

Bayesian linear regression (cont.)

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Outline

- Subjective Bayesian regression
 - Ridge regression
 - Zellner's g-prior
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- Regression with a known covariance matrix
 - Known covariance matrix
 - Covariance matrix known up to a proportionality constant
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Subjective Bayesian regression

Suppose

$$y \sim N(X\beta, \sigma^2 I)$$

and we use a prior for β of the form

$$\beta | \sigma^2 \sim N(b, \sigma^2 B)$$

A few special cases are

- $b = 0$
- B is diagonal
- $B = gI$
- $B = g(X'X)^{-1}$

Ridge regression

Let

$$y = X\beta + e, \quad E[e] = 0, \quad \text{Var}[e] = \sigma^2 I$$

then ridge regression seeks to minimize

$$(y - X\beta)'(y - X\beta) + g\beta'\beta$$

where g is a penalty for $\beta'\beta$ getting too large.

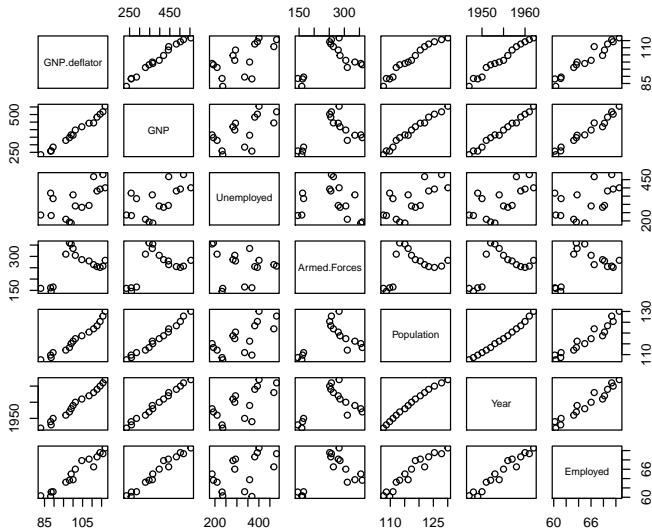
This minimization looks like -2 times the log posterior for a Bayesian regression analysis when using independent normal priors centered at zero with a common variance (c_0) for β :

$$-2\sigma^2 \log p(\beta, \sigma|y) = C + (y - X\beta)'(y - X\beta) + \frac{\sigma^2}{c_0} \beta'\beta$$

where $g = \sigma^2/c_0$. Thus the ridge regression estimate is equivalent to a MAP estimate when

$$y \sim N(X\beta, \sigma^2 I) \quad \beta \sim N(0, c_0 I).$$

Longley data set



Default Bayesian regression (unscaled)

```
summary(lm(GNP.deflator~., longley))
```

Call:

```
lm(formula = GNP.deflator ~ ., data = longley)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-2.0086	-0.5147	0.1127	0.4227	1.5503

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	2946.85636	5647.97658	0.522	0.6144
GNP	0.26353	0.10815	2.437	0.0376 *
Unemployed	0.03648	0.03024	1.206	0.2585
Armed.Forces	0.01116	0.01545	0.722	0.4885
Population	-1.73703	0.67382	-2.578	0.0298 *
Year	-1.41880	2.94460	-0.482	0.6414
Employed	0.23129	1.30394	0.177	0.8631

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.195 on 9 degrees of freedom

Multiple R-squared: 0.9926, Adjusted R-squared: 0.9877

F-statistic: 202.5 on 6 and 9 DF, p-value: 4.426e-09

Default Bayesian regression (scaled)

```
y = longley$GNP.deflator
X = scale(longley[,-1])
summary(lm(y~X))
```

```
Call:
lm(formula = y ~ X)
```

Residuals:

Min	1Q	Median	3Q	Max
-2.0086	-0.5147	0.1127	0.4227	1.5503

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	101.6813	0.2987	340.465	<2e-16 ***
XGNP	26.1933	10.7497	2.437	0.0376 *
XUnemployed	3.4092	2.8263	1.206	0.2585
XArmed.Forces	0.7767	1.0754	0.722	0.4885
XPopulation	-12.0830	4.6871	-2.578	0.0298 *
XYear	-6.7548	14.0191	-0.482	0.6414
XEmployed	0.8123	4.5794	0.177	0.8631

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

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Ridge regression in MASS package

