I02 - Likelihood

STAT 587 (Engineering) - Iowa State University

February 18, 2019

Statistical modeling

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Formally,

- $\mathcal{S} = \{0, 1, 2, \ldots, n\}$
- $\mathcal{P} = \{Bin(n, \theta) : 0 < \theta < 1\}$.

Normal model

Suppose our data are

- a set of real numbers, i.e. between $-\infty$ and ∞,
- the population mean is μ and population variance is σ^2,
- a histogram is reasonably approximated by a bell-shaped curve,
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Formally,

- $\mathcal{S} = \{y_i : y_i \in \mathbb{R}, i \in \{1, 2, \ldots, n\}\}$
- $\mathcal{P} = \{N(\mu, \sigma^2) : -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$ where $\theta = (\mu, \sigma^2)$.

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The likelihood describes the relative support in the data for different values for your parameter, i.e. the larger the likelihood is the more consistent that parameter value is with the data.

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Note: I write $L(\theta)$ without any conditioning, e.g. on y, so that you don't confuse this with a probability mass (or density) function.

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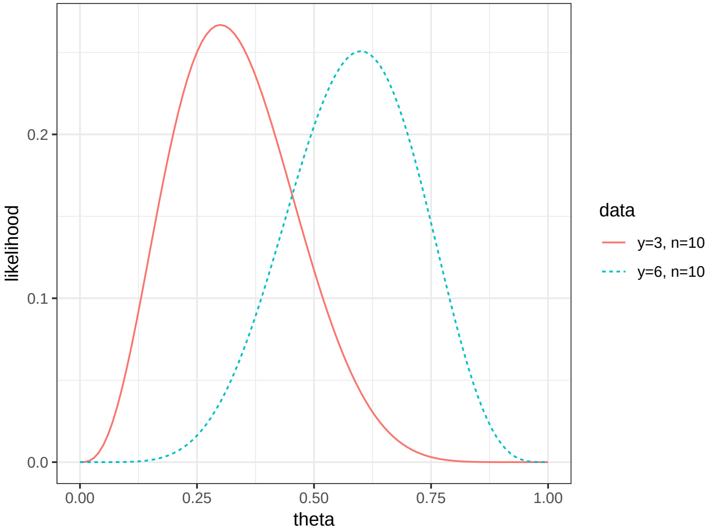

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The likelihood for θ is

$$L(\theta) = p(y|\theta) = \prod_{i=1}^{n} p(y_i|\theta)$$

where we are thinking about this as a function of θ for fixed y.

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$$p(y_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}$$

and

$$\begin{aligned} p(y|\mu,\sigma^2) &= \prod_{i=1}^n p(y_i|\mu,\sigma^2) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2} \end{aligned}$$

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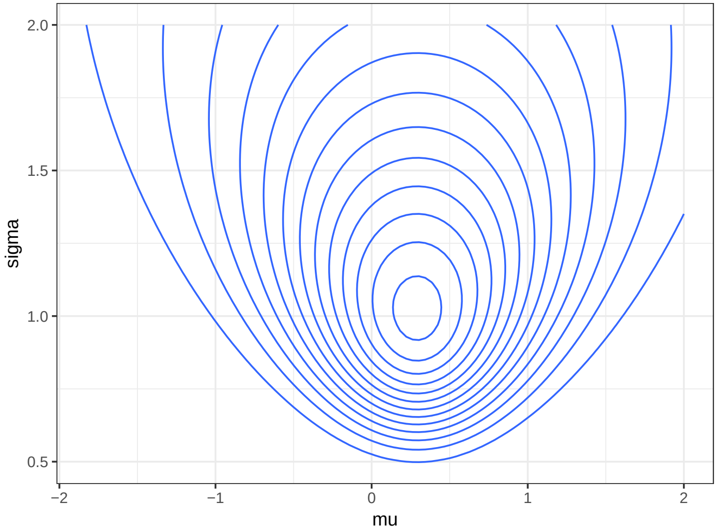
where μ and σ^2 are fixed (but often unknown) and the argument to this function is $y = (y_1, \ldots, y_n)$.

The likelihood is

$$L(\mu, \sigma) = p(y|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2}$$

where y is fixed and known and μ and σ^2 are the arguments to this function.

