STAT 401A - Statistical Methods for Research Workers

Simple linear regression

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Simple Linear Regression

Recall the one-way ANOVA model:

$$Y_{ij} \overset{ind}{\sim} N(\mu_j, \sigma^2)$$

where Y_{ij} is the observation for individual i in group j.

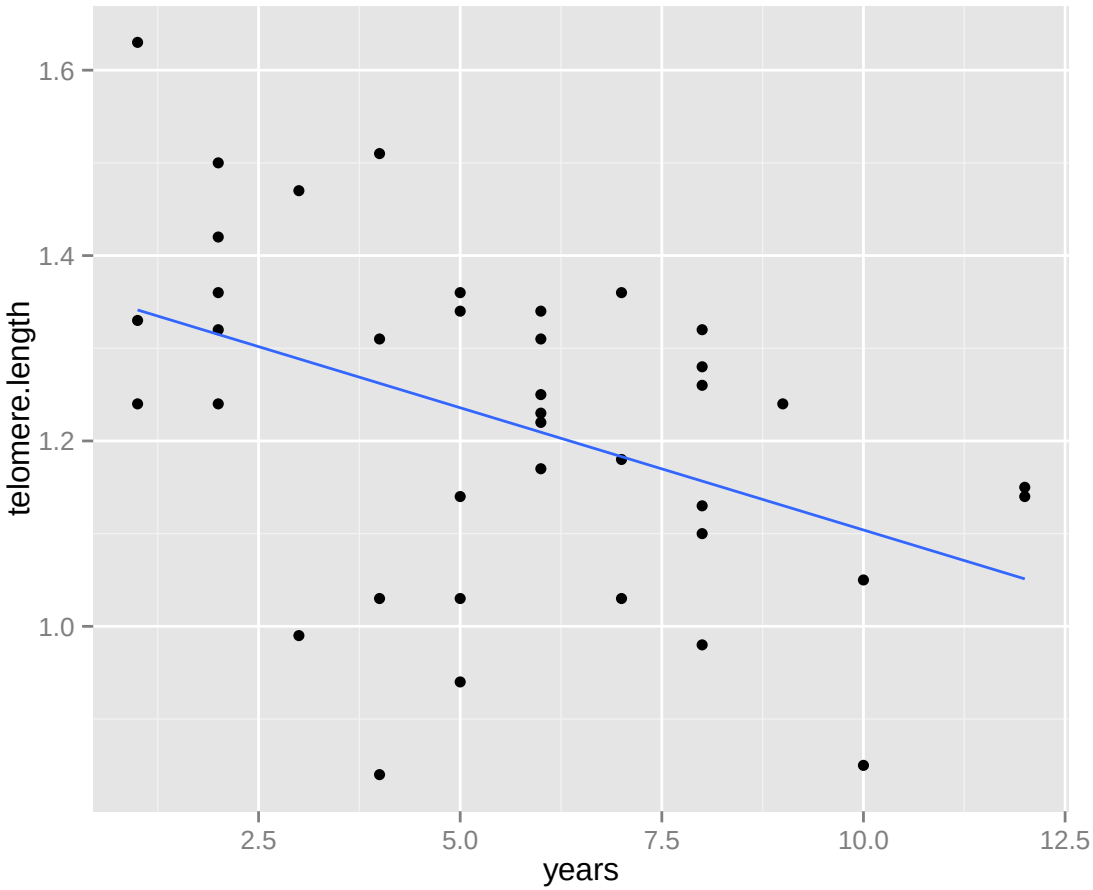
The simple linear regression model is

$$Y_i \overset{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

Terminology (all of these are equivalent):

response	explanatory
outcome	covariate
dependent	independent
endogenous	exogenous


telomere.length
1.6
1.4
1.2
1.0
2.5
5.0
7.5
10.0
12.5
years

Telomere length

http://www.pnas.org/content/101/49/17312

People who are stressed over long periods tend to look haggard, and it is commonly thought that psychological stress leads to premature aging and the earlier onset of diseases of aging.

...

