# Bayesian hypothesis testing

Dr. Jarad Niemi

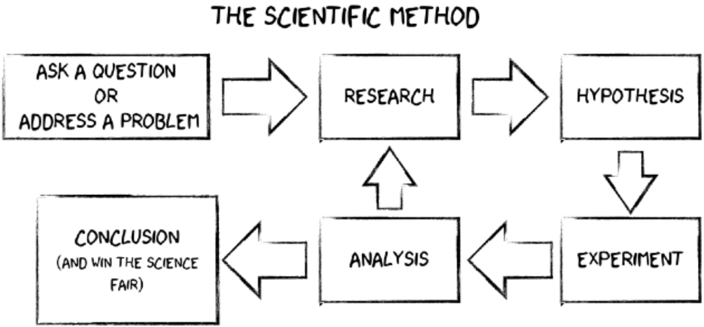
Iowa State University

February 23, 2016

# Outline

- Scientific method
  - Statistical hypothesis testing
  - Simple vs composite hypotheses
- Simple Bayesian hypothesis testing
  - All simple hypotheses
  - All composite hypotheses
- Propriety
  - Posterior
  - Prior predictive distribution
- Bayesian hypothesis testing with mixed hypotheses (models)
  - Prior model probability
  - Prior for parameters in composite hypotheses
    - WARNING: do not use non-informative priors
  - Posterior model probability

# Scientific method




http://www.wired.com/wiredscience/2013/04/whats-wrong-with-the-scientific-method/

# Statistical hypothesis testing

**Definition**

A simple hypothesis specifies the value for all parameters while a composite hypothesis does not.

Let $Y_i \stackrel{ind}{\sim} Ber(\theta)$ and

- $H_0 : \theta = 0.5$ (simple)
- $H_1 : \theta \neq 0.5$ (composite)

# Prior probabilities on simple hypotheses

What is your prior probability for the following hypotheses:

- a coin flip has exactly 0.5 probability of landing heads
- a fertilizer treatment has zero effect on plant growth
- inactivation of a mouse growth gene has zero effect on mouse hair color
- a butterfly flapping its wings in Australia has no effect on temperature in Ames
- guessing the color of a card drawn from a deck has probability 0.5

Many null hypotheses have zero probability *a priori*, so why bother performing the hypothesis test?

# Bayesian hypothesis testing with all simple hypotheses

Let $Y \sim p(y|\theta)$ and $H_j : \theta = \theta_j$ for $j = 1, \ldots, J$. Treat this as a discrete prior on the $\theta_j$, i.e.

$$P(\theta = \theta_j) = p_j.$$

The posterior is then

$$P(\theta = \theta_j|y) = \frac{p_j p(y|\theta_j)}{\sum_{k=1}^J p_k p(y|\theta_k)} \propto p_j p(y|\theta_j).$$

For example, suppose $Y_i \stackrel{ind}{\sim} Ber(\theta)$ and $P(\theta = j/10) = 1/11$ for $j = 0, \ldots, 10$. The posterior is

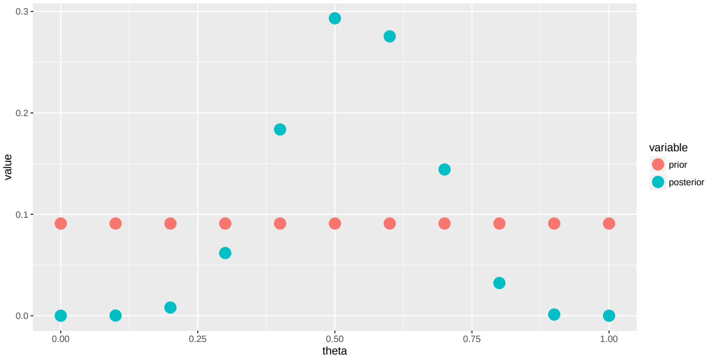
$$P(\theta = j/10|y) \propto \frac{1}{11} \prod_{i=1}^n (j/10)^{y_i} (1-j/10)^{1-y_i} = \frac{1}{11} (j/10)^{n\overline{y}} (1-j/10)^{n(1-\overline{y})}$$

If $j = 0$ ($j = 10$), any $y_i = 1$ ($y_i = 0$) will make the posterior probability zero.

