STAT 401A - Statistical Methods for Research Workers

Nonparametric two-sample tests

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Nonparametric statistics

http://en.wikipedia.org/wiki/Parametric_statistics

Definition

Parametric statistics assumes that the data have come from a certain probability distribution and makes inferences about the parameters of this distribution, e.g. assuming the data come from a normal distribution and estimating the mean μ.

http://en.wikipedia.org/wiki/Nonparametric_statistics

Definition

Nonparametric statistics make no assumptions about the probability distributions of the [data],e.g. randomization and permutation tests.

Central limit theorem

Theorem

Let $X_1, X_2, \ldots$ be a sequence of iid random variables with $E[X_i] = \mu$ and $0 < V[X_i] = \sigma^2 < \infty$. Then

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \stackrel{n\to\infty}{\longrightarrow} N(0,1)$$

where

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

i.e. the sample mean using the first n variables.

Central limit theorem

Lemma

Let $X_1, X_2, \ldots$ be a sequence of iid random variables with $E[X_i] = \mu$ and $0 < V[X_i] = \sigma^2 < \infty$. Then

$$\frac{\overline{X}_n - \mu}{s_n/\sqrt{n}} \stackrel{n\to\infty}{\longrightarrow} N(0,1)$$

where

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$$

i.e. the sample mean and variance using the first n variables.

Bernoulli example

Consider $X_i \overset{iid}{\sim} Ber(p)$, i.e. $X_i = 1$ with probability p and $X_i = 0$ with probability $1-p$. Then $E[X_i] = p$ and $0 < V[X_i] = p(1-p) < \infty$.

```
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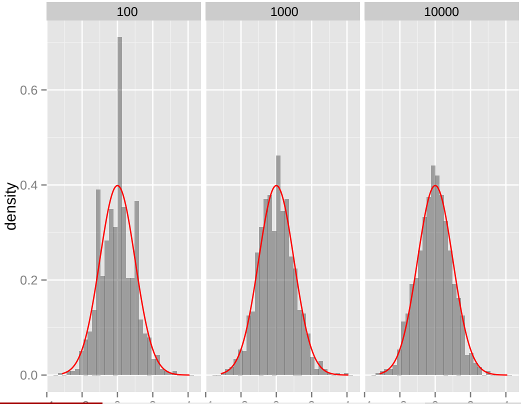
```



Rusty leaves data

year1	year2	diff	diff>0
38	32	6	1
10	16	-6	0
84	57	27	1
36	28	8	1
50	55	-5	0
35	12	23	1
73	61	12	1
48	29	19	1

If there is no effect, then the "diff>0" column should be a 1 or 0 with probability 0.5, i.e. $X_i \stackrel{iid}{\sim} Ber(p)$ and $K = \sum_{i=1}^n X_i \sim Bin(n,p)$.

Sign test

The sign test calculates the probability of observing this many ones (or more extreme) if the null hypothesis is true. For our one-sided hypothesis (removing leaves will decrease rusty leaves), that is the probability of observing 6, 7, or 8 ones. This is

$$\binom{8,6}{0}.5^8 + \binom{8,7}{0}.5^8 + \binom{8,8}{0}.5^8 = 0.14$$

```
K = sum(d[,4])
sum(dbinom(K:8,8,.5))


[1] 0.1445
```

Sign test using normal approximation

Recall that if $K \sim Bin(n, p)$, then $E[K] = np$ and $V[K] = np(1-p)$.
Thus, if $p = 0.5$, then

$$Z = \frac{K - (n/2)}{\sqrt{n/4}} \xrightarrow{n \to \infty} N(0, 1)$$

and we can approximate the pvalue by calculating the area under the normal curve.

```
Z = (K-n/2)/(sqrt(n/4))

Error:  object 'n' not found

1-pnorm(Z)

Error:  object 'Z' not found
```

The continuity correction accounts for the fact that K is discrete:

```
Z = (K-n/2-1/2)/(sqrt(n/4))

Error:  object 'n' not found

1-pnorm(Z)
```

Wilcoxon signed-rank test

Also known as the Wilcoxon signed-rank test:

1. Compute the difference in each pair.
2. Drop zeros from the list.
3. Order the absolute differences from smallest to largest and assign them their ranks.
4. Calculate S: the sum of the ranks from the pairs for which the difference is positive.
5. Calculate $E[S] = n(n+1)/4$ where n is the number of pairs.
6. Calculate $SD[S] = [n(n+1)(2n+1)/24]^{1/2}$.
7. Calculate $Z = (S - E[S] + c)/SD[S]$ where c is the appropriate continuity correction.
8. Calculate the pvalue comparing Z to a standard normal.