```
library(MASS)
gs = seq(from = 0, to = 0.1, by = 0.0001)
m = lm.ridge(GNP.deflator ~ ., longley, lambda = gs)
```

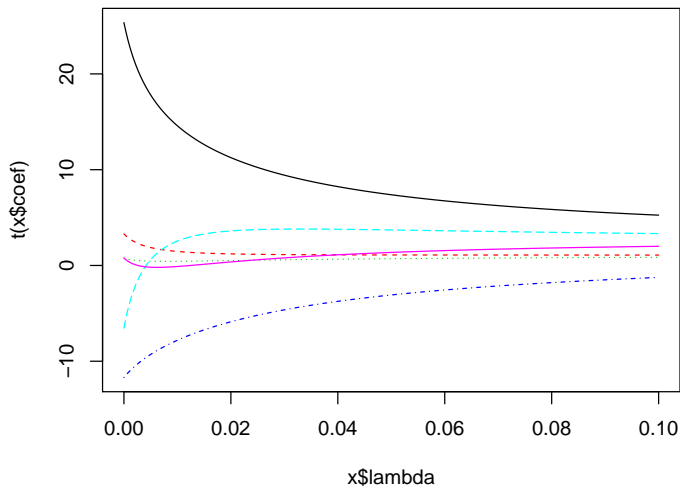
```
# Choose the ridge penalty
select(m)
```

```
modified HKB estimator is 0.006836982
modified L-W estimator is 0.05267247
smallest value of GCV at 0.0057
```

```
# Estimates
est = data.frame(lambda = gs, t(m$coef))
est[round(est$lambda,4) %in% c(.0068,.0053,.0057),]
```

	lambda	GNP	Unemployed	Armed.Forces	Population	Year	Employed
0.0053	0.0053	17.54527	1.831969	0.4501249	-9.184211	0.7946179	-0.1896272
0.0057	0.0057	17.21975	1.785199	0.4453260	-9.047254	1.0213866	-0.1955648
0.0068	0.0068	16.41186	1.675572	0.4369163	-8.692626	1.5486827	-0.1947731

Ridge regression in MASS package



Zellner's g-prior

Suppose

$$y \sim N(X\beta, \sigma^2 I)$$

and you use Zellner's g-prior

$$\beta \sim N(b_0, g\sigma^2(X'X)^{-1}).$$

The posterior is then

$$\begin{aligned}\beta|\sigma^2, y &\sim N\left(\frac{g}{g+1}\left(\frac{b_0}{g} + \hat{\beta}\right), \frac{\sigma^2 g}{g+1}(X'X)^{-1}\right) \\ \sigma^2|y &\sim \text{Inv-}\chi^2\left(n, \frac{1}{n}\left[(n-k)s^2 + \frac{1}{g+1}(\hat{\beta} - b_0)X'X(\hat{\beta} - b_0)\right]\right)\end{aligned}$$

with

$$\begin{aligned}E[\beta|y] &= \frac{g}{g+1}\left(\frac{b_0}{g} + \hat{\beta}\right) \\ E[\sigma^2|y] &= \frac{(n-k)s^2 + \frac{1}{g+1}(\hat{\beta} - b_0)X'X(\hat{\beta} - b_0)}{n-2}\end{aligned}$$

Setting g

In Zellner's g-prior,

$$\beta \sim N(b_0, g\sigma^2(X'X)^{-1}),$$

we need to determine how to set g .

Here are some thoughts:

- $g = 1$ puts equal weight to prior and likelihood
- $g = n$ means prior has the equivalent weight of 1 observation
- $g \rightarrow \infty$ recovers a uniform prior
- Empirical Bayes estimate of g , $\hat{g}_{EG} = \operatorname{argmax}_g p(y|g)$ where

$$p(y|g) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi^{(n+1)/2} n^{1/2}} \|y - \bar{y}\|^{-(n-1)} \frac{(1+g)^{(n-1-k)/2}}{(1+g(1+R^2))^{(n-1)/2}}$$

where R^2 is the usual coefficient of determination.

- Put a prior on g and perform a fully Bayesian analysis.

Zellner's g-prior in R

