STAT 401A - Statistical Methods for Research Workers

Simple linear regression

Jarad Niemi (Dr. J)

Iowa State University

last updated: October 14, 2014

Simple Linear Regression

Recall the one-way ANOVA model:

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

where Y_{ij} is the observation for individual i in group j.

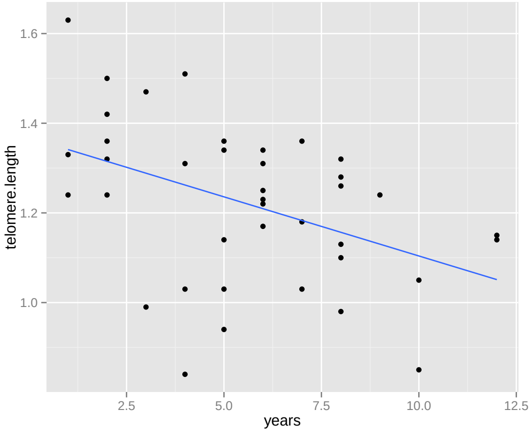
The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

Terminology (all of these are equivalent):

response	explanatory
outcome	covariate
dependent	independent
endogenous	exogenous



1.6
1.4
1.2
1.0
telomere.length
2.5
5.0
7.5
10.0
12.5
years

Telomere length

http://www.pnas.org/content/101/49/17312

> *People who are stressed over long periods tend to look haggard, and it is commonly thought that psychological stress leads to premature aging and the earlier onset of diseases of aging.*
>
> *...*
>
> *This design allowed us to examine the importance of perceived stress and measures of objective stress (caregiving status and chronicity of caregiving stress based on the number of years since a child's diagnosis).*
>
> *...*
>
> *Telomere length values were measured from DNA by a quantitative PCR assay that determines the relative ratio of telomere repeat copy number to single-copy gene copy number (T/S ratio) in experimental samples as compared with a reference DNA sample.*

Interpretation

$$E[Y_i|X_i = x] = \beta_0 + \beta_1 x \qquad V[Y_i|X_i = x] = \sigma^2$$

- If $X_i = 0$, then $E[Y_i|X_i = 0] = \beta_0$.
  β_0 is the expected response when the explanatory variable is zero.

- If X_i increases from x to $x + 1$, then

$$\begin{array}{rl} E[Y_i|X_i = x + 1] & = \beta_0 + \beta_1 x + \beta_1 \\ - \, E[Y_i|X_i = x \quad\;\;\,] & = \beta_0 + \beta_1 x \\ \hline & = \beta_1 \end{array}$$

  β_1 is the expected increase in the response for each unit increase in the explanatory variable.

- σ is the standard deviation of the response for a fixed value of the explanatory variable.

Remove the mean:

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

So the error is

$$e_i = Y_i - (\beta_0 + \beta_1 X_i)$$

which we approximate by the residual

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

The least squares, maximum likelihood, and Bayesian estimators are

$$\begin{array}{ll} \hat{\beta}_1 & = SXY/SXX \\ \hat{\beta}_0 & = \overline{Y} - \hat{\beta}_1 \overline{X} \\ \hat{\sigma}^2 & = SSE/(n-2) \qquad \text{df} = n-2 \end{array}$$

$$\begin{array}{ll} \overline{X} & = \frac{1}{n}\sum_{i=1}^n X_i \\ \overline{Y} & = \frac{1}{n}\sum_{i=1}^n Y_i \end{array}$$

$$\begin{array}{ll} SXY & = \sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) \\ SXX & = \sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^n (X_i - \overline{X})^2 \\ SSE & = \sum_{i=1}^n r_i^2 \end{array}$$

How certain are we about $\hat\beta_0$ and $\hat\beta_1$ being equal to β_0 and β_1?

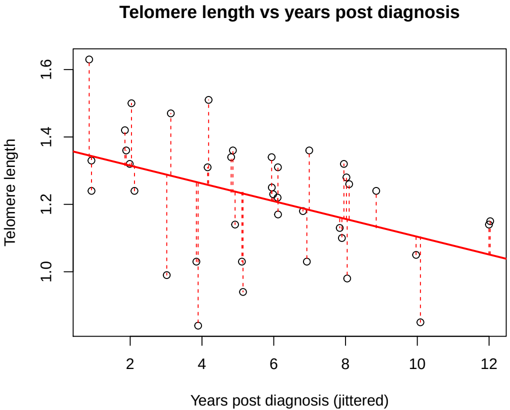
We quantify this uncertainty using their standard errors:

$$
\begin{array}{rll}
SE(\beta_0) & = \hat\sigma\sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} & df = n-2 \\
SE(\beta_1) & = \hat\sigma\sqrt{\frac{1}{(n-1)s_X^2}} & df = n-2
\end{array}
$$

$$
\begin{array}{rl}
s_X^2 & = SXX/(n-1) \\
s_Y^2 & = SYY/(n-1) \\
SYY & = \sum_{i=1}^n (Y_i - \overline{Y})^2
\end{array}
$$

$$
\begin{array}{rlll}
r_{XY} & = \frac{SXY/(n-1)}{s_X s_Y} & & \text{correlation coefficient} \\
R^2 & = r_{XY}^2 & = \frac{SST-SSE}{SST} & \text{coefficient of determination} \\
SST & = SYY = \sum_{i=1}^n (Y_i - \overline{Y})^2 & &
\end{array}
$$

The coefficient of determination (R^2) is the proportion of the total response variation explained by the explanatory variable(s).


Telomere length vs years post diagnosis
Telomere length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
Years post diagnosis (jittered)

Pvalues and confidence interval

We can compute two-sided pvalues via

$$2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_0}{SE(\beta_0)}\right|\right) \qquad \text{and} \qquad 2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_1}{SE(\beta_1)}\right|\right)$$

These test the null hypothesis that the corresponding parameter is zero.

We can construct $100(1-\alpha)\%$ two-sided confidence intervals via

$$\hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\beta_0) \qquad \text{and} \qquad \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\beta_1)$$

These provide ranges of the parameters consistent with the data.

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC CORR DATA=t;
  VAR length;
  WITH years;
  RUN;
```

```
                                        The CORR Procedure

                                    1 With Variables:    years
                                    1      Variables:    length


                                        Simple Statistics

   Variable           N          Mean       Std Dev           Sum       Minimum       Maximum

   years             39       5.58974       2.93543     218.00000       1.00000      12.00000
   length            39       1.22026       0.17977      47.59000       0.84000       1.63000


                          Pearson Correlation Coefficients, N = 39
                                 Prob > |r| under H0: Rho=0

                                                     length

                                        years      -0.43065
                                                     0.0062
```

```
PROC GLM DATA=t;
  MODEL length = years / SOLUTION CLPARM;
  RUN;

                                         The GLM Procedure
                             Number of Observations Read          39
                             Number of Observations Used          39

Dependent Variable: length

                                                    Sum of
       Source                      DF             Squares     Mean Square    F Value    Pr > F
       Model                        1          0.22776588      0.22776588       8.42    0.0062
       Error                       37          1.00033156      0.02703599
       Corrected Total             38          1.22809744

                         R-Square     Coeff Var      Root MSE    length Mean
                         0.185462      13.47473      0.164426       1.220256

       Source                      DF           Type I SS     Mean Square    F Value    Pr > F
       years                        1          0.22776588      0.22776588       8.42    0.0062

       Source                      DF         Type III SS     Mean Square    F Value    Pr > F
       years                        1          0.22776588      0.22776588       8.42    0.0062

                                               Standard
 Parameter           Estimate             Error    t Value    Pr > |t|      95% Confidence Limits
 Intercept        1.367682067        0.05721112      23.91      <.0001     1.251761335   1.483602799
 years           -0.026374315        0.00908674      -2.90      0.0062    -0.044785794 -0.007962836
```

Regression in R

```r
m = lm(telomere.length~years, Telomeres)
with(Telomeres, cor(telomere.length,years))
```

