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Error in library("pscl"):  there is no package called 'pscl'
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Parameter estimation

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Outline

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  - Point estimation
  - Interval estimation
  - Simulation from the posterior
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Parameter estimation

For point or interval estimation of a parameter θ in a model M based on data y, Bayesian inference is based off

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta} \propto p(y|\theta)p(\theta)$$

where

- $p(\theta)$ is the prior distribution for the parameter,
- $p(\theta|y)$ is the posterior distribution for the parameter,
- $p(y|\theta)$ is the statistical model (or likelihood), and
- $p(y)$ is the prior predictive distribution (or marginal likelihood).

Obtaining the posterior

The hard way:

1. Derive $p(y)$.
2. Derive $p(\theta|y) = p(y|\theta)p(\theta)/p(y)$.

The easy way:

1. Derive $f(\theta) \propto p(y|\theta)p(\theta)$.
2. Recognize $f(\theta)$ as the kernel of some distribution.

Definition

The kernel of a probability density (mass) function is the form of the pdf (pmf) with any terms not involving the random variable omitted.

For example, $\theta^{a-1}(1-\theta)^{b-1}$ is the kernel of a beta distribution.

Derive the posterior - the hard way

Suppose $Y \sim Bin(n,\theta)$ and $\theta \sim Be(a,b)$, then

$$\begin{array}{rl} p(y) & = \int p(y|\theta)p(\theta)d\theta \\ & = \int \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\theta^{a-1}(1-\theta)^{b-1}}{\text{Beta}(a,b)}d\theta \\ & = \binom{n}{y}\frac{1}{\text{Beta}(a,b)}\int \theta^{a+y-1}(1-\theta)^{b+n-y-1}d\theta \\ & = \binom{n}{y}\frac{\text{Beta}(a+y,b+n-y)}{\text{Beta}(a,b)} \end{array}$$

which is known as the Beta-binomial distribution.

$$\begin{array}{rl} p(\theta|y) & = p(y|\theta)p(\theta)/p(y) \\ & = \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\theta^{a-1}(1-\theta)^{b-1}}{\text{Beta}(a,b)} \Big/ \binom{n}{y}\frac{\text{Beta}(a+y,b+n-y)}{\text{Beta}(a,b)} \\ & = \frac{\theta^{a+y-1}(1-\theta)^{b+n-y-1}}{\text{Beta}(a+y,b+n-y)} \end{array}$$

Thus $\theta|y \sim Be(a+y, b+n-y)$.

Derive the posterior - the easy way

Suppose $Y \sim Bin(n, \theta)$ and $\theta \sim Be(a, b)$, then

$$\begin{aligned} p(\theta|y) &\propto p(y|\theta)p(\theta) \\ &\propto \theta^y(1-\theta)^{n-y}\theta^{a-1}(1-\theta)^{b-1} \\ &= \theta^{a+y-1}(1-\theta)^{b+n-y-1} \end{aligned}$$

Thus $\theta|y \sim Be(a + y, b + n - y)$.

Interpretation of prior parameters

When constructing the $Be(a, b)$ prior with the binomial likelihood which results in the posterior

$$\theta|y \sim Be(a + y, b + n - y),$$

we can interpret the prior parameters in the following way:

- a: prior successes
- b: prior failures
- $a + b$: prior sample size
- $a/(a + b)$: prior mean

These interpretations may aid in construction of this prior for a given application.

Posterior mean is a weighted average of prior mean and the MLE

The posterior is $\theta|y \sim Be(a+y, b+n-y)$. The posterior mean is

$$\begin{array}{rl} E[\theta|y] & = \frac{a+y}{a+b+n} \\ & = \frac{a}{a+b+n} + \frac{y}{a+b+n} \\ & = \frac{a+b}{a+b+n}\left(\frac{a}{a+b}\right) + \frac{n}{a+b+n}\left(\frac{y}{n}\right) \end{array}$$

Thus, the posterior mean is a weighted average of the prior mean $a/(a+b)$ and the MLE y/n with weights equal to the prior sample size $(a+b)$ and the data sample size (n).

