Set R05 - Multiple Regression

STAT 401 (Engineering) - Iowa State University

March 29, 2017

Multiple regression

Recall the simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

The multiple regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}, \sigma^2)$$

where

- Y_i is the response for observation i and
- $X_{i,p}$ is the p^{th} explanatory variable for observation i.

We may also write

$$Y_i \stackrel{ind}{\sim} N(\mu_i, \sigma^2) \quad \text{or} \quad Y_i = \mu_i + e_i, e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

where

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}.$$

Explanatory variables

There is a lot of flexibility in the mean

$$\mu_i = E[Y_i|X_{i,1}, \ldots, X_{i,p}] = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

as there are many possibilities for the explanatory variables $X_{i,1}, \ldots, X_{i,p}$:

- Higher order terms (X^2)
- Additional explanatory variables ($X_1 + X_2$)
- Dummy variables for categorical variables ($X_1 = \mathrm{I}()$)
- Interactions ($X_1 X_2$)
  - Continuous-continuous
  - Continuous-categorical
  - Categorical-categorical

Interpretation

Model:

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}, \sigma^2)$$

The interpretation is

- β_0 is the expected value of the response Y_i when all explanatory variables are zero.
- $\beta_p,\ p \neq 0$ is the expected increase in the response for a one-unit increase in the p^{th} explanatory variable when all other explanatory variables are held constant.
- R^2 is the proportion of the variance in the response explained by the model

Parameter estimation

Let

$$y = X\beta + \epsilon$$

where

- $y = (y_1, \ldots, y_n)^\top$
- X is $n \times p$ with ith row $X_i = (X_{i,1}, \ldots, X_{i,p})$
- $\beta = (\beta_1, \ldots, \beta_p)^\top$
- $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^\top$

Then we have

$$\begin{aligned} \hat{\beta} &= (X^\top X)^{-1} X^\top y \\ Var(\hat{\beta}) &= \sigma^2 (X^\top X)^{-1} \\ r &= y - X\hat{\beta} \\ \hat{\sigma}^2 &= \frac{1}{n-p} r^\top r \end{aligned}$$

Confidence/credible intervals and pvalues are constructed using

$$\hat{\beta}_j \pm t_{n-p,1-a/2} SE(\hat{\beta}_j) \quad \text{and} \quad \text{pvalue} = \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)}$$

where $SE(\hat{\beta}_j)$ is the jth diagonal element of $\hat{\sigma}^2 (X^\top X)^{-1}$.

Higher order terms (X^2)

Let

- Y_i be the distance for the i^{th} run of the experiment and
- H_i be the height for the i^{th} run of the experiment.

Simple linear regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i \qquad\qquad\qquad\qquad , \sigma^2)$$

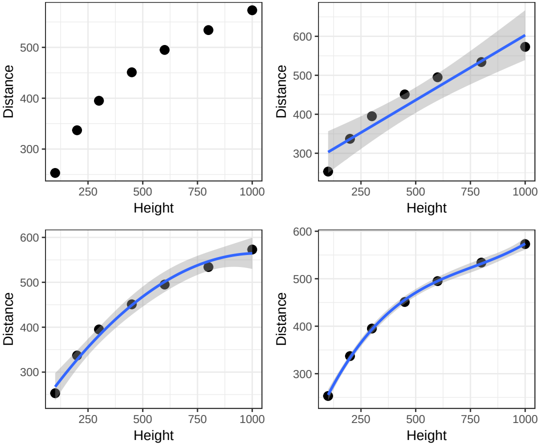
The quadratic multiple regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i + \beta_2 H_i^2 \qquad\qquad , \sigma^2)$$

The cubic multiple regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i + \beta_2 H_i^2 + \beta_3 H_i^3, \sigma^2)$$

