Multiparameter models

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Motivating example

Is Andre Dawkins 3-point percentage higher in 2013-2014 than past years?

Season	Made	Attempts
2009-2010	36	95
2010-2011	64	150
2011-2012	67	171
2013-2014	64	152

Binomial model

Assume an independent binomial model,

$$Y_s \stackrel{ind}{\sim} Bin(n_s, \theta_s), \text{ i.e. }, p(y|\theta) = \prod_{s=1}^{S} p(y_s|\theta_s) = \prod_{s=1}^{S} \binom{n_s}{y_s} \theta_s^{y_s}(1-\theta_s)^{n_s-y_s}$$

where

- y_s is the number of 3-pointers made in season s
- n_s is the number of 3-pointers attempted in season s
- θ_s is the unknown 3-pointer success probability in season s
- S is the number of seasons
- $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)'$ and $y = (y_1, y_2, y_3, y_4)$

and assume independent beta priors distribution:

$$p(\theta) = \prod_{s=1}^{S} p(\theta_s) = \prod_{s=1}^{S} \frac{\theta_s^{a_s-1}(1-\theta_s)^{b_s-1}}{B(a_s, b_s)} \mathrm{I}(0 < \theta_s < 1).$$

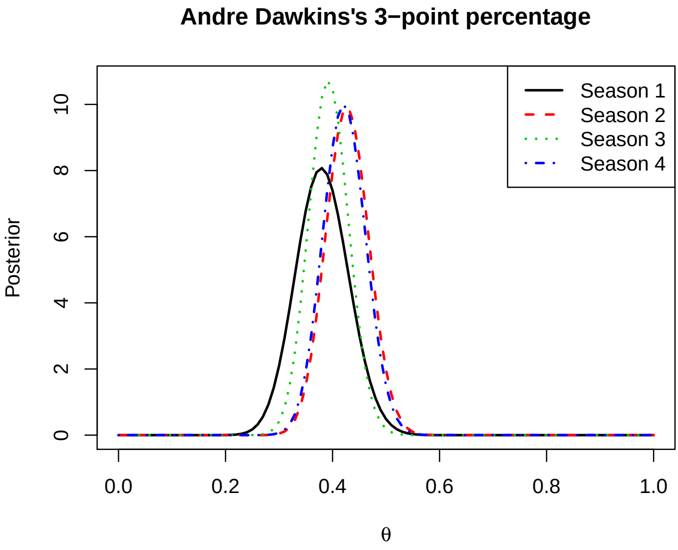
Joint posterior

Derive the posterior according to Bayes rule:

$$\begin{aligned} p(\theta|y) &\propto p(y|\theta)p(\theta) \\ &= \prod_{s=1}^{S} p(y_s|\theta_s) \prod_{s=1}^{S} p(\theta_s) \\ &= \prod_{s=1}^{S} p(y_s|\theta_s)p(\theta_s) \\ &\propto \prod_{s=1}^{S} \text{Beta}(\theta_s|a_s + y_s, b_s + n_s - y_s) \end{aligned}$$

So the posterior for each θ_s is exactly the same as if we treated each season independently.

Joint posterior



Monte Carlo estimates

Estimated means, medians, and quantiles.

```
  year      mean    median       ciL       ciU      hpdL      hpdU
1    1 0.3808984 0.3780401 0.2838980 0.4859991 0.2793826 0.4857247
2    2 0.4298709 0.4283502 0.3516302 0.5121681 0.3497608 0.5077747
3    3 0.3927397 0.3929137 0.3214888 0.4647907 0.3215156 0.4647574
4    4 0.4219583 0.4205461 0.3478506 0.4986194 0.3467727 0.4984729
```

Comparing probabilities across years

The scientific question of interest here is whether Dawkins's 3-point percentage is higher in 2013-2014 than previously. In probability notation this is

$$P(\theta_4 > \theta_s|y) \text{ for } s = 1, 2, 3.$$

which can be approximated via Monte Carlo as

$$P(\theta_4 > \theta_s|y) = E_{\theta|y}[\mathrm{I}(\theta_4 > \theta_s)] \approx \frac{1}{J}\sum_{j=1}^{J} \mathrm{I}\left(\theta_4^{(j)} > \theta_s^{(j)}\right)$$

where

- $\theta_s^{(j)} \stackrel{ind}{\sim} Be(a_s + y_s, b_s + n_s - y_s)$
- $\mathrm{I}(A)$ is in indicator function that is 1 if A is true and zero otherwise.

