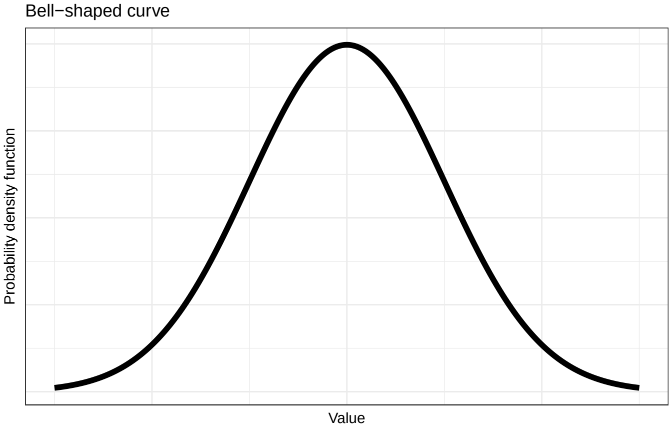
P4 - Central Limit Theorem

STAT 587 (Engineering)
Iowa State University

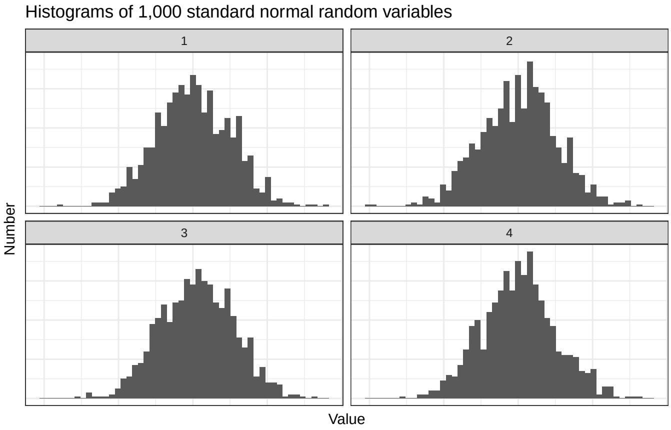
August 27, 2020

Bell-shaped curve

The term bell-shaped curve typically refers to the probability density function for a normal random variable:



Histograms of samples from bell-shaped curves



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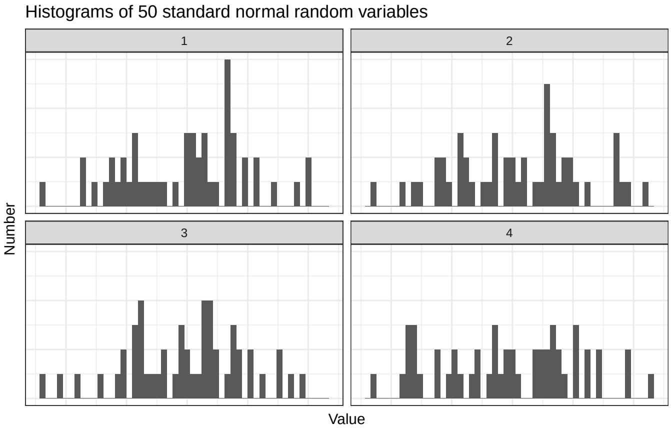
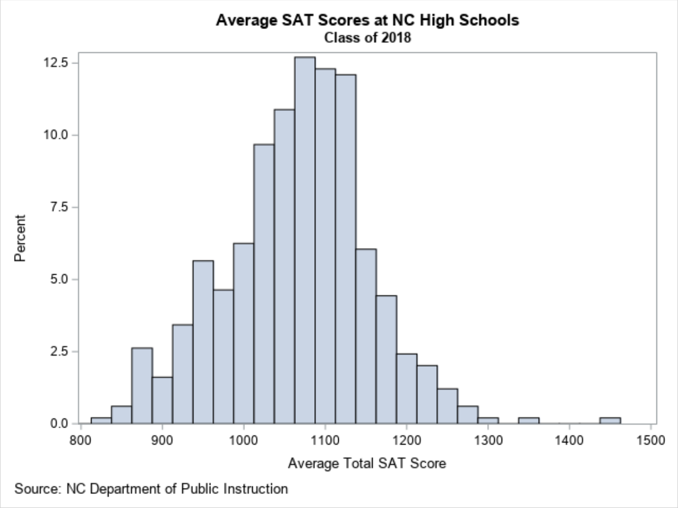


