

STAT 401A - Statistical Methods for Research Workers

Inference Using t -Distributions

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Random variables

From: http://www.stats.gla.ac.uk/steps/glossary/probability_distributions.html

Definition

A **random variable** is a function that associates a unique numerical value with every outcome of an experiment.

Definition

A **discrete random variable** is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4,... Discrete random variables are usually (but not necessarily) counts.

Definition

A **continuous random variable** is one which takes an infinite number of possible values. Continuous random variables are usually measurements.

Random variables

Examples:

- Discrete random variables
 - Coin toss: Heads (1) or Tails (0)
 - Die roll: 1, 2, 3, 4, 5, or 6
 - Number of Ovenbirds at a 10-minute point count
 - RNAseq feature count
- Continuous random variables
 - Pig average daily (weight) gain
 - Corn yield per acre

Statistical notation

Let Y be 1 if the coin toss is heads and 0 if tails, then

$$Y \sim \text{Bin}(n, p)$$

which means

Y is a binomial random variable with n trials and probability of success p

For example, if Y is the number of heads observed when tossing a fair coin ten times, then $Y \sim \text{Bin}(10, 0.5)$.

Later we will be constructing $100(1 - \alpha)\%$ confidence intervals, these intervals are constructed such that if n of them are constructed then $Y \sim \text{Bin}(n, 1 - \alpha)$ will cover the true value.

Statistical notation

Let Y_i be the average daily (weight) gain in pounds for the i th pig, then

$$Y_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$$

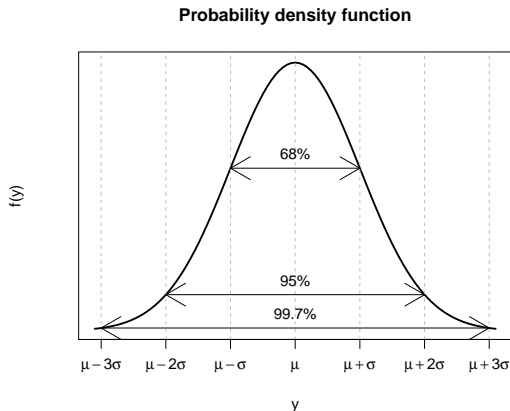
which means

Y_i are independent and identically distributed normal (Gaussian) random variables with expected value $E[Y_i] = \mu$ and variance $V[Y_i] = \sigma^2$ (standard deviation σ).

For example, if a litter of pigs is expected to gain 2 lbs/day with a standard deviation of 0.5 lbs/day and *the knowledge of how much one pig gained does not affect what we think about how much the others have gained*, then $Y_i \stackrel{iid}{\sim} N(2, 0.5^2)$.

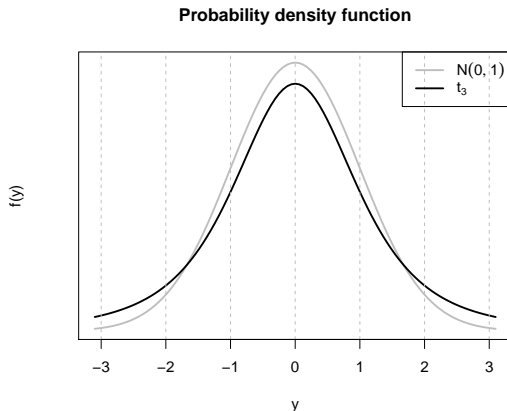
Normal (Gaussian) distribution

A random variable Y has a normal distribution, i.e. $Y \sim N(\mu, \sigma^2)$, with mean μ and variance σ^2 if draws from this distribution follow a bell curve centered at μ with spread determined by σ^2 :



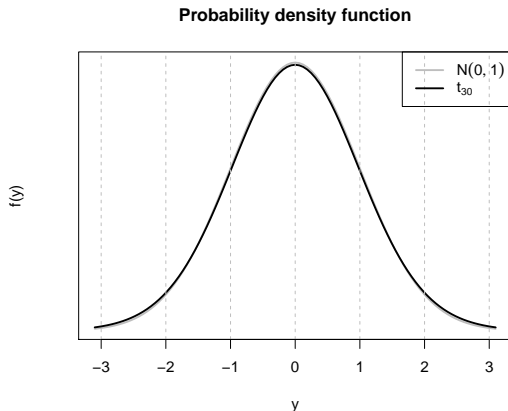
t-distribution

A random variable Y has a t -distribution, i.e. $Y \sim t_\nu$, with degrees of freedom ν if draws from this distribution follow a similar bell shaped pattern:



t-distribution

As $\nu \rightarrow \infty$, then $t_\nu \xrightarrow{d} N(0, 1)$, i.e. as the degrees of freedom increase, a t distribution gets closer and closer to a standard normal distribution, i.e. $N(0, 1)$. If $\nu > 30$, the differences is negligible.