This design allowed us to examine the importance of perceived stress and measures of objective stress (caregiving status and chronicity of caregiving stress based on the number of years since a child's diagnosis).

...

Telomere length values were measured from DNA by a quantitative PCR assay that determines the relative ratio of telomere repeat copy number to single-copy gene copy number (T/S ratio) in experimental samples as compared with a reference DNA sample.

Parameter interpretation

$$E[Y_i|X_i = x] = \beta_0 + \beta_1 x \qquad V[Y_i|X_i = x] = \sigma^2$$

- If $X_i = 0$, then $E[Y_i|X_i = 0] = \beta_0$.
  β_0 is the expected response when the explanatory variable is zero.

- If X_i increases from x to $x + 1$, then

$$\begin{array}{rl} E[Y_i|X_i = x + 1] & = \beta_0 + \beta_1 x + \beta_1 \\ - E[Y_i|X_i = x \quad] & = \beta_0 + \beta_1 x \\ \hline & = \beta_1 \end{array}$$

  β_1 is the expected increase in the response for each unit increase in the explanatory variable.

- σ is the standard deviation of the response for a fixed value of the explanatory variable.

Remove the mean:

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \overset{iid}{\sim} N(0, \sigma^2)$$

So the error is

$$e_i = Y_i - (\beta_0 + \beta_1 X_i)$$

which we approximate by the residual

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

The least squares, maximum likelihood, and Bayesian estimators are

$$\begin{aligned} \hat{\beta}_1 &= SXY/SXX \\ \hat{\beta}_0 &= \overline{Y} - \hat{\beta}_1 \overline{X} \\ \hat{\sigma}^2 &= SSE/(n-2) \qquad \text{df} = n-2 \end{aligned}$$

$$\begin{aligned} \overline{X} &= \tfrac{1}{n}\textstyle\sum_{i=1}^n X_i \\ \overline{Y} &= \tfrac{1}{n}\textstyle\sum_{i=1}^n Y_i \end{aligned}$$

$$\begin{aligned} SXY &= \textstyle\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) \\ SXX &= \textstyle\sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^n (X_i - \overline{X})^2 \\ SSE &= \textstyle\sum_{i=1}^n r_i^2 \end{aligned}$$

How certain are we about $\hat{\beta}_0$ and $\hat{\beta}_1$ being equal to β_0 and β_1?

We quantify this uncertainty using their standard errors:

$$SE(\beta_0) = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} \qquad df = n-2$$

