M5S2 - Confidence Intervals

for population mean with population standard deviation unknown

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Outline

- Confidence intervals for the population mean when the population standard deviation is unknown
  - t distribution
  - Finding t critical values
  - significance level
  - confidence level
  - margin of error

Confidence intervals when σ is known

Recall that by the CLT

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \dot{\sim} N(0,1)$$

where $\overline{X}$ is the (random) sample mean, μ is the population mean, σ is the population standard deviation, and n is the sample size.

When the population standard deviation σ is known, we used this result to construct a $100(1-\alpha)$% confidence interval for the population mean μ using the formula

$$\overline{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

where the z critical value is such that $P(Z > z_{\alpha/2}) = \alpha/2$ for a given significance level α.

If σ is unknown, then we can't use σ to calculate this interval.

Replace σ with s, the sample standard deviation

If $X_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$, we have a similar result when using the sample standard deviation,

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

instead of σ:

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

where t_{n-1} is a Student's t distribution with $n-1$ degrees of freedom.

For a $100(1-\alpha)\%$ confidence interval, we can find a t critical value $t_{n-1,\alpha/2}$ and construct the confidence interval using the following formula:

$$\overline{x} \pm t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

for the observed sample mean $\overline{x}$ and sample standard deviation s.

Student's t-distribution

Student's t-distribution was derived by William Gosset, a statistician working for the Guiness Brewing Company. A random variable T has a (standard) t-distribution with ν degrees of freedom if it has the pdf

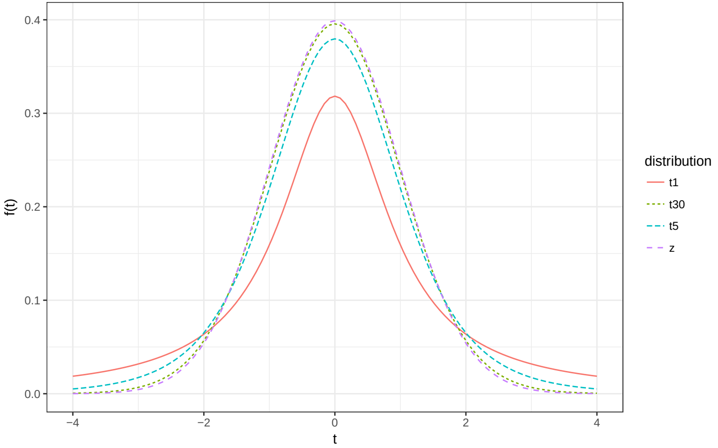
$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\ \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where $\Gamma(x) = \int_0^\infty a^{x-1} e^a da$ and

- $E[T] = 0$ for $\nu > 1$ and
- $Var[T] = \frac{\nu}{\nu-2}$ for $\nu > 2$.

A (standard) t-distribution converges to a standard normal distribution as $\nu \to \infty$.

Student's t-distribution pdf



Finding t critical values

A t critical value $t_{\nu,\alpha/2}$ is the value such that

$$P(T_\nu > t_{\nu,\alpha/2}) = \alpha/2$$

where T_ν is the t-distribution with ν degrees of freedom.

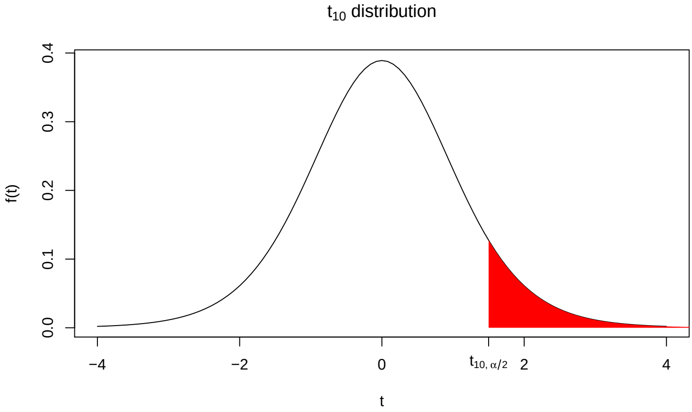


Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

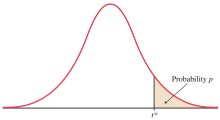


TABLE D t **distribution critical values**

	Upper tail probability p											
df	**.25**	**.20**	**.15**	**.10**	**.05**	**.025**	**.02**	**.01**	**.005**	**.0025**	**.001**	**.0005**
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
z^*	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291
	50%	**60%**	**70%**	**80%**	**90%**	**95%**	**96%**	**98%**	**99%**	**99.5%**	**99.8%**	**99.9%**
	Confidence level C											

Confidence Intervals for μ when σ is unknown

Definition

Let μ be the population mean and σ be the unknown population standard deviation for a normal population. Choose a significance level α which you can convert to a confidence level $C = 100(1-\alpha)\%$ and a t critical value $t_{n-1,\alpha/2}$ where $P(T_{n-1} > t_{n-1,\alpha/2}) = \alpha/2$.

You obtain a random sample of n observations from the population and calculate the sample mean $\overline{x}$ and sample standard deviation s. Then a $C = 100(1-\alpha)\%$ confidence interval for μ is

$$\overline{x} \pm t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} = \left(\overline{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}, \overline{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right)$$

where $t_{n-1,\alpha/2} \cdot s/\sqrt{n}$ is called the margin of error.

Savings account balances

US Bank provides students with savings accounts having no monthly maintenance fee and a low minimum monthly transfer. US Bank is interested in knowing the mean monthly balance of all its student savings accounts. They take a random sample of 23 student savings accounts and record that at the end of the month the sample mean savings was \$105 and the standard deviation was \$20. Assuming savings account balances are normally distributed, construct an 80% confidence interval for the mean monthly balance.

Let X_i be the end of the month balance for student i. Then $E[X_i] = \mu$, the mean monthly balance, is unknown, $SD[X_i] = \sigma$ is unknown. We obtained a sample of size $n = 23$ with a sample mean $\overline{x} = \$105$ and a sample standard deviation of $s = \$20$. For a confidence level of 80%, we have $\alpha = 0.2$, $\alpha/2 = 0.1$ and $t_{n-1,\alpha/2} \approx 1.319$. Then we calculate

$$\overline{x} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}} = \$105 \pm 1.319 \frac{\$20}{\sqrt{23}} = (\$99.5, \$110.5)$$