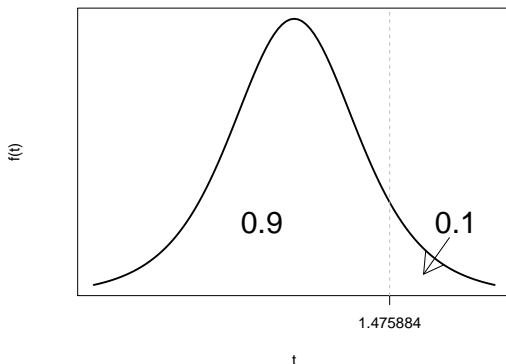


t critical value

Definition

If $T \sim t_v$, a $t_v(1 - \alpha/2)$ critical value is the value such that $P(T < t_v(1 - \alpha/2)) = 1 - \alpha/2$ (or $P(T > t_v(1 - \alpha/2)) = \alpha/2$).

Probability density function t_5



Cedar-apple rust

Cedar-apple rust is a (non-fatal) disease that affects apple trees. Its most obvious symptom is rust-colored spots on apple leaves. Red cedar trees are the immediate source of the fungus that infects the apple trees. If you could remove all red cedar trees within a few miles of the orchard, you should eliminate the problem. In the first year of this experiment the number of affected leaves on 8 trees was counted; the following winter all red cedar trees within 100 yards of the orchard were removed and the following year the same trees were examined for affected leaves.

- Statistical hypothesis:

- H_0 : Removing red cedar trees increases or maintains the same mean number of rusty leaves.

- H_1 : Removing red cedar trees decreases the mean number of rusty leaves.

- Statistical question:

- What is the expected reduction of rusty leaves **in our sample** between year 1 and year 2 (perhaps due to removal of red cedar trees)?

Data

Here are the data

```
library(plyr)
y1 = c(38,10,84,36,50,35,73,48)
y2 = c(32,16,57,28,55,12,61,29)
leaves = data.frame(year1=y1, year2=y2, diff=y1-y2)
leaves
```

	year1	year2	diff
1	38	32	6
2	10	16	-6
3	84	57	27
4	36	28	8
5	50	55	-5
6	35	12	23
7	73	61	12
8	48	29	19

```
summarize(leaves, n=length(diff), mean=mean(diff), sd=sd(diff))
```

	n	mean	sd
1	8	10.5	12.2

Is this a statistically significant difference?

Assumptions

Let

- Y_{1j} be the number of rusty leaves on tree j in year 1
- Y_{2j} be the number of rusty leaves on tree j in year 2

Assume

$$D_j = Y_{1j} - Y_{2j} \stackrel{iid}{\sim} N(\mu, \sigma^2)$$

Then the statistical hypothesis test is

$$H_0: \mu = 0 \ (\mu \leq 0)$$

$$H_1: \mu > 0$$

while the statistical question is 'what is μ '?

Paired t-test pvalue

Test statistic

$$t = \frac{\bar{D} - \mu}{SE(\bar{D})}$$

where $SE(\bar{D}) = s/\sqrt{n}$ with

- n being the number of observations (differences),
- s being the sample standard deviation of the differences, and
- \bar{D} being the average difference.

If H_0 is true, then $\mu = 0$ and $t \sim t_{n-1}$. The pvalue is $P(t_{n-1} > t)$ since this is a one-sided test. By symmetry, $P(t_{n-1} > t) = P(t_{n-1} < -t)$.