$$SE(\beta_1) = \hat{\sigma}\sqrt{\frac{1}{(n-1)s_X^2}} \qquad df = n-2$$

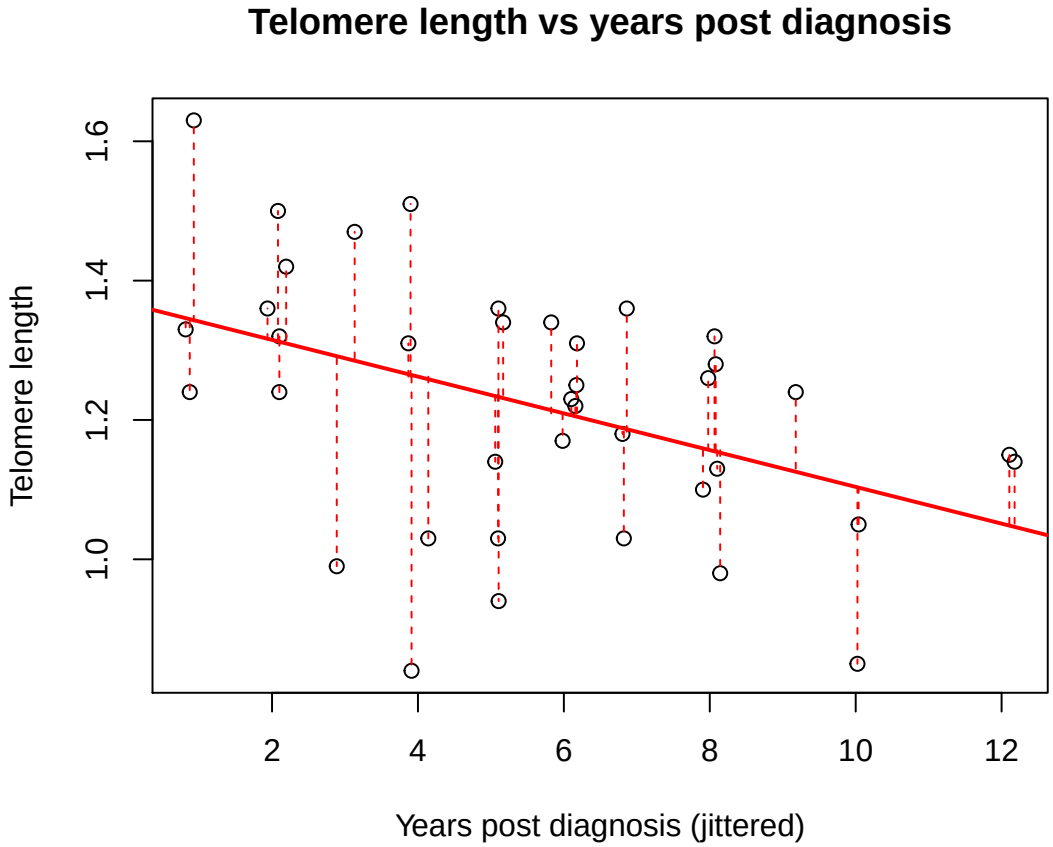
$$s_X^2 = SXX/(n-1)$$
$$s_Y^2 = SYY/(n-1)$$
$$SYY = \sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

$$r_{XY} = \frac{SXY/(n-1)}{s_X s_Y} \qquad \text{correlation coefficient}$$

$$R^2 = r_{XY}^2 = \frac{SST - SSE}{SST} \qquad \text{coefficient of determination}$$

$$SST = SYY = \sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

The coefficient of determination (R^2) is the proportion of the total response variation explained by the explanatory variable(s).


Telomere length vs years post diagnosis
Telomere length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
Years post diagnosis (jittered)

Pvalues and confidence interval

We can compute two-sided pvalues via

$$2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_0}{SE(\beta_0)}\right|\right) \quad \text{and} \quad 2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_1}{SE(\beta_1)}\right|\right)$$

These test the null hypothesis that the corresponding parameter is zero.

We can construct $100(1-\alpha)\%$ two-sided confidence intervals via

$$\hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\beta_0) \quad \text{and} \quad \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\beta_1)$$

These provide ranges of the parameters consistent with the data.

Calculations by hand

```
  n Xbar Ybar   s_X    s_Y    r_XY
1 39 5.59 1.22 2.935 0.1798 -0.4307
```

$$
\begin{aligned}
SXX &= (n-1)s_X^2 = (39-1) \times 2.935427^2 = 327.4358 \\
SYY &= (n-1)s_Y^2 = (39-1) \times 0.1797731^2 = 1.228098 \\
SXY &= (n-1)s_X s_Y r_{XY} = (39-1) \times 2.935427 \times 0.1797731 \times -0.4306534 = -8.635897 \\
\hat{\beta}_1 &= SXY/SXX = -8.635897/327.4358 = -0.02637432 \\
\hat{\beta}_0 &= \overline{Y} - \hat{\beta}_1 \overline{X} = 1.220256 - (-0.02637432) \times 5.589744 = 1.367682 \\
R^2 &= r_{XY}^2 = (-0.4306534)^2 = 0.1854624 \\
SSE &= SYY(1-R^2) = 1.228098(1-0.1854624) = 1.000332 \\
\hat{\sigma}^2 &= SSE/(n-2) = 1.000332/(39-2) = 0.027036 \\
\hat{\sigma} &= \sqrt{\hat{\sigma}^2} = \sqrt{0.027036} = 0.1644263 \\
SE(\hat{\beta}_0) &= \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{39} + \frac{5.589744^2}{327.4358}} = 0.05721115 \\
SE(\hat{\beta}_1) &= \hat{\sigma}\sqrt{\frac{1}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{327.4358}} = 0.009086742 \\
p_{H_0:\beta_0=0} &= 2P\left(t_{n-2} < -\left|\frac{1.367682}{0.05721115}\right|\right) = 2P(t_{37} < -23.90586) < 0.0001 \\
p_{H_0:\beta_1=0} &= 2P\left(t_{n-2} < -\left|\frac{-0.02637432}{0.009086742}\right|\right) = 2P(t_{37} < -2.902506) < 0.0062 \\
CI_{95\%\ \beta_0} &= \hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\hat{\beta}_0) = 1.367682 \pm 2.026192 \times 0.05721115 = (1.251761, 1.483603) \\
CI_{95\%\ \beta_1} &= \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\hat{\beta}_1) = -0.02637432 \pm 2.026192 \times 0.009086742 = (-0.044785804 - 0.007962836)
\end{aligned}
$$

