# STAT 401A - Statistical Methods for Research Workers

## Regression diagnostics

Jarad Niemi (Dr. J)

Iowa State University

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> *This isn't just true in statistics! Maps are a type of model; they are wrong. But good maps are very useful.*

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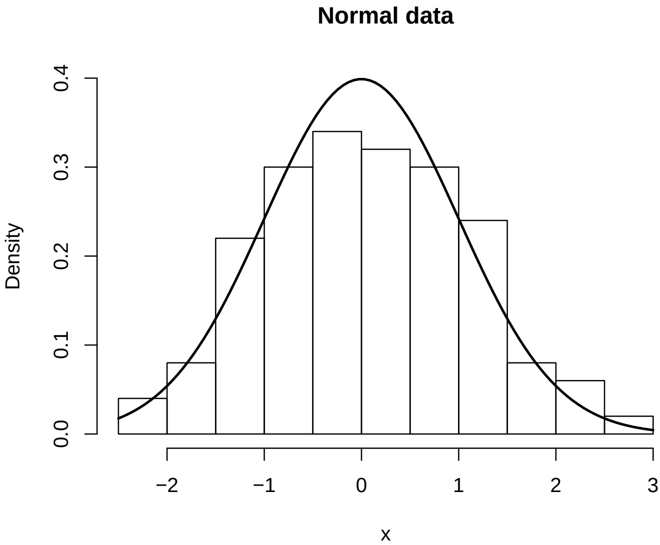
- Linearity between mean response and explanatory variable

# Histograms with best fitting bell curves



# Normal QQ-plot

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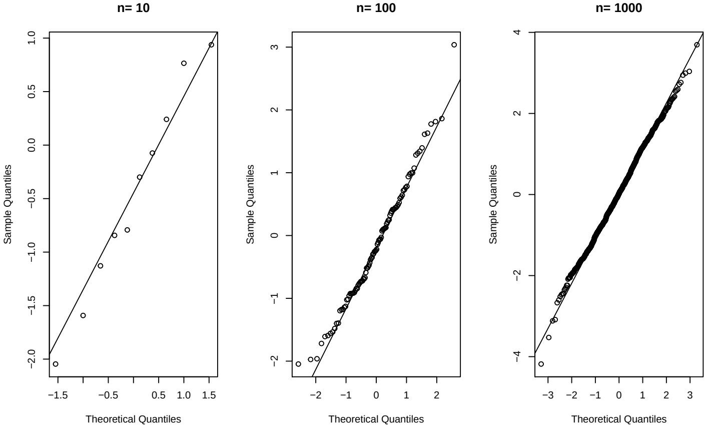
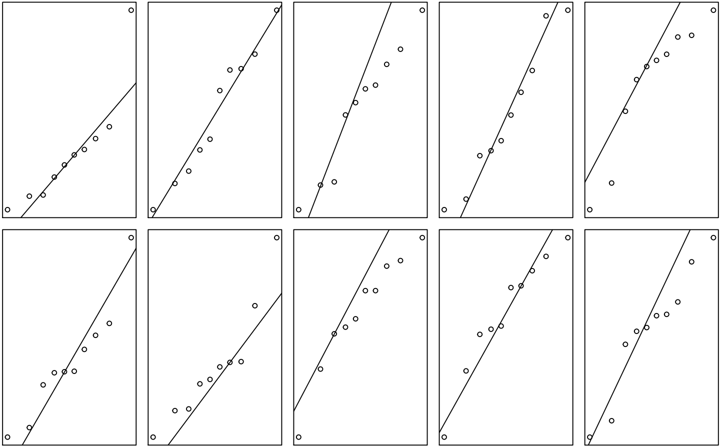
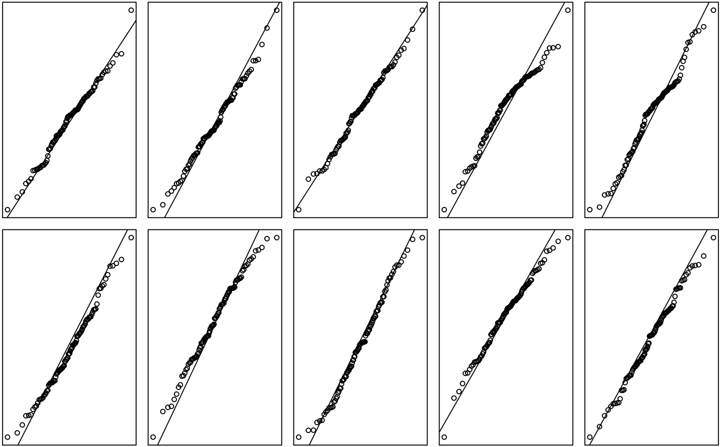

n= 10
n= 100
n= 1000
Sample Quantiles
Theoretical Quantiles
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−1.5
−0.5
0.5
1.0
1.5
3
2
1
0
−1
−2
−2
−1
0
1
2
4
2
0
−2
−4
−3
−2
−1
0
1
2
3

