Set 11 - Normal model

STAT 401 (Engineering) - Iowa State University

February 17, 2017

Outline

- Normal model with known variance
- Normal model with known mean
- Normal model

Corn yield

For the following examples, we will consider measuring corn yield on fields. We will base our analyses of the following values:

- Mean yield per field is 200 bushels per acre
- Standard deviation of yield per field is 20 bushels per acre

In the following analyses, we will be assuming

- Mean is unknown while SD is known to be 20
- Mean is known to be 200 while SD is unknown
- Both are mean and standard deviation are unknown

Normal model with known variance

Suppose $Y_i \overset{ind}{\sim} N(\mu, s^2)$ and we assume the default prior $p(\mu) \propto 1$.

This "prior" is actually not a distribution at all, since its integral is not finite. Nonetheless, we can still use it to derive a posterior.

If you work through the math (lots of algebra and a little calculus), you will find

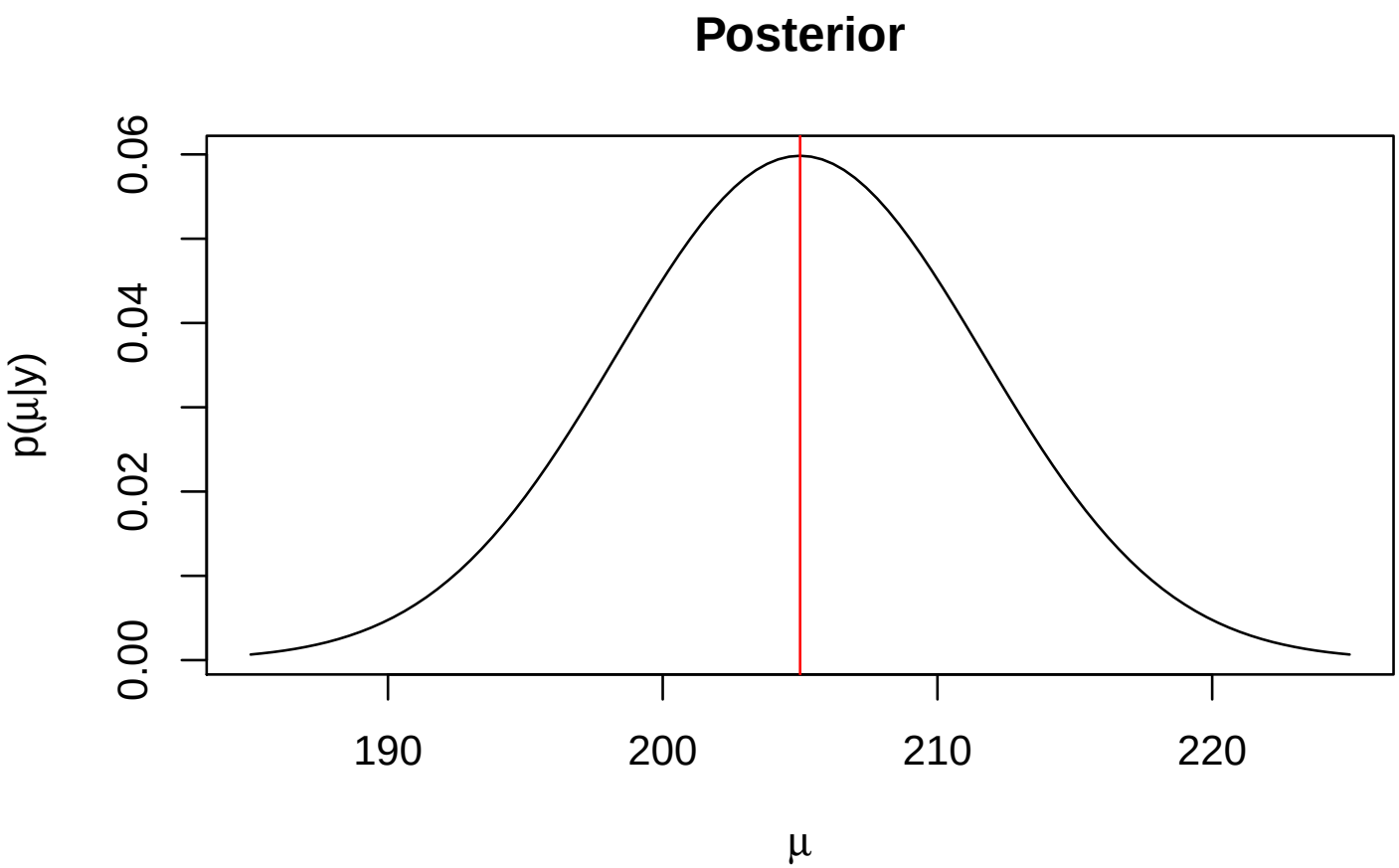
$$\mu|y \sim N(\overline{y}, s^2/n).$$

This looks exactly like the likelihood, but now it is normalized, i.e. it integrates to 1 and therefore it is a valid probability density function.

The Bayes estimator is

$$E[\mu|y] = \overline{y}.$$