Case1001



R code and output

```
# Construct the variables by hand
case1001$Height2 = case1001$Height^2
case1001$Height3 = case1001$Height^3

m1 = lm(Distance~Height,                 case1001)
m2 = lm(Distance~Height+Height2,         case1001)
m3 = lm(Distance~Height+Height2+Height3, case1001)

coefficients(m1)


(Intercept)      Height
 269.712458    0.333337


coefficients(m2)


  (Intercept)        Height       Height2
 1.999128e+02  7.083225e-01 -3.436937e-04


coefficients(m3)


  (Intercept)        Height       Height2       Height3
 1.557755e+02  1.115298e+00 -1.244943e-03  5.477104e-07
```

R code and output

```
# Let R construct the variables for you
m = lm(Distance~poly(Height, 3, raw=TRUE), case1001)
summary(m)


Call:
lm(formula = Distance ~ poly(Height, 3, raw = TRUE), data = case1001)

Residuals:
       1        2        3        4        5        6        7
-2.40359  3.58091  1.89175 -4.46885 -0.08044  2.32159 -0.84138

Coefficients:
                               Estimate Std. Error t value Pr(>|t|)
(Intercept)                   1.558e+02  8.326e+00  18.710 0.000333 ***
poly(Height, 3, raw = TRUE)1  1.115e+00  6.567e-02  16.983 0.000445 ***
poly(Height, 3, raw = TRUE)2 -1.245e-03  1.384e-04  -8.994 0.002902 **
poly(Height, 3, raw = TRUE)3  5.477e-07  8.327e-08   6.577 0.007150 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.011 on 3 degrees of freedom
Multiple R-squared:  0.9994, Adjusted R-squared:  0.9987
F-statistic:  1595 on 3 and 3 DF,  p-value: 2.662e-05
```

Longnose Dace Abundance

From `http://udel.edu/~mcdonald/statmultreg.html`:

> *I extracted some data from the Maryland Biological Stream Survey. ... The dependent variable is the number of Longnose Dace (Rhinichthys cataractae) per 75-meter section of [a] stream. The independent variables are the area (in acres) drained by the stream; the dissolved oxygen (in mg/liter); the maximum depth (in cm) of the 75-meter segment of stream; nitrate concentration (mg/liter); sulfate concentration (mg/liter); and the water temperature on the sampling date (in degrees C).*

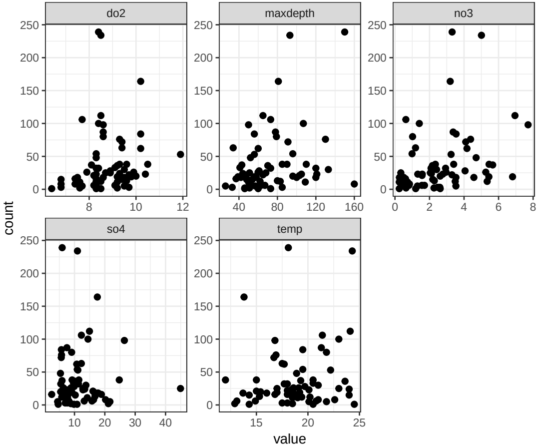
Consider the model

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}, \sigma^2)$$

where

- Y_i: count of Longnose Dace in stream i
- $X_{i,1}$: maximum depth (in cm) of stream i
- $X_{i,2}$: nitrate concentration (mg/liter) of stream i