Estimated probabilities

```r
# Should be able to use dcast
d = data.frame(theta_1 = sim$theta[sim$year==1],
               theta_2 = sim$theta[sim$year==2],
               theta_3 = sim$theta[sim$year==3],
               theta_4 = sim$theta[sim$year==4])

# Probabilities that season 4 percentage is higher than other seasons
mean(d$theta_4 > d$theta_1)
```

```
[1] 0.746
```

```r
mean(d$theta_4 > d$theta_2)
```

```
[1] 0.451
```

```r
mean(d$theta_4 > d$theta_3)
```

```
[1] 0.693
```

Using JAGS

```
library(rjags)
independent_binomials = "
model {
  for (i in 1:N) {
    y[i] ~ dbin(theta[i],n[i])
    theta[i] ~ dbeta(1,1)
  }
}
"

d = list(y=c(36,64,67,64), n=c(95,150,171,152), N=4)
m = jags.model(textConnection(independent_binomials), d)


Compiling model graph
   Resolving undeclared variables
   Allocating nodes
Graph information:
   Observed stochastic nodes: 4
   Unobserved stochastic nodes: 4
   Total graph size: 21

Initializing model


res = coda.samples(m, "theta", 1000)
```

```
summary(res)


Iterations = 1:1000
Thinning interval = 1
Number of chains = 1
Sample size per chain = 1000

1. Empirical mean and standard deviation for each variable,
   plus standard error of the mean:

           Mean      SD Naive SE Time-series SE
theta[1] 0.3816 0.05008 0.001584       0.001584
theta[2] 0.4267 0.04063 0.001285       0.001285
theta[3] 0.3951 0.03725 0.001178       0.001291
theta[4] 0.4205 0.04047 0.001280       0.001280

2. Quantiles for each variable:

           2.5%    25%    50%    75%  97.5%
theta[1] 0.2886 0.3466 0.3825 0.4148 0.4790
theta[2] 0.3491 0.3978 0.4261 0.4545 0.5073
theta[3] 0.3253 0.3685 0.3952 0.4213 0.4683
theta[4] 0.3374 0.3950 0.4206 0.4463 0.4993
```

```r
# Extract sampled theta values
theta = as.matrix(res[[1]]) # with only 1 chain, all values are in the first list element

# Calculate probabilities that season 4 percentage is higher than other seasons
mean(theta[,4] > theta[,1])
```

```
[1] 0.737
```

```r
mean(theta[,4] > theta[,2])
```

```
[1] 0.468
```

```r
mean(theta[,4] > theta[,3])
```

```
[1] 0.677
```

Background probability theory

- Scaled Inv-χ^2 distribution
- Location-scale t-distribution
- Normal-Inv-χ^2 distribution

Scaled-inverse χ^2-distribution

If $\sigma^2 \sim IG(a, b)$, then $\sigma^2 \sim \text{Inv-}\chi^2(v, s^2)$ with

- $a = v/2$ and $b = vs^2/2$, or, equivalently,
- $v = 2a$ and $s^2 = b/a$.

