STAT 401A - Statistical Methods for Research Workers

Simple linear regression

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Simple Linear Regression

Recall the one-way ANOVA model:

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

where Y_{ij} is the observation for individual i in group j.

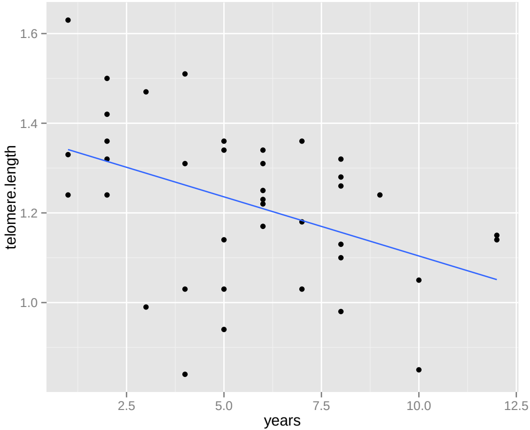
The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

Terminology (all of these are equivalent):

response	explanatory
outcome	covariate
dependent	independent
endogenous	exogenous


1.6
1.4
1.2
1.0
telomere.length
2.5
5.0
7.5
10.0
12.5
years

Telomere length

http://www.pnas.org/content/101/49/17312

> *People who are stressed over long periods tend to look haggard, and it is commonly thought that psychological stress leads to premature aging and the earlier onset of diseases of aging.*
>
> *...*
>
> *This design allowed us to examine the importance of perceived stress and measures of objective stress (caregiving status and chronicity of caregiving stress based on the number of years since a child's diagnosis).*
>
> *...*
>
> *Telomere length values were measured from DNA by a quantitative PCR assay that determines the relative ratio of telomere repeat copy number to single-copy gene copy number (T/S ratio) in experimental samples as compared with a reference DNA sample.*

Parameter interpretation

$$E[Y_i|X_i = x] = \beta_0 + \beta_1 x \qquad V[Y_i|X_i = x] = \sigma^2$$

- If $X_i = 0$, then $E[Y_i|X_i = 0] = \beta_0$.
  β_0 is the expected response when the explanatory variable is zero.

- If X_i increases from x to $x + 1$, then

$$\begin{array}{rl} E[Y_i|X_i = x+1] & = \beta_0 + \beta_1 x + \beta_1 \\ -\ E[Y_i|X_i = x] & = \beta_0 + \beta_1 x \\ \hline & = \beta_1 \end{array}$$

  β_1 is the expected increase in the response for each unit increase in the explanatory variable.

- σ is the standard deviation of the response for a fixed value of the explanatory variable.

Remove the mean:

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \stackrel{iid}{\sim} N(0,\sigma^2)$$

So the error is

$$e_i = Y_i - (\beta_0 + \beta_1 X_i)$$

which we approximate by the residual

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

The least squares, maximum likelihood, and Bayesian estimators are

$$\begin{array}{ll} \hat{\beta}_1 & = SXY/SXX \\ \hat{\beta}_0 & = \overline{Y} - \hat{\beta}_1 \overline{X} \\ \hat{\sigma}^2 & = SSE/(n-2) \qquad \text{df} = n-2 \end{array}$$

$$\begin{array}{ll} \overline{X} & = \frac{1}{n}\sum_{i=1}^n X_i \\ \overline{Y} & = \frac{1}{n}\sum_{i=1}^n Y_i \end{array}$$

$$\begin{array}{ll} SXY & = \sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) \\ SXX & = \sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^n (X_i - \overline{X})^2 \\ SSE & = \sum_{i=1}^n r_i^2 \end{array}$$

How certain are we about $\hat{\beta}_0$ and $\hat{\beta}_1$ being equal to β_0 and β_1?

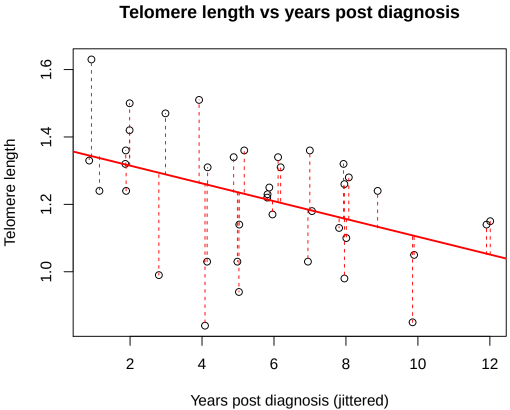
We quantify this uncertainty using their standard errors:

$$
\begin{aligned}
SE(\beta_0) &= \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} && df = n-2 \\
SE(\beta_1) &= \hat{\sigma}\sqrt{\frac{1}{(n-1)s_X^2}} && df = n-2
\end{aligned}
$$

$$
\begin{aligned}
s_X^2 &= SXX/(n-1) \\
s_Y^2 &= SYY/(n-1) \\
SYY &= \textstyle\sum_{i=1}^n (Y_i - \overline{Y})^2
\end{aligned}
$$

$$
\begin{aligned}
r_{XY} &= \frac{SXY/(n-1)}{s_X s_Y} && && \text{correlation coefficient} \\
R^2 &= r_{XY}^2 && = \frac{SST-SSE}{SST} && \text{coefficient of determination} \\
SST &= SYY = \textstyle\sum_{i=1}^n (Y_i - \overline{Y})^2
\end{aligned}
$$

The coefficient of determination (R^2) is the proportion of the total response variation explained by the explanatory variable(s).


Telomere length vs years post diagnosis
Telomere length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
Years post diagnosis (jittered)

Pvalues and confidence interval

We can compute two-sided pvalues via

$$2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_0}{SE(\beta_0)}\right|\right) \qquad \text{and} \qquad 2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_1}{SE(\beta_1)}\right|\right)$$

These test the null hypothesis that the corresponding parameter is zero.

We can construct $100(1-\alpha)\%$ two-sided confidence intervals via

$$\hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\beta_0) \qquad \text{and} \qquad \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\beta_1)$$

These provide ranges of the parameters consistent with the data.

Calculations by hand

```
   n Xbar Ybar   s_X    s_Y    r_XY
1 39 5.59 1.22 2.935 0.1798 -0.4307
```

$$
\begin{aligned}
SXX &= (n-1)s_X^2 = (39-1) \times 2.935427^2 = 327.4358 \\
SYY &= (n-1)s_Y^2 = (39-1) \times 0.1797731^2 = 1.228098 \\
SXY &= (n-1)s_X s_Y r_{XY} = (39-1) \times 2.935427 \times 0.1797731 \times -0.4306534 = -8.635897 \\
\hat\beta_1 &= SXY/SXX = -8.635897/327.4358 = -0.02637432 \\
\hat\beta_0 &= \overline{Y} - \hat\beta_1 \overline{X} = 1.220256 - (-0.02637432) \times 5.589744 = 1.367682 \\
R^2 &= r_{XY}^2 = (-0.4306534)^2 = 0.1854624 \\
SSE &= SYY(1-R^2) = 1.228098(1-0.1854624) = 1.000332 \\
\hat\sigma^2 &= SSE/(n-2) = 1.000332/(39-2) = 0.027036 \\
\hat\sigma &= \sqrt{\hat\sigma^2} = \sqrt{0.027036} = 0.1644263 \\
SE(\hat\beta_0) &= \hat\sigma\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{39} + \frac{5.589744^2}{327.4358}} = 0.05721115 \\
SE(\hat\beta_1) &= \hat\sigma\sqrt{\frac{1}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{327.4358}} = 0.009086742 \\
p_{H_0:\beta_0=0} &= 2P\left(t_{n-2} < -\left|\frac{1.367682}{0.05721115}\right|\right) = 2P(t_{37} < -23.90586) < 0.0001 \\
p_{H_0:\beta_1=0} &= 2P\left(t_{n-2} < -\left|\frac{-0.02637432}{0.009086742}\right|\right) = 2P(t_{37} < -2.902506) < 0.0062 \\
CI_{95\%}\ \beta_0 &= \hat\beta_0 \pm t_{n-2}(1-\alpha/2)SE(\hat\beta_0) = 1.367682 \pm 2.026192 \times 0.05721115 = (1.251761, 1.483603) \\
CI_{95\%}\ \beta_1 &= \hat\beta_1 \pm t_{n-2}(1-\alpha/2)SE(\hat\beta_1) = -0.02637432 \pm 2.026192 \times 0.009086742 = (-0.044785804 - 0.007962836)
\end{aligned}
$$

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC CORR DATA=t;
  VAR length;
  WITH years;
  RUN;
```

```
                                        The CORR Procedure

                                    1 With Variables:    years
                                    1      Variables:    length


                                        Simple Statistics

   Variable           N          Mean       Std Dev           Sum       Minimum       Maximum

   years             39       5.58974       2.93543     218.00000       1.00000      12.00000
   length            39       1.22026       0.17977      47.59000       0.84000       1.63000