Exploratory



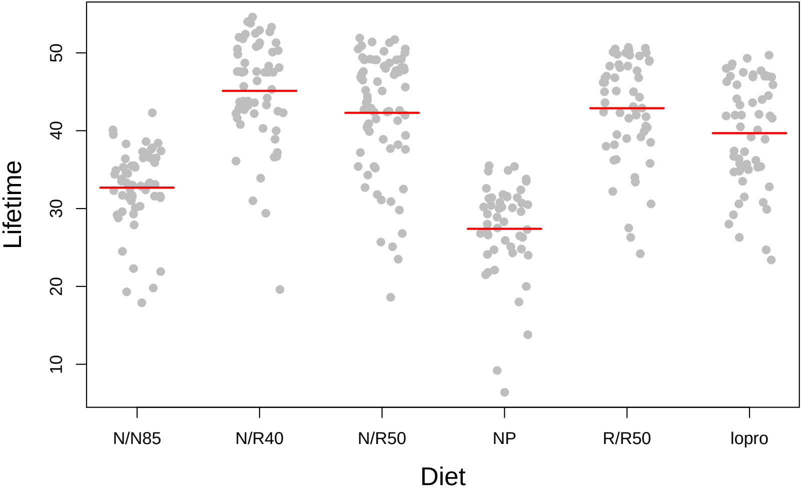
R code and output

```
m <- lm(count~no3+maxdepth, longnosedace)
summary(m)


Call:
lm(formula = count ~ no3 + maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-55.060 -27.704  -8.679  11.794 165.310

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.5550    15.9586  -1.100  0.27544
no3           8.2847     2.9566   2.802  0.00671 **
maxdepth      0.4811     0.1811   2.656  0.00997 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.39 on 64 degrees of freedom
Multiple R-squared:  0.1936, Adjusted R-squared:  0.1684
F-statistic: 7.682 on 2 and 64 DF,  p-value: 0.001022
```

Interpretation

- Intercept (β_0): The expected count of Longnose Dace when maximum depth and nitrate concentration are both zero is -18.
- Coefficient for maxdepth (β_1): Holding nitrate concentration constant, each cm increase in maximum depth is associated with an additional 0.48 Longnose Dace counted on average.
- Coefficient for no3 (β_2): Holding maximum depth constant, each mg/liter increase in nitrate concentration is associated with an addition 8.3 Longnose Dace counted on average.
- Coefficient of determination (R^2): The model explains 19% of the variability in the count of Longnose Dace.

Using a categorical variable as an explanatory variable.



Regression with a categorical variable

- Choose one of the levels as the reference level, e.g. N/N85
- Construct dummy variables using indicator functions, i.e.

$$\mathrm{I}(A) = \begin{cases} 1 & A \text{ is TRUE} \\ 0 & A \text{ is FALSE} \end{cases}$$

for the other levels, e.g.

$$\begin{aligned}
X_{i,1} &= \mathrm{I}(\text{diet for observation } i \text{ is N/R40}) \\
X_{i,2} &= \mathrm{I}(\text{diet for observation } i \text{ is N/R50}) \\
X_{i,3} &= \mathrm{I}(\text{diet for observation } i \text{ is NP}) \\
X_{i,4} &= \mathrm{I}(\text{diet for observation } i \text{ is R/R50}) \\
X_{i,5} &= \mathrm{I}(\text{diet for observation } i \text{ is lopro})
\end{aligned}$$

- Estimate the parameters of a multiple regression model using these dummy variables.

R code and output

```
# by default, R uses the alphabetically first group as the reference level
case0501$Diet = relevel(case0501$Diet, ref='N/N85')

m = lm(Lifetime~Diet, case0501)
summary(m)


Call:
lm(formula = Lifetime ~ Diet, data = case0501)

Residuals:
     Min       1Q   Median       3Q      Max
-25.5167  -3.3857   0.8143   5.1833  10.0143

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  32.6912     0.8846  36.958  < 2e-16 ***
DietN/R40    12.4254     1.2352  10.059  < 2e-16 ***
DietN/R50     9.6060     1.1877   8.088 1.06e-14 ***
DietNP       -5.2892     1.3010  -4.065 5.95e-05 ***
DietR/R50    10.1945     1.2565   8.113 8.88e-15 ***
Dietlopro     6.9945     1.2565   5.567 5.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.678 on 343 degrees of freedom
Multiple R-squared:  0.4543, Adjusted R-squared:  0.4463
F-statistic:  57.1 on 5 and 343 DF,  p-value: < 2.2e-16
```