For these data,

$$\bar{D} = 10.5, SE(\bar{D}) = 4.31, t_7 = 2.43, \text{ and } p = 0.02$$

Confidence interval for μ

The $100(1-\alpha)\%$ confidence interval has lower endpoint

$$\bar{D} - t_{n-1}(1 - \alpha)SE(\bar{D})$$

and upper endpoint at infinity

For these data at 95% confidence, $t_7(0.9) = 1.89$ and thus the lower endpoint is

$$10.5 - 1.89 \cdot 4.31 = 2.33$$

So we are 95% confident that the true difference in the number of rusty leaves is greater than 2.33.

SAS code for paired t-test

```
DATA leaves;
  INPUT tree year1 year2;
  DATALINES;
1 38 32
2 10 16
3 84 57
4 36 28
5 50 55
6 35 12
7 73 61
8 48 29
;

PROC TTEST DATA=leaves SIDES=U;
  PAIRED year1*year2;
  RUN;
```

SAS output for paired t-test

```

The TTEST Procedure

Difference:  year1 - year2

N          Mean      Std Dev    Std Err    Minimum    Maximum
8         10.5000     12.2007     4.3136     -6.0000     27.0000

      Mean      95% CL Mean      Std Dev    95% CL Std Dev
10.5000     2.3275 Infty         12.2007     8.0668 24.8317

      df    t Value    Pr > t
      7      2.43     0.0226

```


R output for paired t-test

```
t.test(leaves$year1, leaves$year2, paired=TRUE, alternative="greater")
```

Paired t-test

data: leaves\$year1 and leaves\$year2

t = 2.434, df = 7, p-value = 0.02257

alternative hypothesis: true difference in means is greater than 0

95 percent confidence interval:

2.328 Inf

sample estimates:

mean of the differences

10.5

Statistical Conclusion

Removal of red cedar trees within 100 yards is associated with a significant reduction in rusty apple leaves (paired t-test $t_7=2.43$, $p=0.023$). The mean reduction in rust color leaves is 10.5 [95% CI (2.33, ∞)].

Do Japanese cars get better mileage than American cars?

- Statistical hypothesis:

H_0 : Mean mpg of Japanese cars is the same as mean mpg of American cars.

H_1 : Mean mpg of Japanese cars is different than mean mpg of American cars.

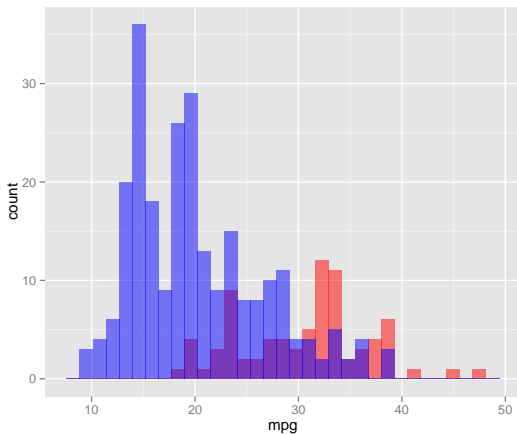
- Statistical question:

What is the difference in mean mpg between Japanese and American cars?

- Data collection:

- Collect a random sample of Japanese/American cars

```
mpg = read.csv("mpg.csv")  
library(ggplot2)  
ggplot(mpg, aes(x=mpg))+  
  geom_histogram(data=subset(mpg, country=="Japan"), fill="red", alpha=0.5)+  
  geom_histogram(data=subset(mpg, country=="US"), fill="blue", alpha=0.5)
```



Assumptions

Let

- Y_{1j} represent the j th Japanese car
- Y_{2j} represent the j th American car

Assume

$$Y_{1j} \stackrel{iid}{\sim} N(\mu_1, \sigma^2) \quad Y_{2j} \stackrel{iid}{\sim} N(\mu_2, \sigma^2)$$

Restate the hypotheses using this notation

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 \neq \mu_2$$

Alternatively

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 - \mu_2 \neq 0$$

Test statistic

The test statistic we use here is

$$\frac{\bar{Y}_1 - \bar{Y}_2 - (\mu_1 - \mu_2)}{SE(\bar{Y}_1 - \bar{Y}_2)}$$

where

- \bar{Y}_1 is the sample average mpg of the Japanese cars
- \bar{Y}_2 is the sample average mpg of the American cars

and

$$SE(\bar{Y}_1 - \bar{Y}_2) = s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{(n_1 + n_2 - 2)}}$$

where

- s_1 is the sample standard deviation of the mpg of the Japanese cars
- s_2 is the sample standard deviation of the mpg of the American cars

