Set S03 - Random effects

STAT 401 (Engineering) - Iowa State University

April 21, 2017

Regression models

For continuous Y_i, we have linear regression

$$Y_i \stackrel{ind}{\sim} N(\mu_i, \sigma^2), \quad \mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

For binary or count with an upper maximum Y_i, we have logistic regression

$$Y_i \stackrel{ind}{\sim} Bin(n_i, \theta_i), \quad \log\left(\frac{\theta_i}{1-\theta_i}\right) = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

For count data with no upper maximum, we have Poisson regression

$$Y_i \stackrel{ind}{\sim} Po(\lambda_i), \quad \log(\lambda_i) = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

But what if our observations cannot reasonably be assumed to be independent given these explanatory variables?

Random effect model

Suppose we have continuous observations Y_{ij} for individual i from group j. A random effects model (with a common variance) assumes

$$Y_{ij} = \mu + \alpha_j + \epsilon_{ij} \quad \epsilon_{ij} \stackrel{ind}{\sim} N(0, \sigma_\epsilon^2)$$

and, to make the α_i random effects, independent of ϵ_{ij}

$$\alpha_j \stackrel{ind}{\sim} N(0, \sigma_\alpha^2).$$

This makes observations within the group correlated since

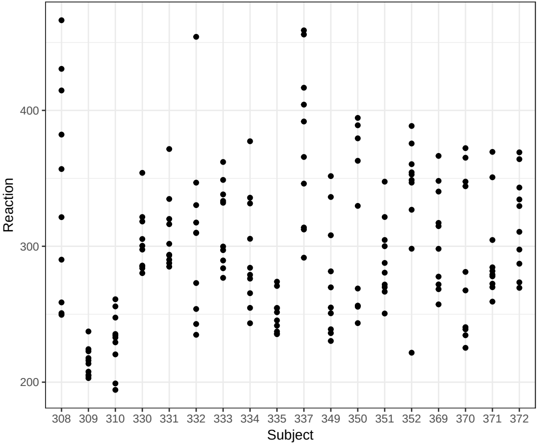
$$\begin{aligned} Cov[Y_{ij}, Y_{i'j}] &= Cov[\alpha_j + \epsilon_{ij}, \alpha_j + \epsilon_{i'j}] \\ &= Var[\alpha_j] = \sigma_\alpha^2 \end{aligned}$$

and

$$Cor[Y_{ij}, Y_{i'j}] = \frac{Cov[Y_{ij}, Y_{i'j}]}{\sqrt{Var[Y_{ij}]Var[Y_{i'j}]}} = \frac{\sigma_\alpha^2}{\sigma_\alpha^2 + \sigma_\epsilon^2}.$$

Sleep study example

```r
ggplot(sleepstudy, aes(Subject, Reaction)) + geom_point() + theme_bw()
```



Sleep study example

```
summary(me <- lmer(Reaction ~ (1|Subject), sleepstudy))


Linear mixed model fit by REML ['lmerMod']
Formula: Reaction ~ (1 | Subject)
   Data: sleepstudy

REML criterion at convergence: 1904.3

Scaled residuals:
    Min      1Q  Median      3Q     Max
-2.4983 -0.5501 -0.1476  0.5123  3.3446

Random effects:
 Groups   Name        Variance Std.Dev.
 Subject  (Intercept) 1278     35.75
 Residual             1959     44.26
Number of obs: 180, groups:  Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept)   298.51       9.05   32.98
```

Mixed effect model

Suppose we have continuous observations Y_{ij} for individual i from group j and an associated explanatory variable X_{ij}. A mixed effect model assumes

$$Y_{ij} = \beta_0 + \beta_1 X_{ij} + \alpha_j + \epsilon_{ij} \quad \epsilon_{ij} \stackrel{ind}{\sim} N(0, \sigma^2_\epsilon)$$

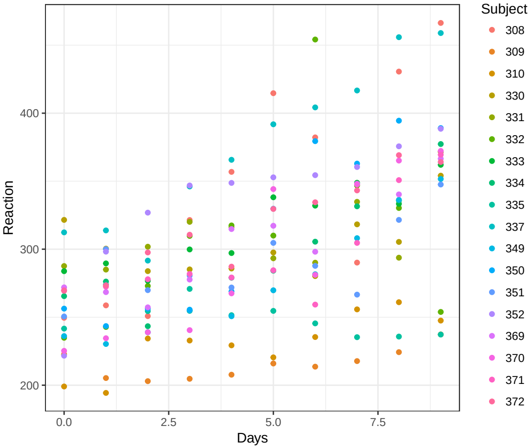
and, to make the α_i random effects, independent of ϵ_{ij}

$$\alpha_j \stackrel{ind}{\sim} N(0, \sigma^2_\alpha).$$

Again, this enforces a correlation between the observations within a group. This model is often referred to as a random intercept model because each group has its own intercept $(\beta_0 + \alpha_j)$ and these are *random* since α_j has a distribution. Thus this model is related to a model that includes a fixed effect for each subject. But here those subject specific effects are shrunk toward an overall mean (β_0).

