P4 - Central Limit Theorem

STAT 587 (Engineering) - Iowa State University

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Central Limit Theorem (CLT)

Main Idea: Sums and averages of random variables from any distribution have approximate normal distributions for sufficiently large sample sizes.

Theorem (Central Limit Theorem)

Suppose $X_1, X_2, \ldots$ are iid random variables with

$$E[X_i] = \mu \quad Var[X_i] = \sigma^2.$$

Define

$$\begin{aligned} \textit{Sample Sum: } S_n &= X_1 + X_2 + \cdots + X_n \\ \textit{Sample Average: } \overline{X}_n &= S_n/n. \end{aligned}$$

Then

$$\lim_{n\to\infty} \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad \textit{and} \quad \lim_{n\to\infty} \frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1).$$

Approximating distributions

Rather than considering the limit, I typically think of the following approximations as n gets large.

For the sample average,

$$\overline{X}_n \dot{\sim} N(\mu, \sigma^2/n).$$

where $\dot{\sim}$ indicates *approximately distributed*. Note that

$$E\left[\overline{X}_n\right] = \mu \qquad \text{and} \qquad Var\left[\overline{X}_n\right] = \sigma^2/n.$$

For the sample sum,

$$S_n \dot{\sim} N(n\mu, n\sigma^2).$$

Note that

$$\begin{aligned} E[S_n] &= E\left[n\overline{X}_n\right] = nE\left[\overline{X}_n\right] = n\mu \\ Var[S_n] &= Var\left[n\overline{X}_n\right] = n^2 Var\left[\overline{X}_n\right] = n^2\frac{\sigma^2}{n} = n\sigma^2. \end{aligned}$$

Averages and sums of uniforms

Let $X_i \stackrel{ind}{\sim} Unif(0,1)$. Then

$$\mu = E[X_i] = \frac{1}{2} \quad \text{and} \quad \sigma^2 = Var[X_i] = \frac{1}{12}.$$

Thus

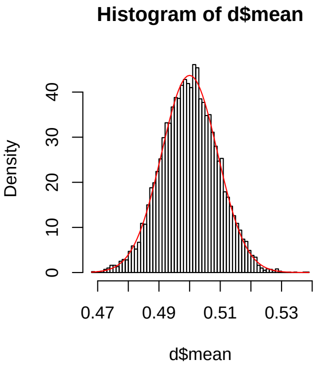
$$\overline{X}_n \dot{\sim} N\left(\frac{1}{2}, \frac{1}{12n}\right)$$

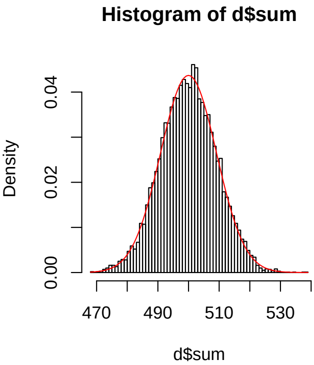
and

$$S_n \dot{\sim} N\left(\frac{n}{2}, \frac{n}{12}\right).$$

```
n_sims <- 10000
n_obs  <- 1000
d <- data.frame(rep = rep(1:n_sims, each = n_obs),
                x   = runif(n_sims * n_obs)) %>%
  group_by(rep) %>%
  summarize(mean = mean(x),
            sum  = sum(x))

opar = par(mfrow=c(1,2))
hist(d$mean, 50, probability = TRUE)
curve(dnorm(x, mean = 1/2, sqrt(1/12/n_obs)), add = TRUE, col = "red")
hist(d$sum,  50, probability = TRUE)
curve(dnorm(x, mean = n_obs/2, sqrt(n_obs/12)), add = TRUE, col = "red")
```