Deriving from the inverse gamma, the scaled-inverse χ^2 has

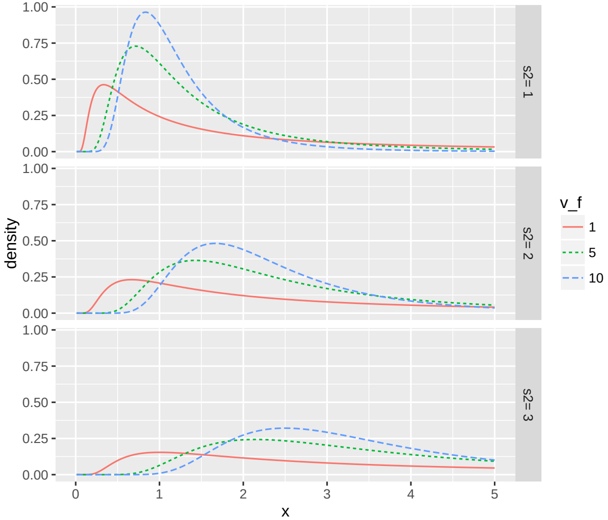
- Mean: $vs^2/(v-2)$ for $v > 2$
- Mode: $vs^2/(v+2)$
- Variance: $2v^2(s^2)^2/[(v-2)^2(v-4)]$ for $v > 4$

So s^2 is a point estimate and $v \to \infty$ means the variance decreases, since, for large v,

$$\frac{2v^2(s^2)^2}{(v-2)^2(v-4)} \approx \frac{2v^2(s^2)^2}{v^3} = \frac{2(s^2)^2}{v}.$$

Scaled-inverse χ^2-distribution

```
dinvgamma = function(x, a, b, ...) dgamma(1/x, a, b,...)/x^2
dsichisq = function(x, v, s2, ...) dinvgamma(x, v/2, v*s2/2, ...)
```



Location-scale t-distribution

The t-distribution is a location-scale family (Casella & Berger Thm 3.5.6), i.e. if T_v has a standard t-distribution with v degrees of freedom and pdf

$$f_t(t) = \frac{\Gamma([v+1]/2)}{\Gamma(v/2)\sqrt{v\pi}} \left(1 + t^2/v\right)^{-(v+1)/2},$$

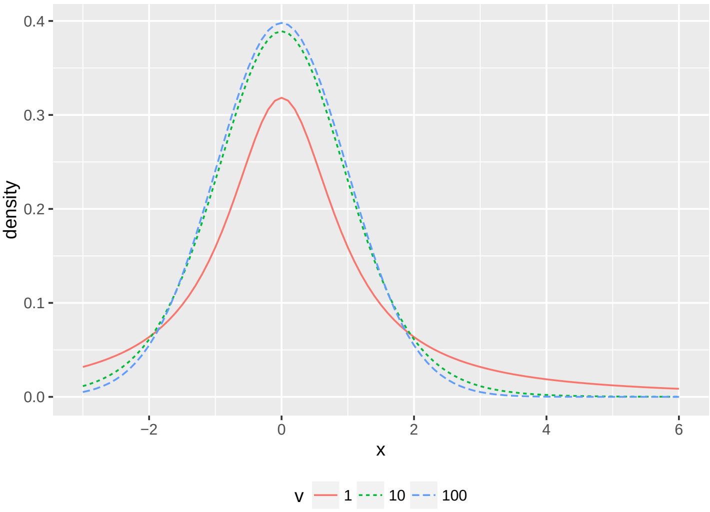
then $X = m + sT_v$ has pdf

$$f_X(x) = f_t([x-m]/s)/s = \frac{\Gamma([v+1]/2)}{\Gamma(v/2)\sqrt{v\pi}s} \left(1 + \frac{1}{v}\left[\frac{x-m}{s}\right]^2\right)^{-(v+1)/2}.$$

This is referred to as a t distribution with v degrees of freedom, location m, and scale s; it is written as $t_v(m, s^2)$. Also,

$$t_v(m, s^2) \stackrel{v\to\infty}{\longrightarrow} N(m, s^2).$$

t distribution as v changes



Normal-Inv-χ^2 distribution

Let $\mu|\sigma^2 \sim N(m, \sigma^2/k)$ and $\sigma^2 \sim \text{Inv-}\chi^2(v, s^2)$, then the kernel of this joint density is

$$\begin{aligned} p(\mu,\sigma^2) &= p(\mu|\sigma^2)p(\sigma^2) \\ &\propto (\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2/k}(\mu-m)^2} (\sigma^2)^{-\frac{v}{2}-1} e^{-\frac{vs^2}{2\sigma^2}} \\ &= (\sigma^2)^{-(v+3)/2} e^{-\frac{1}{2\sigma^2}\left[k(\mu-m)^2+vs^2\right]} \end{aligned}$$