Example data

Assume $Y \sim Bin(n, \theta)$ and $\theta \sim Be(1, 1)$ (which is equivalent to Unif(0,1)). If we observe three successes ($y = 3$) out of ten attempts ($n = 10$). Then our posterior is $\theta|y \sim Be(1 + 3, 1 + 10 - 3) \stackrel{d}{=} Be(4, 8)$. The posterior mean is

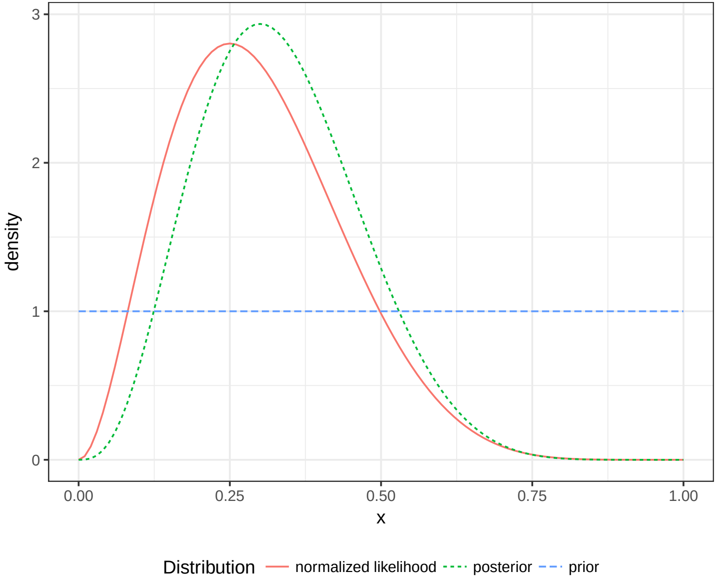
$$E[\theta|y] = \frac{2}{12}\frac{1}{2} + \frac{10}{12}\frac{3}{10} = \frac{4}{12}.$$

Remark Note that a $Be(1, 1)$ is equivalent to $p(\theta) = \mathrm{I}(0 < \theta < 1)$, i.e.

$$p(\theta|y) \propto p(y|\theta)p(\theta) = p(y|\theta)\mathrm{I}(0 < \theta < 1)$$

so it may seem that a reasonable approach to a default prior is to replace $p(\theta)$ by a 1 (times the parameter constraint). We will see later that this depends on the parameterization.

Posterior distribution



Point and interval estimation

Nothing inherently Bayesian about obtaining point and interval estimates.

Point estimation requires specifying a loss (or utility) function.

A $100(1-a)\%$ credible interval is any interval in the posterior that contains the parameter with probability $(1-a)$.

Point estimation

Define a loss (or utility) function $L\left(\theta, \hat{\theta}\right) = -U\left(\theta, \hat{\theta}\right)$ where

- θ is the parameter of interest
- $\hat{\theta} = \hat{\theta}(y)$ is the estimator of θ.

Find the estimator that minimizes the expected loss:

$$\hat{\theta}_{Bayes} = \text{argmin}_{\hat{\theta}}\, E\left[L\left(\theta, \hat{\theta}\right) \middle| y \right]$$

or maximizes expected utility.

Common estimators:

- Mean: $\hat{\theta}_{Bayes} = E[\theta|y]$ minimizes $L\left(\theta, \hat{\theta}\right) = \left(\theta - \hat{\theta}\right)^2$
- Median: $\int_{\hat{\theta}_{Bayes}}^{\infty} p(\theta|y)d\theta = \frac{1}{2}$ minimizes $L\left(\theta, \hat{\theta}\right) = \left|\theta - \hat{\theta}\right|$
- Mode: $\hat{\theta}_{Bayes} = \text{argmax}_\theta\, p(\theta|y)$ is obtained by minimizing $L\left(\theta, \hat{\theta}\right) = -\mathrm{I}\left(|\theta - \hat{\theta}| < \epsilon\right)$ as $\epsilon \to 0$, also called maximum a posterior (MAP) estimator.

Mean minimizes squared-error loss

Theorem

The mean minimizes expected squared-error loss.