                         Pearson Correlation Coefficients, N = 39
                                Prob > |r| under H0: Rho=0

                                                length

                                        years      -0.43065
                                                     0.0062
```

```
PROC GLM DATA=t;
  MODEL length = years / CLPARM;
  RUN;

                                          The GLM Procedure
                                Number of Observations Read          39
                                Number of Observations Used          39

Dependent Variable: length

                                                 Sum of
       Source                      DF           Squares      Mean Square    F Value     Pr > F
       Model                        1        0.22776588       0.22776588       8.42     0.0062
       Error                       37        1.00033156       0.02703599
       Corrected Total             38        1.22809744

                      R-Square     Coeff Var      Root MSE    length Mean
                      0.185462      13.47473      0.164426       1.220256

       Source                      DF         Type I SS      Mean Square    F Value     Pr > F
       years                        1        0.22776588       0.22776588       8.42     0.0062

       Source                      DF       Type III SS      Mean Square    F Value     Pr > F
       years                        1        0.22776588       0.22776588       8.42     0.0062

                                       Standard
 Parameter          Estimate              Error    t Value    Pr > |t|      95% Confidence Limits
 Intercept       1.367682067         0.05721112      23.91      <.0001     1.251761335   1.483602799
 years          -0.026374315         0.00908674      -2.90      0.0062    -0.044785794  -0.007962836
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
with(Telomeres, cor(telomere.length,years))
```

```
[1] -0.4307
```

```
anova(m)
```

```
Analysis of Variance Table

Response: telomere.length
          Df Sum Sq Mean Sq F value Pr(>F)
years      1  0.228   0.228    8.42 0.0062 **
Residuals 37  1.000   0.027
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
summary(m)



Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
    Min      1Q  Median      3Q     Max
-0.4222 -0.0854  0.0206  0.1074  0.2887

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.36768    0.05721    23.9   <2e-16 ***
years       -0.02637    0.00909    -2.9   0.0062 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.164 on 37 degrees of freedom
Multiple R-squared:  0.185, Adjusted R-squared:  0.163
F-statistic: 8.42 on 1 and 37 DF,  p-value: 0.0062


confint(m)


               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

Conclusion

Telomere length at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% confidence interval of (1.25, 1.48). For each year since diagnosis, the telomere length decreases by 0.026 with a 95% confidence interval of (0.008, 0.045) on average. The proportional of variability in telomere length described by years since diagnosis is 18.5%.

`http://www.pnas.org/content/101/49/17312`

> *The zero-order correlation between chronicity of caregiving [years] and mean telomere length, r,is* -0.445 *(P <0.01). [*$R^2 = 0.198$ *was shown in the plot.]*

Remark I'm guessing our analysis and that reported in the paper don't match exactly due to a discrepancy in the data.

Summary

- The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

  where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

- Know how to use SAS/R to obtain $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$, R^2, pvalues, CIs, etc.
- Interpret SAS output
  - At a value of zero for the explanatory variable ($X_i = 0$), β_0 is the expected value for the response (Y_i).
  - For each unit increase in the explanatory variable value, β_1 is the expected increase in the response.
  - At a constant value of the explanatory variable, σ^2 is the variance of the responses.
  - The coefficient of determination (R^2) is the percentage of the total response variation explained by the explanatory variable(s).

What is $E[Y|X=x]$?

We know $\beta_0 = E[Y|X=0]$, but what about $X = x$?

$$E[Y|X=x] = \beta_0 + \beta_1 x$$

which we can estimate via

$$E[\widehat{Y|X=}\,x] = \hat{\beta}_0 + \hat{\beta}_1 x$$

but there is uncertainty in both β_0 and β_1. So the standard error of $E[Y|X=x]$ is

$$SE(E[Y|X=x]) = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(\overline{X}-x)^2}{(n-1)s_X^2}}$$

and a $100(1-\alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1-\alpha/2)SE(E[Y|X=x])$$

What do we predict about Y at $X = x$?