```
m <- 200
s <- 20
n <- 9
y <- rnorm(n, mean = m, sd = s)
curve(dnorm(x, mean = mean(y), sd = s/sqrt(n)), mean(y)-3*s/sqrt(n), mean(y)+3*s/sqrt(n),
      xlab = expression(mu),
      ylab = expression(paste("p(",mu,"|y)")),
      main = "Posterior")
abline(v=mean(y), col='red')
```



Credible intervals

We can obtain credible intevals directly.

```
a <- .05
qnorm(c(a/2,1-a/2), mean(y), sd = sqrt(1/n))
```

```
[1] 204.3474 205.6540
```

Or we can use the fact that

$$\frac{\mu - \overline{y}}{s/\sqrt{n}} = Z \sim N(0,1)$$

to construct the interval using

$$\overline{y} \pm z_{a/2} s/\sqrt{n}$$

where $a/2 = \int_{z_{a/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, i.e. the area to the right of $z_{a/2}$ under the pdf of a standard normal is $a/2$.

```
mean(y) + c(-1,1)*qnorm(.975)*sqrt(1/n)
```

```
[1] 204.3474 205.6540
```

Normal model with known mean

Suppose $Y_i \overset{ind}{\sim} N(m, \sigma^2)$ and we assume the default prior $p(\sigma^2) \propto \frac{1}{\sigma^2}\mathrm{I}(\sigma^2 > 0)$.

Again, this "prior" is actually not a distribution at all, since its integral is not finite. Nonetheless, we can still use it to derive a posterior.

If you work through the math (lots of algebra and a little calculus), you will find