```
library(BMS)
m = zlm(GNP.deflator~., longley, g='UIP') # g=n
summary(m)
```

Coefficients

	Exp.Val.	St.Dev.
(Intercept)	2779.49311839	NA
GNP	0.24802564	0.26104901
Unemployed	0.03433686	0.07300367
Armed.Forces	0.01050452	0.03730077
Population	-1.63485161	1.62641807
Year	-1.33533979	7.10751875
Employed	0.21768268	3.14738044

```
Log Marginal Likelihood:
-44.07653
g-Prior: UIP
Shrinkage Factor: 0.941
```

Bayes Factors for regression model comparison

Consider two models with design matrices X^1 and X^2 (not including an intercept) and corresponding dimensions (n, p_1) and (n, p_2) . Zellner's g-prior provides a relatively simple way to construct default priors for model comparison. Formally, we compare

$$\begin{aligned}y &\sim N(\alpha 1_n + X^1 \beta^1, \sigma^2 I) \\ \beta &\sim N(b_1, g_1 \sigma^2 [(X^1)'(X^1)]^{-1}) \\ p(\alpha, \sigma^2) &\propto 1/\sigma^2\end{aligned}$$

and

$$\begin{aligned}y &\sim N(\alpha 1_n + X^2 \beta^2, \sigma^2 I) \\ \beta &\sim N(b_2, g_2 \sigma^2 [(X^2)'(X^2)]^{-1}) \\ p(\alpha, \sigma^2) &\propto 1/\sigma^2\end{aligned}$$

Bayes Factors for regression model comparison

The Bayes Factor for comparing these two models is

$$B_{12}(y) = \frac{(g_1 + 1)^{-p_1/2} \left[(n - p_1 - 1)s_1^2 + (\hat{\beta}_1 - b_1)' (X^1)' X^1 (\hat{\beta}_1 - b_1) / (g_1 + 1) \right]^{-(n-1)/2}}{(g_2 + 1)^{-p_2/2} \left[(n - p_2 - 1)s_2^2 + (\hat{\beta}_2 - b_2)' (X^2)' X^2 (\hat{\beta}_2 - b_2) / (g_2 + 1) \right]^{-(n-1)/2}}$$

Now, we can set $g_1 = g_2$ and calculate Bayes Factors.

```
library(bayess)
m = BayesReg(longley$GNP.deflator, longley[, -1], g = nrow(longley))
```

	PostMean	PostStError	Log10bf	EvidAgaH0
Intercept	101.6813	0.7431		
x1	23.8697	25.1230	-0.3966	
x2	3.1068	6.6053	-0.5603	
x3	0.7078	2.5134	-0.5954	
x4	-11.0111	10.9543	-0.3714	
x5	-6.1556	32.7640	-0.6064	
x6	0.7402	10.7025	-0.614	

