

# M5S3 - Interpretation of Confidence Intervals

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# Outline

- Interpretation of probability
  - Frequentist
  - Bayesian

# Frequentist interpretation of probability

## Interpretation

The *frequentist interpretation of probability* is that probability is the long-run relative frequency of an event.

Thus, if we have a sequence of independent and identically distributed binary random variables  $I_1, I_2, \dots$  where  $I_i$  is the indicator of the event occurring in the  $i$ th trial, i.e.

$$I_i = \begin{cases} 1 & \text{if the event occurs in the } i\text{th trial} \\ 0 & \text{if the event does not occur in the } i\text{th trial,} \end{cases}$$

the probability  $p$  is defined as

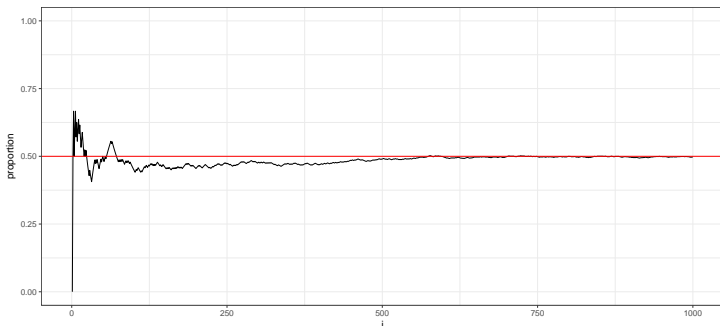
$$p = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n I_i}{n}.$$

# Coin flipping example

Let  $I_i$  be the indicator that the  $i$ th coin flip is heads, i.e.

$$I_i = \begin{cases} 1 & \text{if the } i\text{th coin flip is heads} \\ 0 & \text{if the } i\text{th coin flip is not heads,} \end{cases}$$

Now we define the probability as the proportion of heads as the number of flips tends to infinity.

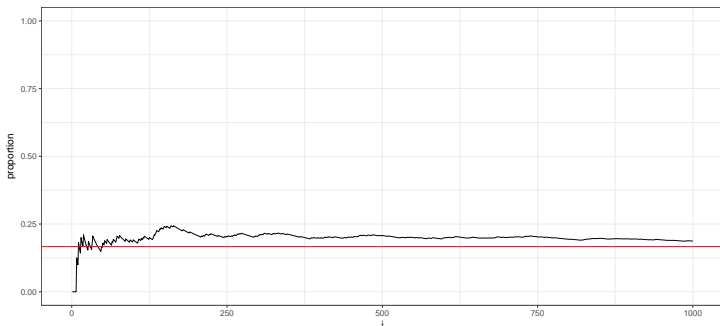


# Die rolling example

Let  $I_i$  be the indicator of the  $i$ th die roll being a 1, i.e.

$$I_i = \begin{cases} 1 & \text{if the } i\text{th die roll is 1} \\ 0 & \text{if the } i\text{th die roll is not 1,} \end{cases}$$

Now we define the probability as the proportion of 1s as the number of rolls tends to infinity.



# Construction of confidence intervals

Recall that the formula for a  $100(1 - \alpha)\%$  confidence interval (CI) based on the standard normal is

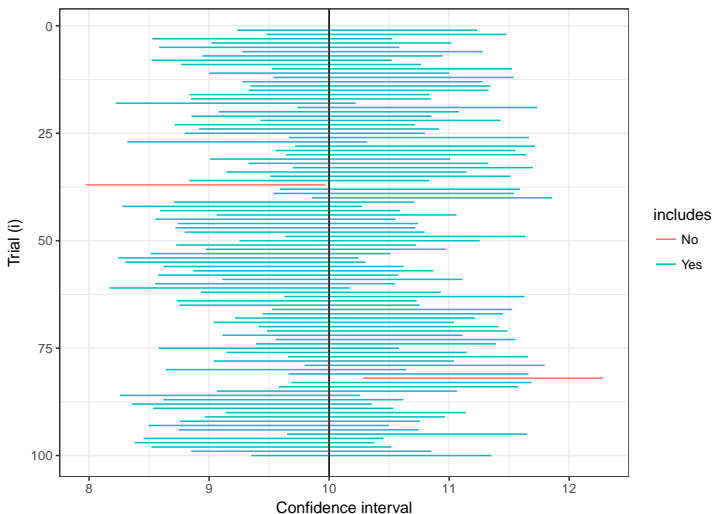
$$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

We obtained this interval by calculating the following probability

$$P\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

Thus a confidence interval has random endpoints since  $\bar{X}$  is random. We can imagine performing this procedure repeatedly and calculating the proportion of times the CI includes  $\mu$ .

Let  $X_i \stackrel{iid}{\sim} N(10, 1^2)$  with  $n = 4$ . Then a 95% CI based on the Empirical Rule is  $\bar{X} \pm 2 \cdot 1/\sqrt{4} = \bar{X} \pm 1$ .



