M3S1 - Binomial Distribution

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Outline

- Random variables
  - Probability distribution function
  - Expectation (mean)
  - Variance
- Discrete distributions
  - Bernoulli
  - Binomial

Probability

Definition

A probability is a mathematical function, $P(E)$, that describes how likely an event E is to occur. This function adheres to two basic rules:

1. $0 \leq P(E) \leq 1$
2. For mutually exclusive events $E_1, \ldots, E_K$,

$$P(E_1 \text{ or } E_2 \text{ or } \cdots \text{ or } E_K) = P(E_1) + P(E_2) + \cdots + P(E_K).$$

Flipping a coin

Suppose we are flipping an unbiased coin that has two sides: heads (H) and tails (T). Then

$$P(H) = 0.5 \qquad P(T) = 0.5.$$

which adheres to rule 1) and

$$P(H \text{ or } T) = P(H) + P(T) = 0.5 + 0.5 = 1$$

which adheres to rule 2). So this is a valid probability.

Rolling a 6-sided die

Suppose we are rolling an unbiased 6-sided die. If we count the number of pips on the upturned face, then the possible events are 1, 2, 3, 4, 5, and 6. Then

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$$

which adheres to 1). What is

$$P(1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6) = 1.$$

To verify 2), we would need to calculate the probability of the 2^6 possible colections of mutually exclusive events and find that their probability is the sum of the individual probabilities.

Random variable

Definition

A random variable is the uncertain, numeric outcome of a random process. A discrete random variable takes on one of a list of possible values. A continuous random variable takes on any value in an interval.

A random variable is denoted by a capital letter, e.g. X or Y.

Discrete random variables:

- result of a coin flip
- the number of pips on the upturned face of a 6-sided die roll
- whether or not a company beats its earnings forecast
- the number of HR incidents next month

Continuous random variables:

- my height
- how far away a 6-sided die lands
- a company's next quarterly earnings
- a company's closing stock price tomorrow

Probability distribution function

Definition

A probability distribution function describes all possible outcomes for a random variable and the probability of those outcomes.

For example,

- Coin flipping:

$$P(H) = P(T) = 1.$$

- Unbiased 6-sided die rolling

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6.$$

- Company earnings compared to forecasts

$$\begin{aligned} P(\text{Earnings within 5\% of forecast}) &= 0.6 \\ P(\text{Earnings less than 5\% of forecast}) &= 0.1 \\ P(\text{Earnings greater than 5\% of forecast}) &= 0.3 \end{aligned}$$

Events

Definition

An event is a set of possible outcomes of a random variable.

Discrete random variables:

- a coin flipping heads is heads
- the number of pips on the upturned face of a 6-sided die roll is less than 3
- a company beats its earnings forecast
- the number of HR incidents next month is less between 5 and 10

Continuous random variables:

- my height is greater than 6 feet
- how far away a 6-sided die lands is less than 3 feet
- a company's next quarterly earnings is within 5% of forecast
- a company's closing stock price tomorrow is less than today's

Die rolling

Suppose we roll an unbiased 6-sided die. Determine the probabilities of the following events. The number of pips is

- exactly 3
- less than 3
- is greater than or equal to 3
- is odd
- is even and less than 5

Bernoulli random variable

Definition

A Bernoulli random variable has two possible outcomes:

- 1 (success)
- 0 (failure)

A Bernoulli random variable is completey characterized by a single probability p, the probability of success (1). We write $X \sim Ber(p)$ to indicate that X is a random variable that has a Bernoulli distribution with probability of success p. If $X \sim Ber(p)$, then we know $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

Examples:

- a coin flip landing heads
- a 6-sided die landing on 1
- a 6-sided die landing on 1 or 2
- a company beating its earnings forecast
- a company's stock price closing higher tomorrow

Coin flipping

Suppose we are flipping an unbiased coin and we let

$$X = \begin{cases} 0 & \text{if coin flip lands on tails} \\ 1 & \text{if coin flip lands on heads} \end{cases}$$

Then $X \sim Ber(0.5)$ which means $p = 0.5$ is the probability of success (heads) and $P(X = 1) = 0.5$ and $P(X = 0) = 0.5$.