# Normal (n=10)



# Normal(n=100)



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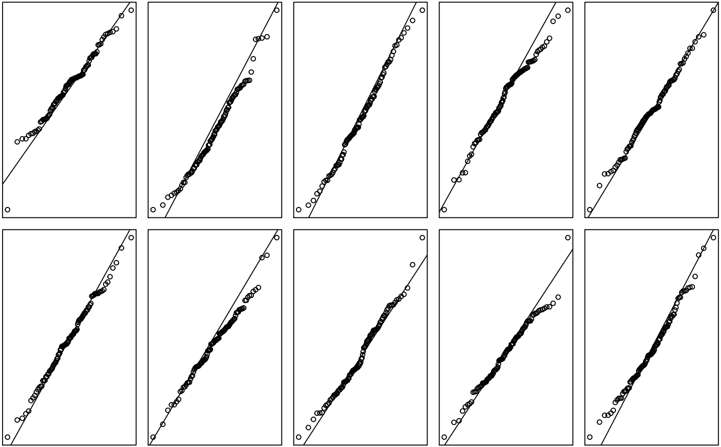
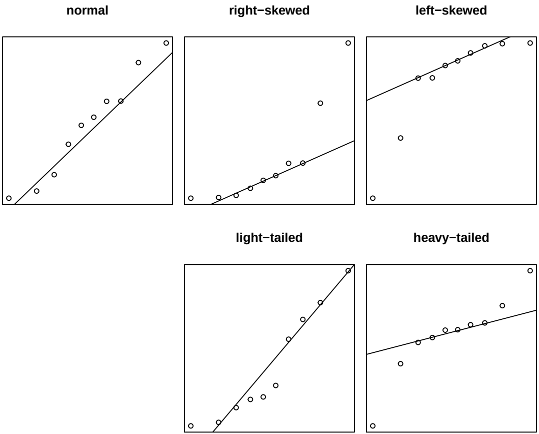


# Not normal (n=10)



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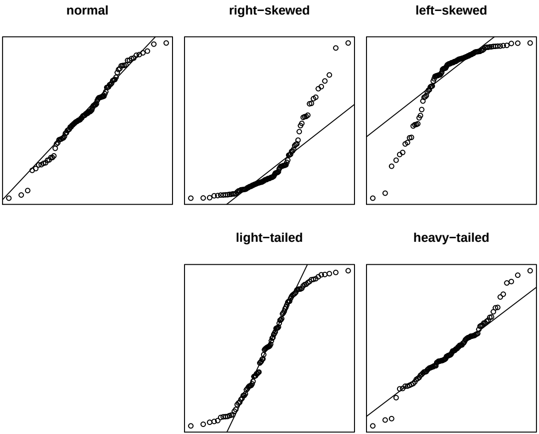

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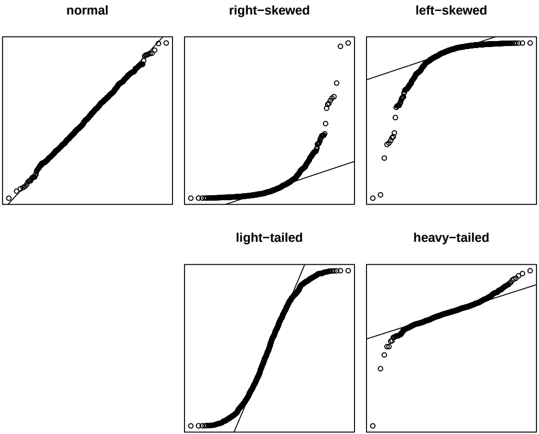

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Other patterns are certainly possible, but these are the most common.