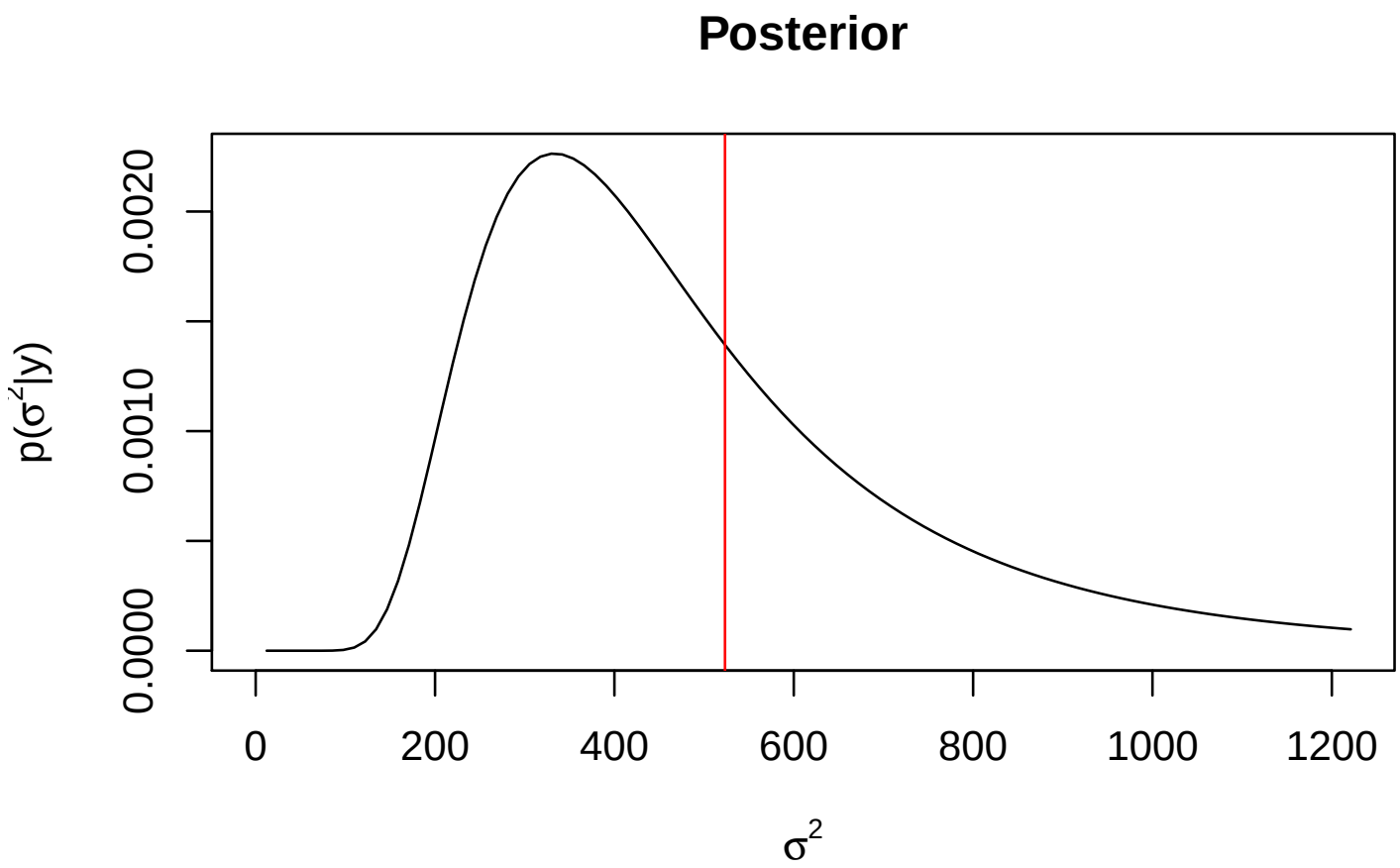
$$\sigma^2|y \sim IG\left(\frac{n}{2}, \frac{\sum_{i=1}^n (y_i - m)^2}{2}\right)$$

where IG indicates an inverse gamma distribution.

The Bayes estimator is

$$E[\sigma^2|y] = \frac{\frac{\sum_{i=1}^n (y_i - m)^2}{2}}{\frac{n}{2} - 1} = \frac{\sum_{i=1}^n (y_i - m)^2}{n-2} \text{ for } n > 2$$

```
S <- sum((y-m)^2)
curve(MCMCpack::dinvgamma(x, shape = n/2, scale = S/2), 0, 3*S/n,
      xlab = expression(sigma^2),
      ylab = expression(paste("p(",sigma^2,"|y)")),
      main = "Posterior")
abline(v = (S/2)/((n/2)-1), col='red')
```



Credible intervals for variance

We don't have a quantile function for this inverse gamma distribution. So we'll obtain estimates of the interval endpoints by taking a bunch of simulated draws from the inverse gamma distribution and finding their sample quantiles.

```
draws <- MCMCpack::rinvgamma(1e5, shape = n/2, scale = S/2)
quantile(draws, c(a/2, 1-a/2))
```

```
     2.5%     97.5%
 192.6423 1353.9686
```

If you don't have the MCMCpack library, you can draw from the gamma distribution and then invert the draws. It is slightly confusing because the 'scale' parameter for the inverse gamma is the 'rate' parameter for the gamma.

```
draws <- rgamma(1e5, shape = n/2, rate = S/2)
quantile( 1/draws, c(a/2, 1-a/2))
```