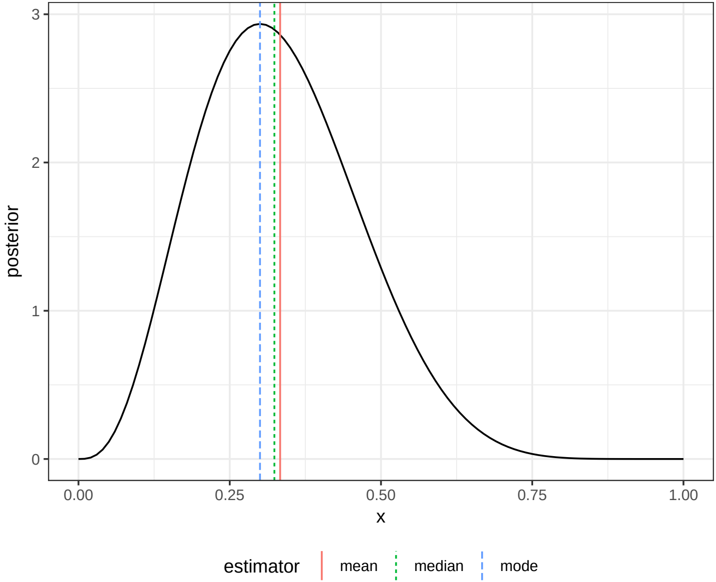
Proof.

Suppose $L\left(\theta,\hat{\theta}\right) = \left(\theta - \hat{\theta}\right)^2 = \theta^2 - 2\theta\hat{\theta} + \hat{\theta}^2$, then

$$\begin{aligned}
E\left[L\left(\theta,\hat{\theta}\right)\middle| y\right] &= E\left[\theta^2|y\right] - 2\hat{\theta}E[\theta|y] + \hat{\theta}^2 \\
\frac{d}{d\hat{\theta}}E\left[L\left(\theta,\hat{\theta}\right)\middle| y\right] &= -2E[\theta|y] + 2\hat{\theta} \overset{set}{=} 0 \implies \hat{\theta} = E[\theta|y] \\
\frac{d^2}{d\hat{\theta}^2}E\left[L\left(\theta,\hat{\theta}\right)\middle| y\right] &= 2
\end{aligned}$$

So $\hat{\theta} = E[\theta|y]$ minimizes expected squared-error loss. □

Point estimation



Interval estimation

Definition

A $100(1-a)\%$ credible interval is any interval (L,U) such that

$$1-a=\int_L^U p(\theta|y)d\theta.$$

Some typical intervals are

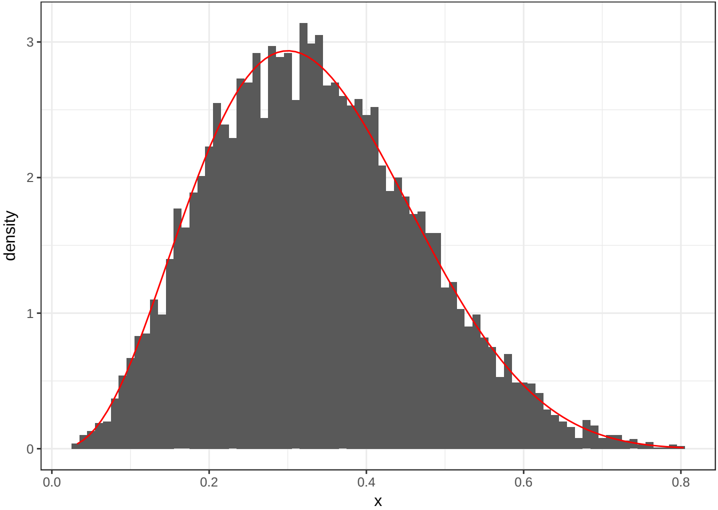
- Equal-tailed: $a/2=\int_{-\infty}^{L} p(\theta|y)d\theta=\int_U^{\infty} p(\theta|y)d\theta$
- One-sided: either $L=-\infty$ or $U=\infty$
- Highest posterior density (HPD): $p(L|y)=p(U|y)$ for a uni-modal posterior which is also the shortest interval

Interval estimation

```
Error in library("pscl"):  there is no package called 'pscl'
Error in betaHPD(a + y, b + n - y, 0.95):  could not find function "betaHPD"
Error in data.frame(type = c("equal-tail", "lower tail", "higher tail", :  object 'hpd' not found
Error in fortify(data):  object 'interval' not found
```

Simulation from the posterior

An estimate of the full posterior can be obtained via simulation, i.e.



Estimates via simulation

We can also obtain point and interval estimates using these simulations

```
c(mean = mean(sim$x), median = median(sim$x))
```

```
     mean    median 
0.3351243 0.3270852 
```

```
quantile(sim$x, c(.025,.975)) # Equal-tail
```

```
     2.5%     97.5% 
0.1095259 0.6112486 
```

```
quantile(sim$x, .05) # Upper
```

```
       5% 
0.1347931 
```

```
quantile(sim$x, .95) # Lower
```

```
      95% 
0.5653678 
```

Guess the probability

- A coin spins heads.
- Seattle Seahawks win 2015 Super Bowl.
- The first base pair on my genome is A.