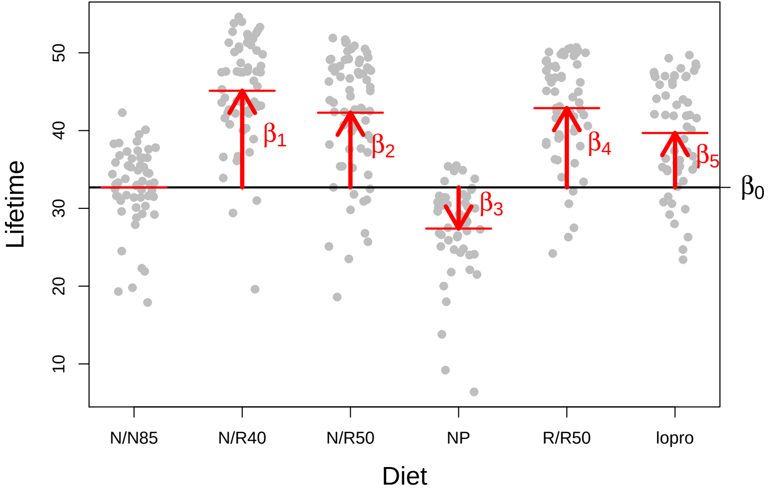
Interpretation

- $\beta_0 = E[Y_i|\text{reference level}]$, i.e. expected response for the reference level
  Note: the only way $X_{i,1} = \cdots = X_{i,p} = 0$ is if all indicators are zero, i.e. at the reference level.
- $\beta_p, p > 0$: expected change in the response moving from the reference level to the level associated with the p^{th} dummy variable
  Note: the only way for $X_{i,p}$ to increase by one and all other indicators to stay constant is if initially $X_{i,1} = \cdots = X_{i,p} = 0$ and now $X_{i,p} = 1$

For example,

- The expected lifetime for mice on the N/N85 diet is 32.7 weeks.
- The expected increase in lifetime for mice on the N/R40 diet compared to the N/N85 diet is 12.4 weeks.
- The model explains 45% of the variability in mice lifetimes.

Using a categorical variable as an explanatory variable.



Interactions

Why an interaction?

> *Two explanatory variables are said to interact if the effect that one of them has on the mean response depends on the value of the other.*

For example,

- Longnose dace: The effect of nitrate (no3) on longnose dace count depends on the maxdepth. (Continuous-continuous)
- Case1002: The effect of mass on energy depends on the species type. (Continuous-categorical)
- Yield: the effect of tillage method depends on the fertilizer brand (Categorical-categorical)

Continuous-continuous interaction

For observation i, let

- Y_i be the response
- $X_{i,1}$ be the first explanatory variable and
- $X_{i,2}$ be the second explanatory variable.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}.$$

The mean with the interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,1} X_{i,2}.$$

Intepretation - main effects only

Let $X_{i,1} = x_1$ and $X_{i,2} = x_2$, then we can rewrite the line (μ) as

$$\mu = (\beta_0 + \beta_2 x_2) + \beta_1 x_1$$

which indicates that the intercept of the line for x_1 depends on the value of x_2.

Similarly,

$$\mu = (\beta_0 + \beta_1 x_1) + \beta_2 x_2$$

which indicates that the intercept of the line for x_2 depends on the value of x_1.

Intepretation - with an interaction

Let $X_{i,1} = x_1$ and $X_{i,2} = x_2$, then we can rewrite the mean (μ) as

$$\mu = (\beta_0 + \beta_2 x_2) + (\beta_1 + \beta_3 x_2) x_1$$

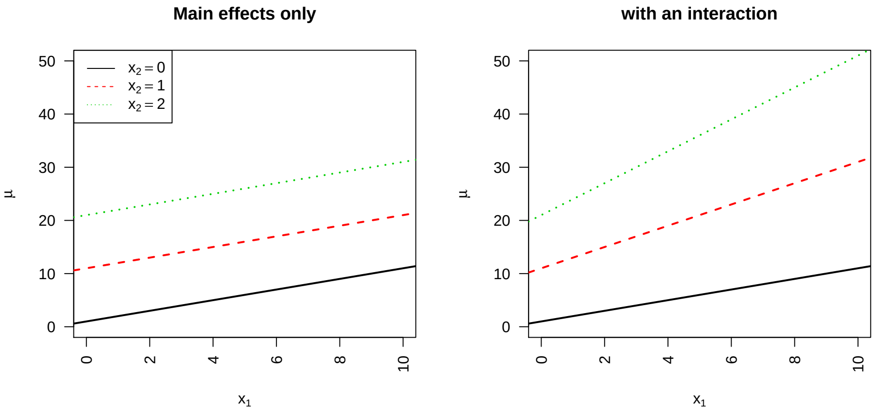
which indicates that both the intercept and slope for x_1 depend on the value of x_2.