Normal likelihood



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When the data are discrete, the MLE the is parameter value that maximizes the probability of the observed data.

Binomial MLE via derivatives

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$$\begin{array}{rl} \ell(\theta) & = \log \binom{n}{y} + y\log(\theta) + (n-y)\log(1-\theta) \\ \frac{d}{d\theta}\ell(\theta) & = \frac{y}{\theta} - \frac{n-y}{1-\theta} \end{array}$$

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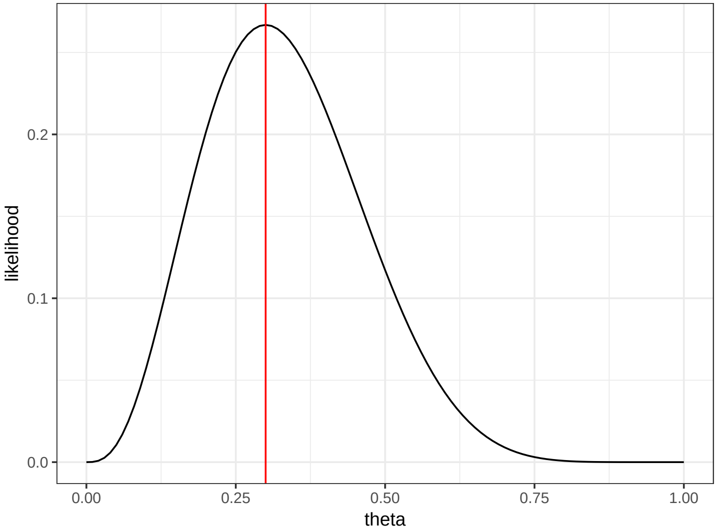
Take the second derivative of $\ell(\theta)$ with respect to θ and check to make sure it is negative.

Binomial MLE graphically



Numerical maximization

```
log_likelihood <- function(theta) {
  dbinom(3, size = 10, prob = theta, log = TRUE)
}

optim(0.5, log_likelihood,
      method='L-BFGS-B',            # this method to use bounds
      lower = 0.001, upper = .999,  # cannot use 0 and 1 exactly
      control = list(fnscale = -1)) # maximize


$par
[1] 0.3000006

$value
[1] -1.321151

$counts
function gradient
       7        7

$convergence
[1] 0

$message
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

Normal MLE

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$, then

$$L(\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2}$$

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&= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y}+\overline{y}-\mu)^2}
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&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 + -\frac{n}{2\sigma^2}(\overline{y}-\mu)^2\right) \quad \text{since } \sum_{i=1}^n (y_i-\overline{y}) = 0
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&= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y}+\overline{y}-\mu)^2} \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n \left[(y_i-\overline{y})^2 + 2(y_i-\overline{y})(\overline{y}-\mu) + (\overline{y}-\mu)^2\right]\right) \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 + -\frac{n}{2\sigma^2}(\overline{y}-\mu)^2\right) \quad \text{since } \sum_{i=1}^n (y_i-\overline{y}) = 0 \\
\ell(\mu,\sigma^2) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 - \frac{1}{2\sigma^2} n(\overline{y}-\mu)^2 \\
\frac{\partial}{\partial\mu}\ell(\mu,\sigma^2) &= \frac{n}{\sigma^2}(\overline{y}-\mu) \stackrel{set}{=} 0 \implies \hat{\mu}_{MLE} = \overline{y} \\
\frac{\partial}{\partial\sigma^2}\ell(\mu,\sigma^2) &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (y_i-\overline{y})^2 \stackrel{set}{=} 0
\end{aligned}
$$

Normal MLE

If $Y_i \overset{ind}{\sim} N(\mu, \sigma^2)$, then

$$\begin{aligned}
L(\mu, \sigma^2) &= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2} \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y}+\overline{y}-\mu)^2} \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n \left[(y_i-\overline{y})^2 + 2(y_i-\overline{y})(\overline{y}-\mu) + (\overline{y}-\mu)^2\right]\right) \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 + -\frac{n}{2\sigma^2}(\overline{y}-\mu)^2\right) \quad \text{since } \sum_{i=1}^n (y_i-\overline{y}) = 0 \\
\ell(\mu, \sigma^2) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 - \frac{1}{2\sigma^2} n(\overline{y}-\mu)^2 \\
\frac{\partial}{\partial\mu}\ell(\mu, \sigma^2) &= \frac{n}{\sigma^2}(\overline{y}-\mu) \overset{set}{=} 0 \implies \hat{\mu}_{MLE} = \overline{y} \\
\frac{\partial}{\partial\sigma^2}\ell(\mu, \sigma^2) &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (y_i-\overline{y})^2 \overset{set}{=} 0 \\
&\implies \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^n (y_i-\overline{y})^2
\end{aligned}$$

Normal MLE

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$, then

$$
\begin{aligned}
L(\mu,\sigma^2) &= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2} \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y}+\overline{y}-\mu)^2} \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n \left[(y_i-\overline{y})^2 + 2(y_i-\overline{y})(\overline{y}-\mu) + (\overline{y}-\mu)^2\right]\right) \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 + -\frac{n}{2\sigma^2}(\overline{y}-\mu)^2\right) \quad \text{since } \sum_{i=1}^n (y_i-\overline{y}) = 0 \\
\ell(\mu,\sigma^2) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\overline{y})^2 - \frac{1}{2\sigma^2} n(\overline{y}-\mu)^2 \\
\frac{\partial}{\partial\mu}\ell(\mu,\sigma^2) &= \frac{n}{\sigma^2}(\overline{y}-\mu) \stackrel{set}{=} 0 \implies \hat{\mu}_{MLE} = \overline{y} \\
\frac{\partial}{\partial\sigma^2}\ell(\mu,\sigma^2) &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (y_i-\overline{y})^2 \stackrel{set}{=} 0 \\
&\implies \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^n (y_i-\overline{y})^2 = \frac{n-1}{n}S^2
\end{aligned}
$$

Thus, the MLE for a normal model is

$$
\hat{\mu}_{MLE} = \overline{y}, \quad \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^n (y_i-\overline{y})^2
$$

Numerical maximization