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC CORR DATA=t;
  VAR length;
  WITH years;
  RUN;
```

The CORR Procedure

1 With Variables: years
1 Variables: length

Simple Statistics

Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
years	39	5.58974	2.93543	218.00000	1.00000	12.00000
length	39	1.22026	0.17977	47.59000	0.84000	1.63000

Pearson Correlation Coefficients, N = 39
Prob > |r| under H0: Rho=0

	length
years	-0.43065
	0.0062

```
PROC GLM DATA=t;
  MODEL length = years / SOLUTION CLPARM;
  RUN;
```

The GLM Procedure

Number of Observations Read	39
Number of Observations Used	39

Dependent Variable: length

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	0.22776588	0.22776588	8.42	0.0062
Error	37	1.00033156	0.02703599		
Corrected Total	38	1.22809744			

R-Square	Coeff Var	Root MSE	length Mean
0.185462	13.47473	0.164426	1.220256

Source	DF	Type I SS	Mean Square	F Value	Pr > F
years	1	0.22776588	0.22776588	8.42	0.0062

Source	DF	Type III SS	Mean Square	F Value	Pr > F
years	1	0.22776588	0.22776588	8.42	0.0062

Parameter	Estimate	Standard Error	t Value	Pr > |t|	95% Confidence Limits	
Intercept	1.367682067	0.05721112	23.91	<.0001	1.251761335	1.483602799
years	-0.026374315	0.00908674	-2.90	0.0062	-0.044785794	-0.007962836

Regression in R

```
m = lm(telomere.length~years, Telomeres)
with(Telomeres, cor(telomere.length,years))


[1] -0.4307


anova(m)


Analysis of Variance Table

Response: telomere.length
          Df Sum Sq Mean Sq F value Pr(>F)
years      1  0.228   0.228    8.42 0.0062 **
Residuals 37  1.000   0.027
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
summary(m)
```

```

Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
    Min      1Q  Median      3Q     Max
-0.4222 -0.0854  0.0206  0.1074  0.2887

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.36768    0.05721    23.9   <2e-16 ***
years       -0.02637    0.00909    -2.9   0.0062 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.164 on 37 degrees of freedom
Multiple R-squared:  0.185, Adjusted R-squared:  0.163
F-statistic: 8.42 on 1 and 37 DF,  p-value: 0.0062
```

```
confint(m)
```

```
               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

Conclusion

Telomere length at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% confidence interval of (1.25, 1.48). For each year increase since diagnosis, the length decreases by 0.026 with a 95% confidence interval of (0.008, 0.045). The proportional of variability in telomere length described by years since diagnosis is 18.5%.

http://www.pnas.org/content/101/49/17312

> *The zero-order correlation between chronicity of caregiving [years] and mean telomere length, r,is* -0.445 *(*$P < 0.01$*).*
> *[*$R^2 = 0.198$ *was shown in the plot.]*

Remark I'm guessing our analysis and that reported in the paper don't match exactly due to a discrepancy in the data.