# Discrete prior example

```
n = 13; y = rbinom(n,1,.45); sum(y)
```

```
[1] 7
```



# Bayesian hypothesis testing with all composite hypotheses

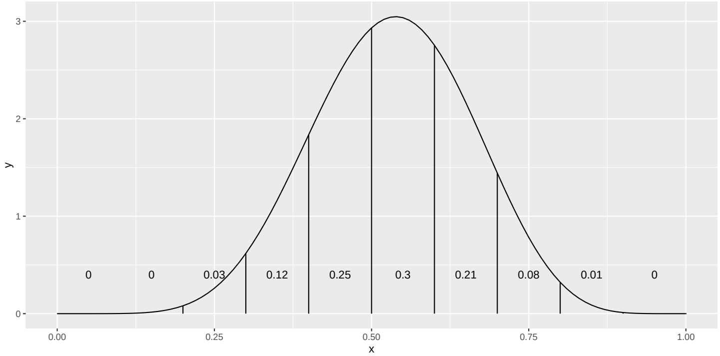
Let $Y \sim p(y|\theta)$ and $H_j : \theta \in (E_{j-1}, E_j]$ for $j = 1, \ldots, J$. Just calculate the area under the curve, i.e.

$$P(H_j|y) = \int_{E_{j-1}}^{E_j} p(\theta|y) d\theta$$

For example, suppose $Y_i \stackrel{ind}{\sim} Ber(\theta)$ and $E_j = j/10$ for $j = 0, \ldots, 10$. Now, assume

$$\theta \sim Be(1,1) \quad \text{and thus} \quad \theta|y \sim Be(1 + n\overline{y}, 1 + n[1 - \overline{y}]).$$

# Beta example



# Tonelli's Theorem (successor to Fubini's Theorem)

### Theorem

*Tonelli's Theorem states that if $\mathcal{X}$ and $\mathcal{Y}$ are $\sigma$-finite measure spaces and $f$ is non-negative and measureable, then*

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} f(x,y) dy dx = \int_{\mathcal{Y}} \int_{\mathcal{X}} f(x,y) dx dy$$

*i.e. you can interchange the integrals (or sums).*

On the following slides, the use of this theorem will be indicated by TT.

# Proper priors with discrete data

**Theorem**

*If the prior is proper and the data are discrete, then the posterior is always proper.*

**Proof.**

Let $p(\theta)$ be the prior and $p(y|\theta)$ be the statistical model. Thus, we need to show that

$$p(y) = \int_\Theta p(y|\theta)p(\theta)d\theta < \infty \quad \forall y.$$

For discrete $y$, we have

$$\begin{aligned} p(y) &\le \sum_{z\in\mathcal{Y}} p(z) = \sum_{z\in\mathcal{Y}} \int_\Theta p(z|\theta)p(\theta)d\theta \overset{TT}{=} \int_\Theta \sum_{z\in\mathcal{Y}} p(z|\theta)p(\theta)d\theta \\ &= \int_\Theta p(\theta)d\theta = 1. \end{aligned}$$

Thus the posterior is always proper if $y$ is discrete and the prior is proper. $\square$

# Proper priors with continuous data

### Theorem

*If the prior is proper and the data are continuous, then the posterior is almost always proper.*

### Proof.

Let $p(\theta)$ be the prior and $p(y|\theta)$ be the statistical model. Thus, we need to show that

$$p(y) = \int_\Theta p(y|\theta)p(\theta)d\theta < \infty \quad \text{for almost all } y.$$

For continuous $y$, we have

$$\int_\mathcal{Y} p(z)dz = \int_\mathcal{Y} \int_\Theta p(z|\theta)p(\theta)d\theta dz \overset{TT}{=} \int_\Theta \int_\mathcal{Y} p(z|\theta)dz\, p(\theta)d\theta = \int_\Theta p(\theta)d\theta = 1$$

thus $p(y)$ is finite except on a set of measure zero, i.e. $p(y)$ is almost always proper. □

# Proper prior predictive distributions

In the previous derivations, we showed that

$$\sum_{z\in\mathcal{Y}} p(z) = 1 \qquad \text{and} \qquad \int_{\mathcal{Y}} p(z)dz = 1$$

for discrete and continuous data, respectively.

Thus, when the prior is proper, the prior predictive distribution is also proper.

