I07 - Posterior model probability

STAT 401 (Engineering) - Iowa State University

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One-sided alternative hypotheses

For "one-sided alternative hypotheses" analyzed from a Bayesian perspective, I think we should ignore the "hypothesis test" completely. That is, we should obtain a posterior and calculate the posterior probabilities from that posterior.

For example, let $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ with hypotheses

- $H_0 : \mu \le 0$
- $H_A : \mu > 0$.

Assume $p(\mu, \sigma^2) \propto 1/\sigma^2$ and obtain the posterior, i.e.

$$\mu|y \sim t_{n-1}(\overline{y}, s^2/n).$$

Then calculate

$$P(\mu \le 0|y) = P\left(T_{n-1} \le \frac{-\overline{y}}{s/\sqrt{n}}\right).$$

Posterior model probabilities

From a Bayesian perspective, each hypothesis is a different model that describes how the world works. There is no limit to two hypotheses and hypotheses are all treated the same, i.e. there is no "special" hypothesis.

We are interested in determine our belief about each model after we see the data. Thus, we calculate posterior model probabilities over some set of J models i.e,

$$p(H_j|y) = \frac{p(y|H_j)p(H_j)}{p(y)} = \frac{p(y|H_j)p(H_j)}{\sum_{k=1}^{J} p(y|H_k)p(H_k)}.$$

In order to accomplish this, we need to determine

- prior model probabilities: $p(H_j)\,\forall\, j = 1, \ldots, J$ and
- priors over parameters in each model:

$$p(y|H_j) = \int p(y|\theta)p(\theta|H_j)d\theta.$$

Two hypotheses

A pvalue can loosely be interpreted as “the probability of observing these data if the null hypothesis is true”,i.e.

$$p(y|H_0),$$

But what we really want is “the probability the null hypothesis is true, given that we observed this data”. Of course, we can never know the truth (or even the probability of the truth), but we can make statements about our belief that the null hypothesis is true, i.e.

$$p(H_0|y) = \frac{p(y|H_0)p(H_0)}{p(y)}.$$

If there are only two hypotheses (say H_0 and H_A), then we have

$$p(H_0|y) = \frac{p(y|H_0)p(H_0)}{p(y|H_0)p(H_0) + p(y|H_A)p(H_A)} = \left[1 + \frac{p(y|H_A)p(H_A)}{p(y|H_0)p(H_0)}\right]^{-1}.$$

Point null hypotheses

If $H_0 : \theta = \theta_0$ and $H_A : \theta \neq \theta_0$, then

$$\begin{aligned} p(y|H_0) &= p(y|\theta_0) \\ p(y|H_A) &= \int p(y, \theta|H_A) d\theta \\ &= \int p(y|\theta, H_A) p(\theta|H_A) d\theta \\ &= \int p(y|\theta) p(\theta|H_A) d\theta \end{aligned}$$

where $p(\theta|H_A)$ is the distribution of the parameter θ when the alternative hypothesis is true.

Example

If $Y_i \stackrel{ind}{\sim} N(\mu, 1)$ and we have the hypotheses $H_0 : \mu = 0$ vs $H_A : \mu \neq 0$ with $\mu|H_A \sim N(0, 1)$, then

$$\begin{aligned} y|H_0 &\sim N(0, 1) \\ y|H_A &\sim N(0, 2). \end{aligned}$$

Bayesian hypothesis tests

To conduct a Bayesian hypothesis test, you need to specify

- $p(H_j)$ and
- $p(\theta|H_j)$

for every hypothesis $j = 1, \ldots, J$. Then, you can calculate

$$p(H_j|y) = \frac{p(y|H_j)p(H_j)}{\sum_{k=1}^{J} p(y|H_k)p(H_k)} = \left[1 + \sum_{k \neq j} \frac{p(y|H_k)}{p(y|H_j)} \frac{p(H_k)}{p(H_j)}\right]^{-1}$$

where

$$BF(H_k : H_j) = \frac{p(y|H_k)}{p(y|H_j)}$$

are the Bayes factor for hypothesis H_k compared to hypothesis H_j and

$$p(y|H_j) = \int p(y|\theta)p(\theta|H_j)d\theta$$

for all j.