```
x
```

```
[1] -0.8969145  0.1848492  1.5878453
```

```
log_likelihood <- function(theta) {
  sum(dnorm(x, mean = theta[1], sd = exp(theta[2]), log = TRUE))
}

o <- optim(c(0,0), log_likelihood,
           control = list(fnscale = -1))
o$convergence # make sure this is 0 indicating convergence
```

```
[1] 0
```

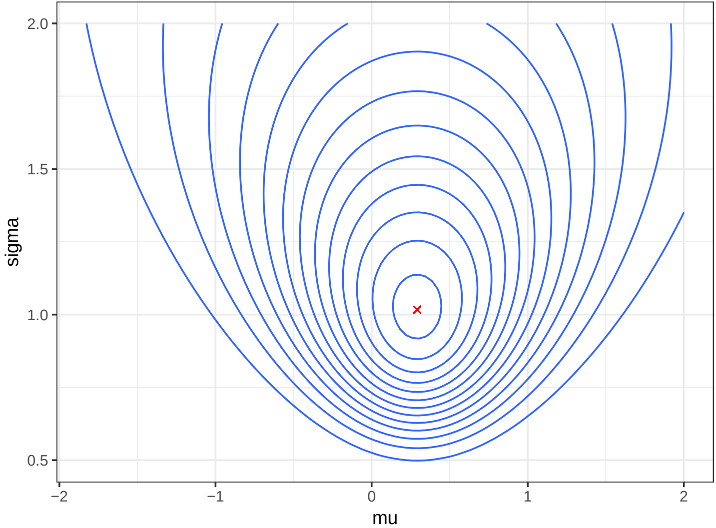
```
o$par[1]; exp(o$par[2])^2 # mean and variance
```

```
[1] 0.2918674
[1] 1.03446
```

```
n <- length(x)
mean(x); (n-1)/n*var(x) # var uses n-1 in the denominator
```

```
[1] 0.2919267
[1] 1.034738
```

Normal likelihood



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- For independent observations, the joint probability mass (density) function is the product of the marginal probability mass (density) functions.
- The likelihood is the joint probability mass (density) function when the argument of the function is the parameter (vector).
- The maximum likelihood estimator (MLE) is the value of the parameter (vector) that maximizes the likelihood.