What are priors?

Definition

A prior probability distribution, often called simply the prior, of an uncertain quantity θ is the probability distribution that would express one's uncertainty about θ before the "data" is taken into account.

http://en.wikipedia.org/wiki/Prior_distribution

Priors

Definition

A prior $p(\theta)$ is conjugate if for $p(\theta) \in \mathcal{P}$ and $p(y|\theta) \in \mathcal{F}$, $p(\theta|y) \in \mathcal{P}$ where $\mathcal{F}$ and $\mathcal{P}$ are families of distributions.

For example, the beta distribution ($\mathcal{P}$) is conjugate to the binomial distribution with unknown probability of success ($\mathcal{F}$) since

$$\theta \sim \text{Be}(a,b) \qquad \text{and} \qquad \theta|y \sim \text{Be}(a+y, b+n-y).$$

Definition

A natural conjugate prior is a conjugate prior that has the same functional form as the likelihood.

For example, the beta distribution is a natural conjugate prior since

$$p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1} \qquad \text{and} \qquad L(\theta) \propto \theta^{y}(1-\theta)^{n-y}.$$

Discrete priors are conjugate

Theorem

Discrete priors are conjugate.

Proof.

Suppose $p(\theta)$ is discrete, i.e.

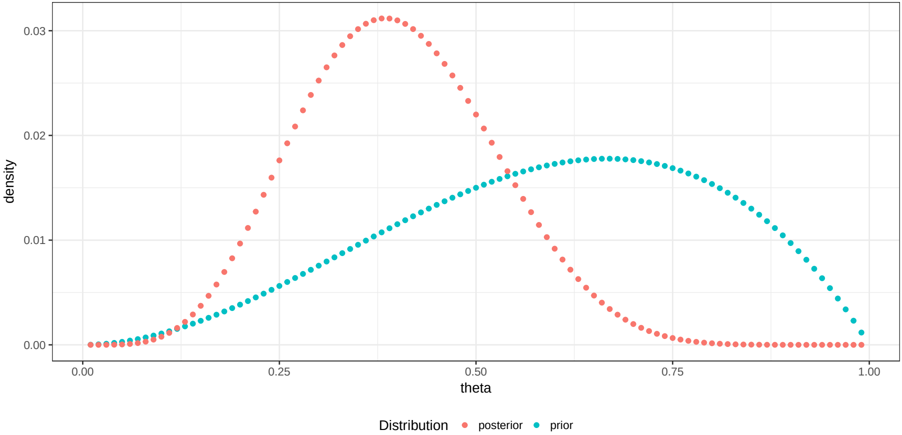
$$P(\theta = \theta_i) = p_i \qquad \sum_{i=1}^{\mathrm{I}} p_i = 1$$

and $p(y|\theta)$ is the model. Then, $P(\theta = \theta_i|y) = p_i'$ is the posterior with

$$p_i' = \frac{p_i p(y|\theta_i)}{\sum_{j=1}^{\mathrm{I}} p_j p(y|\theta_j)} \propto p_i p(y|\theta_i).$$

□

Discrete prior



Mixtures of conjugate priors are conjugate

Theorem

Mixtures of conjugate priors are conjugate.

Proof.

Let $p_i = P(H_i)$ and $p_i(\theta) = p(\theta|H_i)$,

$$\theta \sim \sum_{i=1}^{\mathrm{I}} p_i p_i(\theta) \qquad \sum_{i=1}^{\mathrm{I}} p_i = 1,$$

and $p_i(y) = \int p(y|\theta)p_i(\theta)d\theta$, then

$$\begin{aligned} p(\theta|y) &= \frac{1}{p(y)}p(y|\theta)p(\theta) = \frac{1}{p(y)}p(y|\theta)\sum_{i=1}^{\mathrm{I}} p_i p_i(\theta) \\ &= \frac{1}{p(y)}\sum_{i=1}^{\mathrm{I}} p_i p(y|\theta)p_i(\theta) = \frac{1}{p(y)}\sum_{i=1}^{\mathrm{I}} p_i p_i(y) p_i(\theta|y) \\ &= \sum_{i=1}^{\mathrm{I}} \frac{p_i p_i(y)}{p(y)} p_i(\theta|y) = \sum_{i=1}^{\mathrm{I}} \frac{p_i p_i(y)}{\sum_{j=1}^{\mathrm{I}} p_j p_j(y)} p_i(\theta|y) \end{aligned}$$