# Improper prior predictive distributions

**Theorem**

*If $p(\theta)$ is improper, then $p(y) = \int p(y|\theta)p(\theta)d\theta$ is improper.*

**Proof.**

$$\begin{aligned} \int p(y)dy &= \int \int p(y|\theta)p(\theta)d\theta dy \overset{TT}{=} \int p(\theta) \int p(y|\theta)dyd\theta \\ &= \int p(\theta)d\theta \end{aligned}$$

since $p(\theta)$ is improper, so is $p(y)$. A similar result holds for discrete $y$ replacing the integral with a sum. □

# Bayesian hypothesis testing

To evaluate the relative plausibility of a hypothesis (model), we use the posterior model probability:

$$p(H_j|y) = \frac{p(y|H_j)p(H_j)}{p(y)} = \frac{p(y|H_j)p(H_j)}{\sum_{k=1}^{J} p(y|H_k)p(H_k)} \propto p(y|H_j)p(H_j).$$

where $p(H_j)$ is the prior model probability and

$$p(y|H_j) = \int p(y|\theta)p(\theta|H_j)d\theta$$

is the marginal likelihood under model $H_j$ and $p(\theta|H_j)$ is the prior for parameters $\theta$ when model $H_j$ is true.

# Marginal likelihood

The marginal likelihood calculation differs for simple vs composite hypotheses:

- Simple hypotheses can be considered to have a Dirac delta function for a prior, e.g. if $H_0 : \theta = \theta_0$ then $\theta|H_0 \sim \delta_{\theta_0}$. Then the marginal likelihood is

$$p(y|H_0) = \int p(y|\theta)p(\theta|H_0)d\theta = p(y|\theta_0).$$

- Composite hypotheses have a continuous prior and thus

$$p(y|H_j) = \int p(y|\theta)p(\theta|H_j)d\theta.$$

# Two models

If we only have two models: $H_0$ and $H_1$, then

$$p(H_0|y) = \frac{p(y|H_0)p(H_0)}{p(y|H_0)p(H_0) + p(y|H_1)p(H_1)} = \frac{1}{1 + \frac{p(y|H_1)}{p(y|H_0)}\frac{p(H_1)}{p(H_0)}}$$

where

$$\frac{p(H_1)}{p(H_0)} = \frac{p(H_1)}{1 - p(H_1)}$$

is the prior odds in favor of $H_1$ and

$$BF(H_1 : H_0) = \frac{p(y|H_1)}{p(y|H_0)} = \frac{1}{BF(H_0 : H_1)}$$

is the Bayes Factor for model $H_1$ relative to $H_0$.

# Binomial model

Consider a coin flipping experiment so that $Y_i \stackrel{ind}{\sim} Ber(\theta)$ and the null hypothesis $H_0 : \theta = 0.5$ versus the alternative $H_1 : \theta \neq 0.5$ and $\theta | H_1 \sim Be(a, b)$.

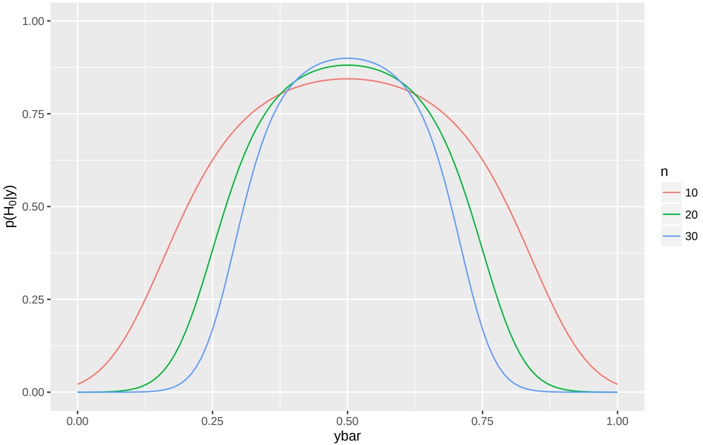
$$
\begin{aligned}
BF(H_0 : H_1) &= \frac{0.5^n}{\int_0^1 \theta^{n\overline{y}}(1-\theta)^{n(1-\overline{y})}\frac{\theta^{a-1}(1-\theta)^{b-1}}{Beta(a,b)}d\theta} \\
&= \frac{0.5^n}{\frac{1}{Beta(a,b)}\int_0^1 \theta^{a+n\overline{y}-1}(1-\theta)^{b+n-n\overline{y}-1}\theta} \\
&= \frac{0.5^n}{\frac{Beta(a+n\overline{y},b+n-n\overline{y})}{Beta(a,b)}} \\
&= \frac{0.5^n Beta(a,b)}{Beta(a+n\overline{y},b+n-n\overline{y})}
\end{aligned}
$$

and with $p(H_0) = p(H_1)$ the posterior model probability is

$$
P(H_0|y) = \frac{1}{1 + \frac{1}{BF(H_0:H_1)}}.
$$

# Sample size and sample average

$P(H_0) = P(H_1) = 0.5$ and $\theta|H_1 \sim Be(1,1)$:



# "Non-informative" prior

Recall that $\theta \sim Be(a,b)$ has

- $a$ prior successes and
- $b$ prior failures.