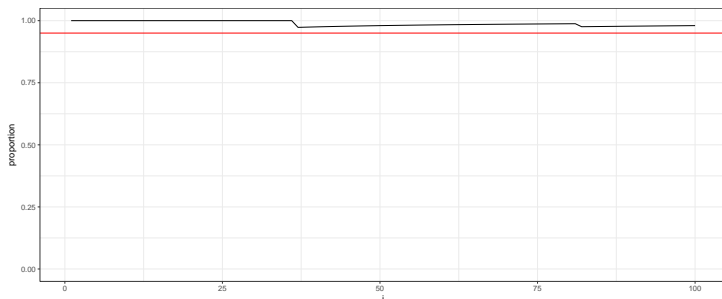
# Interpretation of Confidence Intervals

Let  $I_i$  be the indicator that the  $i$ th  $100(1 - \alpha)\%$  confidence interval (CI) for the population mean  $\mu$  contains  $\mu$ , i.e.

$$I_i = \begin{cases} 1 & \text{if the } i\text{th CI includes } \mu \\ 0 & \text{if the } i\text{th CI does not include } \mu, \end{cases}$$

Then

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n I_i}{n} = 1 - \alpha.$$





## Relation to binomial distribution

Recall that a random variable  $Y$  has a binomial distribution if

$$Y = \sum_{i=1}^n I_i$$

where  $I_i$  are independent and identically distributed (iid) Bernoulli random variables with a common probability of success  $p$ . Here

$$I_i = \begin{cases} 1 & \text{if CI } i \text{ includes } \mu \\ 0 & \text{if CI } i \text{ does not include } \mu \end{cases}$$

Since each CI has probability  $1 - \alpha$  of including  $\mu$ , we have  $I_i \stackrel{iid}{\sim} \text{Ber}(1 - \alpha)$  and  $Y \sim \text{Bin}(n, 1 - \alpha)$ .

## Expected number of CIs that cover $\mu$

If we construct  $n$  CIs each with probability  $1 - \alpha$  of including  $\mu$ , then how many CIs do we expect will include  $\mu$ . Since  $Y$  is the random number of CIs that **will include the truth** and  $Y \sim \text{Bin}(n, 1 - \alpha)$ , we have

$$E[Y] = n(1 - \alpha).$$

Calculate the expected number that **will include the truth** for the following scenarios:

- $n = 100, 1 - \alpha = 0.95$  then  $E[Y] = 100 \cdot 0.95 = 95$
- $n = 1000, 1 - \alpha = 0.7$  then  $E[Y] = 1000 \cdot 0.70 = 700$
- $n = 77, 1 - \alpha = 0.66$  then  $E[Y] = 77 \cdot 0.66 = 50.82$

If we are interested in how many will not cover the truth, this is the random variable  $n - Y$  and  $E[n - Y] = n - E[Y]$ . Calculate the expected number that **will not include the truth** for the same scenarios:

- $n = 100, 1 - \alpha = 0.95$  then  $E[n - Y] = 100 - 95 = 5$
- $n = 1000, 1 - \alpha = 0.7$  then  $E[n - Y] = 1000 - 700 = 300$
- $n = 77, 1 - \alpha = 0.66$  then  $E[n - Y] = 77 - 50.82 = 26.18$

# Summary

Here are the interpretation statements for a  $100(1 - \alpha)\%$  confidence interval for the population mean  $\mu$ :

- Out of  $n$   $100(1 - \alpha)\%$  confidence interval for  $\mu$ , we expect  $n(1 - \alpha)$  confidence intervals to include/cover  $\mu$  (and  $n\alpha$  to not cover  $\mu$ ).
- We are  $100(1 - \alpha)\%$  confident that  $\mu$  falls within the bounds of the constructed interval.

I really **hate** the second statement as I believe it gives you a false impression of what you have actually learned. The second interpretation **DOES NOT** tell you what you should believe, it is really a succinct version of the previous interpretation.

When you see the words confidence or confident, think in your head, the word **frequency**.

# Issues with a frequentist interpretation of probability

How can you interpret the following probability statements:

- What is the probability it will rain tomorrow?
- What is the probability the Vikings will their next game?
- What is the probability my unborn child has Down syndrome?
- What is the probability humans are the main cause of climate change on Earth?

# Bayesian interpretation of probability

## Interpretation

*The **Bayesian interpretation of probability** is that probability is a statement about your degree of belief that an event will (or has) occurred.*

### Advantages:

- Can interpret probability for one time events.
- States what you should believe.
- Natural to make decisions based on your belief.
- Everyone has their own probability.

### Disadvantages:

- Requires more math (integration).
- Requires you to specify your belief before seeing data.
- Has no relation to relative frequency.
- Everyone has their own probability.

# Credible intervals

The Bayesian analog to confidence intervals are credible intervals. These intervals tell you where **you** should believe the parameter to be. Thus a  $100(1 - \alpha)\%$  credible interval for  $\mu$  tells you that you should believe that the true population mean  $\mu$  is in the interval with probability  $1 - \alpha$ .

It turns out, under a particular prior, the confidence intervals that we construct are exactly the same as credible intervals. Thus, you will actually be correct even when you misinterpret confidence intervals.