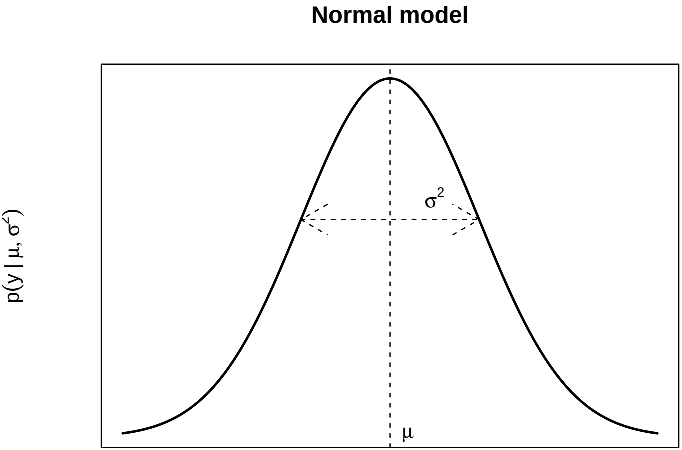
In addition, the marginal distribution for μ is

$$\begin{aligned} p(\mu) &= \int p(\mu|\sigma^2)p(\sigma^2)d\sigma^2 \\ &= \frac{\Gamma([v+1]/2)}{\Gamma(v/2)\sqrt{v\pi}s/\sqrt{k}} \left(1 + \frac{1}{v}\left[\frac{\mu-m}{s/\sqrt{k}}\right]^2\right)^{-(v+1)/2}. \end{aligned}$$

Thus $\mu \sim t_v(m, s^2/k)$.

Univariate normal model

Suppose $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$.



Confidence interval for μ

Let

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \qquad \text{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2.$$

Then,

$$T = \frac{\overline{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

and an equal-tail $100(1-\alpha)\%$ confidence interval can be constructed via

$$1-\alpha = P\left(-c \le T \le c\right) = P\left(\overline{Y} - \frac{cS}{\sqrt{n}} \le \mu \le \overline{Y} + \frac{cS}{\sqrt{n}}\right)$$

and thus $\overline{y} \pm cS/\sqrt{n}$ is an equal-tail 95% confidence interval where $c = t_{n-1}(1-\alpha/2)$ is the t-critical value.

Default priors

Jeffreys prior can be shown to be $p(\mu, \sigma^2) \propto (1/\sigma^2)^{3/2}$. But alternative methods, e.g. reference prior, find that $p(\mu, \sigma^2) \propto 1/\sigma^2$ is a more appropriate prior.

The posterior under the reference prior is

$$\begin{aligned} p(\mu, \sigma^2|y) &\propto (\sigma^2)^{-n/2} \exp\left(-\tfrac{1}{2\sigma^2}\textstyle\sum_{i=1}^n (y_i - \mu)^2\right) \times \tfrac{1}{\sigma^2} \\ &= (\sigma^2)^{-n/2} \exp\left(-\tfrac{1}{2\sigma^2}\textstyle\sum_{i=1}^n (y_i - \overline{y} + \overline{y} - \mu)^2\right) \times \tfrac{1}{\sigma^2} \\ &\vdots \\ &= (\sigma^2)^{-(n-1+3)/2} \exp\left(-\tfrac{1}{2\sigma^2}\left[n(\mu - \overline{y})^2 + (n-1)S^2\right]\right) \end{aligned}$$

Thus

$$\mu|\sigma^2, y \sim N(\overline{y}, \sigma^2/n) \qquad \sigma^2 \sim \text{Inv-}\chi^2(n-1, S^2).$$

Marginal posterior for μ

The marginal posterior for μ is

$$\mu|y \sim t_{n-1}(\overline{y}, S^2/n).$$

An equal-tailed $100(1-\alpha)\%$ credible inteval can be obtained via

$$\overline{y} \pm cS/\sqrt{n}$$

where $c = t_{n-1}(1-\alpha)$ is the t-critical value from before. This is exactly the same as the $100(1-\alpha)\%$ confidence inteval.

