

Conjugate prior for a covariance matrix

The natural conjugate prior for a covariance matrix is the inverse-Wishart distribution, which has density

$$p(\Sigma) \propto |\Sigma|^{-(\nu+d+1)/2} \exp\left(-\frac{1}{2}\mathrm{tr}\left(S\Sigma^{-1}\right)\right)$$

with $\nu > d-1$ and S is a positive definite matrix. The expected value is

$$E[\Sigma] = \frac{S}{\nu-d-1}$$

for $\nu > d+1$. We write $\Sigma \sim IW(\nu, S^{-1})$.

Special cases:

- If $\nu = d+1$ and S is diagonal, then each of the correlations in Σ has a marginal uniform prior.
- Jeffreys prior

$$p(\Sigma) = |\Sigma|^{-(d+1)/2}$$

Issues with the inverse-Wishart distribution

If $\Sigma \sim IW(\nu, S)$, then $\Sigma_{ii} \sim IG([\nu - (d-1)]/2, S_{ii}/2)$. In particular, if $\nu = d+1$ and $S = \mathrm{I}$ (to ensure marginally uniform priors on the correlations), then $\Sigma_{ii} \sim IG(1, 1/2)$.

The problems

- although the correlations are marginally uniform, they are not independent *a priori* of the variances (diagonal elements of Σ),
- the inverse gamma distribution has a region near zero of extremely low density that can cause extreme bias toward larger values for truly small variances,
- this in turn causes the correlation to be shrunk toward zero.

Deconstructing the covariance matrix

Let

$$\Sigma = \text{diag}(\sigma)\Omega\text{diag}(\sigma)$$

where

- σ is a vector of standard deviations
- Ω is a correlation matrix

which results in the standard deviations and correlations being independent *a priori*.

Now we can put whatever prior we want on σ and Ω, e.g. $\sigma_i \stackrel{ind}{\sim} Ca^+(0, ?)$.

LKJ correlation matrix prior

The LKJ (Lewandowski, Kurowicka, and Joe 2009) distribution is

$$p(\Omega) = |\Omega|^{\eta-1}$$

where Ω is a correlation matrix with implicit dimension d and $\eta > 0$ is the shape parameter.

- if $\eta = 1$, then the density is uniform over correlation matrices of dimension d
- if $\eta > 1$, the identity matrix is the modal correlation matrix with a sharper peak in the density for larger values of η
- if $\eta < 1$, the density has a trough at the identity matrix.

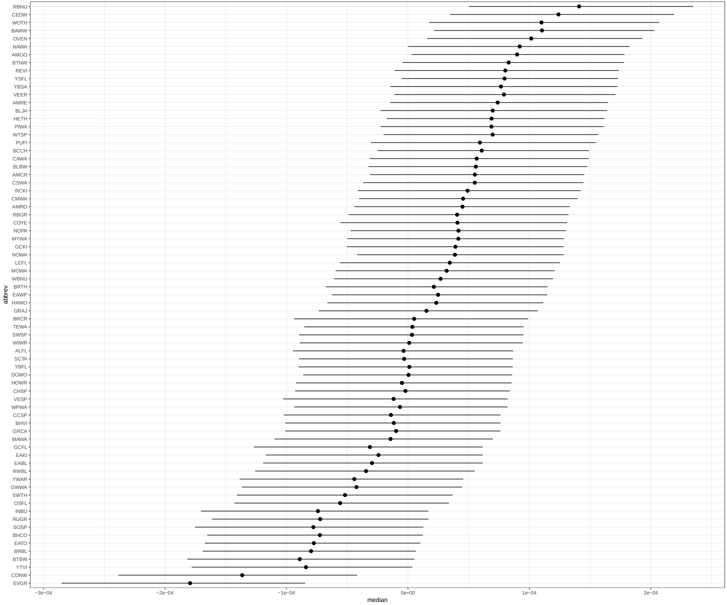
```
model = "
data {
  int<lower=1> n_species;
  int<lower=1> n_years;
  vector[n_years] y[n_species];
  matrix[n_years,2] X;
}
parameters {
  real<lower=0> sigma;
  vector[2] beta[n_species];
  vector[2] mu_beta;
  vector<lower=0>[2] sigma_beta;
  corr_matrix[2] L;
}
model {
  sigma ~ cauchy(0,1);
  sigma_beta ~ cauchy(0,1);
  L ~ lkj_corr(1.0);
  beta ~ multi_normal(mu_beta, diag_matrix(sigma_beta) * L * diag_matrix(sigma_beta));
  for (s in 1:n_species) y[s] ~ normal(X*beta[s], sigma);
}
"
```

```
tmp = reshape2::dcast(d[,c('year','abbrev','y')], abbrev~year, value.var='y')
dat = list(n_species = nrow(tmp),
           n_years   = ncol(tmp)-1,
           y         = tmp[,-1],
           X         = cbind(1, as.numeric(names(tmp)[-1])-2005),
           prior_scale = 0.01)
m = stan_model(model_code=model)
```

```
In file included from C:/Users/jarad/Documents/R/win-library/3.3/BH/include/boost/config.hpp:39:0,
                 from C:/Users/jarad/Documents/R/win-library/3.3/BH/include/boost/math/tools/config.hpp:13,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/core/var.hpp
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/core/gevv_vv
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/core.hpp:12,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/mat.hpp:4,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math.hpp:4,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/src/stan/model/model_heade
                 from file27a469f93cfb.cpp:8:
C:/Users/jarad/Documents/R/win-library/3.3/BH/include/boost/config/compiler/gcc.hpp:186:0: warning: "BOOST_NO_C
 #  define BOOST_NO_CXX11_RVALUE_REFERENCES
 ^
<command-line>:0:0: note: this is the location of the previous definition
cc1plus.exe: warning: unrecognized command line option "-Wno-ignored-attributes"
```

```
r = sampling(m, dat, refresh=0)
```

```
 Elapsed Time: 30.079 seconds (Warm-up)
               28.136 seconds (Sampling)
               58.215 seconds (Total)

[1] "The following numerical problems occurred the indicated number of times on chain 1"
                                                                          count
Exception thrown at line 18: lkj_corr_log: y is not positive definite.     9
```