```
[1] -0.4307
```

```r
anova(m)
```

```
Analysis of Variance Table

Response: telomere.length
          Df Sum Sq Mean Sq F value Pr(>F)
years      1  0.228   0.228    8.42 0.0062 **
Residuals 37  1.000   0.027
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Regression in R

```
m = lm(telomere.length~years, Telomeres)
summary(m)
```

```
Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
    Min      1Q  Median      3Q     Max
-0.4222 -0.0854  0.0206  0.1074  0.2887

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.36768    0.05721    23.9   <2e-16 ***
years       -0.02637    0.00909    -2.9   0.0062 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.164 on 37 degrees of freedom
Multiple R-squared:  0.185, Adjusted R-squared:  0.163
F-statistic: 8.42 on 1 and 37 DF,  p-value: 0.0062
```

```
confint(m)
```

```
               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

Conclusion

Telomere length at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% confidence interval of (1.25, 1.48). For each year increase since diagnosis, the length decreases by 0.026 with a 95% confidence interval of (0.008, 0.045). The proportional of variability in telomere length described by years since diagnosis is 18.5%.

http://www.pnas.org/content/101/49/17312

> *The zero-order correlation between chronicity of caregiving [years] and mean telomere length, r,is* -0.445 *(P <0.01). [*$R^2 = 0.198$ *was shown in the plot.]*

Remark I'm guessing our analysis and that reported in the paper don't match exactly due to a discrepancy in the data.

Summary

- The simple linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

  where Y_i and X_i are the response and explanatory variable, respectively, for individual i.

- Know how to use SAS/R to obtain $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$, R^2, pvalues, CIs, etc.

- Interpret SAS output
  - At a value of zero for the explanatory variable ($X_i = 0$), β_0 is the expected value for the response (Y_i).
  - For each unit increase in the explanatory variable value, β_1 is the expected increase in the response.
  - At a constant value of the explanatory variable, σ^2 is the variance of the responses.
  - The coefficient of determination (R^2) is the percentage of the total response variation explained by the explanatory variable(s).

Testing Composite hypotheses

Comparing two models

- H_0 : (reduced)
- H_1 : (full)

Do the following

1. Calculate extra sum of squares.
2. Calculate extra degrees of freedom
3. Calculate

$$\text{F-statistic} = \frac{\text{Extra sum of squares / Extra degrees of freedom}}{\hat{\sigma}^2_{full}}$$

4. Compare this to an F-distribution with
   - numerator degrees of freedom = extra degrees of freedom
   - denominator degrees of freedom = degrees of freedom in estimating $\hat{\sigma}^2_{full}$

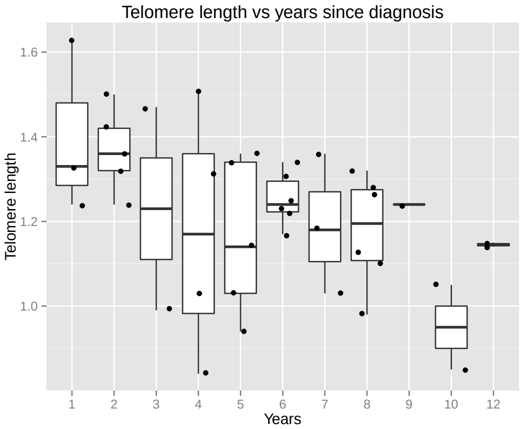
Simple Linear Regression

Two models:

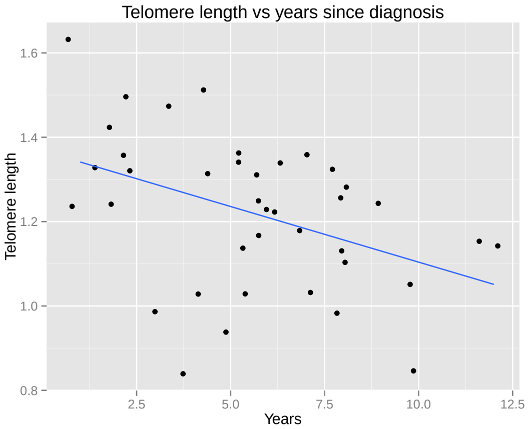
ANOVA:	$Y_{ij} \stackrel{ind}{\sim} N(\mu_i, \sigma^2)$	(full)
Regression:	$Y_{ij} \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$	(reduced)

- Regression model is reduced:
  - ANOVA has J parameters for the mean
  - Regression has 2 parameters for the mean
  - $\mu_i = \beta_0 + \beta_1 X_i$
- Small pvalues indicate a lack-of-fit, i.e. the reduced model is not adequate.
- Lack-of-fit F-test requires multiple observations at a few X_i values.

Telomere length



Telomere length



SAS code

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years / CLB LACKFIT;
  RUN;
```

```
                              The REG Procedure
                                Model: MODEL1
                         Dependent Variable: length

                   Number of Observations Read          39
                   Number of Observations Used          39


                             Analysis of Variance

                                   Sum of           Mean
  Source                   DF     Squares         Square    F Value    Pr > F

  Model                     1     0.22777        0.22777       8.42    0.0062
  Error                    37     1.00033        0.02704
    Lack of Fit             9     0.18223        0.02025       0.69    0.7093
    Pure Error             28     0.81810        0.02922
  Corrected Total          38     1.22810
```

Indicates no evidence for a lack of fit, i.e. regression seems adequate.