Predictive distribution

Let $\tilde{y} \sim N(\mu, \sigma^2)$. The predictive distribution is

$$\int \int p(\tilde{y}|\mu, \sigma^2) p(\mu|\sigma^2, y) p(\sigma^2|y) d\mu d\sigma^2$$

The easiest way to derive this is to write $\tilde{y} = \mu + \epsilon$ with

$$\mu|\sigma^2 \sim N(\overline{y}, \sigma^2/n) \qquad \epsilon|\sigma^2 \sim N(0, \sigma^2)$$

independent of each other. Thus

$$\tilde{y}|\sigma^2 \sim N(\overline{y}, \sigma^2[1 + 1/n]).$$

with $\sigma^2 \sim \text{Inv-}\chi^2(n-1, S^2)$. Now, we can use the Normal-Inv-χ^2 theory, to find that

$$\tilde{y} \sim t_{n-1}(\overline{y}, S^2[1 + 1/n]).$$

Conjugate prior for μ and σ^2

The joint conjugate prior for μ and σ^2 is

$$\mu|\sigma^2 \quad \sim N(m, \sigma^2/k) \qquad \sigma^2 \quad \sim \text{Inv-}\chi^2(v, s^2)$$

where s^2 serves as a prior guess about σ^2 and v controls how certain we are about that guess.

The posterior under this prior is

$$\mu|\sigma^2, y \sim N(m', \sigma^2/k') \qquad \sigma^2|y \sim \text{Inv-}\chi^2(v', (s')^2)$$

where

$$\begin{aligned} k' &= k + n \\ m' &= [km + n\overline{y}]/k' \\ v' &= v + n \\ v'(s')^2 &= vs^2 + (n-1)S^2 + \tfrac{kn}{k'}(\overline{y} - m)^2 \end{aligned}$$

Marginal posterior for μ

The marginal posterior for μ is

$$\mu|y \sim t_{v'}(m', (s')^2/k').$$

An equal-tailed $100(1-\alpha)\%$ credible inteval can be obtained via

$$m' \pm cs'/\sqrt{k'}$$

where $c = t_{v'}(1-\alpha)$ is the t-critical value from before.

Marginal posterior via simulation

An alternative to deriving the closed form posterior for μ is to simulate from the distribution. Recall that

$$\mu|\sigma^2, y \sim N(m', \sigma^2/k') \qquad \sigma^2|y \sim \text{Inv-}\chi^2(v', (s')^2)$$

To obtain a simulation from the posterior distribution $p(\mu, \sigma^2|y)$, do the following

1. Calculate m', k', v', and s'.
2. Simulate $\sigma^2 \sim \text{Inv-}\chi^2(v', (s')^2)$.
3. Using this value, simulate $\mu \sim N(m', \sigma^2/k')$.

Not only does this provide a sample from the joint distribution for μ, σ but it also (therefore) provides a sample from the marginal distribution for μ. The integral was suggestive:

$$p(\mu|y) = \int p(\mu|\sigma^2, y)p(\sigma^2|y)d\sigma^2$$

Predictive distribution via simulation

Similarly, we can obtain the predictive distribution via simulation. Recall that

$$p(\tilde{y}|y) = \int \int p(\tilde{y}|\mu, \sigma^2) p(\mu|\sigma^2, y) p(\sigma^2|y) d\mu d\sigma^2$$

To obtain a simulation from the predictive distribution $p(\tilde{y}|y)$, do the following

1. Calculate m', k', v', and s'.
2. Simulate $\sigma^2 \sim \text{Inv-}\chi^2(v', (s')^2)$.
3. Using this value, simulate $\mu \sim N(m', \sigma^2/k')$.
4. Using μ and σ^2 from above, simulate $\tilde{y} \sim N(\mu, \sigma^2)$.