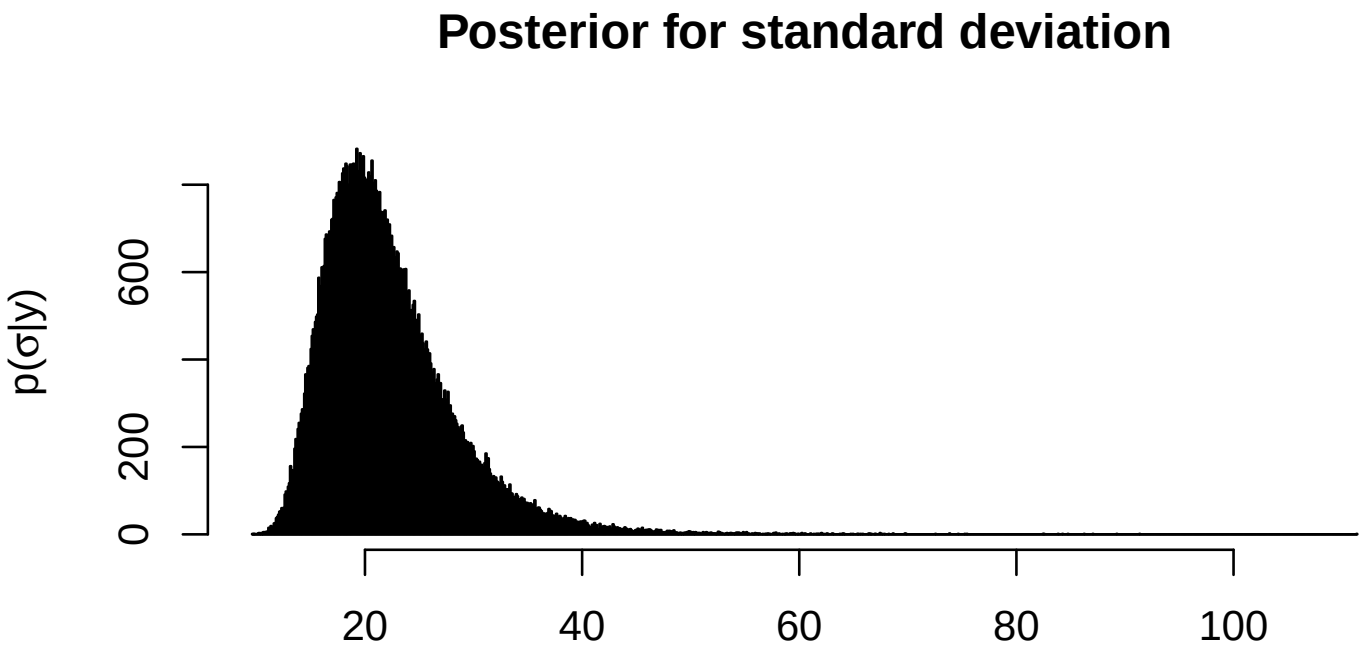
```
     2.5%     97.5%
 193.2657 1364.9407
```

Posterior and credible intervals for standard deviation

```
draws <- sqrt( 1/rgamma(1e5, shape = n/2, rate = S/2) )
quantile( draws, c(a/2, 1-a/2))
```

```
    2.5%    97.5%
13.89069 36.85157
```

```
hist(draws, 1001,
     xlab = expression(sigma),
     ylab = expression(paste("p(",sigma,"|y)")),
     main = "Posterior for standard deviation")
```



Normal model

Suppose $Y_i \overset{ind}{\sim} N(\mu, \sigma^2)$ and we assume the default prior $p(\mu, \sigma^2) \propto \frac{1}{\sigma^2} \mathrm{I}(\sigma^2 > 0)$.

Again, this "prior" is actually not a distribution at all, since its integral is not finite. Nonetheless, we can still use it to derive a posterior.

If you work through the math (lots of algebra and a little calculus), you will find

$$\begin{aligned} \mu|\sigma^2, y &\sim N(\overline{y}, \sigma^2/n) \\ \sigma^2|y &\sim IG\left(\tfrac{n-1}{2}, \tfrac{\sum_{i=1}^n (y_i - \overline{y})^2}{2}\right) \end{aligned}$$

The joint posterior is obtained using

$$p(\mu, \sigma^2|y) = p(\mu|\sigma^2, y) p(\sigma^2|y).$$

The Bayes estimator is

$$\begin{aligned} E[\mu|y] &= \overline{y} \\ E[\sigma^2|y] &= \frac{\frac{\sum_{i=1}^n (y_i - \overline{y})^2}{2}}{\frac{n-1}{2} - 1} = \frac{\sum_{i=1}^n (y_i - \overline{y})^2}{n-3} \text{ for } n > 3 \end{aligned}$$

