R07 - Contrasts

STAT 587 (Engineering) - Iowa State University

April 19, 2019

Scientific questions

Here are a few example scientific questions:

1. What is the effect of pre-wean calorie restriction on mean lifetimes? With these data, we can ask what is the difference in mean lifetimes for N/R50 and R/R50 diet?
2. What is the difference in mean lifetimes between mice on a 40 kcal diet compared to those on a 50 kcal diet? With these data, we can ask what is the difference in mean lifetimes on N/R40 diet compared to N/R50 and R/R50 combined?
3. What is the effect of high calorie vs low calorie diets on mean lifetimes? With these data, we can ask what is the difference in mean lifetimes for high calorie (NP and N/N85) diets compared to low calorie diets (N/R40, N/R50, R/R50, lopro)?

We can compute contrasts:

$$\gamma_1 = \mu_{R/R50} - \mu_{N/R50}$$

Converting scientific questions into mathematical quantities

Consider the one-way ANOVA model: $Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$ where $j = 1, \ldots, J$.

Here are a few simple alternative hypotheses:

1. What is the difference in mean lifetimes for N/R50 and R/R50 diet?
2. What is the difference in mean lifetimes on N/R40 diet compared to N/R50 and R/R50 combined?
3. What is the difference in mean lifetimes for high calorie (NP and N/N85) diets compared to low calorie diets (N/R40, N/R50, R/R50, lopro)?

We can compute contrasts:

$$\begin{aligned}
\gamma_1 &= \mu_{R/R50} - \mu_{N/R50} \\
\gamma_2 &= \mu_{N/R40} - \frac{1}{2}(\mu_{N/R50} + \mu_{R/R50}) \\
\gamma_3 &= \frac{1}{4}(\mu_{N/R50} + \mu_{R/R50} + \mu_{N/R40} + \mu_{lopro}) - \frac{1}{2}(\mu_{NP} + \mu_{N/N85})
\end{aligned}$$

Contrasts

Definition

A linear combination of group means has the form

$$\gamma = C_1\mu_1 + C_2\mu_2 + \ldots + C_J\mu_J$$

where C_j are known coefficients and μ_j are the unknown population means.

Definition

A linear combination with $C_1 + C_2 + \cdots + C_J = 0$ is a contrast.

Remark Contrast interpretation is usually best if $|C_1| + |C_2| + \cdots + |C_J| = 2$, i.e. the positive coefficients sum to 1 and the negative coefficients sum to -1.

Inference on contrasts

Contrast

$$\gamma = C_1\mu_1 + C_2\mu_2 + \cdots + C_J\mu_J$$

Estimated by

$$g = C_1\overline{Y}_1 + C_2\overline{Y}_2 + \cdots + C_J\overline{Y}_J$$

with standard error

$$SE(g) = \hat{\sigma}\sqrt{\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \cdots + \frac{C_J^2}{n_J}}.$$

Two-sided p-values for $H_0 : g = g_0$ (typically $g_0 = 0$) and posterior tail probabilities (i.e. $2P(\gamma > 0|y)$ or $2P(\gamma < 0|y)$):

$$t = \frac{g - g_0}{SE(g)}, \quad p = 2P(T_{n-J} < -|t|).$$

Two-sided equal-tail $100(1-\alpha)$% confidence/credible intervals:

$$g \pm t_{n-J,1-\alpha/2}SE(g).$$

Contrasts for mice lifetime dataset

For these contrasts:

1. Mean lifetimes for N/R50 and R/R50 diet are different.
2. Mean lifetimes for N/R40 is different than for N/R50 and R/R50 combined.
3. Mean lifetimes for high calorie (NP and N/N85) diets is different than for low calorie diets combined.

$H_0 : \gamma = 0 \qquad H_1 : \gamma \neq 0 :$

$$
\begin{aligned}
\gamma_1 &= \mu_{R/R50} - \mu_{N/R50} \\
\gamma_2 &= \mu_{N/R40} - \frac{1}{2}(\mu_{N/R50} + \mu_{R/R50}) \\
\gamma_3 &= \frac{1}{4}(\mu_{N/R50} + \mu_{R/R50} + \mu_{N/R40} + \mu_{lopro}) - \frac{1}{2}(\mu_{NP} + \mu_{N/N85})
\end{aligned}
$$