Similarly,

$$\mu = (\beta_0 + \beta_1 x_1) + (\beta_2 + \beta_3 x_1) x_2$$

which indicates that both the intercept and slope for x_2 depend on the value of x_1.

Visualizing the models



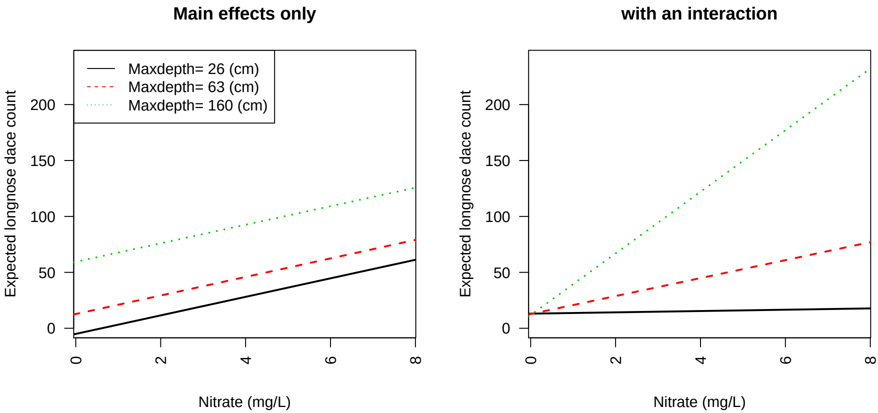
R code and output - main effects only

```
mM = lm(count ~ no3+maxdepth, longnosedace)
summary(mM)


Call:
lm(formula = count ~ no3 + maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max 
-55.060 -27.704  -8.679  11.794 165.310 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)   
(Intercept) -17.5550    15.9586  -1.100  0.27544   
no3           8.2847     2.9566   2.802  0.00671 **
maxdepth      0.4811     0.1811   2.656  0.00997 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.39 on 64 degrees of freedom
Multiple R-squared:  0.1936, Adjusted R-squared:  0.1684 
F-statistic: 7.682 on 2 and 64 DF,  p-value: 0.001022
```

R code and output - with an interaction

```
mI = lm(count ~ no3*maxdepth, longnosedace)
summary(mI)


Call:
lm(formula = count ~ no3 * maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-65.111 -21.399  -9.562   5.953 151.071

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  13.321043  23.455710   0.568   0.5721
no3          -4.646272   7.856932  -0.591   0.5564
maxdepth     -0.009338   0.329180  -0.028   0.9775
no3:maxdepth  0.201219   0.113576   1.772   0.0813 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 42.68 on 63 degrees of freedom
Multiple R-squared:  0.2319, Adjusted R-squared:  0.1953
F-statistic: 6.339 on 3 and 63 DF,  p-value: 0.0007966
```

Visualizing the model



Continuous-categorical interaction

Let category A be the reference level. For observation i, let

- Y_i be the response
- $X_{i,1}$ be the continuous explanatory variable,
- B_i be a dummy variable for category B, and
- C_i be a dummy variable for category C.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i.$$

The mean with the interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i + \beta_4 X_{i,1} B_i + \beta_5 X_{i,1} C_i.$$

Think about this model as a different line for each level of the categorical explanatory variable.

Interpretation for the main effect model

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i.$$

For each category, the line is

Category	Line (μ)		
A	β_0	$+$	$\beta_1 X$
B	$(\beta_0 + \beta_2)$	$+$	$\beta_1 X$
C	$(\beta_0 + \beta_3)$	$+$	$\beta_1 X$

Each category has a different intercept, but a common slope.