Focusing on μ

Typically, the main quantity of interest in the normal model is the mean, μ. Thus, we are typically interested in the marginal posterior for μ:

$$p(\mu|y) = \int p(\mu|\sigma^2, y)p(\sigma^2|y)d\sigma^2.$$

If

$$\mu|\sigma^2, y \sim N(\overline{y}, \sigma^2/n) \quad \text{and} \quad \sigma^2|y \sim IG\left(\frac{n-1}{2}, \frac{\sum_{i=1}^n (y_i - \overline{y})^2}{2}\right),$$

then

$$\mu|y \sim t_{n-1}(\overline{y}, S^2/n) \quad \text{where} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \overline{y})^2$$

that is, $\mu|y$ has a t distribution with $n-1$ degrees of freedom, location parameter $\overline{y}$ and scale parameter S^2/n.

t distribution

Definition

A t distributed random variable, $T \sim t_v(m, s^2)$ has probability density function

$$f_T(t) = \frac{\Gamma([v+1]/2)}{\Gamma(v/2)\sqrt{v\pi}s}\left(1 + \frac{1}{v}\left[\frac{x-m}{s}\right]^2\right)^{-(v+1)/2}$$

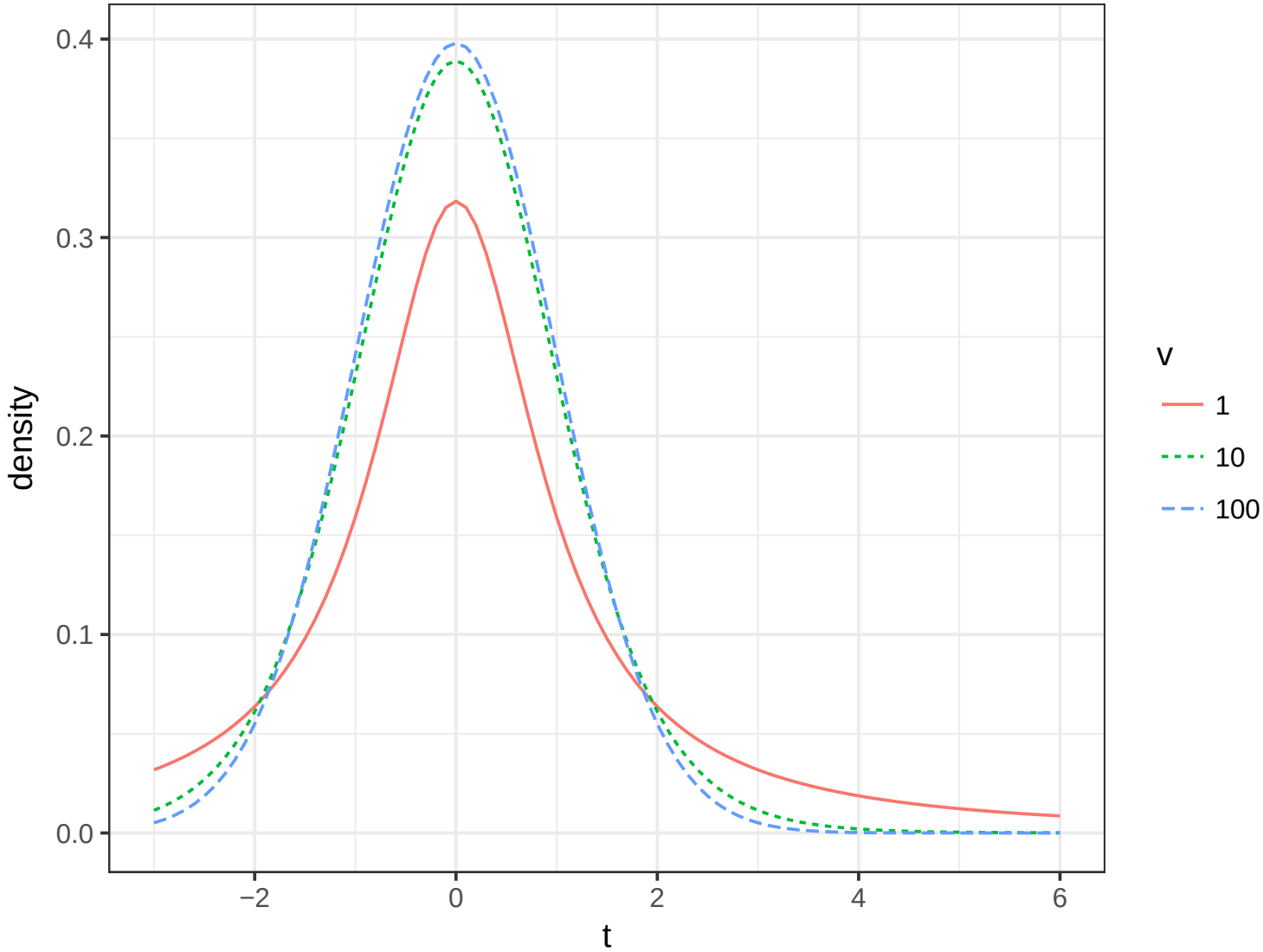
with degrees of freedom v, location m, and scale s^2. It has