	N/N85	N/R40	N/R50	NP	R/R50	lopro
early rest - none @ 50kcal	0.00	0.00	-1.00	0.00	1.00	0.00
40kcal/week - 50kcal/week	0.00	1.00	-0.50	0.00	-0.50	0.00
lo cal - hi cal	-0.50	0.25	0.25	-0.50	0.25	0.25

Mice lifetime examples

	Diet	n	mean	sd
1	N/N85	57	32.69	5.13
2	N/R40	60	45.12	6.70
3	N/R50	71	42.30	7.77
4	NP	49	27.40	6.13
5	R/R50	56	42.89	6.68
6	lopro	56	39.69	6.99

Contrasts:

	g	SE(g)	t	p	L	U
early rest - none @ 50kcal	0.59	1.19	0.49	0.62	-1.76	2.94
40kcal/week - 50kcal/week	2.53	1.05	2.41	0.02	0.46	4.59
lo cal - hi cal	12.45	0.78	15.96	0.00	10.92	13.98

Fit the multiple regression model

```
m = lm(Lifetime ~ Diet, data = Sleuth3::case0501)
summary(m)


Call:
lm(formula = Lifetime ~ Diet, data = Sleuth3::case0501)

Residuals:
     Min       1Q   Median       3Q      Max
-25.5167  -3.3857   0.8143   5.1833  10.0143

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  32.6912     0.8846  36.958  < 2e-16 ***
DietN/R40    12.4254     1.2352  10.059  < 2e-16 ***
DietN/R50     9.6060     1.1877   8.088 1.06e-14 ***
DietNP       -5.2892     1.3010  -4.065 5.95e-05 ***
DietR/R50    10.1945     1.2565   8.113 8.88e-15 ***
Dietlopro     6.9945     1.2565   5.567 5.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.678 on 343 degrees of freedom
Multiple R-squared:  0.4543,Adjusted R-squared:  0.4463
F-statistic:  57.1 on 5 and 343 DF,  p-value: < 2.2e-16
```

Construct contrasts

```
K = rbind("early rest - none @ 50kcal"=c( 0, 0,-1, 0, 1, 0),
          "40kcal/week - 50kcal/week" =c( 0, 2,-1, 0,-1, 0) / 2, # note the denominator here
          "lo cal - hi cal"           =c(-2, 1, 1,-2, 1, 1) / 4) # and here
colnames(K) = levels(case0501$Diet)
K
```

```
                           N/N85 N/R40 N/R50   NP R/R50 lopro
early rest - none @ 50kcal   0.0  0.00 -1.00  0.0  1.00  0.00
40kcal/week - 50kcal/week    0.0  1.00 -0.50  0.0 -0.50  0.00
lo cal - hi cal             -0.5  0.25  0.25 -0.5  0.25  0.25
```

```
# (Complicated) code to construct list from data.frame by row
# https://stackoverflow.com/questions/3492379/data-frame-rows-to-a-list
# you could just construct lists from the beginning, but the K data.frame is
# used previously in the code to construct the contrasts by hand
K_list <- split(K, seq(nrow(K)))
K_list <- setNames(split(K, seq(nrow(K))), rownames(K))

K_list
```

```
$`early rest - none @ 50kcal`
[1]  0  0 -1  0  1  0

$`40kcal/week - 50kcal/week`
[1]  0.0  1.0 -0.5  0.0 -0.5  0.0

$`lo cal - hi cal`
[1] -0.50  0.25  0.25 -0.50  0.25  0.25
```

```r
library("emmeans")
em = emmeans(m, ~ Diet)
em
```

```
 Diet   emmean    SE  df lower.CL upper.CL
 N/N85    32.7 0.885 343     31.0     34.4
 N/R40    45.1 0.862 343     43.4     46.8
 N/R50    42.3 0.793 343     40.7     43.9
 NP       27.4 0.954 343     25.5     29.3
 R/R50    42.9 0.892 343     41.1     44.6
 lopro    39.7 0.892 343     37.9     41.4

Confidence level used: 0.95
```

```r
co = contrast(em, K_list)

# p-values (and posterior tail probabilities)
co
```

```
 contrast                   estimate   SE  df t.ratio p.value
 early rest - none @ 50kcal    0.589 1.19 343   0.493  0.6223
 40kcal/week - 50kcal/week     2.525 1.05 343   2.408  0.0166
 lo cal - hi cal              12.450 0.78 343  15.961  <.0001
```

```r
# confidence/credible intervals
confint(co)
```

```
 contrast                   estimate   SE  df lower.CL upper.CL
 early rest - none @ 50kcal    0.589 1.19 343   -1.759     2.94
 40kcal/week - 50kcal/week     2.525 1.05 343    0.463     4.59
 lo cal - hi cal              12.450 0.78 343   10.915    13.98
```