On the last slide, we calculated $E[Y|X = x]$ and it's uncertainty, but if we are trying to predict a new observation, we need to account for the sampling variablity σ^2. Thus a prediction about Y at a new $X = x$ is still

$$Pred\{Y|X = x\} = \hat{\beta}_0 + \hat{\beta}_1 x$$

but the uncertainty includes the variability due to σ^2. So the standard error of $Pred\{Y|X = x\}$ is

$$SE(Pred\{Y|X = x\}) = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(\overline{X} - x)^2}{(n-1)s_X^2}}$$

and a $100(1 - \alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1 - \alpha/2) SE(Pred\{Y|X = x\}).$$

```
DATA tnew;
  INPUT years;
  DATALINES;
  4
  ;

DATA combined;
  SET t tnew;
  RUN;

PROC PRINT DATA=combined;
  WHERE years=4;
  RUN;
```

Obs	years	length
10	4	1.51
11	4	1.31
15	4	1.03
16	4	0.84
40	4	.

```
PROC GLM DATA=combined;
 MODEL length = years;
 OUTPUT OUT=combinedreg PREDICTED=predicted  LCLM=lclm UCLM=uclm LCL=lcl UCL=ucl;
 RUN;

PROC PRINT DATA=combinedreg;
  WHERE length=.;             /* . is missing data in SAS */
  RUN;
```

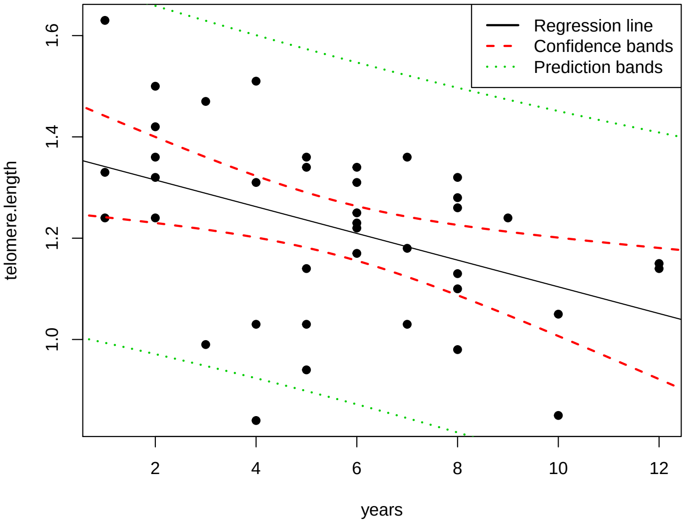
Obs	years	length	predicted	lclm	uclm	lcl	ucl
40	4	.	1.26218	1.20133	1.32303	0.92351	1.60086

```
m = lm(telomere.length~years, Telomeres)
new = data.frame(years=4)
predict(m, new, interval="confidence")
```

```
    fit   lwr   upr
1 1.262 1.201 1.323
```

```
predict(m, new, interval="prediction")
```

```
    fit    lwr   upr
1 1.262 0.9235 1.601
```


Regression line
Confidence bands
Prediction bands
telomere.length
1.0
1.2
1.4
1.6
years
2
4
6
8
10
12

Shifting the intercept

The intercept (β_0) is the expected response when the explanatory variable is zero.

So, if we change our explanatory variable, we change the interpretation of our intercept, e.g. if, instead of using number of years since diagnosis, we use "number of years since diagnosis minus 4", then our intercept is the expected response at 4 years since diagnosis.

Let x be number of years since diagnosis, then

$$E[Y|X=x] = \tilde{\beta}_0 + \tilde{\beta}_1(x-4) = (\beta_0 - 4\beta_1) + \beta_1 x$$

so our new parameters for the mean are

- intercept $\tilde{\beta}_0 = (\beta_0 - 4\beta_1)$and
- slope $\tilde{\beta}_1 = \beta_1$ (unchanged).

```
DATA t;
  INFILE "telomeres.csv" DSD FIRSTOBS=2;
  INPUT years length;
  years4 = years-4;

PROC GLM DATA=combined;
 MODEL length = years;
 RUN;

PROC GLM DATA=combined;
 MODEL length = years4;
 RUN;
```

```
                                   Standard
 Parameter          Estimate          Error    t Value    Pr > |t|      95% Confidence Limits

 Intercept       1.367682067     0.05721112      23.91      <.0001     1.251761335  1.483602799
 years          -0.026374315     0.00908674      -2.90      0.0062    -0.044785794 -0.007962836

 Intercept       1.262184808     0.03003174      42.03      <.0001     1.201334726  1.323034890
 years4         -0.026374315     0.00908674      -2.90      0.0062    -0.044785794 -0.007962836
```

```
m0 = lm(telomere.length ~   years   , Telomeres)
m4 = lm(telomere.length ~ I(years-4), Telomeres)

coef(m0)


(Intercept)       years
    1.36768    -0.02637


coef(m4)


 (Intercept) I(years - 4)
     1.26218     -0.02637


confint(m0)


               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963


confint(m4)


                2.5 %    97.5 %
(Intercept)   1.20133  1.323035
I(years - 4) -0.04479 -0.007963
```