$$\begin{aligned} E[T] &= m & v > 1 \\ Var[T] &= s^2 \tfrac{v}{v-2} & v > 2. \end{aligned}$$

In addition,

$$t_v(m, s^2) \xrightarrow{d} N(m, s^2) \quad \text{as} \quad v \to \infty.$$

t distribution as v changes



Credible intervals

In R, there is no way to obtain t credible intervals directly. Thus we can use the fact that

$$\frac{\mu - \overline{y}}{S/\sqrt{n}} = t \sim t_{n-1}(0,1)$$

to construct the interval using

$$\overline{y} \pm t_{n-1,a/2} S/\sqrt{n}$$

where the area to the right of $t_{n-1,a/2}$ under the pdf of a standard t is $a/2$.

```
mean(y) + c(-1,1)*qt(.975, df=n-1)*sd(y)/sqrt(n)
```

```
[1] 189.0652 220.9362
```

Corn yield

In evaluating corn yield for a particular year, the yield on a number of fields is measured. (For simplicity, assume that fields are standardized in size.) We measure 9 randomly selected fields in Iowa and find the average is 205 bushels per acre and the sample standard deviation is 21 bushels per acre. Provide a 90% credible interval for the mean yield across all fields in Iowa.

Let Y_i be the yield in field i and assume

$$Y_i \overset{ind}{\sim} N(\mu, \sigma^2).$$

If we assume the default prior $p(\mu, \sigma^2) \propto 1/\sigma^2$, then we have

$$\mu | y \sim t_{n-1}(\overline{y}, S^2/n).$$

A 90% interval is

```
a    <- 0.1

mean(y) +c(-1,1)*qt(1-a/2, df=n-1)*sd(y)/sqrt(n)
```

```
[1] 192.1504 217.8510
```