Summary

- The simple linear regression model is

$$Y_i \overset{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

- Know how to use SAS/R to obtain $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$, R^2, pvalues, CIs, etc.
- Interpret SAS output
  - At a value of zero for the explanatory variable ($X_i = 0$), β_0 is the expected value for the response (Y_i).
  - For each unit increase in the explanatory variable value, β_1 is the expected increase in the response.
  - At a constant value of the explanatory variable, σ^2 is the variance of the responses.
  - The coefficient of determination (R^2) is the percentage of the total response variation explained by the explanatory variable(s).

What is $E[Y|X = x]$?

We know $\beta_0 = E[Y|X = 0]$, but what about $X = x$?

$$E[Y|X = x] = \beta_0 + \beta_1 x$$

which we can estimate via

$$\widehat{E[Y|X = x]} = \hat{\beta}_0 + \hat{\beta}_1 x$$

but there is uncertainty in both β_0 and β_1. So the standard error of $E[Y|X = x]$ is

$$SE(E[Y|X = x]) = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(\overline{X} - x)^2}{(n-1)s_X^2}}$$

and a $100(1-\alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1 - \alpha/2)SE(E[Y|X = x])$$

What do we predict about Y at $X = x$?

On the last slide, we calculated $E[Y|X = x]$ and it's uncertainty, but if we are trying to predict a new observation, we need to account for the sampling variablity σ^2. Thus a prediction about Y at a new $X = x$ is still

$$Pred\{Y|X = x\} = \hat{\beta}_0 + \hat{\beta}_1 x$$

but the uncertainty includes the variability due to σ^2. So the standard error of $Pred\{Y|X = x\}$ is

$$SE(Pred\{Y|X = x\}) = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(\overline{X} - x)^2}{(n-1)s_X^2}}$$

and a $100(1-\alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1-\alpha/2)SE(Pred\{Y|X = x\}).$$

```
DATA tnew;
  INPUT years;
  DATALINES;
  4
  ;

DATA combined;
  SET t tnew;
  RUN;

PROC PRINT DATA=combined;
  WHERE years=4;
  RUN;
```

Obs	years	length
10	4	1.51
11	4	1.31
15	4	1.03
16	4	0.84
40	4	.

```
PROC GLM DATA=combined;
 MODEL length = years;
 OUTPUT OUT=combinedreg PREDICTED=predicted  LCLM=lclm UCLM=uclm LCL=lcl UCL=ucl;
 RUN;

PROC PRINT DATA=combinedreg;
  WHERE length=.;              /* . is missing data in SAS */
  RUN;
```