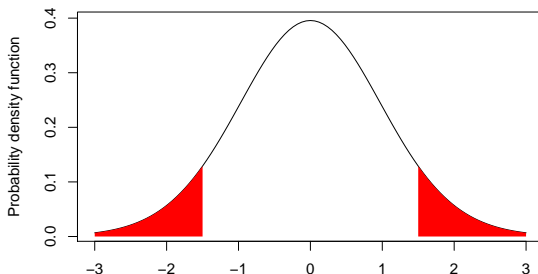
Pvalue

If H_0 is true, then $\mu_1 = \mu_2$ and the test statistic

$$t = \frac{\bar{Y}_1 - \bar{Y}_2 - (\mu_1 - \mu_2)}{SE(\bar{Y}_1 - \bar{Y}_2)} \sim t_{n_1+n_2-2}$$

where t_ν is a t-distribution with ν degrees of freedom.

Pvalue is $P(|t_{n_1+n_2-2}| > |t|) = P(t_{n_1+n_2-2} > |t|) + P(t_{n_1+n_2-2} < -|t|)$ or as a picture



Hand calculation

To calculate the quantity by hand, we need 6 numbers:

```
library(plyr)
ddply(mpg, .(country), summarize, n=length(mpg), mean=mean(mpg), sd=sd(mpg))
```

	country	n	mean	sd
1	Japan	79	30.48	6.108
2	US	249	20.14	6.415

Calculate

$$\begin{aligned}
 s_p &= \sqrt{\frac{(79-1) \cdot 6.11^2 + (249-1) \cdot 6.41^2}{79+249-2}} = 6.34 \\
 SE(\bar{Y}_1 - \bar{Y}_2) &= 6.34 \sqrt{\frac{1}{79} + \frac{1}{249}} = 0.82 \\
 t &= \frac{30.5 - 20.1}{0.82} = 12.6
 \end{aligned}$$

Finally, we are interested in finding

$P(|t_{326}| > |12.6|) = 2P(t_{326} < -|12.6|) < 0.0001$ which is found using a table or software.

Confidence interval

Alternatively, we can construct a $100(1-\alpha)\%$ confidence interval. The formula is

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{n_1+n_2-2}(1-\alpha/2)SE(\bar{Y}_1 - \bar{Y}_2)$$

where \pm indicates plus and minus and $t_{\nu}(1-\alpha/2)$ is the value such that $P(t_{\nu} < t_{\nu}(1-\alpha/2)) = 1-\alpha/2$. If $\alpha = 0.05$ and $\nu = 326$, then $t_{\nu}(1-\alpha/2) = 1.97$.

The 95% confidence interval is

$$30.5 - 20.1 \pm 1.97 \cdot 0.82 = (8.73, 11.9)$$

We are 95% confident that, on average, Japanese cars get between 8.73 and 11.9 more mpg than American cars.

SAS code for two-sample t-test

```
DATA mpg;  
  INFILE 'mpg.csv' DELIMITER=', ' FIRSTOBS=2;  
  INPUT mpg country $;  
  
PROC TTEST DATA=mpg;  
  CLASS country;  
  VAR mpg;  
  RUN;
```

The TTEST Procedure

Variable: mpg

country	N	Mean	Std Dev	Std Err	Minimum	Maximum
Japan	79	30.4810	6.1077	0.6872	18.0000	47.0000
US	249	20.1446	6.4147	0.4065	9.0000	39.0000
Diff (1-2)		10.3364	6.3426	0.8190		

country	Method	Mean	95% CL Mean		Std Dev	95% CL Std Dev	
Japan		30.4810	29.1130	31.8491	6.1077	5.2814	7.2429
US		20.1446	19.3439	20.9452	6.4147	5.8964	7.0336
Diff (1-2)	Pooled	10.3364	8.7252	11.9477	6.3426	5.8909	6.8699
Diff (1-2)	Satterthwaite	10.3364	8.7576	11.9152			

Method	Variances	df	t Value	Pr > t
Pooled	Equal	326	12.62	<.0001
Satterthwaite	Unequal	136.87	12.95	<.0001