Sleep study example

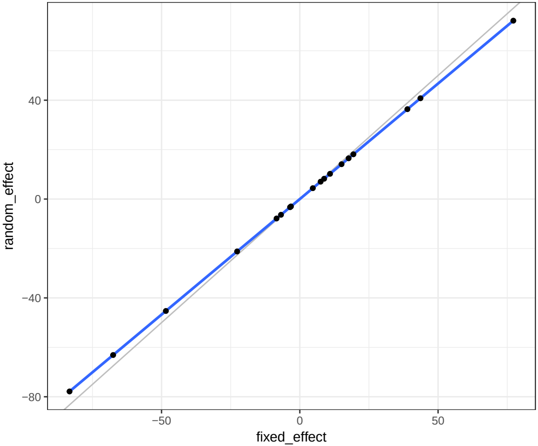
```
ggplot(sleepstudy, aes(Days, Reaction, color = Subject)) +
  geom_point() + theme_bw()
```



Sleep study example

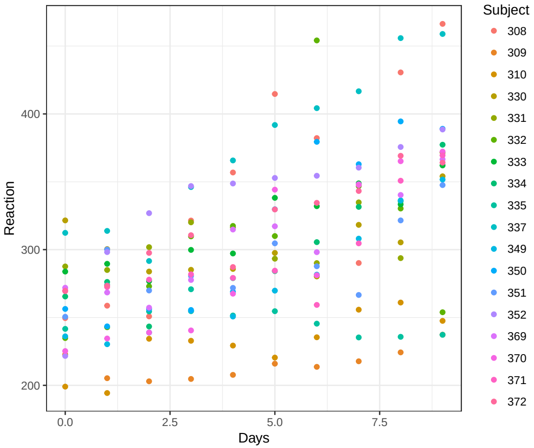
```
summary(me <- lmer(Reaction ~ Days + (1|Subject), sleepstudy))

Linear mixed model fit by REML ['lmerMod']
Formula: Reaction ~ Days + (1 | Subject)
   Data: sleepstudy

REML criterion at convergence: 1786.5

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.2257 -0.5529  0.0109  0.5188  4.2506

Random effects:
 Groups   Name        Variance Std.Dev.
 Subject  (Intercept) 1378.2   37.12
 Residual              960.5   30.99
Number of obs: 180, groups:  Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept) 251.4051     9.7467   25.79
Days         10.4673     0.8042   13.02

Correlation of Fixed Effects:
     (Intr)
Days -0.371
```

Shrinkage



Sleep study example

```r
ggplot(sleepstudy, aes(Days, Reaction, color = Subject)) +
  geom_point() + theme_bw()
```



Random slope model

Suppose we have continuous observations Y_{ij} for individual i from group j. A mixed effect model with group specific slopes assumes

$$Y_{ij} = \beta + 0 + \beta_1 X_{ij} + \alpha_{0j} + \alpha_{1j} X_{ij} + \epsilon_{ij} \quad \epsilon_{ij} \overset{ind}{\sim} N(0, \sigma_\epsilon^2)$$

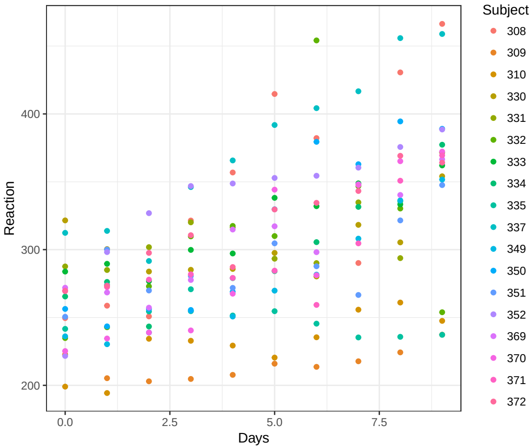
and, independent of ϵ_{ij},

$$\begin{pmatrix} \alpha_{0j} \\ \alpha_{1j} \end{pmatrix} \overset{ind}{\sim} N(0, \Sigma_\alpha)$$

$N(0, \Sigma_\alpha)$ represents a bivariate normal with mean 0 and covariance matrix Σ_α. This model is often referred to as a random slope model because each group has its own slope $(\beta_1 + \alpha_{1j})$ and these are *random* since α_{1j} has a distribution. Thus this model is related to a model that includes an interaction between the group and the explanatory variable, but here those subject specific slopes are shrunk toward an overall slope (β_1).

Sleep study example

```
ggplot(sleepstudy, aes(Days, Reaction, color = Subject)) +
  geom_point() + theme_bw()
```



Sleep study example

```
summary(me <- lmer(Reaction ~ Days + (Days|Subject), sleepstudy))


Linear mixed model fit by REML ['lmerMod']
Formula: Reaction ~ Days + (Days | Subject)
   Data: sleepstudy

REML criterion at convergence: 1743.6

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.9536 -0.4634  0.0231  0.4634  5.1793

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 Subject  (Intercept) 612.09   24.740
          Days         35.07    5.922   0.07
 Residual             654.94   25.592
Number of obs: 180, groups:  Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept)  251.405      6.825   36.84
Days          10.467      1.546    6.77

Correlation of Fixed Effects:
     (Intr)
Days -0.138
```