```
Posterior Mean of Sigma2: 8.8342
Posterior StError of Sigma2: 13.0037
```

Known covariance matrix

Suppose $y \sim N(X\beta, S)$ where S is a known covariance matrix, then $p(\beta) \propto 1$ is a non-informative prior.

Let L be a Cholesky factor of S , i.e. $LL^\top = S$, then the model can be rewritten as

$$L^{-1}y \sim N(L^{-1}X\beta, I).$$

The posterior, $p(\beta|y)$, is the same as for ordinary linear regression replacing y with $L^{-1}y$, X with $L^{-1}X$ and σ^2 with 1 where L^{-1} is inverse of L . Thus

$$\begin{aligned}\beta|y &\sim N(\hat{\beta}, V_\beta) \\ V_\beta &= ([L^{-1}X]^\top L^{-1}X)^{-1} = (X^\top S^{-1}X)^{-1} \\ \hat{\beta} &= ([L^{-1}X]^\top L^{-1}X)^{-1} [L^{-1}X]^\top L^{-1}y = V_\beta X^\top S^{-1}y\end{aligned}$$

So rather than computing these, just transform your data using $L^{-1}y$ and $L^{-1}X$ and force $\sigma^2 = 1$.

Autoregressive process of order 1

A mean zero, stationary autoregressive process of order 1 assumes

$$\epsilon_t = r\epsilon_{t-1} + \delta_t$$

with $-1 < r < 1$ and $\delta_t \stackrel{ind}{\sim} N(0, t^2)$.

Suppose

$$y_t = X_t' \beta + \epsilon_t$$

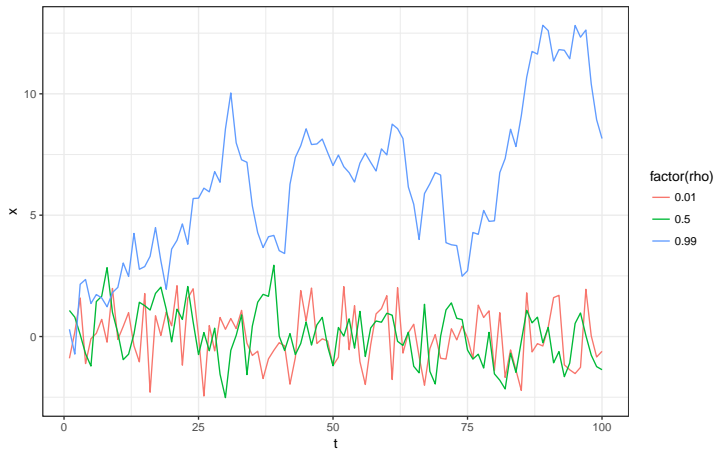
or, equivalently,

$$y = N(X\beta, S)$$

where $S = s^2 R$ with

- stationary variance $s^2 = t^2 / [1 - r^2]$ and
- correlation matrix R with elements $R_{ij} = r^{|i-j|}$.

Example autoregressive processes



Calculate posterior

```

ar1_covariance = function(n, r, s) {
  V = diag(n)
  s^2/(1-r^2) * r^(abs(row(V)-col(V)))
}

# Covariance
n = 100
S = ar1_covariance(n,.9,2)

# Simulate data
set.seed(1)
library(MASS)
k = 50
X = matrix(rnorm(n*k), n, k)
beta = rnorm(k)
y = mvrnorm(1,X%*%beta, S)

# Estimate beta
Linv = solve(t(chol(S)))
Linvy = Linv%*%y
LinvX = Linv%*%X
m = lm(Linvy ~ 0+LinvX)

# Force sigma=1
Vb = vcov(m)/summary(m)$sigma^2

```

Credible intervals

```
# Credible intervals
sigma = sqrt(diag(Vb))
ci = data.frame(lcl=coefficients(m)-qnorm(.975)*sigma,
               ucl=coefficients(m)+qnorm(.975)*sigma,
               truth=beta)
head(ci,10)
```

		lcl	ucl	truth
Lin vX1	-2.17072576	-1.1397374	-1.5163733	
Lin vX2	0.24668568	1.3350616	0.6291412	
Lin vX3	-1.95292532	-1.0391643	-1.6781940	
Lin vX4	0.47043115	1.5172772	1.1797811	
Lin vX5	1.05482123	1.8256552	1.1176545	
Lin vX6	-1.80257089	-0.6888599	-1.2377359	
Lin vX7	-1.58638219	-0.7773177	-1.2301645	
Lin vX8	0.01345063	0.9211755	0.5977909	
Lin vX9	0.04529412	1.0160908	0.2988644	
Lin vX10	-0.31334691	0.7148231	-0.1101394	

```
all.equal(Vb[1:k^2], solve(t(X)%*%solve(S)%*%X)[1:k^2])
```

```
[1] TRUE
```

```
all.equal(as.numeric(coefficients(m)), as.numeric(Vb%*%t(X)%*%solve(S)%*%y))
```