RBNU
CEDW
WOTH
BAWW
OVEN
NAWA
AMGO
BTNW
REVI
YSFL
YBSA
VEER
AMRE
BLJA
HETH
PIWA
WTSP
PUFI
BCCH
CAWA
BLBW
AMCR
CSWA
RCKI
CMWA
AMRO
RBGR
COYE
NOPA
MYWA
GCKI
NOWA
LEFL
MOWA
WBNU
BRTH
EAWP
HAWO
GRAJ
BRCR
TEWA
SWSP
WIWR
ALFL
SCTA
YBFL
DOWO
HOWR
CHSP
VESP
WPWA
CCSP
BHVI
GRCA
MAWA
GCFL
EAKI
EABL
RWBL
YWAR
GWWA
SWTH
OSFL
INBU
RUGR
SOSP
BHCO
EATO
BRBL
BTBW
YTVI
CONW
EVGR
abbrev
−3e−04
−2e−04
−1e−04
0e+00
1e−04
2e−04
median

Hierarchical model for the variances

The model

$$\begin{aligned}
y_{st} &\stackrel{ind}{\sim} N(\beta_{s,0} + x_{st}\beta_{s,1}, \sigma_s^2) \\
\beta_s &\stackrel{ind}{\sim} N(\mu_\beta, \Sigma_\beta) \\
\sigma_s &\stackrel{ind}{\sim} LN(\mu_\sigma, \tau_\sigma)
\end{aligned}$$

and a prior

$$p(\mu_\sigma, \tau_\sigma, \mu_\beta, \Sigma_\beta) \propto p(\mu_\sigma)p(\tau_\sigma)p(\mu_\beta)p(\Sigma_\beta)$$

and

- $p(\mu_\sigma) \propto 1$,
- $\tau_\sigma \sim Ca^+(0, 1)$,
- $p(\mu_\beta) \propto 1$, and
- Σ_β as before