Thus, in some sense $a, b \longrightarrow 0$ puts minimal prior data into the analysis.

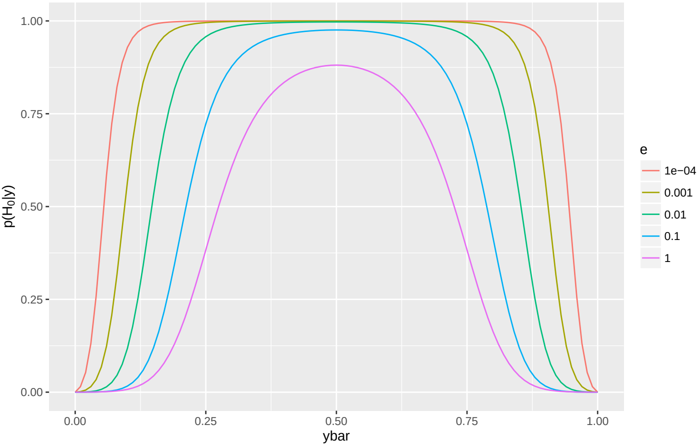
If $\theta|H_1 \sim Be(e,e)$, then

$$BF(H_0 : H_1) = \frac{0.5^n Be(e,e)}{Be(e+n\overline{y}, b+n-n\overline{y})} \xrightarrow{e \longrightarrow 0} \infty \quad \text{for any } \overline{y} \in (0,1)$$

since $Be(e,e) \xrightarrow{e \longrightarrow 0} \infty$.

# Limit of proper prior

$P(H_0) = P(H_1) = 0.5$ and $\theta | H_1 \sim Be(e, e)$:



# Normal example

Consider the model $Y \stackrel{ind}{\sim} N(\theta, 1)$ and the hypothesis test

- $H_0 : \theta = 0$ versus
- $H_1 : \theta \neq 0$ with prior $\theta | H_1 \sim N(0, C)$.

The predictive distribution under $H_1$ is

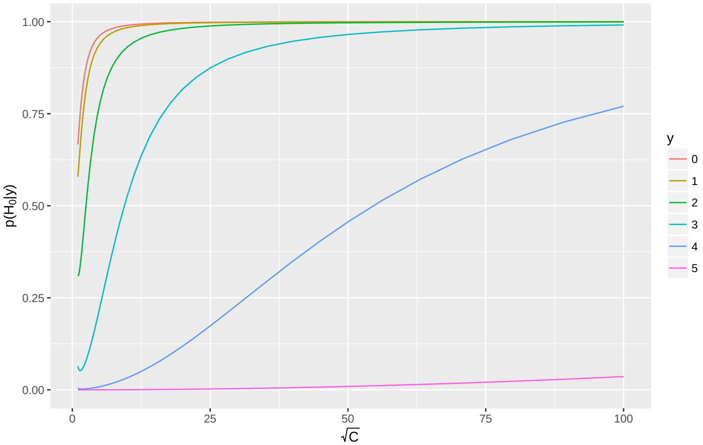
$$p(y|H_1) = \int p(y|\theta)p(\theta|H_1)d\theta = N(y; 0, 1 + C)$$

and the Bayes factor is

$$BF(H_0 : H_1) = \frac{N(y; 0, 1)}{N(y; 0, 1 + C)}.$$

The Bayes factor will increase as $C \to \infty$ for any $y$ and this only gets worse if you use an improper prior.

# Normal example



# Summary

- Treat hypothesis testing as parameter estimation
  - All simple hypotheses: discrete prior
  - All composite hypotheses: continuous prior
- Formal Bayesian hypothesis testing (simple and composite hypotheses)
  - Specify prior model probabilities
  - Specify parameter priors for composite hypotheses
    WARNING: Do not use non-informative priors!
  - Calculate Bayes Factors or posterior model probabilities

# Scientific method updated

*All models are wrong, but some are useful.*

George Box 1987