Summary of normal inference

- Default analysis
  - Prior: (think $\mu \sim N(0, \infty)$ and $\sigma^2 \sim \text{Inv-}\chi^2(0, 0)$)
    $$p(\mu, \sigma^2) \propto 1/\sigma^2$$
  - Posterior:
    $$\mu|\sigma^2, y \sim N(\overline{y}, \sigma^2/n),\ \sigma^2|y \sim \text{Inv-}\chi^2(n-1, S^2),\ \mu|y \sim t_{n-1}(\overline{y}, S^2/n)$$
- Conjugate analysis
  - Prior:
    $$\mu|\sigma^2 \sim N(m, \sigma^2/k),\ \sigma^2 \sim \text{Inv-}\chi^2(v, s^2),\ \mu \sim t_v(m, s^2/k)$$
  - Posterior:
    $$\mu|\sigma^2, y \sim N(m', \sigma^2/k'),\ \sigma^2|y \sim \text{Inv-}\chi^2(v', (s')^2),\ \mu|y \sim t_{v'}(m', (s')^2/k')$$
    with
    $$k' = k + n, m' = [km + n\overline{y}]/k', v' = v + n,$$
    $$v'(s')^2 = vs^2 + (n-1)S^2 + \frac{kn}{k'}(\overline{y} - m)^2$$

Monte Carlo integration

Consider evaluating the integral

$$E_X[h(X)] = \int_{\mathcal{X}} h(x)f(x)dx$$

using the Monte Carlo estimate

$$\hat{h}_J = \frac{1}{J}\sum_{j=1}^{J} h\left(x^{(j)}\right)$$

where $x^{(j)} \overset{ind}{\sim} f(x)$. We know

- SLLN: $\hat{h}_J$ converges almost surely to $E_X[h(X)]$.
- CLT: if h^2 has finite expectation, then

  $$\hat{h}_J \xrightarrow{d} N(E_X[h(X)], v_J)$$

  where

  $$v_J = \frac{1}{J}\widehat{V_X[h(X)]} \approx \frac{1}{J^2}\sum_{s=1}^{J}\left[h\left(x^{(j)}\right) - \hat{h}_J\right]^2.$$

Definite integral

Suppose you are interested in evaluating

$$\mathrm{I} = \int_0^1 e^{-x^2/2} dx.$$

Then set

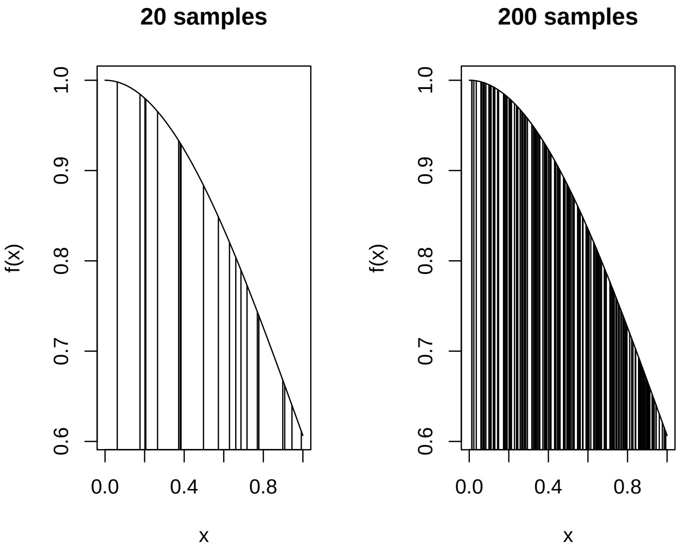
- $h(x) = e^{-x^2/2}$ and
- $f(x) = 1$, i.e. $x \sim \text{Unif}(0,1)$.