Signed rank test

year1	year2	diff	diff>0	absdiff	rank
50	55	-5	0	5	1.0
38	32	6	1	6	2.5
10	16	-6	0	6	2.5
36	28	8	1	8	4.0
73	61	12	1	12	5.0
48	29	19	1	19	6.0
35	12	23	1	23	7.0
84	57	27	1	27	8.0

- $S = 32.5$
- $E[S] = 18$
- $SD[S] = 7.14$
- $Z = 1.96$ (with continuity correction of -0.5)
- $p = 0.02$

Signed-rank test in R

```
# By hand
S = sum(d$rank[d$"diff>0"==1])
n = nrow(d)
ES = n*(n+1)/4
SDS = sqrt(n*(n+1)*(2*n+1)/24)
z = (S-0.5-ES)/SDS
1-pnorm(z)

[1] 0.02497

# Using a function
wilcox.test(d$year1, d$year2, paired=T)

Warning:  cannot compute exact p-value with ties


Wilcoxon signed rank test with continuity correction

data:  d$year1 and d$year2
V = 32.5, p-value = 0.04967
alternative hypothesis: true location shift is not equal to 0
```

Divide this two-sided pvalue by 2 since the data are in agreement with the alternative hypothesis (fewer rusty leaves after removal).

SAS code for paired nonparametric test

```
DATA leaves;
  INPUT tree year1 year2;
  diff = year1-year2;
  DATALINES;
1 38 32
2 10 16
3 84 57
4 36 28
5 50 55
6 35 12
7 73 61
8 48 29
;

PROC UNIVARIATE DATA=leaves;
    VAR diff;
    RUN;
```

SAS code for paired nonparametric tests

```
                     The UNIVARIATE Procedure
                          Variable:  diff

                              Moments

N                           8    Sum Weights                  8
Mean                     10.5    Sum Observations            84
Std Deviation      12.2007026    Variance            148.857143
Skewness           -0.1321468    Kurtosis            -1.2476273
Uncorrected SS           1924    Corrected SS              1042
Coeff Variation    116.197167    Std Error Mean      4.31359976


                    Basic Statistical Measures

        Location                    Variability

    Mean     10.50000     Std Deviation            12.20070
    Median   10.00000     Variance                148.85714
    Mode       .          Range                    33.00000
                          Interquartile Range      20.50000


                    Tests for Location: Mu0=0

         Test           -Statistic-    -----p Value------

         Student's t    t  2.434162    Pr > |t|    0.0451
         Sign           M         2    Pr >= |M|   0.2891
         Signed Rank    S      14.5    Pr >= |S|   0.0469
```

Conclusion

Removal of red cedar trees within 100 yards is associated with a significant reduction in rusty apple leaves (Wilcoxon signed rank test, p=0.023).

Do these data look normal?

```
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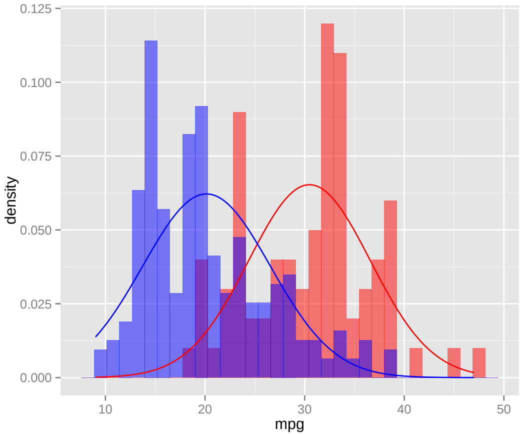


Rank-sum test

Also referred to as the Wilcoxon rank-sum test and the Mann-Whitney U test:

1. Transform the data to ranks
2. Calculate U, the sum of ranks of the group with a smaller sample size
3. Calculate $E[U] = n_1\overline{R}$
    1. n_1: sample size of the smaller group
    2. $\overline{R}$: average rank
4. Calculate $SD(U) = s_R\sqrt{\frac{n_1 n_2}{(n_1+n_2)}}$
    1. n_2: sample size of the larger group
    2. s_R: standard deviation of the ranks
5. Calculate $Z = (U + c - E[U])/SD(U)$ where c, the continuity correction, is either 0.5 or -0.5.
6. Determine the pvalue using a standard normal distribution.

Example on a small dataset

mpg	country	rank
13	US	1.0
15	US	2.0
17	US	3.0
22	US	4.0
26	Japan	5.5
26	US	5.5
28	US	7.0
32	Japan	8.0
33	Japan	9.0

- $U = 22.5$
- $E[U] = 15$
- $SD[U] = 3.86$
- $z = 1.81$ (appropriate continuity correction is -0.5)
- $p = 0.07$

Example on a small dataset