Mixtures of conjugate priors are conjugate

Bottom line: if

$$\theta \sim \sum_{i=1}^{\mathrm{I}} p_i p_i(\theta) \qquad \sum_{i=1}^{\mathrm{I}} p_i = 1$$

and $p_i(y) = \int p(y|\theta)p_i(\theta)d\theta$, then

$$\theta|y \sim \sum_{i=1}^{\mathrm{I}} p_i' p_i(\theta|y) \qquad p_i' \propto p_i p_i(y)$$

where $p_i(\theta|y) = p(y|\theta)p_i(\theta)/p_i(y)$.

Mixture of beta distributions

Recall, if $Y \sim Bin(n,\theta)$ and $\theta \sim \text{Be}(a,b)$, then

$$\begin{array}{rl} p(y) & = \int p(y|\theta)p(\theta)d\theta \\ & = \int \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\theta^{a-1}(1-\theta)^{b-1}}{\text{Beta}(a,b)} \\ & = \binom{n}{y}\frac{1}{\text{Beta}(a,b)}\int \theta^{a+y-1}(1-\theta)^{b+n-y-1}d\theta \\ & = \binom{n}{y}\frac{\text{Beta}(a+y,b+n-y)}{\text{Beta}(a,b)} \qquad y=0,\ldots,n \end{array}$$

If $Y \sim Bin(n,\theta)$ and

$$\theta \sim p\,\text{Be}(a_1,b_1) + (1-p)\text{Be}(a_2,b_2),$$

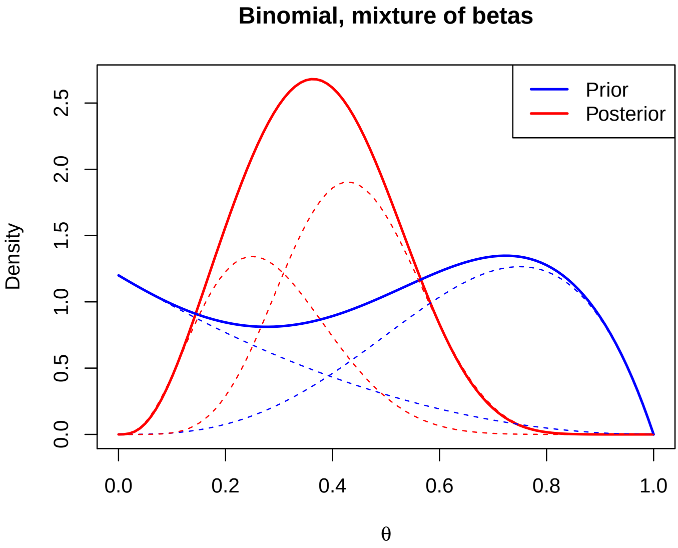
then

$$\theta|y \sim p'\,\text{Be}(a_1+y,b_1+n-y) + (1-p')\text{Be}(a_2+y,b_2+n-y)$$

with

$$p' = \frac{p\,p_1(y)}{p\,p_1(y)+(1-p)p_2(y)} \qquad p_i(y) = \binom{n}{y}\frac{\text{Beta}(a_i+y,b_i+n-y)}{\text{Beta}(a_i,b_i)}$$

Mixture priors



Default priors

Definition

A default prior is used when a data analyst is unable or unwilling to specify an informative prior distribution.

Default priors

Can we always use $p(\theta) \propto 1$?

Suppose we use $\phi = \log(\theta/[1-\theta])$, the log odds as our parameter, and set $p(\phi) \propto 1$, then the implied prior on θ is

$$\begin{array}{rl} p_\theta(\theta) \propto & 1 \left| \frac{d}{d\theta} \log(\theta/[1-\theta]) \right| \\ & = \frac{1-\theta}{\theta} \left[\frac{1}{1-\theta} + \frac{\theta}{[1-\theta]^2} \right] \\ & = \frac{1-\theta}{\theta} \left[\frac{[1-\theta]+\theta}{[1-\theta]^2} \right] \\ & = \theta^{-1}[1-\theta]^{-1} \end{array}$$

a Be(0,0), if that were a proper distribution, and is different from setting $p(\theta) \propto 1$ which results in the Be(1,1) prior.

