Bayesian hypothesis testing (cont.)

Dr. Jarad Niemi

Iowa State University

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Outline

- Review of formal Bayesian hypothesis testing
- Likelihood ratio tests
- Jeffrey-Lindley paradox

Bayes tests = evaluate predictive models

Consider a standard hypothesis test scenario:

$$H_0 : \theta = \theta_0, \qquad H_1 : \theta \neq \theta_0$$

A Bayesian measure of the support for the null hypothesis is the Bayes Factor:

$$BF(H_0 : H_1) = \frac{p(y|H_0)}{p(y|H_1)} = \frac{p(y|\theta_0)}{\int p(y|\theta)p(\theta|H_1)d\theta}$$

where $p(\theta|H_1)$ is the prior distribution for θ under the alternative hypothesis. Thus the Bayes Factor measures the predictive ability of the two Bayesian models. Both models say $p(y|\theta)$ are the data model if we know θ, but

1. Model 0 says $\theta = \theta_0$ and thus $p(y|\theta_0)$ is our predictive distribution for y while
2. Model 1 says $p(\theta|H_1)$ is our uncertainty about θ and thus

   $$p(y|H_1) = \int p(y|\theta)p(\theta|H_1)d\theta$$

   is our predictive distribution for y.

Normal example

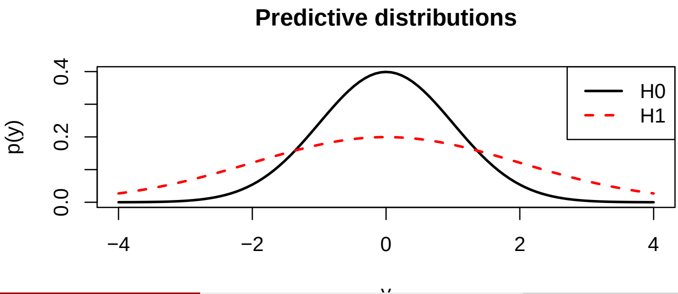
Consider $y \sim N(\theta, 1)$ and

$$H_0 : \theta = 0, \qquad H_1 : \theta \neq 0$$

and we assume $\theta | H_1 \sim N(0, C)$. Thus,

$$BF(H_0 : H_1) = \frac{p(y|H_0)}{p(y|H_1)} = \frac{p(y|\theta_0)}{\int p(y|\theta)p(\theta|H_1)d\theta} = \frac{N(y; 0, 1)}{N(y; 0, 1 + C)}.$$

Now, as $C \to \infty$, our predictions about y become less sharp.



Likelihood Ratio Tests

Consider a likelihood $L(\theta) = p(y|\theta)$, then the liklihood ratio test statistic for testing $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_0^c$ with $\Theta = \Theta_0 \cup \Theta_0^c$ is

$$\lambda(y) = \frac{\sup_{\Theta_0} L(\theta)}{\sup_{\Theta} L(\theta)} = \frac{L(\hat{\theta}_{0,MLE})}{L(\hat{\theta}_{MLE})}$$

where $\hat{\theta}_{MLE}$ and $\hat{\theta}_{0,MLE}$ are the (restricted) MLEs. The likelihood ratio test (LRT) is any test that has a rejection region of the form $\{y : \lambda(y) \le c\}$. (Casella & Berger Def 8.2.1)

Under certain conditions (see Casella & Berger 10.3.3), as $n \to \infty$

$$-2 \log \lambda(y) \to \chi^2_\nu$$

where ν us the difference between the number of free parameters specified by $\theta \in \theta_0$ and the number of free parameters specified by $\theta \in \Theta$.

Binomial example

Consider a coin flipping experiment so that $Y_i \stackrel{iid}{\sim} Ber(\theta)$ and the null hypothesis $H_0 : \theta = 0.5$ versus the alternative $H_1 : \theta \neq 0.5$. Then

$$\lambda(y) = \frac{\sup_{\Theta_0} L(\theta)}{\sup_{\Theta} L(\theta)} = \frac{0.5^n}{\hat{\theta}_{MLE}^{n\overline{y}}(1-\hat{\theta}_{MLE})^{n-n\overline{y}}} = \frac{0.5^n}{\overline{y}^{n\overline{y}}(1-\overline{y})^{n-n\overline{y}}}$$

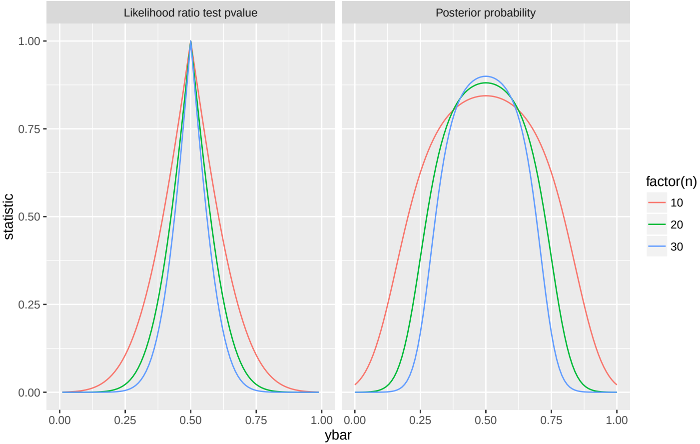
and $-2\log\lambda(y) \to \chi^2_1$ as $n \to \infty$ so

$$pvalue \approx P(\chi^2_1 > -2\log\lambda(y)).$$

If $pvalue < \alpha$, then we reject H_0 at level α. Typically $\alpha = 0.05$.