Summary

- Contrasts are linear combinations of means where the coefficients sum to zero
- t-test tools are used to calculate pvalues and confidence intervals

Sulfur effect on scab disease in potatoes

The experiment was conducted to investigate the effect of sulfur on controlling scab disease in potatoes. There were seven treatments: control, plus spring and fall application of 300, 600, 1200 lbs/acre of sulfur. The response variable was percentage of the potato surface area covered with scab averaged over 100 random selected potatoes. A completely randomized design was used with 8 replications of the control and 4 replications of the other treatments.

Cochran and Cox. (1957) Experimental Design (2nd ed). pg96 and Agron. J. 80:712-718 (1988)

Scientific question:

- Does sulfur have any impact at all?
- What is the difference between spring and fall application of sulfur?
- What is the effect of increased sulfur application?

Data

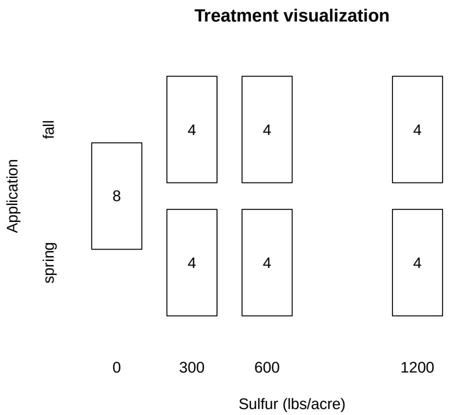
```
   inf trt row col sulfur application treatment
1    9  F3   4   1    300        fall        F3
2   12   0   4   2      0   (Missing)         0
3   18  S6   4   3    600      spring        S6
4   10 F12   4   4   1200        fall       F12
5   24  S6   4   5    600      spring        S6
6   17 S12   4   6   1200      spring       S12
7   30  S3   4   7    300      spring        S3
8   16  F6   4   8    600        fall        F6
9   10   0   3   1      0   (Missing)         0
10   7  S3   3   2    300      spring        S3
11   4 F12   3   3   1200        fall       F12
12  10  F6   3   4    600        fall        F6
13  21  S3   3   5    300      spring        S3
14  24   0   3   6      0   (Missing)         0
15  29   0   3   7      0   (Missing)         0
16  12  S6   3   8    600      spring        S6
17   9  F3   2   1    300        fall        F3
18   7 S12   2   2   1200      spring       S12
19  18  F6   2   3    600        fall        F6
20  30   0   2   4      0   (Missing)         0
21  18  F6   2   5    600        fall        F6
22  16 S12   2   6   1200      spring       S12
23  16  F3   2   7    300        fall        F3
24   4 F12   2   8   1200        fall       F12
25   9  S3   1   1    300      spring        S3
26  18   0   1   2      0   (Missing)         0
27  17 S12   1   3   1200      spring       S12
28  19  S6   1   4    600      spring        S6
29  32   0   1   5      0   (Missing)         0
30   5 F12   1   6   1200        fall       F12
```

Design

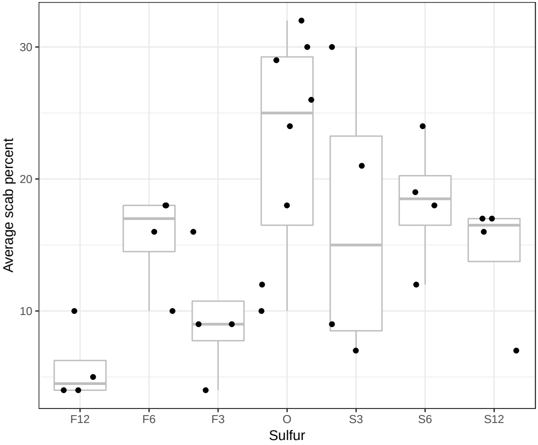
Completely randomized design
potato scab experiment

row \ col	1	2	3	4	5	6	7	8
4	F3	O	S6	F12	S6	S12	S3	F6
3	O	S3	F12	F6	S3	O	O	S6
2	F3	S12	F6	O	F6	S12	F3	F12
1	S3	O	S12	S6	O	F12	O	F3