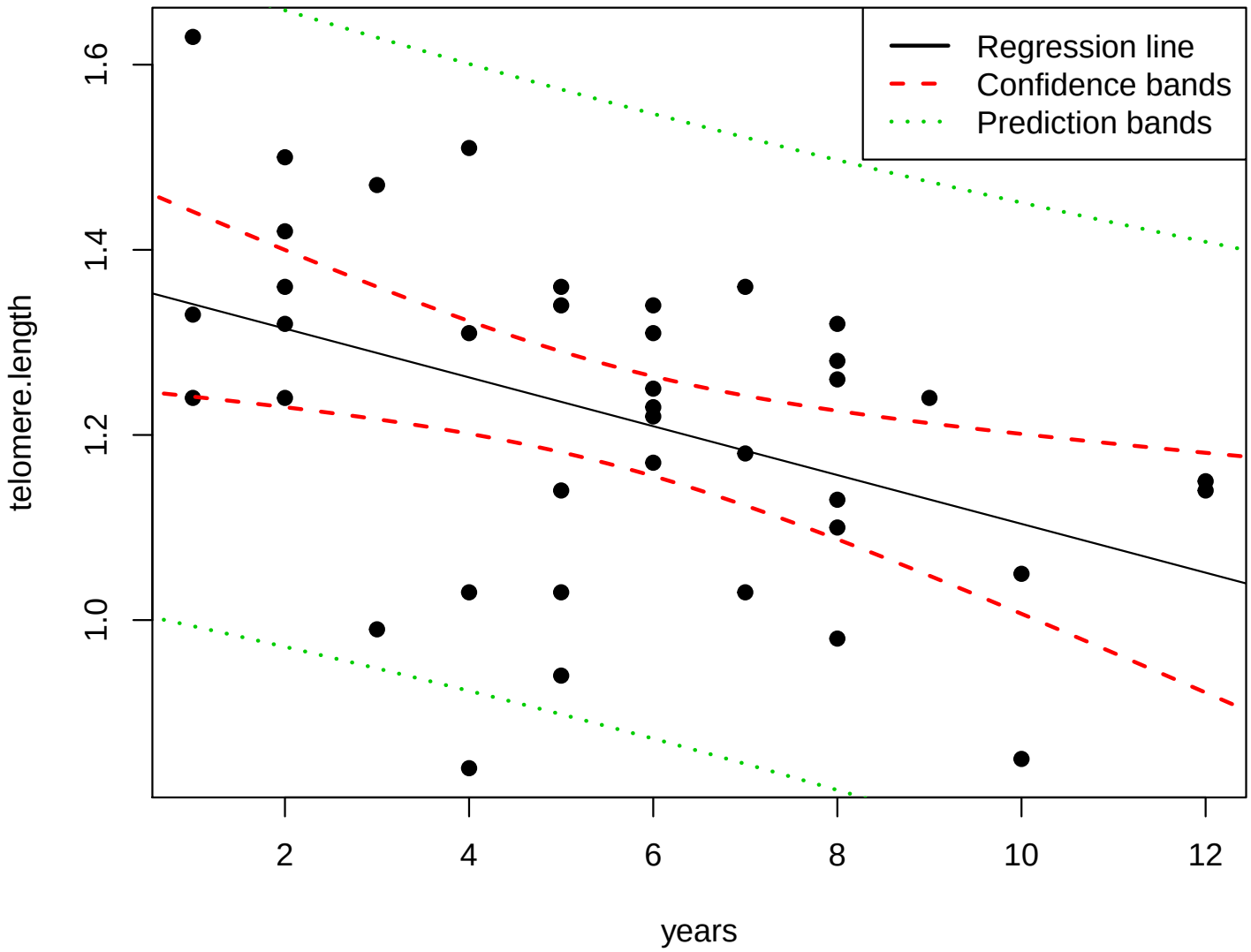
Obs	years	length	predicted	lclm	uclm	lcl	ucl
40	4	.	1.26218	1.20133	1.32303	0.92351	1.60086

```
m = lm(telomere.length~years, Telomeres)
new = data.frame(years=4)
predict(m, new, interval="confidence")
```

```
    fit   lwr   upr
1 1.262 1.201 1.323
```

```
predict(m, new, interval="prediction")
```

```
    fit    lwr   upr
1 1.262 0.9235 1.601
```


Regression line
Confidence bands
Prediction bands
telomere.length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
years

Testing Composite hypotheses

Comparing two models

- H_0 : (reduced)
- H_1 : (full)

Do the following

1. Calculate extra sum of squares.
2. Calculate extra degrees of freedom
3. Calculate

$$\text{F-statistic} = \frac{\text{Extra sum of squares} \ / \ \text{Extra degrees of freedom}}{\hat{\sigma}^2_{full}}$$

4. Compare this to an F-distribution with
   - numerator degrees of freedom = extra degrees of freedom
   - denominator degrees of freedom = degrees of freedom in estimating $\hat{\sigma}^2_{full}$

Lack-of-fit F-test

Let Y_{ij} be the i^{th} observation from the j^{th} group where the group is defined by those observations having the same explanatory variable value (X_j).