Equality of Variances

Method	Num df	Den df	F Value	Pr > F
Folded F	248	78	1.10	0.6194

R code/output for two-sample t-test

```
t.test(mpg~country, data=mpg, var.equal=TRUE)
```

Two Sample t-test

data: mpg by country

t = 12.62, df = 326, p-value < 2.2e-16

alternative hypothesis: true difference in means is not equal to 0

95 percent confidence interval:

8.725 11.948

sample estimates:

mean in group Japan	mean in group US
30.48	20.14

Conclusion

Mean miles per gallon of Japanese cars is significantly different than mean miles per gallon of American cars (two-sample t-test $t=12.62$, $p < 0.0001$). Japanese cars get an average of 10.3 [95% CI (8.7,11.9)] more miles per gallon than American cars.

Hypotheses

Three key features:

- a test statistic calculated from data
- a sampling distribution for the test statistic under the null hypothesis
- a region that is as or more extreme (one-sided vs two-sided hypotheses)

Calculate probability of being in the region:

Definition

A **pvalue** is the probability of observing a test statistic as or more extreme than that observed, if the null hypothesis is true.

- If pvalue is less than or equal to α , we reject the null hypothesis.
- If pvalue is greater than α , we fail to reject the null hypothesis.

Hypothesis framework

Let's assume, we have

- $Y_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$ for $i = 1, \dots, n$ and have
- calculated a test statistic t , and
- if the null hypothesis is true, t has a t_ν sampling distribution.

Now, we can have one of three types of hypotheses:

- Two-sided ($H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$):

$$\text{pvalue} = P(|t_\nu| > |t|) = P(t_\nu > |t|) + P(t_\nu < -|t|) = 2P(t_\nu < -|t|)$$

- One-sided ($H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$):

$$\text{pvalue} = P(t_\nu > t) = P(t_\nu < -t)$$

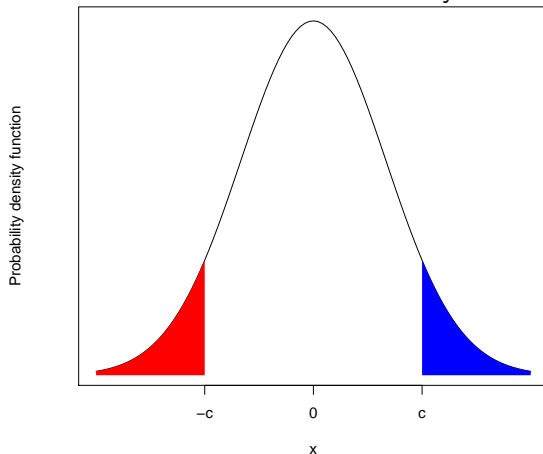
- One-sided ($H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$):

$$\text{pvalue} = P(t_\nu < t)$$

$F(c) = P(t_\nu < c)$ is the **cumulative distribution function** for a t distribution with ν degrees of freedom.

Symmetric distributions

The standard normal and t distributions are both symmetric around zero.



$$P(T_\nu > c) = P(t_\nu < -c) \quad \text{blue area is equal to red area}$$

Paired t-test example

In the paired t-test example, we had a test statistic $t = 2.43$ with a t_7 sampling distribution if the null hypothesis is true.

Consider the following hypotheses (μ is the expected difference):

- Two-sided ($H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$):

$$\text{pvalue} = 2P(t_7 < -2.43) = 0.0454$$

- One-sided ($H_0 : \mu \leq 0$ vs $H_1 : \mu > 0$):

$$\text{pvalue} = P(t_7 < -2.43) = 0.0227$$

- One-sided ($H_0 : \mu \geq 0$ vs $H_1 : \mu < 0$):

$$\text{pvalue} = P(t_7 < 2.43) = 0.9773$$

Two-sample t-test example

In a two-sample t-test, we might have a test statistic $t = -2$ with a t_{30} sampling distribution if the null hypothesis is true.