Interpretation for the model with an interaction

The model with an interaction is

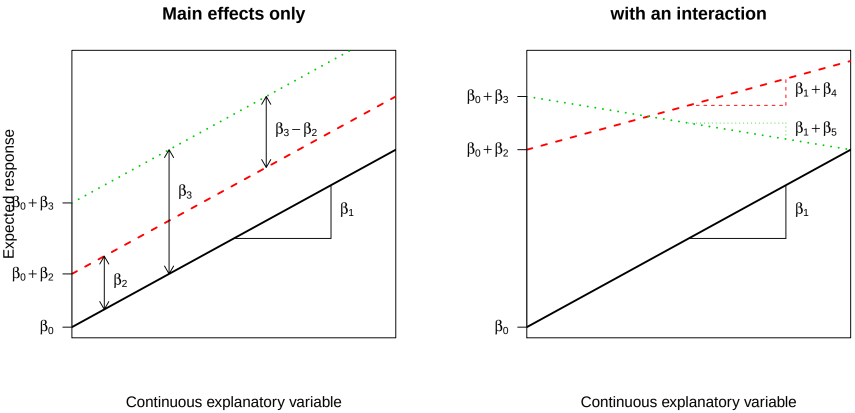
$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i + \beta_4 X_{i,1} B_i + \beta_5 X_{i,1} C_i$$

For each category, the line is

Category	Line (μ)		
A	β_0	$+\ \beta_1$	X
B	$(\beta_0 + \beta_2)$	$+(\beta_1 + \beta_4)$	X
C	$(\beta_0 + \beta_3)$	$+(\beta_1 + \beta_5)$	X

Each category has its own intercept and its own slope.

Visualizing the models



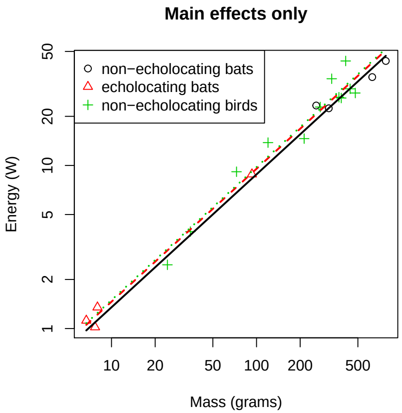
R code and output - main effects only

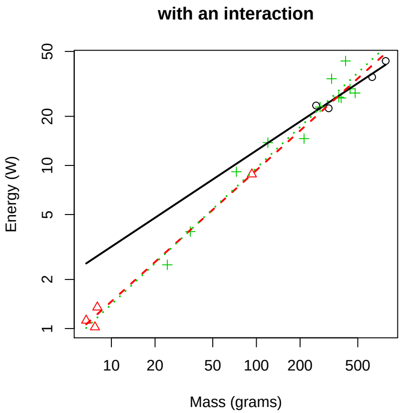
```
case1002$Type = relevel(case1002$Type, ref='non-echolocating bats') # match SAS
summary(mM <- lm(log(Energy)~log(Mass)+Type, case1002))


Call:
lm(formula = log(Energy) ~ log(Mass) + Type, data = case1002)

Residuals:
     Min       1Q   Median       3Q      Max
-0.23224 -0.12199 -0.03637  0.12574  0.34457

Coefficients:
                           Estimate Std. Error t value Pr(>|t|)
(Intercept)                -1.57636    0.28724  -5.488 4.96e-05 ***
log(Mass)                   0.81496    0.04454  18.297 3.76e-12 ***
Typeecholocating bats       0.07866    0.20268   0.388    0.703
Typenon-echolocating birds  0.10226    0.11418   0.896    0.384
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.186 on 16 degrees of freedom
Multiple R-squared:  0.9815, Adjusted R-squared:  0.9781
F-statistic: 283.6 on 3 and 16 DF,  p-value: 4.464e-14
```