Normal example

Let $Y \sim N(\mu, 1)$ and consider the hypotheses $H_0 : \mu = 0$ and $H_A : \mu \neq 0$ with $\mu | H_A \sim N(0, C)$ and, for simplicity, $p(H_0) = p(H_A) = 0.5$. Then the two hypotheses are really

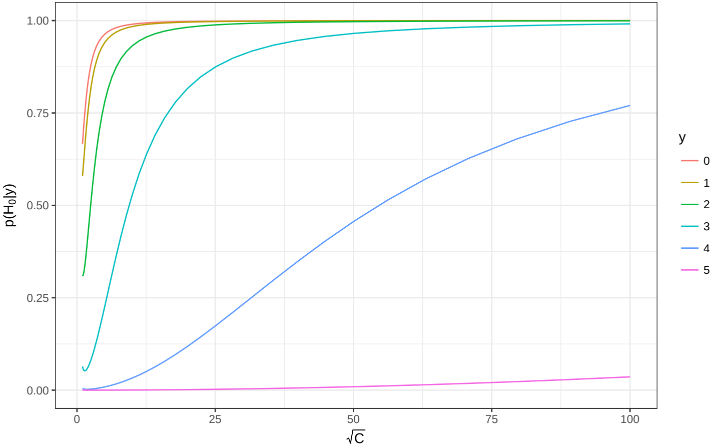
- $H_0 : Y \sim N(0, 1)$ and
- $H_A : Y \sim N(0, 1 + C)$.

Thus

$$p(H_0|y) = \left[1 + \frac{p(y|H_A)}{p(y|H_0)}\right]^{-1} = \left[1 + \frac{N(y; 0, 1 + C)}{N(y; 0, 1)}\right]^{-1}$$

where $N(y; \mu, \sigma^2)$ is evaluating the probability density function for a normal distribution with mean μ and variance σ^2 at the value y.

Normal example



Do pvalues and posterior probabilities agree?

Suppose $\sim Bin(n, \theta)$ and we have the hypotheses $H_0 : \theta = 0.5$ and $H_A : \theta \neq 0.5$ We observe $n = 10,000$ and $y = 4,900$ and find the pvalue is

$$pvalue \approx 2P(Y \leq 4900) = 0.0466$$

so we would reject H_0 at the 0.05 level.

The posterior probability of H_0 if we assume $\theta|H_A \sim Unif(0,1)$ and $p(H_0) = p(H_A) = 0.5$ is

$$p(H_0|y) \approx \frac{1}{1 + 1/10.8} = 0.96,$$

so the probability of H_0 being true is 96%.

It appears the posterior model probability and pvalue completely disagree!

Jeffrey-Lindley Paradox

Definition

The Jeffrey-Lindley Paradox concerns a situation when comparing two hypotheses H_0 and H_1 given data y and find

- a frequentist test result is significant leading to rejection of H_0, but
- the posterior probability of H_0 is high.

This can happen when

- the effect size is small,
- n is large,
- H_0 is relatively precise,
- H_1 is relative diffuse, and
- the prior model odds is ≈ 1.

No real paradox

Pvalues:

- Pvalues measure how incompatible your data are with the null hypothesis.
- The smaller the pvalue, the more incompatible.
- But they say nothing about how likely the alternative is.

Posterior model probabilities:

- Posterior model probabilities measure how likely the data are under the predictive distribution for each hypothesis.
- The larger the posterior probability, the more predictive that hypothesis was compared to the other hypotheses.
- But this requires you to have at least two well-thought out models, i.e. no vague priors.

Thus, these two statistics provide completely different measures of model adequecy.