```
n1 = sum(sm$country=="Japan")
n2 = sum(sm$country=="US")
U = sum(sm$rank[sm$country=="Japan"])
EU = n1*mean(sm$rank)
SDU = sd(sm$rank) * sqrt(n1*n2/(n1+n2))
Z = (U-.5-EU)/SDU
2*pnorm(-Z)

[1] 0.06953

wilcox.test(mpg~country, sm)

Warning:  cannot compute exact p-value with ties

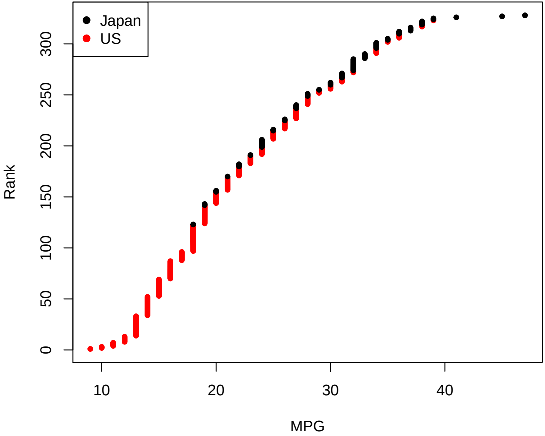

Wilcoxon rank sum test with continuity correction

data:  mpg by country
W = 16.5, p-value = 0.06953
alternative hypothesis: true location shift is not equal to 0
```

Visual representation of Rank Sum Test

```
ordr = order(mpg$mpg)
mpg.ordered = mpg[ordr,]

par(mar=c(5,4,0,0)+.1)
plot(mpg.ordered$mpg, 1:nrow(mpg), col=mpg.ordered$country, pch=19, xlab="MPG", cex=0.7, ylab="Rank")
legend("topleft", c("Japan","US"), col=1:2, pch=19)
```



R code and output for Rank Sum Test

```
wilcox.test(mpg~country,mpg)
```

```
Wilcoxon rank sum test with continuity correction

data:  mpg by country
W = 17150, p-value < 2.2e-16
alternative hypothesis: true location shift is not equal to 0
```

SAS code for Wilcoxon rank sum test

```
DATA mpg;
    INFILE 'mpg.csv' DELIMITER=',' FIRSTOBS=2;
    INPUT mpg country $;

PROC NPAR1WAY DATA=mpg WILCOXON;
    CLASS country;
    VAR mpg;
    RUN;
```

```
                    The NPAR1WAY Procedure

          Wilcoxon Scores (Rank Sums) for Variable mpg
                 Classified by Variable country

                       Sum of      Expected       Std Dev          Mean
country         N      Scores      Under H0      Under H0         Score
-----------------------------------------------------------------------
US            249    33646.50      40960.50    733.579091    135.126506
Japan          79    20309.50      12995.50    733.579091    257.082278

                Average scores were used for ties.


                   Wilcoxon Two-Sample Test

                   Statistic             20309.5000

                   Normal Approximation
                   Z                         9.9696
                   One-Sided Pr >  Z         <.0001
                   Two-Sided Pr > |Z|        <.0001

                   t Approximation
                   One-Sided Pr >  Z         <.0001
                   Two-Sided Pr > |Z|        <.0001

              Z includes a continuity correction of 0.5.


                     Kruskal-Wallis Test

                   Chi-Square               99.4068
                   DF                             1
```

Conclusion

Average miles per gallon of Japanese cars are significantly different than average miles per gallon of American cars (Wilcoxon rank sum test, $p < 0.0001$).