```
[1] TRUE
```

Variance known up to a proportionality constant

Consider the model

$$y \sim N(X\beta, \sigma^2 S)$$

for a known S with default prior $p(\beta, \sigma^2) \propto 1/\sigma^2$.

The posterior is

$$p(\beta, \sigma^2 | y) = p(\beta | \sigma^2, y) p(\sigma^2 | y)$$

$$\beta | \sigma^2, y \sim N(\hat{\beta}, \sigma^2 V_\beta)$$

$$\sigma^2 | y \sim \text{Inv-}\chi^2(n - k, s^2)$$

$$\beta | y = t_{n-k}(\hat{\beta}, s^2 V_\beta)$$

$$\hat{\beta} = (X^\top S^{-1} X)^{-1} X^\top S^{-1} y$$

$$V_\beta = (X^\top S^{-1} X)^{-1}$$

$$\begin{aligned} s^2 &= \frac{1}{n-k} (L^{-1} y - L^{-1} X \hat{\beta})^\top (L^{-1} y - L^{-1} X \hat{\beta}) \\ &= \frac{1}{n-k} (y - X \hat{\beta})^\top S^{-1} (y - X \hat{\beta}) \end{aligned}$$

where $LL' = S$.

AR1 process

Consider the model

$$y \sim N(X\beta, \sigma^2 R)$$

where R is the correlation matrix from an AR1 process.

This is exactly what we had before, except we do not assume $\sigma = 1$.

Posterior with unknown σ^2

```

m      = lm(Linvy ~ 0+LinvX)
Vb     = vcov(m)
bhat   = coefficients(m)
df      = n-k
s2     = sum(residuals(m)^2)/df

# Credible intervals
cbind(confint(m), Truth=beta)[1:10,]

```

	2.5 %	97.5 %	Truth
LinvX1	-2.20156617	-1.1088970	-1.5163733
LinvX2	0.21412861	1.3676187	0.6291412
LinvX3	-1.98025906	-1.0118306	-1.6781940
LinvX4	0.43911637	1.5485920	1.1797811
LinvX5	1.03176293	1.8487135	1.1176545
LinvX6	-1.83588583	-0.6555450	-1.2377359
LinvX7	-1.61058410	-0.7531158	-1.2301645
LinvX8	-0.01370255	0.9483287	0.5977909
LinvX9	0.01625424	1.0451307	0.2988644
LinvX10	-0.34410302	0.7455792	-0.1101394

Parameterized covariance matrix

Suppose

$$y \sim N(X\beta, S(\theta))$$

where $S(\theta)$ is now unknown, but can be characterized by a low dimensional θ , e.g.

- Autoregressive process of order 1:

$$S(\theta) = \sigma^2 R(\rho), R_{ij}(\rho) = \rho^{|i-j|}$$

- Gaussian process with exponential covariance function:

$$S(\theta) = \tau^2 R(\rho) + \sigma^2 I, R_{ij}(\rho) = \exp(-\rho d_{ij})$$

- Conditionally autoregressive (CAR) model:

$$S(\theta) = \sigma^2 (D_w - \rho W)^{-1}$$

MCMC for parameterized covariance matrices

Suppose

$$y \sim N(X\beta, S(\theta))$$

then an MCMC strategy is

1. Sample $\beta|\theta, y$, i.e. regression with a known covariance matrix.
2. Sample $\theta|\beta, y$.

Alternatively, if

$$y \sim N(X\beta, \sigma^2 R(\theta))$$

then an MCMC strategy is

1. Sample $\beta, \sigma^2|\theta, y$, i.e. regression when variance is known up to a proportionality constant..
2. Sample $\theta|\beta, \sigma^2, y$.

Since θ exists in a low dimension, many of the methods we have learned can be used, e.g. ARS, MH, slice sampling, etc.

Summary

- Subjective Bayesian regression
 - Ridge regression
 - Zellner's g-prior
 - Bayes' Factors for model comparison
- Regression with a known covariance matrix
 - Known covariance matrix
 - Covariance matrix known up to a proportionality constant
 - MCMC for parameterized covariance matrix
 - Time series
 - Spatial analysis