R code and output - with an interaction

```
summary(mI <- lm(log(Energy)~log(Mass)*Type, case1002))


Call:
lm(formula = log(Energy) ~ log(Mass) * Type, data = case1002)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.25152 -0.12643 -0.00954  0.08124  0.32840 

Coefficients:
                                     Estimate Std. Error t value Pr(>|t|)  
(Intercept)                           -0.2024     1.2613  -0.161   0.8748  
log(Mass)                              0.5898     0.2061   2.861   0.0126 *
Typeecholocating bats                 -1.2681     1.2854  -0.987   0.3406  
Typenon-echolocating birds            -1.3784     1.2952  -1.064   0.3053  
log(Mass):Typeecholocating bats        0.2149     0.2236   0.961   0.3529  
log(Mass):Typenon-echolocating birds   0.2456     0.2134   1.151   0.2691  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1899 on 14 degrees of freedom
Multiple R-squared:  0.9832, Adjusted R-squared:  0.9771 
F-statistic: 163.4 on 5 and 14 DF,  p-value: 6.696e-12
```

Visualizing the models





Categorical-categorical

Let category A and type 0 be the reference level. For observation i, let

- Y_i be the response,
- 1_i be a dummy variable for type 1,
- B_i be a dummy variable for category B, and
- C_i be a dummy variable for category C.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i.$$

The mean with an interaction is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i + \beta_4 1_i B_i + \beta_5 1_i C_i.$$

Interpretation for the main effects model

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i.$$

- β_0 is the expected response for category A and type 0
- β_1 is the change in response for moving from type 0 to type 1
- β_2 is the change in response for moving from category A to category B
- β_3 is the change in response for moving from category A to category C

The means are then

	Category		
Type	A	B	C
0	β_0	$\beta_0 + \beta_2$	$\beta_0 + \beta_3$
1	$\beta_0 + \beta_1$	$\beta_0 + \beta_1 + \beta_2$	$\beta_0 + \beta_1 + \beta_3$

Interpretation for the model with an interaction

The mean with an interaction is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i + \beta_4 1_i B_i + \beta_5 1_i C_i.$$

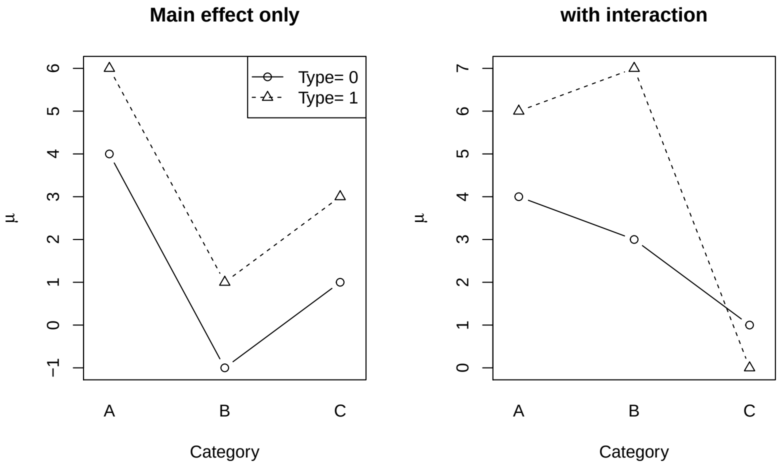
- β_0 is the expected response for category A and type 0
- β_1 is the change in response for moving from type 0 to type 1 for category A
- β_2 is the change in response for moving from category A to category B for type 0
- β_3 is the change in response for moving from category A to category C for type 0
- β_4 is the difference in change in response for moving from category A to category B for type 1 compared to type 0
- β_5 is the difference in change in response for moving from category A to category C for type 1 compared to type 0

The means are then

	Category		
Type	A	B	C
0	β_0	$\beta_0 + \beta_2$	$\beta_0 + \beta_3$
1	$\beta_0 + \beta_1$	$\beta_0 + \beta_1 + \beta_2 + \beta_4$	$\beta_0 + \beta_1 + \beta_3 + \beta_5$

This is referred to as the cell-means model.