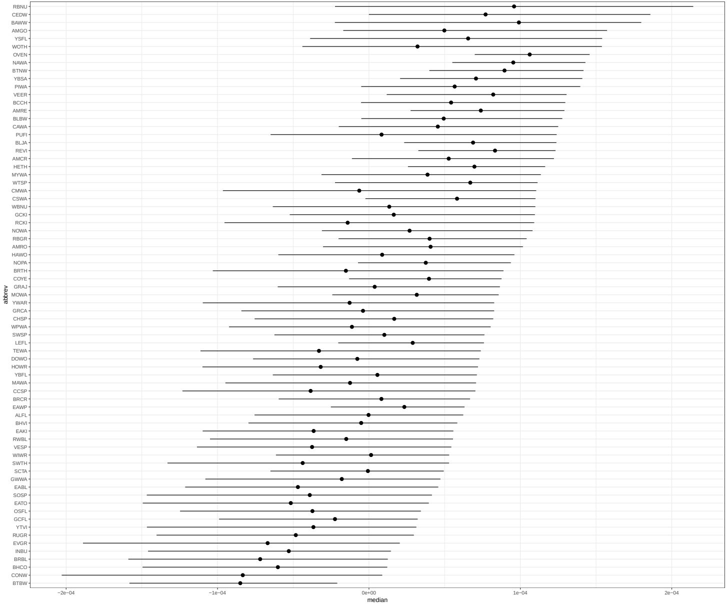
```
model2 = "
data {
  int<lower=1> n_species;
  int<lower=1> n_years;
  vector[n_years] y[n_species];
  matrix[n_years,2] X;
}
parameters {
  real<lower=0> sigma[n_species];
  real mu;
  real<lower=0> tau;
  vector[2] beta[n_species];
  vector[2] mu_beta;
  vector<lower=0>[2] sigma_beta;
  corr_matrix[2] L;
}
model {
  tau ~ cauchy(0,1);
  sigma ~ lognormal(mu,tau);
  sigma_beta ~ cauchy(0,1);
  L ~ lkj_corr(1.0);
  beta ~ multi_normal(mu_beta, diag_matrix(sigma_beta) * L * diag_matrix(sigma_beta));
  for (s in 1:n_species) y[s] ~ normal(X*beta[s], sigma[s]);
}
"
```

```
m2 = stan_model(model_code=model2)
```

```
In file included from C:/Users/jarad/Documents/R/win-library/3.3/BH/include/boost/config.hpp:39:0,
                 from C:/Users/jarad/Documents/R/win-library/3.3/BH/include/boost/math/tools/config.hpp:13,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/core/var.hpp
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/core/gevv_vv
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/core.hpp:12,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math/rev/mat.hpp:4,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/stan/math.hpp:4,
                 from C:/Users/jarad/Documents/R/win-library/3.3/StanHeaders/include/src/stan/model/model_heade
                 from file27a44a6b4ad3.cpp:8:
C:/Users/jarad/Documents/R/win-library/3.3/BH/include/boost/config/compiler/gcc.hpp:186:0: warning: "BOOST_NO_C
 #  define BOOST_NO_CXX11_RVALUE_REFERENCES
 ^
<command-line>:0:0: note: this is the location of the previous definition
cc1plus.exe: warning: unrecognized command line option "-Wno-ignored-attributes"
```

```
r2 = sampling(m2, dat)
```

```
SAMPLING FOR MODEL '6f90fb3aa950384573fb103b06a9d4ff' NOW (CHAIN 1).

Chain 1, Iteration:    1 / 2000 [  0%]  (Warmup)
Chain 1, Iteration:  200 / 2000 [ 10%]  (Warmup)
Chain 1, Iteration:  400 / 2000 [ 20%]  (Warmup)
Chain 1, Iteration:  600 / 2000 [ 30%]  (Warmup)
Chain 1, Iteration:  800 / 2000 [ 40%]  (Warmup)
Chain 1, Iteration: 1000 / 2000 [ 50%]  (Warmup)
Chain 1, Iteration: 1001 / 2000 [ 50%]  (Sampling)
Chain 1, Iteration: 1200 / 2000 [ 60%]  (Sampling)
Chain 1, Iteration: 1400 / 2000 [ 70%]  (Sampling)
Chain 1, Iteration: 1600 / 2000 [ 80%]  (Sampling)
Chain 1, Iteration: 1800 / 2000 [ 90%]  (Sampling)
```


RBNU
CEDW
BAWW
AMGO
YSFL
WOTH
OVEN
NAWA
BTNW
YBSA
PIWA
VEER
BCCH
AMRE
BLBW
CAWA
PUFI
BLJA
REVI
AMCR
HETH
MYWA
WTSP
CMWA
CSWA
WBNU
GCKI
RCKI
NOWA
RBGR
AMRO
HAWO
NOPA
BRTH
COYE
GRAJ
MOWA
YWAR
GRCA
CHSP
WPWA
SWSP
LEFL
TEWA
DOWO
HOWR
YBFL
MAWA
CCSP
BRCR
EAWP
ALFL
BHVI
EAKI
RWBL
VESP
WIWR
SWTH
SCTA
GWWA
EABL
SOSP
EATO
OSFL
GCFL
YTVI
RUGR
EVGR
INBU
BRBL
BHCO
CONW
BTBW
abbrev
−2e−04
−1e−04
0e+00
1e−04
2e−04
median