and approximate by a Monte Carlo estimate via

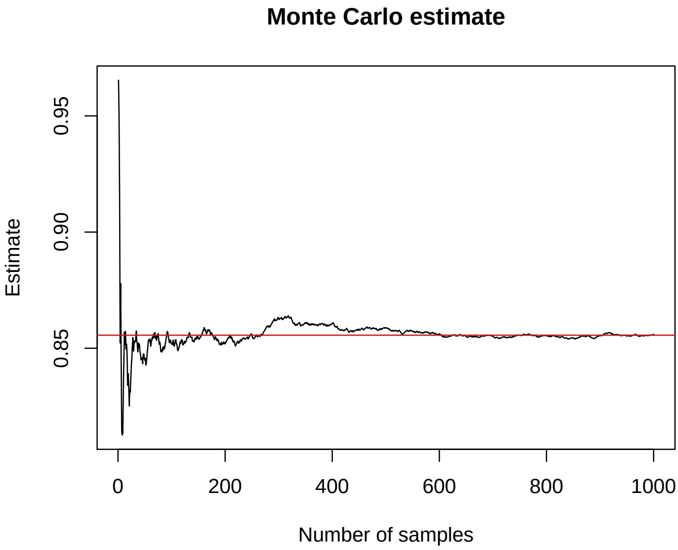
1. For $j = 1, \ldots, J$,
   a. sample $x^{(j)} \sim Unif(0,1)$ and
   b. calculate $h\left(x^{(j)}\right)$.
2. Calculate

$$\mathrm{I} \approx \frac{1}{J} \sum_{j=1}^{J} h(x^{(j)}).$$

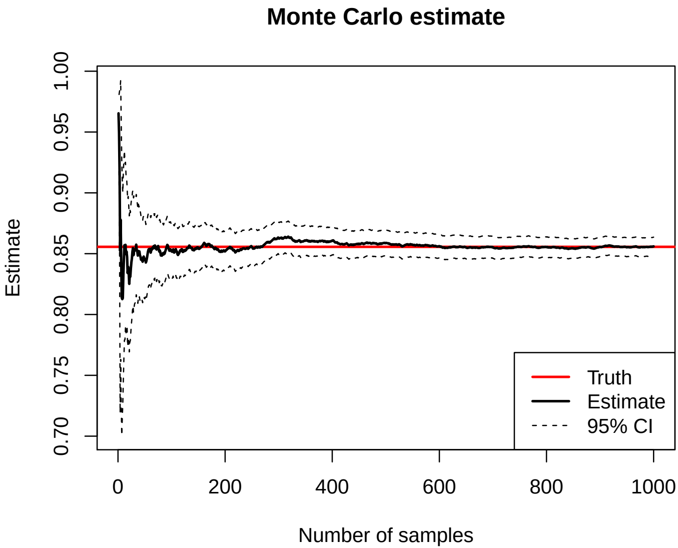
Strong law of large numbers



Strong law of large numbers



Central limit theorem



Infinite bounds

Suppose $X \sim N(0,1)$ and you are interested in evaluating

$$E_X[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Then set

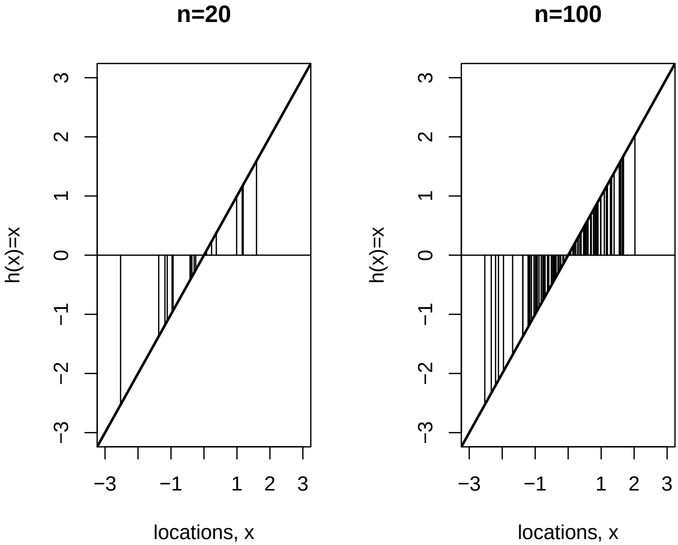
- $h(x) = x$ and
- $f(x) = \phi(x)$, i.e. $x \sim N(0,1)$.

and approximate by a Monte Carlo estimate via

1. For $j = 1, \ldots, J$,
   a. sample $x^{(j)} \sim N(0,1)$ and
   b. calculate $h\left(x^{(j)}\right)$.
2. Calculate

$$E_X[X] \approx \frac{1}{J} \sum_{j=1}^{J} h(x^{(j)}).$$

Non-uniform sampling



Monte Carlo estimate