Hypothesis tests

with binomial example

STAT 587 (Engineering)
Iowa State University

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- reject the null hypothesis or
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What if you observed $y = 2$? Well, $P(Y = 2) \approx 0.01$.

Large data

Let $Y \sim Bin(n, \theta)$ with

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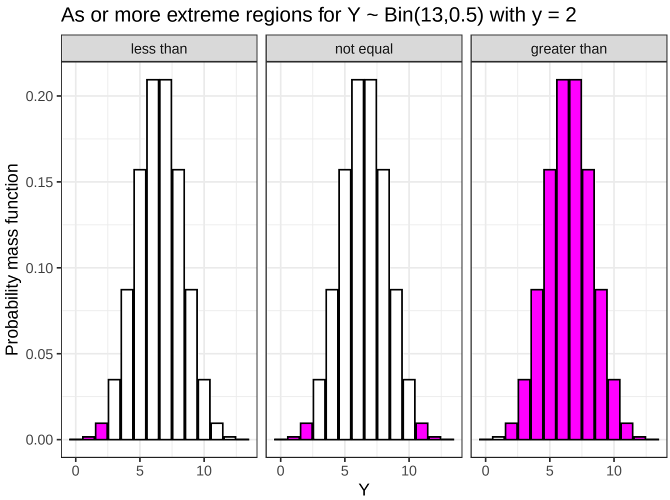
or

$$H_A : \theta \neq \theta_0 \implies |Y - n\theta_0| \geq |y - n\theta_0|.$$

as or more extreme regions



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pbinom(y, size = n, prob = theta0)
```

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[1] 0.01123047
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1-pbinom(y-1, size = n, prob = theta0)
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[1] 0.998291
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$$H_A : \theta \neq 0.5 \implies p\text{-value} = P(|Y - n\theta_0| \leq |y - n\theta_0|)$$

```
2*pbinom(y, size = n, prob = theta0)

[1] 0.02246094
```

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binom.test(y, n, p = theta0, alternative = "less")$p.value
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$$H_A : \theta \neq 0.5$$

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binom.test(y, n, p = theta0, alternative = "two.sided")$p.value

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- reject if p-value $< a$ otherwise
- fail to reject.

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reject H_0	type I error	correct
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If we fail to reject H_0, insufficient evidence to say that the data are incompatible with this model.

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binom.test(3, 6, p = 1/6, alternative = "greater")$p.value

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With a signficance level of $a = 0.05$, you fail to reject the null hypothesis.

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- Use p-values to determine whether to
  - reject the null hypothesis or
  - fail to reject the null hypothesis.
- More assessment is required to determine if other model assumptions hold.