Jeffreys prior

Definition

Jeffreys prior is a prior that is invariant to parameterization and is obtained via

$$p(\theta) \propto \sqrt{\det \mathcal{I}(\theta)}$$

where $\mathcal{I}(\theta)$ is the Fisher information.

For example, for a binomial distribution $\mathcal{I}(\theta) = \frac{n}{\theta[1-\theta]}$, so

$$p(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2} = \theta^{1/2-1}(1-\theta)^{1/2-1}$$

a Be(1/2,1/2) distribution.

Fisher information

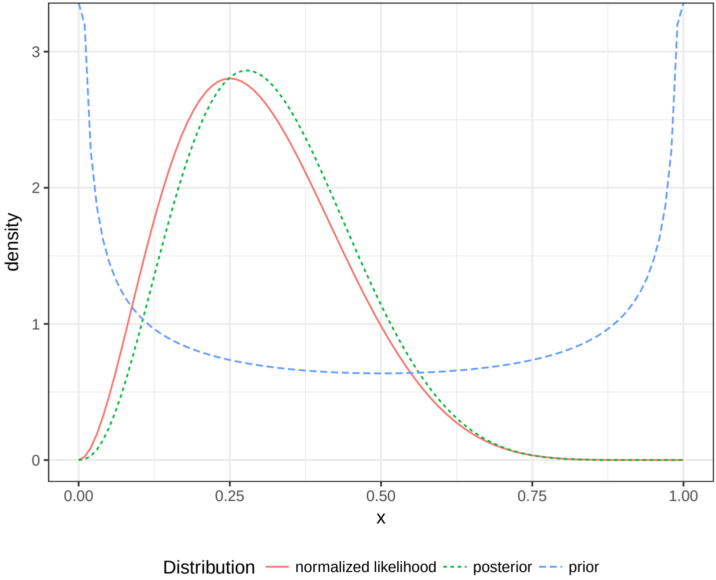
Theorem

The Fisher information for $Y \sim Bin(n, \theta)$ *is* $\mathcal{I}(\theta) = \frac{n}{\theta(1-\theta)}$.

Proof.

Since the binomial is an exponential family, we can use Lemma 7.3.11 of Casella and Berger (2nd ed).

$$
\begin{aligned}
\mathcal{I}(\theta) &= -E_{y|\theta}\left[\frac{\partial^2}{\partial\theta^2}\log p(y|\theta)\right] \\
&= -E_{y|\theta}\left[\frac{\partial^2}{\partial\theta^2}\log\binom{n}{y} + y\log\theta + (n-y)\log(1-\theta)\right] \\
&= -E_{y|\theta}\left[\frac{\partial}{\partial\theta}\frac{y}{\theta} - \frac{n-y}{1-\theta}\right] \\
&= -E_{y|\theta}\left[-\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}\right] \\
&= -\left[-\frac{n\theta}{\theta^2} - \frac{n-n\theta}{(1-\theta)^2}\right] = \frac{n}{\theta} + \frac{n}{(1-\theta)} \\
&= \frac{n}{\theta(1-\theta)}
\end{aligned}
$$


density
3
2
1
0
0.00
0.25
0.50
0.75
1.00
x
Distribution
normalized likelihood
posterior
prior

Non-conjugate priors

If $Y \sim Bin(n, \theta)$ and $p(\theta) = e^\theta/(e-1)$, then

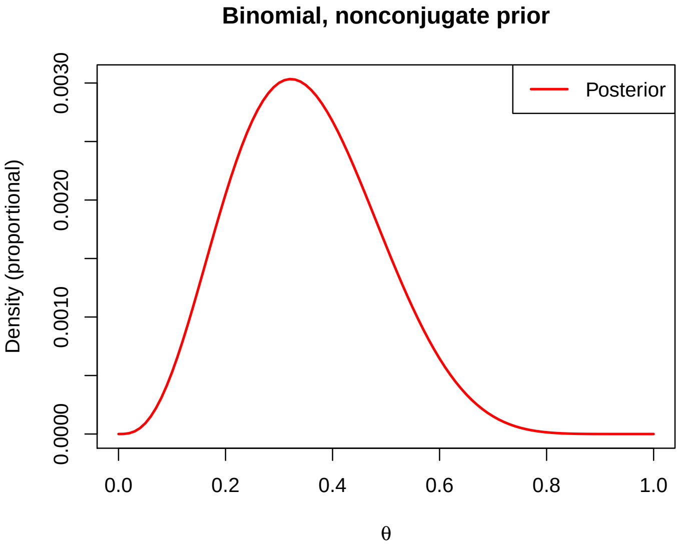
$$p(\theta|y) \propto f(\theta) = \theta^y(1-\theta)^{n-y}e^\theta$$

which is not a known distribution.