Informative Bayesian analysis when variance is known

Let Y_i be the corn yield (in bushels/ac) from field i. Assume

$$Y_i \overset{ind}{\sim} N(\mu, s^2) \quad \text{and} \quad \mu \sim N(m, C).$$

Then

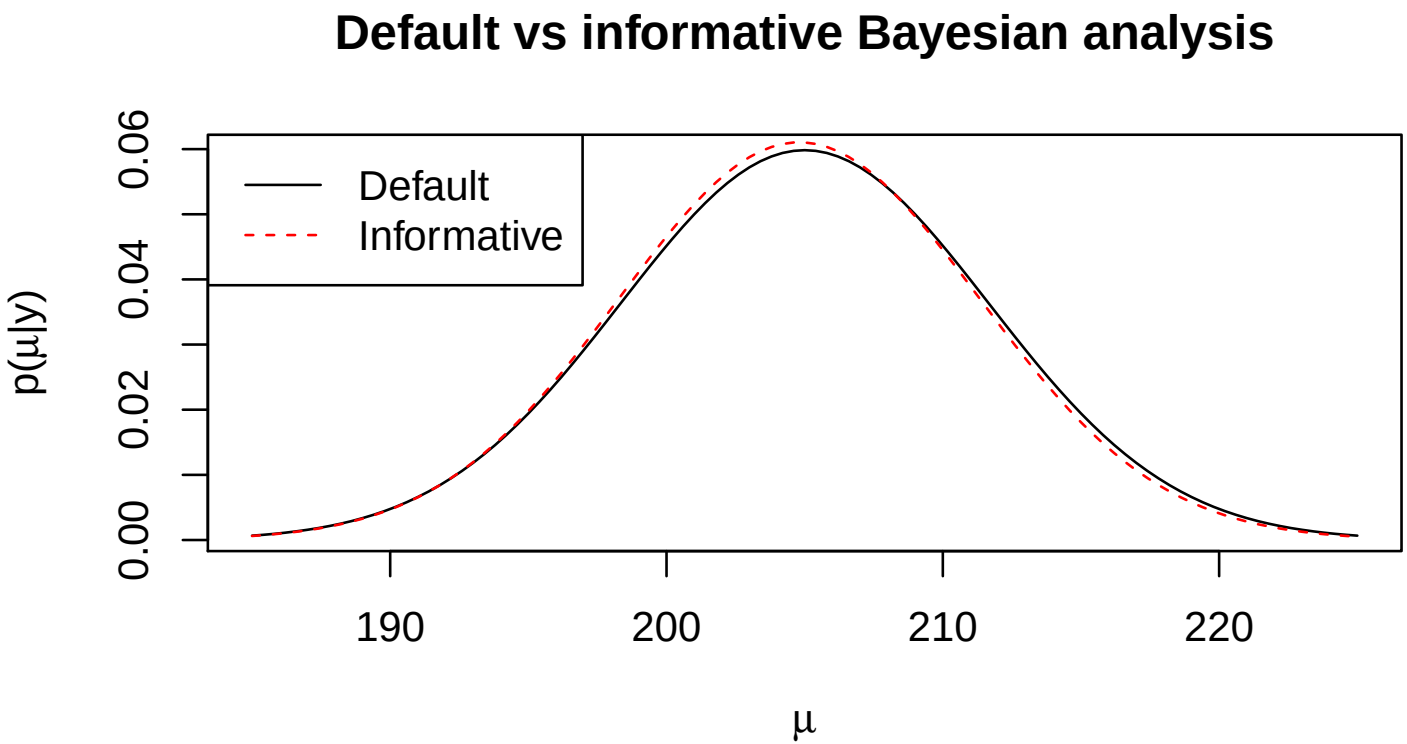
$$\begin{aligned} \mu|y &\sim N(m', C') \\ C' &= \left[\frac{1}{C} + \frac{n}{s^2}\right]^{-1} \\ m' &= C'\left[\frac{1}{C}m + \frac{n}{s^2}\overline{y}\right] = \frac{1/C}{1/C+n/s^2}m + \frac{n/s^2}{1/C+n/s^2}\overline{y} \end{aligned}$$

```
m = 200
C = 33^2
Cp = 1/(1/C+n/s^2)
mp = Cp*(m/C+n*mean(y)/s^2)
```

So if we assume $m = 200$ and $C = 33^2$ and combine this with our observed data n =9 and $\overline{y}$ =205, then we have the posterior $\mu|y \sim N(205, 7^2)$.

Comparison of default vs informative Bayesian analysis

```
ybar = mean(y)
se = s/sqrt(n)
curve(dnorm(x, mean=ybar, sd=se), ybar-3*se, ybar+3*se,
      xlab=expression(mu),
      ylab=expression(paste("p(",mu,"|y)")),
      main="Default vs informative Bayesian analysis")
curve(dnorm(x, mean=mp, sd=sqrt(Cp)), col='red', lty=2, add=TRUE)
legend("topleft", c("Default","Informative"), col=c("black","red"),
       lty = 1:2)
```



Informative Bayesian analysis

The joint conjugate prior for μ and σ^2 is

$$\mu|\sigma^2 \sim N(m, \sigma^2/k) \qquad \sigma^2 \sim \text{Inv-}\chi^2(v, s^2)$$

where s^2 serves as a prior guess about σ^2 and v controls how certain we are about that guess.

The posterior under this prior is

$$\mu|\sigma^2, y \sim N(m', \sigma^2/k') \qquad \sigma^2|y \sim \text{Inv-}\chi^2(v', (s')^2)$$

where

$$\begin{aligned} k' &= k + n \\ m' &= [km + n\overline{y}]/k' \\ v' &= v + n \\ v'(s')^2 &= vs^2 + (n-1)S^2 + \tfrac{kn}{k'}(\overline{y} - m)^2 \end{aligned}$$