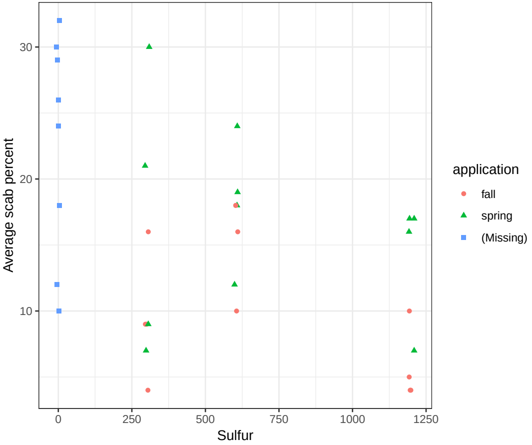
Design


Treatment visualization
Application
fall
spring
8
4
4
4
4
4
4
0
300
600
1200
Sulfur (lbs/acre)

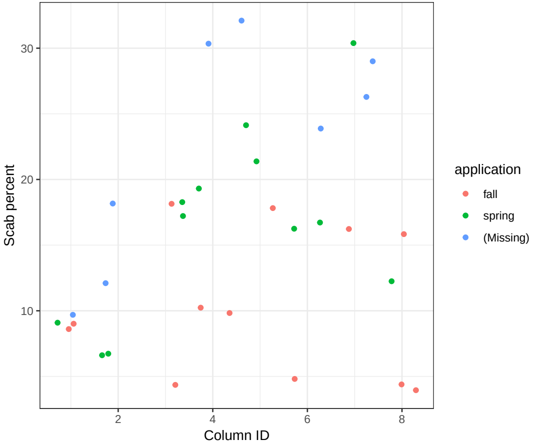
Data



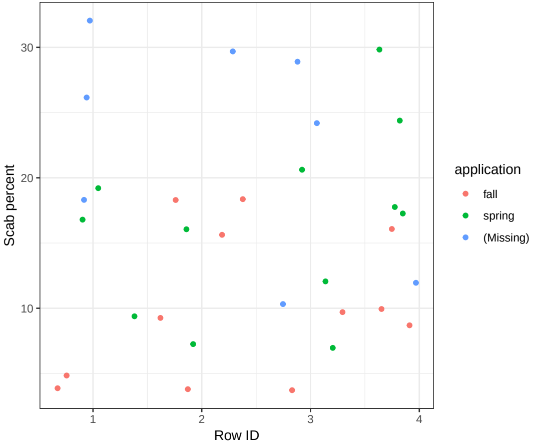
Data


Average scab percent
30
20
10
0
250
500
750
1000
1250
Sulfur
application
fall
spring
(Missing)

Data


Scab percent
30
20
10
2
4
6
8
Column ID
application
fall
spring
(Missing)

Data


Scab percent
30
20
10
1
2
3
4
Row ID
application
fall
spring
(Missing)

Model

Y_{ij}: avg % of surface area covered with scab for plot i in treatment j for $j = 1, \ldots, 7$.

Assume $Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$.

Hypotheses:

- Difference amongst any means: One-way ANOVA F-test
- *Any effect*: Control vs sulfur
- *Fall vs spring*: Contrast comparing fall vs spring applications
- *Sulfur level*: Linear trend contrast

Contrasts

- *Sulfur effect*: Any sulfur vs none

$$\begin{aligned}\gamma &= \tfrac{1}{6}(\mu_{F12}+\mu_{F6}+\mu_{F3}+\mu_{S3}+\mu_{S6}+\mu_{S12})-\mu_O \\ &= \tfrac{1}{6}(\mu_{F12}+\mu_{F6}+\mu_{F3}+\mu_{S3}+\mu_{S6}+\mu_{S12}-6\mu_O)\end{aligned}$$

- *Fall vs spring*: Contrast comparing fall vs spring applications

$$\begin{aligned}\gamma &= \tfrac{1}{3}(\mu_{F12}+\mu_{F6}+\mu_{F3})+0\mu_O-\tfrac{1}{3}(\mu_{S3}+\mu_{S6}+\mu_{S12}) \\ &= \tfrac{1}{3}\left[1\mu_{F12}+1\mu_{F6}+1\mu_{F3}+0\mu_O-1\mu_{S3}-1\mu_{S6}-1\mu_{S12}\right]\end{aligned}$$

Contrasts (cont.)