Options

- Plot $f(\theta)$ (possibly multiplying by a constant).
- Find $i = \int f(\theta)d\theta$, so that $p(\theta|y) = f(\theta)/i$.
- Evaluate $f(\theta)$ on a grid and normalize by the grid spacing.

Plot of $f(\theta)$



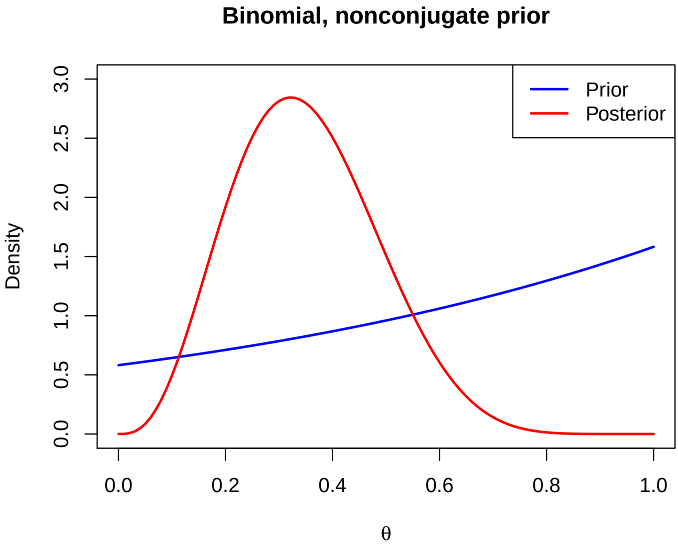
Numerical integration

Find $i = \int f(\theta) d\theta$, so that $p(\theta|y) = f(\theta)/i$.

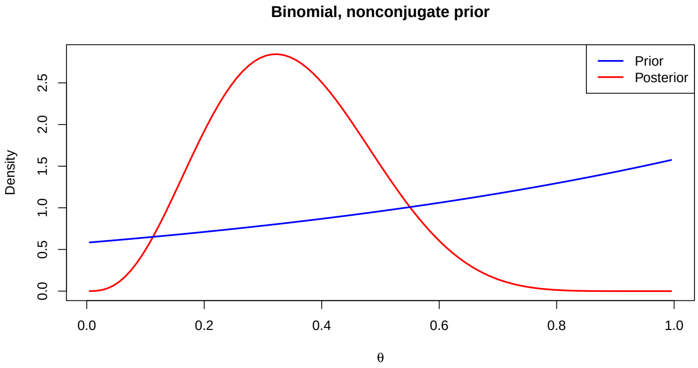
```
(i = integrate(f, 0, 1))

0.001066499 with absolute error < 1.2e-17
```

Nonconjugate prior, numerical integration



Nonconjugate prior, evaluated on a grid




```
theta[c(which(cumsum(d)*w>0.025)[1]-1, which(cumsum(d)*w>0.975)[1])] # 95\% CI

[1] 0.105 0.625
```

Improper priors

Definition

An unnormalized density, $f(\theta)$, is proper if $\int f(\theta)d\theta = c < \infty$, and otherwise it is improper.

To create a normalized density from a proper unnormalized density, use

$$p(\theta|y) = \frac{f(\theta)}{c}$$

to see that $p(\theta|y)$ is a proper normalized density note that $c = \int f(\theta)d\theta$ is not a function of θ, then

$$\int p(\theta|y)d\theta = \int \frac{f(\theta)}{\int f(\theta)d\theta}d\theta = \int \frac{f(\theta)}{c}d\theta = \frac{1}{c}\int f(\theta)d\theta = \frac{c}{c} = 1$$

Be(0,0) prior

Recall that Be(a, b) is a proper probability distribution if $a > 0, b > 0$.

Suppose $Y \sim Bin(n, \theta)$ and $p(\theta) \propto \theta^{-1}(1-\theta)^{-1}$, i.e. the kernel of a $Be(0, 0)$ distribution. This is an improper distribution.

The posterior, $\theta|y \sim Be(y, n-y)$, is proper if $0 < y < n$.