Die rolling

Suppose we are rolling an unbiased 6-sided die and we let

$$X = \begin{cases} 0 & \text{if die lands on 3, 4, 5, or 6} \\ 1 & \text{if die lands on 1 or 2} \end{cases}$$

Then $X \sim Ber(1/3)$ which means $p = 1/3$ is the probability of success (a 1 or 2) and $P(X = 1) = 1/3$ and $P(X = 0) = 2/3$.

Mean of a random variable

Definition

The mean of a random variable is a probability weighted average of the outcomes of that random variable. This mean is also called the expectation of the random variable and for a random variable X is denoted $E[X]$ (or $E(X)$).

For a Bernoulli random variable $X \sim Ber(p)$, we have

$$E[X] = (1 - p) \times 0 + p \times 1 = p.$$

Variance of a random variable

Definition

The variance of a random variable is the probability-weighted average of the squared difference from the mean. The variance of a random variable X is denoted $Var[X]$ (or $Var(X)$).

For a Bernoulli random variable $X \sim Ber(p)$, we have

$$\begin{aligned} Var[X] \quad &= (1-p) \times (0-p)^2 + p \times (1-p)^2 \\ &= (1-p) \times p^2 + p \times (1-2p+p^2) \\ &= p^2 - p^3 + p - 2p^2 + p^3) \\ &= p - p^2 \\ &= p(1-p). \end{aligned}$$

Coin flipping

If $X \sim Ber(0.5)$, then

- $E[X] = 1/2$
- $Var[X] = 1/2 \times (1 - 1/2) = 1/2 \times 1/2 = 1/4.$

If $X \sim Ber(1/3)$, then

- $E[X] = 1/3$
- $Var[X] = 1/3 \times (1 - 1/3) = 1/3 \times 2/3 = 2/9.$

If $X \sim Ber(2/9)$, then

- $E[X] = 2/9$
- $Var[X] = 2/9 \times (1 - 2/9) = 2/9 \times 7/9 = 14/81.$

Die rolling

Let X be the number of pips on the upturned face of an unbiased 6-sided die. Find the probability distribution function, the expected value (mean), and the variance.

Then the probability distribution function is

$$P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = 1/6.$$

The expected value, $E[X]$, is

$$\begin{aligned} E[X] &= 1/6 \times 1 + 1/6 \times 2 + 1/6 \times 3 + 1/6 \times 4 + 1/6 \times 5 + 1/6 \times 6 \\ &= 3.5. \end{aligned}$$

The variance, $Var[X]$, is

$$\begin{aligned} Var[X] &= 1/6 \times (1 - 3.5)^2 + 1/6 \times (2 - 3.5)^2 + 1/6 \times (3 - 3.5)^2 \\ &\quad + 1/6 \times (4 - 3.5)^2 + 1/6 \times (5 - 3.5)^2 + 1/6 \times (6 - 3.5)^2 \\ &= 2.91\overline{6}. \end{aligned}$$

Independence

Definition

Two random variables are independent if the outcome of one random variable does not affect the probabilities of the outcomes of the other random variable.

For independent random variables X and Y and constants a, b, and c, we have the following properties

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

and

$$Var[aX + bY + c] = a^2 Var[X] + b^2 Var[Y].$$

Sum of independent Bernoulli random variables

Let X_i, for $i = 1, \ldots, n$ be independent Bernoulli random variable with a common probability of success p. We write

$$X_i \stackrel{ind}{\sim} Ber(p).$$

Then the sum

$$Y = \sum_{i=1}^{n} X_i$$

is a binomial random variable.

Binomial

Definition

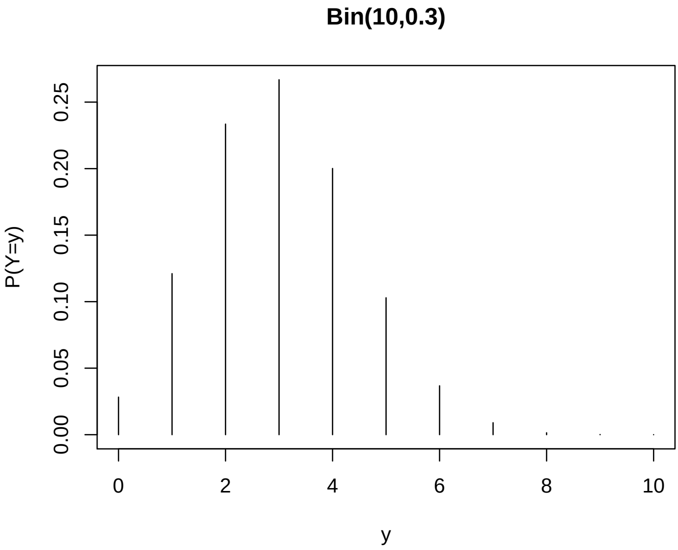
A binomial random variable with n attempts and probability of success p has a probability distribution function

$$P(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y}$$

for $0 \leq p \leq 1$ and $y = 0, 1, \ldots, n$ where

$$\binom{n}{y} = \frac{n!}{(n-y)!y!}.$$

We write $Y \sim Bin(n, p)$.