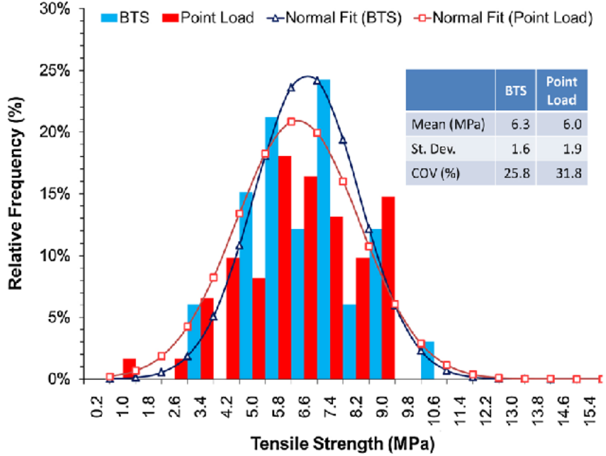
SAT scores

https://blogs.sas.com/content/iml/2019/03/04/visualize-sat-scores-nc.html



Tensile strength

https://www.researchgate.net/figure/Comparison-of-histograms-for-BTS-and-tensile-strength-estimated-from-point-load_fig5_260617256

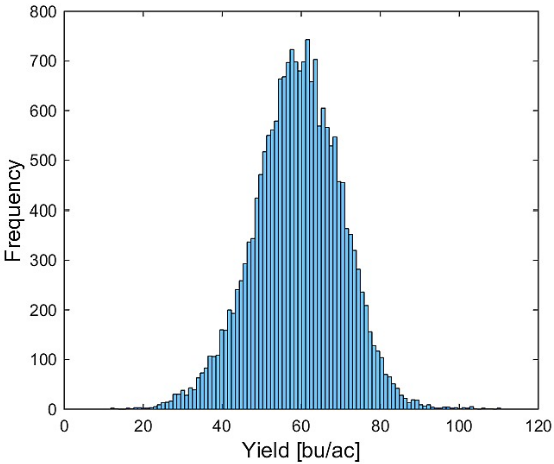


	BTS	Point Load
Mean (MPa)	6.3	6.0
St. Dev.	1.6	1.9
COV (%)	25.8	31.8

Yield

https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0184198

S1 Fig. Histogram of yield.



Sums and averages of iid random variables

Suppose $X_1, X_2, \ldots$ are iid random variables with

$$E[X_i] = \mu \quad Var[X_i] = \sigma^2.$$

Define

$$\begin{aligned} \text{Sample Sum: } S_n &= X_1 + X_2 + \cdots + X_n \\ \text{Sample Average: } \overline{X}_n &= S_n/n. \end{aligned}$$

For S_n, we know

$$E[S_n] = n\mu, \quad Var[S_n] = n\sigma^2, \quad \text{and} \quad SD[S_n] = \sqrt{n}\sigma.$$

For $\overline{X}_n$, we know

$$E[\overline{X}_n] = \mu, \quad Var[\overline{X}_n] = \sigma^2/n, \quad \text{and} \quad SD[\overline{X}_n] = \sigma/\sqrt{n}.$$

Central Limit Theorem (CLT)

Suppose $X_1, X_2, \ldots$ are iid random variables with

$$E[X_i] = \mu \quad Var[X_i] = \sigma^2.$$

Define

$$\begin{aligned} \text{Sample Sum: } S_n &= X_1 + X_2 + \cdots + X_n \\ \text{Sample Average: } \overline{X}_n &= S_n/n. \end{aligned}$$

Then the Central Limit Theorem says

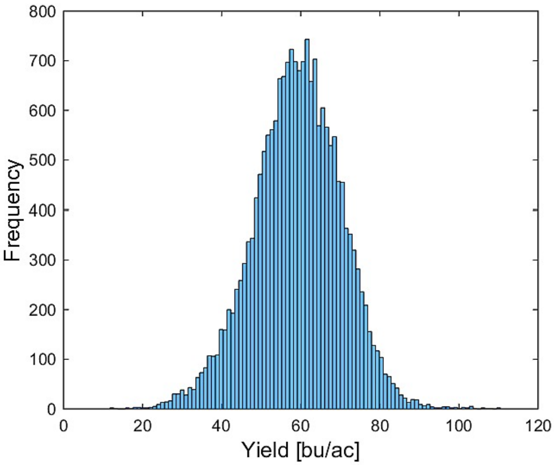
$$\lim_{n\to\infty} \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad \text{and} \quad \lim_{n\to\infty} \frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1).$$

Main Idea: Sums and averages of iid random variables from any distribution have approximate normal distributions for sufficiently large sample sizes.