Consider the following hypotheses ($\mu_1 - \mu_2$ is the expected difference):

- Two-sided ($H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 - \mu_2 \neq 0$):

$$\text{pvalue} = 2P(t_{30} < -2) = 0.0546$$

- One-sided ($H_0 : \mu_1 - \mu_2 \leq 0$ vs $H_1 : \mu_1 - \mu_2 > 0$):

$$\text{pvalue} = P(t_{30} < 2) = 0.9727$$

- One-sided ($H_0 : \mu_1 - \mu_2 \geq 0$ vs $H_1 : \mu_1 - \mu_2 < 0$):

$$\text{pvalue} = P(t_{30} < -2) = 0.0273$$

Confidence interval construction

Key steps in confidence interval construction:

- 1 Calculate point estimate
- 2 Calculate standard error of the statistic
- 3 Set error level α
- 4 Find the appropriate critical value
- 5 Construct the $100(1 - \alpha)\%$ confidence interval

- Two-sided ($H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$): (L, U)

$$(L, U) = \text{estimate} \pm \text{critical value}(1 - \alpha/2) \times \text{standard error}$$

- One-sided ($H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$): (L, ∞)

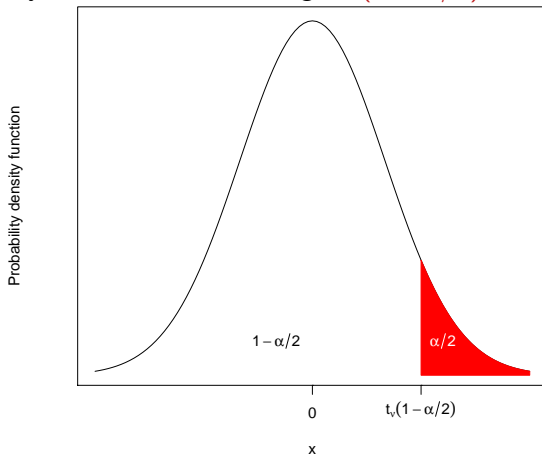
$$L = \text{estimate} - \text{critical value}(1 - \alpha) \times \text{standard error}$$

- One-sided ($H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$): $(-\infty, U)$

$$U = \text{estimate} + \text{critical value}(1 - \alpha) \times \text{standard error}$$

Critical values

A related quantity are critical values, e.g. $t_\nu(1 - \alpha/2)$.



Let $c = t_\nu(1 - \alpha/2)$, then we need $P(t_\nu < c) = 1 - \alpha/2$, i.e. the inverse of the cumulative distribution function.

Paired t-test example

In paired t-test example, we had an estimate $\hat{\mu} = 10.5$ and a standard error of 4.3136 with 7 degrees of freedom.

The 95%, i.e. $\alpha = 0.05$, confidence intervals for μ are

- Two-sided ($t_7(.975) = 2.364624$)

$$10.5 \pm 2.364624 \cdot 4.3136 = (0.30, 20.7)$$

- One-sided (positive) ($t_7(.95) = 1.894579$)

$$(10.5 - 1.894579 \cdot 4.3136, \infty) = (2.33, \infty)$$

- One-sided (negative) ($t_7(.95) = 1.894579$)

$$(-\infty, 10.5 + 1.894579 \cdot 4.3136) = (-\infty, 18.7)$$

Two-sample t-test example

In the two-sample t-test example, we had an estimate $\mu_1 \hat{=} \mu_2 = 10.33643$ and a pooled standard error of 0.8190 with 326 degrees of freedom.

The 90%, i.e. $\alpha = 0.10$, confidence intervals for μ are

- Two-sided ($t_{326}(.95) = 1.649541$)

$$10.33643 \pm 1.649541 \cdot 0.8190 = (9.0, 11.7)$$

- One-sided (positive) ($t_{326}(.90) = 1.285149$)

$$(10.33643 - 1.285149 \cdot 0.8190, \infty) = (9.3, \infty)$$

- One-sided (negative) ($t_{326}(.90) = 1.285149$)

$$(-\infty, 10.33643 + 1.285149 \cdot 0.8190) = (-\infty, 11.4)$$

Find critical values using SAS or R

If $\alpha = 0.05$, then $1 - \alpha/2 = 0.975$.

In SAS,

```
PROC IML;  
  q = QUANTILE('T', 0.975, 7);  
  PRINT q;  
  QUIT;
```

In R,

```
q = qt(0.975,7)
```

Both obtain $q=2.364$.

Summary

Two main approaches to statistical inference:

- Statistical hypothesis (hypothesis test)
- Statistical question (confidence interval)