Summary

- Lack-of-fit F-test tests the assumption of linearity
- Needs multiple observations at various explanatory variable values
- Small pvalue indicates a lack-of-fit, i.e. means are not linear
  - Transform response, e.g. log
  - Transform explanatory variable
  - Add other explanatory variables

Regression

The simpler linear regression model is

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

this can be rewritten as

$$Y_i = \beta_0 + \beta_1 X_i + e_i \qquad e_i \stackrel{ind}{\sim} N(0, \sigma^2)$$

where we estimate the errors via the residuals

$$r_i = \hat{e}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i).$$

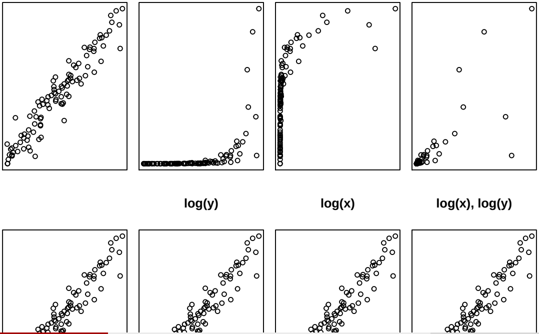
Key assumptions are:

- Linearity between mean response and explanatory variable
- Normality of the errors
- Constant variance of the errors
- Independence between observations

Linearity

Assess using scatterplots of transformed response vs transformed explanatory variable:

```
Error:  argument "main" is missing, with no default
Error:  argument "main" is missing, with no default
Error:  argument "main" is missing, with no default
Error:  argument "main" is missing, with no default
Error:  argument "main" is missing, with no default
```

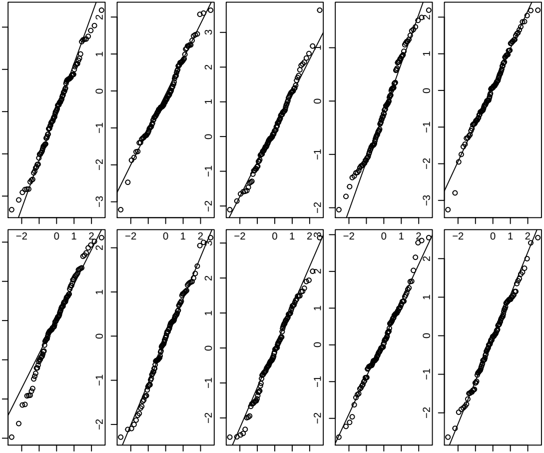


Normality

These are normal.
SAS swaps the x and y axes

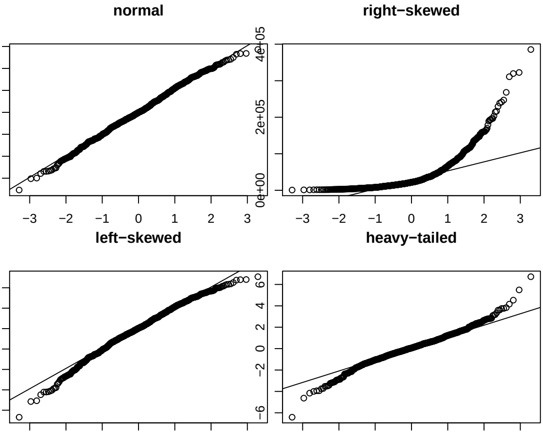
Normality

These are normal.



SAS swaps the x and y axes

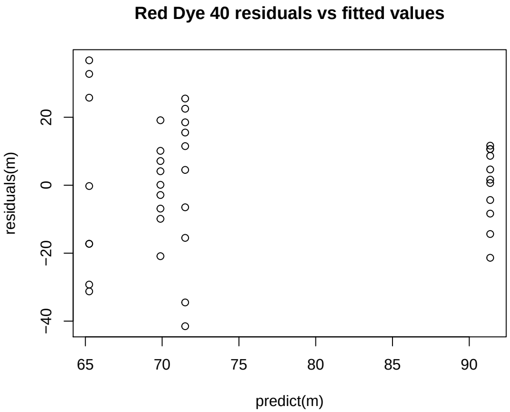
Normality



SAS swaps the x and y axes

Constant variance

Most common non-constant variance is when the variance increases with the mean



Independence

Lack of independence includes

- Cluster effect
- Serial correlation
- Spatial association

Make plots of residuals vs relevant explanatory variables and look for patterns, e.g.

- Residuals vs groups (prefer blocking)
- Residuals vs time (or observation number)
- Residuals vs spatial variable

Summary

Often the best strategy is graphical exploration of the data, here are some relevant graphs:

- transformed response vs transformed explanatory
- transformed response vs transformed explanatory
- qqplot of residuals
- residual vs fitted value
- residual vs explanatory
- residual vs observation number
- residual vs any other variable