- Sulfur linear trend
  - The group sulfur levels (X_j) are 12, 6, 3, 0, 3, 6, and 12 (100 lbs/acre)
  - and a linear trend contrast is $X_j - \overline{X}$

X_i	12	6	3	0	3	6	12
$X_i - \overline{X}$	6	0	-3	-6	-3	0	6

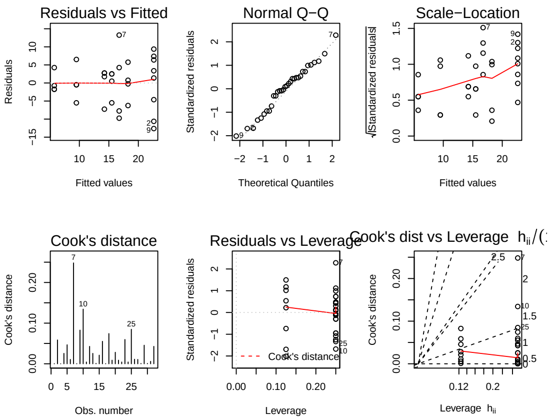
$$\gamma \quad = 6\mu_{F12} + 0\mu_{F6} - 3\mu_{F3} - 6\mu_{O} - 3\mu_{S3} + 0\mu_{S6} + 6\mu_{S12}$$

Trt	F12	F6	F3	O	S3	S6	S12	Div
Sulfur v control	1	1	1	-6	1	1	1	6
Fall v Spring	1	1	1	0	-1	-1	-1	3
Linear Trend	-6	0	-3	-6	-3	0	6	1

```
K =
#                                        F12 F6 F3  0 S3 S6 S12
               list("sulfur - control" = c( 1, 1, 1,-6, 1, 1,  1)/6,
                    "fall - spring"    = c( 1, 1, 1, 0,-1,-1, -1)/3,
                    "linear trend"     = c( 6, 0,-3,-6,-3, 0,  6)/1)

m = lm(inf ~ trt, data = d)
anova(m)
```

```
Analysis of Variance Table

Response: inf
          Df  Sum Sq Mean Sq F value  Pr(>F)
trt        6  972.34 162.057  3.6081 0.01026 *
Residuals 25 1122.88  44.915
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
par(mfrow=c(2,3))
plot(m,1:6)
```



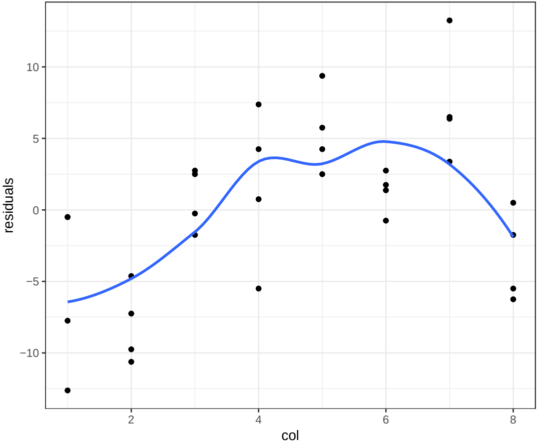
```
em <- emmeans(m, ~trt); em
```

```
 trt emmean   SE df lower.CL upper.CL
 F12   5.75 3.35 25    -1.15     12.7
 F3    9.50 3.35 25     2.60     16.4
 F6   15.50 3.35 25     8.60     22.4
 O    22.62 2.37 25    17.74     27.5
 S12  14.25 3.35 25     7.35     21.2
 S3   16.75 3.35 25     9.85     23.7
 S6   18.25 3.35 25    11.35     25.2

Confidence level used: 0.95
```

```
co <- contrast(em, K)
confint(co)
```

```
 contrast         estimate    SE df lower.CL upper.CL
 sulfur - control    -9.29  2.74 25    -14.9   -3.657
 fall - spring       -6.17  2.74 25    -11.8   -0.532
 linear trend       -81.00 34.82 25   -152.7   -9.279

Confidence level used: 0.95
```

```r
d$residuals <- residuals(m)
ggplot(d, aes(col, residuals)) + geom_point() + stat_smooth(se=FALSE) + theme_bw()
```



Summary

For this particular data analysis

- Significant differences in means between the groups (ANOVA $F_{6,25} = 3.61$ p=0.01)
- Having sulfur was associated with a reducted scab % of 9 (4,15) compared to no sulfur
- Fall application reduced scab % by 6 (0.5,12) compared to spring application
- Linear trend in sulfur was significant (p=0.01)

- Concerned about spatial correlation among columns
- Consider a transformation of the response
  - CI for F12 (-1.2, 12.7) (not shown)
  - Non-constant variance (residuals vs predicted, sulfur, application)