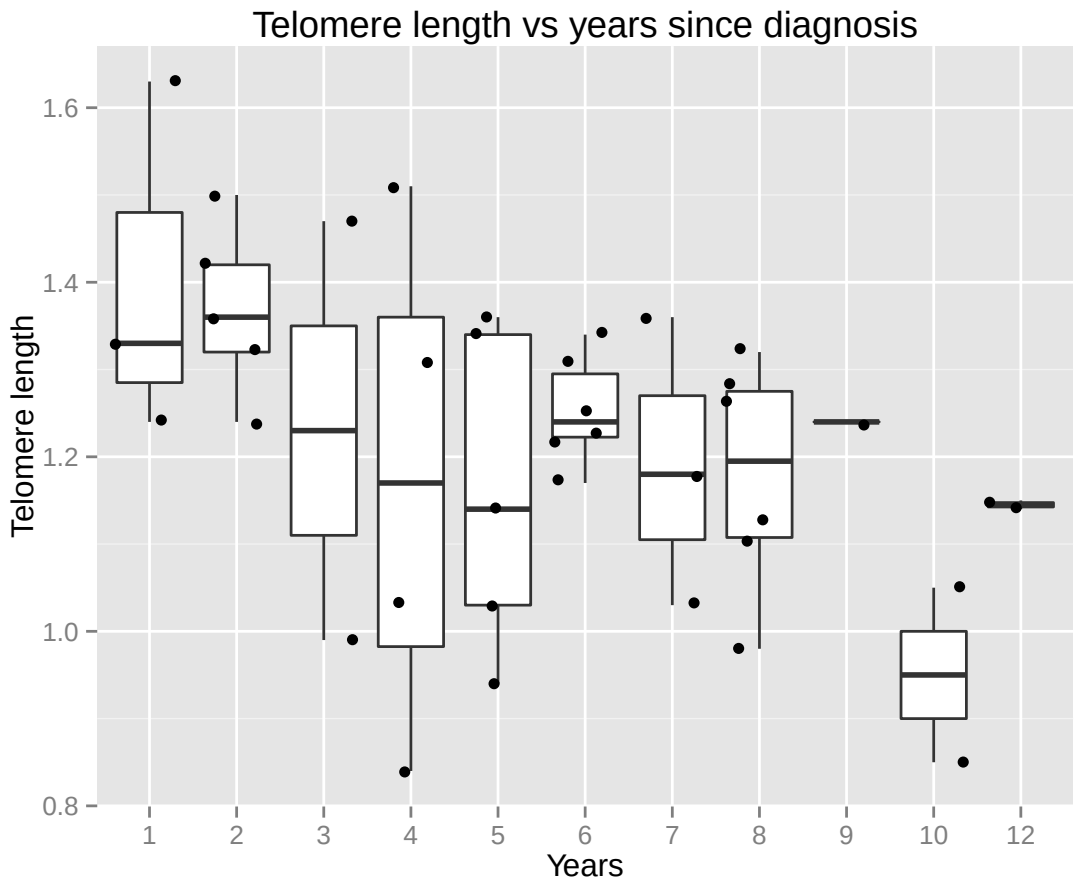
Two models:

ANOVA: $Y_{ij} \overset{ind}{\sim} N(\mu_j, \sigma^2)$ (full)