Visualizing the models



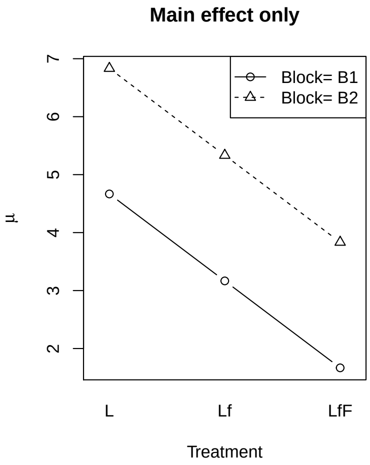
R code and output - main effects only

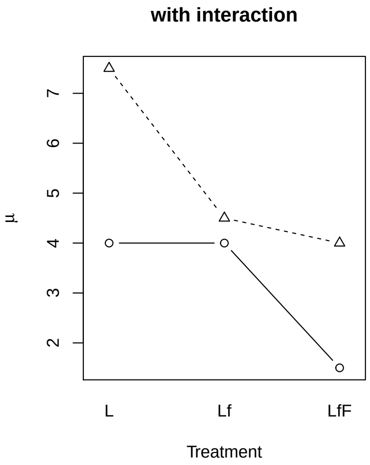
```
# Set the reference levels
case1301$Block = relevel(case1301$Block, ref='B1')
case1301$Treat = relevel(case1301$Treat, ref='L' )
summary(mM <- lm(Cover~Block+Treat, case1301, subset=Block %in% c("B1","B2") & Treat %in% c("L","Lf","LfF")))


Call:
lm(formula = Cover ~ Block + Treat, data = case1301, subset = Block %in%
    c("B1", "B2") & Treat %in% c("L", "Lf", "LfF"))

Residuals:
    Min      1Q  Median      3Q     Max
-2.3333 -0.6667  0.0000  0.7917  1.8333

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   4.6667     0.7683   6.074 0.000298 ***
BlockB2       2.1667     0.7683   2.820 0.022491 *
TreatLf      -1.5000     0.9410  -1.594 0.149578
TreatLfF     -3.0000     0.9410  -3.188 0.012838 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.331 on 8 degrees of freedom
Multiple R-squared:  0.6937, Adjusted R-squared:  0.5788
F-statistic: 6.039 on 3 and 8 DF,  p-value: 0.01881
```

R code and output - with an interaction

```
summary(mI <- lm(Cover~Block*Treat, case1301, subset=Block %in% c("B1","B2") & Treat %in% c("L","Lf","LfF")))


Call:
lm(formula = Cover ~ Block * Treat, data = case1301, subset = Block %in%
    c("B1", "B2") & Treat %in% c("L", "Lf", "LfF"))

Residuals:
   Min     1Q Median     3Q    Max
-1.500 -0.625  0.000  0.625  1.500

Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)        4.0000     0.8898   4.496  0.00412 **
BlockB2            3.5000     1.2583   2.782  0.03193 *
TreatLf            0.0000     1.2583   0.000  1.00000
TreatLfF          -2.5000     1.2583  -1.987  0.09413 .
BlockB2:TreatLf   -3.0000     1.7795  -1.686  0.14280
BlockB2:TreatLfF  -1.0000     1.7795  -0.562  0.59450
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.258 on 6 degrees of freedom
Multiple R-squared:  0.7946, Adjusted R-squared:  0.6234
F-statistic: 4.642 on 5 and 6 DF,  p-value: 0.04429
```

Visualizing the models





When to include interaction terms

From The Statistical Sleuth (3rd ed) page 250:

- when a question of interest pertains to an interaction
- when good reason exists to suspect an interaction or
- when interactions are proposed as a more general model for the purpose of examining the goodness of fit of a model without interaction.

Multiple regression explanatory variables

The possibilities for explanatory variables are

- Higher order terms (X^2)
- Additional explanatory variables (X_1 and X_2)
- Dummy variables for categorical variables ($X_1 = \mathrm{I}()$)
- Interactions ($X_1 X_2$)
  - Continuous-continuous
  - Continuous-categorical
  - Categorical-categorical

We can also combine these explanatory variables, e.g.

- including higher order terms for continuous variables along with dummy variables for categorical variables and
- including higher order interactions ($X_1 X_2 X_3$).