Binomial example

$Y \sim Bin(n, \theta)$ and, for the Bayesian analysis, $\theta | H_1 \sim Be(1, 1)$ and $p(H_0) = p(H_1) = 0.5$:



Do pvalues and posterior probabilities agree?

Suppose $n = 10,000$ and $y = 4,900$, then the pvalue is

$$pvalue \approx P(\chi^2_1 > -2\log(0.135)) = 0.045$$

so we would reject H_0 at the 0.05 level.

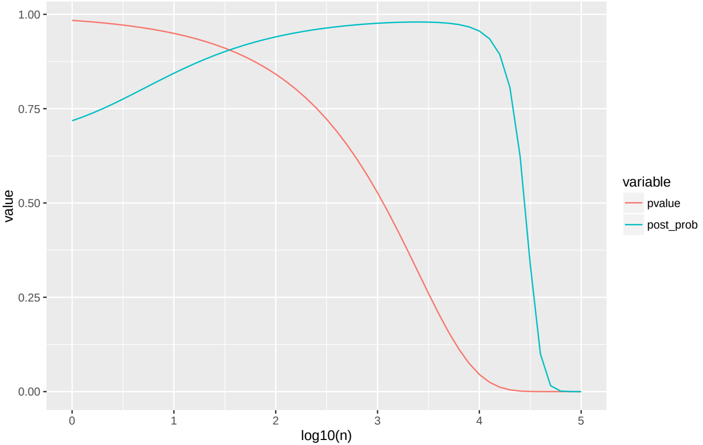
The posterior probability of H_0 is

$$p(H_0|y) \approx \frac{1}{1 + 1/10.8} = 0.96,$$

so the probability of H_0 being true is 96%.

It appears the Bayesian and LRT pvalue completely disagree!

Binomial $\bar{y} = 0.49$ with $n \to \infty$



Jeffrey-Lindley Paradox

Definition

The Jeffrey-Lindley Paradox concerns a situation when comparing two hypotheses H_0 and H_1 given data y and find

- a frequentist test result is significant leading to rejection of H_0, but
- the posterior probability of H_0 is high.

This can happen when

- the effect size is small,
- n is large,
- H_0 is relatively precise,
- H_1 is relative diffuse, and
- the prior model odds is ≈ 1.

Comparison

The test statistic with point null hypotheses:

$$\lambda(y) = \frac{p(y|\theta_0)}{p\left(y|\hat{\theta}_{MLE}\right)}$$

$$BF(H_0 : H_1) = \frac{p(y|\theta_0)}{\int p(y|\theta)p(\theta|H_1)d\theta} = \frac{p(y|H_0)}{p(y|H_1)}$$

A few comments:

- The LRT chooses the best possible alternative value.
- The Bayesian test penalizes for vagueness in the prior.
- The LRT can be interpreted as a Bayesian point mass prior exactly at the MLE.
- Generally, pvalues provide a measure of lack-of-fit of the null model.
- Bayesian tests compare predictive performance of two Bayesian models (model+prior).

Normal mean testing

Let $y \sim N(\theta, 1)$ and we are testing

$$H_0 : \theta = 0 \qquad \text{vs} \qquad H_1 : \theta \neq 0$$

We can compute a two-sided pvalue via

$$\text{pvalue} = 2\Phi(-|y|)$$

where $\Phi(\cdot)$ is the cumulative distribution function for a standard normal.

Typically, we set our Type I error rate at level α, i.e.

$$P(\text{reject } H_0 | H_0 \text{ true}) = \alpha.$$

But, if the pvalue is less than α, we should be interested in

$$P(H_0 \text{ true} | \text{reject } H_0).$$

Pvalue interpretation

Let $y \sim N(\theta, 1)$ and we are testing

$$H_0 : \theta = 0 \qquad \text{vs} \qquad H_1 : \theta \neq 0$$

For the following activity, you need to tell me

1. the observed pvalue,
2. the relative frequencies of null and alternative hypotheses, and
3. the distribution for θ under the alternative.

Then this pvalue app below will calculate (via simulation) the probability the null hypothesis is true.

```
shiny::runGitHub('jarad/pvalue')
```

Pvalue app approach

The idea is that a scientist performs a series of experiments. For each experiment,

- whether H_0 or H_1 is true is randomly determined,
- θ is sampled according to which hypothesis is true, and
- the pvalue is calculated.

This process is repeated until a pvalue of the desired value is achieved, e.g. pvalue=0.05, and the true hypothesis is recorded. Thus,

$$P(H_0 \text{ true} \mid \text{pvalue } = 0.05) \approx \frac{1}{K} \sum_{k=1}^{K} \mathrm{I}(H_0 \text{ true} \mid \text{pvalue } \approx 0.05).$$

Thus, there is nothing Bayesian happening here except that the probability being calculated has the unknown quantity on the left and the known quantity on the right.

Prosecutor's Fallacy

It is common for those using statistics to equate the following

$$\text{pvalue} \stackrel{?}{=} P(\text{data}|H_0 \text{ true}) \neq P(H_0 \text{ true}|\text{data}).$$

but we can use Bayes rule to show us that these probabilities cannot be equated

$$p(H_0|y) = \frac{p(y|H_0)p(H_0)}{p(y)} = \frac{p(y|H_0)p(H_0)}{p(y|H_0)p(H_0) + p(y|H_0)p(H_0)}$$

This situation is common enough that it is called The Prosecutor's Fallacy.