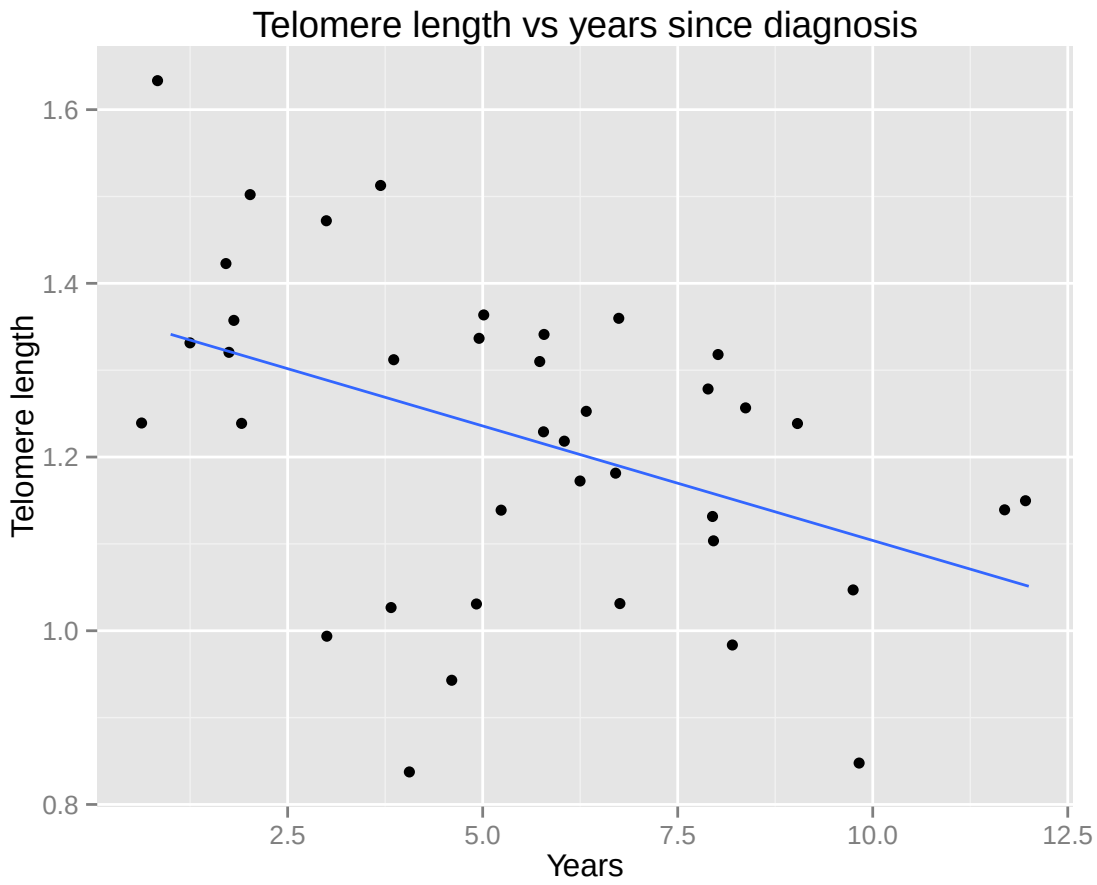
Regression: $Y_{ij} \overset{ind}{\sim} N(\beta_0 + \beta_1 X_j, \sigma^2)$ (reduced)

- Regression model is reduced:
  - ANOVA has J parameters for the mean
  - Regression has 2 parameters for the mean
  - Set $\mu_j = \beta_0 + \beta_1 X_j$.
- Small pvalues indicate a lack-of-fit, i.e. the reduced model is not adequate.
- Lack-of-fit F-test requires multiple observations at a few X_j values!

Telomere length



Telomere length



SAS code

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years / CLB LACKFIT;
  RUN;
```

The REG Procedure
Model: MODEL1
Dependent Variable: length

Number of Observations Read	39
Number of Observations Used	39

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	0.22777	0.22777	8.42	0.0062
Error	37	1.00033	0.02704		
Lack of Fit	9	0.18223	0.02025	0.69	0.7093
Pure Error	28	0.81810	0.02922		
Corrected Total	38	1.22810			

Indicates no evidence for a lack of fit, i.e. regression seems adequate.

```
# Use as.factor to turn a continuous variable into a categorical variable
m_anova = lm(telomere.length ~ as.factor(years), Telomeres)
m_reg   = lm(telomere.length ~           years , Telomeres)
anova(m_reg, m_anova)


Analysis of Variance Table

Model 1: telomere.length ~ years
Model 2: telomere.length ~ as.factor(years)
  Res.Df   RSS Df Sum of Sq    F Pr(>F)
1     37 1.000
2     28 0.818  9     0.182 0.69   0.71
```

No evidence of a lack of fit.

Summary

- Lack-of-fit F-test tests the assumption of linearity
- Needs multiple observations at various explanatory variable values
- Small pvalue indicates a lack-of-fit, i.e. means are not linear
  - Transform response, e.g. log
  - Transform explanatory variable
  - Add other explanatory variables