Normal approximation to a binomial

Recall that a binomial distribution can be considered as a sum of iid Bernouli random variables, i.e. if $Y_n = \sum_{i=1}^{n} X_i$ where $X_i \overset{ind}{\sim} Ber(p)$, then

$$Y_n \sim Bin(n, p).$$

For a binomial random variable, we have

$$E[Y_n] = np \qquad \text{and} \qquad Var[Y_n] = np(1-p).$$

By the CLT,

$$\lim_{n\to\infty} \frac{Y_n - np}{\sqrt{np(1-p)}} \to N(0,1),$$

if n is large,

$$Y_n \dot{\sim} N(np, np[1-p]).$$

Roulette example

A European roulette wheel has 39 slots: one green, 19 black, and 19 red. If I play black everytime, what is the probability that I will have won more than I lost after 99 spins of the wheel?

Let Y indicate the total number of wins and assume $Y \sim Bin(n, p)$ with $n = 99$ and $p = 19/39$. The desired probability is $P(Y \geq 50)$. Then

$$P(Y \geq 50) = 1 - P(Y < 50) = 1 - P(Y \leq 49)$$

```
n = 99
p = 19/39
1-pbinom(49, n, p)
```

```
[1] 0.399048
```

We can approximate Y using $X \sim N(np, np(1-p))$.

$$P(Y \geq 50) \approx 1 - P(X < 50)$$

```
1-pnorm(50, n*p, sqrt(n*p*(1-p)))
```

```
[1] 0.3610155
```

Astronomy example

An astronomer wants to measure the distance, d, from the observatory to a star. The astronomer takes 30 measurements that she believes are unbiased and finds the average of these measurements to be 29.4 parsecs and you know the variance of each measurement to be 4 parsecs^2. What is the probability the average is within 0.5 parsecs?

Let X_i be the i^{th} measurement. The astronomer assumes that $X_1, X_2, \ldots X_n$ are iid with $E[X_i] = d$ (unbiased) and $Var[X_i] = \sigma^2 = 4$. The estimate of d is

$$\overline{X}_n = \frac{(X_1 + X_2 + \cdots + X_n)}{n} = 29.4.$$

and, by the Central Limit Theorem, we believe $\overline{X}_n \sim N(d, \sigma^2/n)$ where $n = 30$. We want to find

$$\begin{aligned} P\left(|\overline{X}_n - d| < 0.5\right) &= P\left(-0.5 < \overline{X}_n - d < 0.5\right) \\ &= P\left(\tfrac{-0.5}{\sigma/\sqrt{n}} < \tfrac{\overline{X}_n - d}{\sigma/\sqrt{n}} < \tfrac{0.5}{\sigma/\sqrt{n}}\right) \\ &= P\left(\tfrac{-0.5}{\sigma/\sqrt{n}} < Z < \tfrac{0.5}{\sigma/\sqrt{n}}\right) \\ &\approx P(-1.37 < Z < 1.37) \\ &= P(Z < 1.37) - P(Z < -1.37) \\ &\approx 0.915 - 0.085 = 0.830 \end{aligned}$$

Astronomy example (cont.)

Suppose the astronomer wants to be within 0.5 parsecs with at least 95% probability. How many more samples would she need to take?

We solve

$$\begin{aligned} 0.95 \le P\left(\left|\overline{X}_n - d\right| < .5\right) &= P\left(-0.5 < \overline{X}_n - d < 0.5\right) \\ &= P\left(\frac{-0.5}{\sigma/\sqrt{n}} < \frac{\overline{X}_n - d}{\sigma/\sqrt{n}} < \frac{0.5}{\sigma/\sqrt{n}}\right) \\ &= P(-z < Z < z) \\ &= 1 - [P(Z < -z) + P(Z > z)) \\ &= 1 - 2P(Z < -z) \end{aligned}$$

where $z = 0.5/(\sigma/\sqrt{n}) = 1.96$ since

```
-qnorm(.025)
```

```
[1] 1.959964
```

and thus $n = 61.47$ which we round up to $n = 62$ to ensure the probability is *at least* 0.95.