Yield

https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0184198

S1 Fig. Histogram of yield.



Approximating distributions

Rather than considering the limit, I typically think of the following approximations as n gets large.

For the sample average,

$$\overline{X}_n \dot{\sim} N(\mu, \sigma^2/n).$$

where $\dot{\sim}$ indicates *approximately distributed* because

$$E\left[\overline{X}_n\right] = \mu \qquad \text{and} \qquad Var\left[\overline{X}_n\right] = \sigma^2/n.$$

For the sample sum,

$$S_n \dot{\sim} N(n\mu, n\sigma^2)$$

because

$$\begin{aligned} E[S_n] &= n\mu \\ Var[S_n] &= n\sigma^2. \end{aligned}$$

Averages and sums of uniforms

Let $X_i \stackrel{ind}{\sim} Unif(0,1)$. Then

$$\mu = E[X_i] = \frac{1}{2} \quad \text{and} \quad \sigma^2 = Var[X_i] = \frac{1}{12}.$$

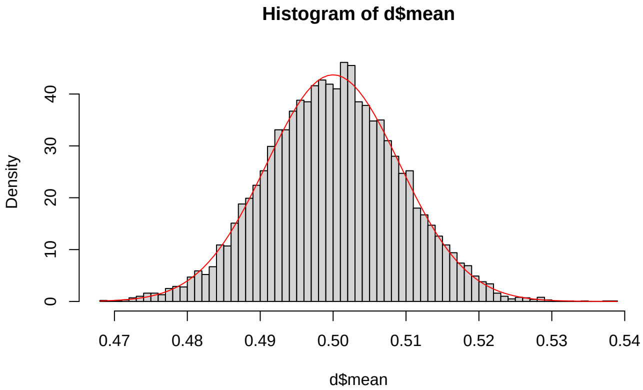
Thus

$$\overline{X}_n \dot{\sim} N\left(\frac{1}{2}, \frac{1}{12n}\right)$$

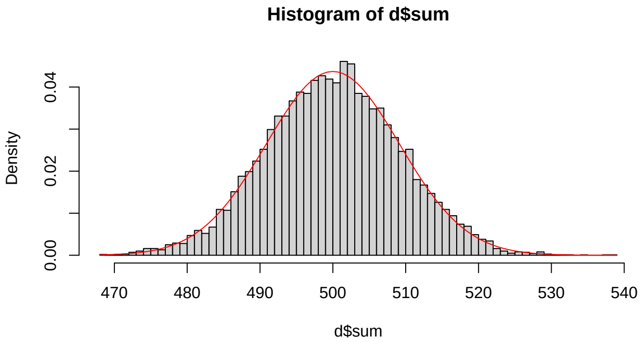
and

$$S_n \dot{\sim} N\left(\frac{n}{2}, \frac{n}{12}\right).$$

Averages of uniforms



Sums of uniforms



Normal approximation to a binomial

Recall if $Y_n = \sum_{i=1}^{n} X_i$ where $X_i \overset{ind}{\sim} Ber(p)$, then

$$Y_n \sim Bin(n, p).$$

For a binomial random variable, we have

$$E[Y_n] = np \qquad \text{and} \qquad Var[Y_n] = np(1-p).$$

By the CLT,

$$\lim_{n\to\infty} \frac{Y_n - np}{\sqrt{np(1-p)}} \to N(0,1),$$

if n is large,

$$Y_n \dot{\sim} N(np, np[1-p]).$$

Roulette example

A European roulette wheel has 39 slots: one green, 19 black, and 19 red. If I play black every time, what is the probability that I will have won more than I lost after 99 spins of the wheel?