Bin(10,0.3)
P(Y=y)
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
y

Binomial expected value and variance

The expected value (mean) is

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \cdots + X_n] \\ &= E[X_1] + E[X_2] + \cdots + E[X_n] \\ &= p + p + \cdots + p \\ &= np. \end{aligned}$$

The variance is

$$\begin{aligned} Var[Y] &= Var[X_1 + X_2 + \cdots + X_n] \\ &= Var[X_1] + Var[X_2] + \cdots + Var[X_n] \\ &= p(1-p) + p(1-p) + \cdots + p(1-p) \\ &= np(1-p). \end{aligned}$$

Examples

If $Y \sim Bin(10, .3)$, then

$$E[Y] = 10 \times 0.3 = 3$$

and

$$Var[Y] = 10 \times 0.3 \times (1 - 0.3) = 10 \times 0.3 \times 0.7 = 2.1.$$

If $Y \sim Bin(65, 1/4)$, then

$$E[Y] = 65 \times 1/4 = 16.25$$

and

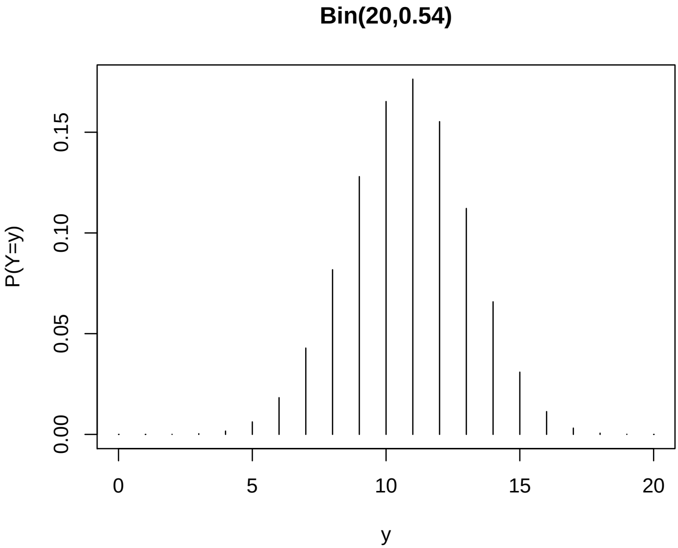
$$Var[Y] = 65 \times 1/4 \times (1 - 1/4) = 65 \times 1/4 \times 3/4 = 12.1875.$$

AVP Example

In the 2018 AVP Gold Series Championships in Chicago, IL, Alex Klineman and April Ross beat Sara Hughes and Summer Ross in 2 sets with scores 25-23, 21-16. Suppose that these scores actually determine the probability that Klineman/Ross will score a point against Hughes/Ross, i.e. $p = (25 + 21)/(25 + 23 + 21 + 16) = 0.54$ and that each point is independent.

Let Y be the number of points Klineman/Ross will win (against Hughes/Ross) over the next 20 points. Based on our assumptions $Y \sim Bin(20, 0.54)$.

AVP Example (cont.)



AVP Example (cont.)

Here are some questions we can answer:

- How many points do we expect Klineman/Ross to score?

$$E[Y] = 20 \times .54 = 10.8 \text{ points}$$

- What is the variance around this number?

$$Var[Y] = 20 \times .54 \times (1 - .54) = 4.966 \text{ points}^2$$

- What is the standard deviation around this number?

$$SD[Y] = \sqrt{Var[Y]} = \sqrt{4.966} = 2.23 \text{ points}$$

- What is the probability that Klineman/Ross will win at least 10 points?

$$P(Y >= 10) = P(Y = 10) + P(Y = 11) + \cdots + P(Y = 20) = 0.72$$

AVP Example (cont.)