Let Y indicate the total number of wins and assume $Y \sim Bin(n,p)$ with $n = 99$ and $p = 19/39$. The desired probability is $P(Y \geq 50)$. Then

$$P(Y \geq 50) = 1 - P(Y < 50) = 1 - P(Y \leq 49)$$

```
n = 99
p = 19/39
1-pbinom(49, n, p)

[1] 0.399048
```

Roulette example

A European roulette wheel has 39 slots: one green, 19 black, and 19 red. If I play black every time, what is the probability that I will have won more than I lost after 99 spins of the wheel?

Let Y indicate the total number of wins. We can approximate Y using $X \sim N(np, np(1-p))$.

$$P(Y \geq 50) \approx 1 - P(X < 50)$$

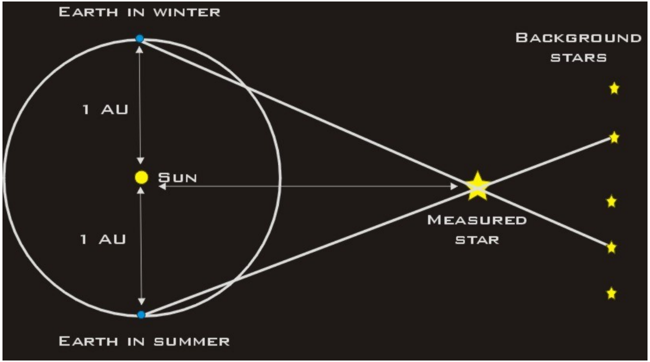
```
1-pnorm(50, n*p, sqrt(n*p*(1-p)))

[1] 0.3610155
```

A better approximation can be found using a continuity correction.

Astronomy example

An astronomer wants to measure the distance, d, from Earth to a star. Suppose the procedure has a known standard deviation of 2 parsecs. The astronomer takes 30 iid measurements and finds the average of these measurements to be 29.4 parsecs. What is the probability the average is within 0.5 parsecs?

http://planetary-science.org/astronomy/distance-and-magnitudes/



Astronomy example

Let X_i be the i^{th} measurement. The astronomer assumes that $X_1, X_2, \ldots X_n$ are iid with $E[X_i] = d$ and $Var[X_i] = \sigma^2 = 4$. The estimate of d is

$$\overline{X}_n = \frac{(X_1 + X_2 + \cdots + X_n)}{n} = 29.4.$$

and, by the Central Limit Theorem, we believe $\overline{X}_n \dot{\sim} N(d, \sigma^2/n)$ where $n = 30$. We want to find

$$\begin{aligned} P\left(|\overline{X}_n - d| < 0.5\right) \quad &= P\left(-0.5 < \overline{X}_n - d < 0.5\right) \\ &= P\left(\frac{-0.5}{\sigma/\sqrt{n}} < \frac{\overline{X}_n - d}{\sigma/\sqrt{n}} < \frac{0.5}{\sigma/\sqrt{n}}\right) \\ &\approx P(-1.37 < Z < 1.37) \end{aligned}$$

```r
diff(pnorm(c(-1.37,1.37)))
```

```
[1] 0.8293131
```

Astronomy example (cont.)

Suppose the astronomer wants to be within 0.5 parsecs with at least 95% probability. How many more samples would she need to take?

We solve

$$\begin{aligned}
0.95 \le P\left(\left|\overline{X}_n - d\right| < .5\right) &= P\left(-0.5 < \overline{X}_n - d < 0.5\right) \\
&= P\left(\frac{-0.5}{\sigma/\sqrt{n}} < \frac{\overline{X}_n - d}{\sigma/\sqrt{n}} < \frac{0.5}{\sigma/\sqrt{n}}\right) \\
&= P(-z < Z < z) \\
&= 1 - [P(Z < -z) + P(Z > z)] \\
&= 1 - 2P(Z < -z)
\end{aligned}$$

where $z = 0.5/(\sigma/\sqrt{n}) = 1.96$ since

```
-qnorm(.025)

[1] 1.959964
```

and thus $n = 61.47$ which we round up to $n = 62$ to ensure

Summary

- Central Limit Theorem
  - Sums
  - Averages
- Examples
  - Uniforms
  - Binomial
  - Roulette
- Sample size
  - Astronomy