# Constant variance

Recall the model

$$Y_i = \beta_0 + \beta_1 X_i + e_i \quad e_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

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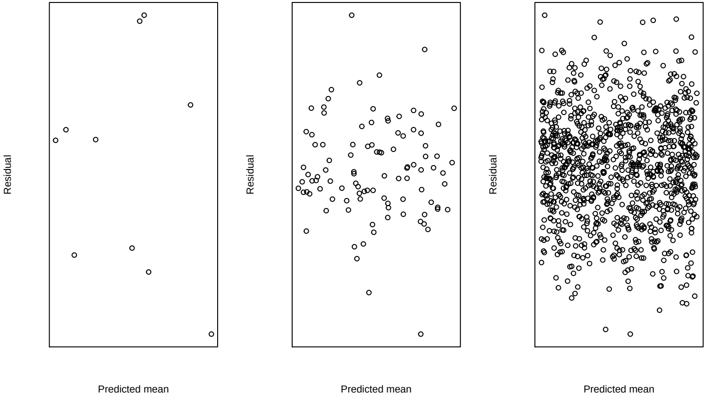

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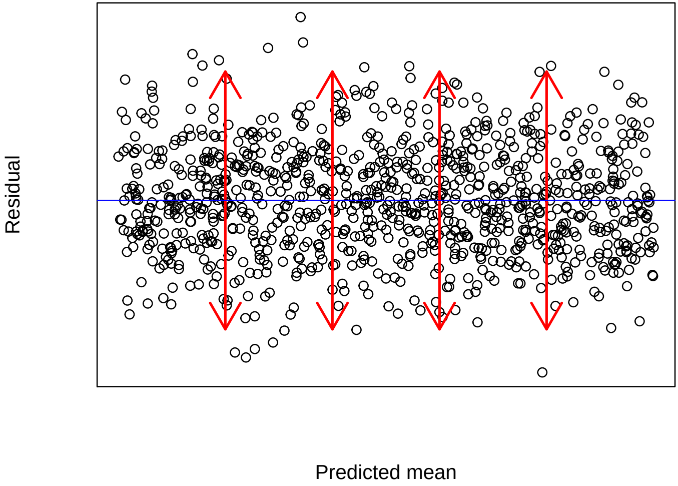

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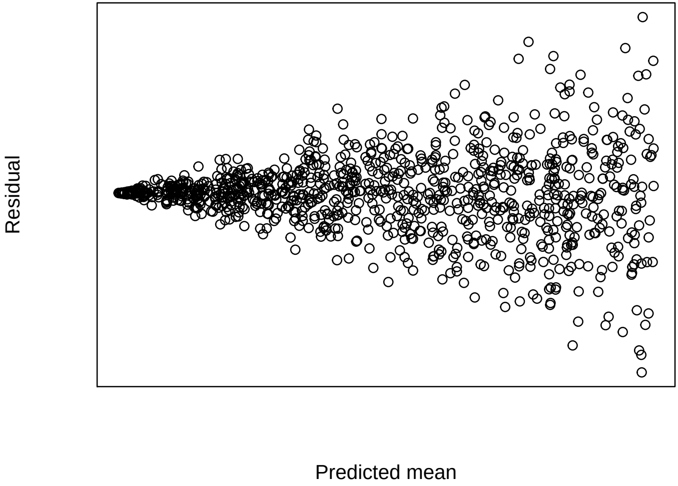

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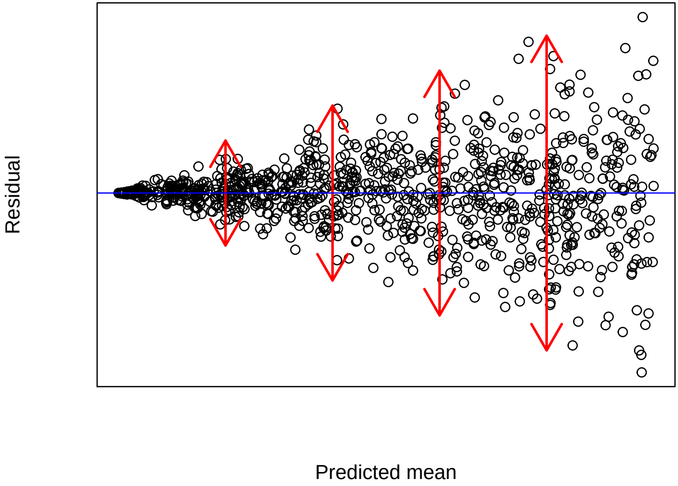
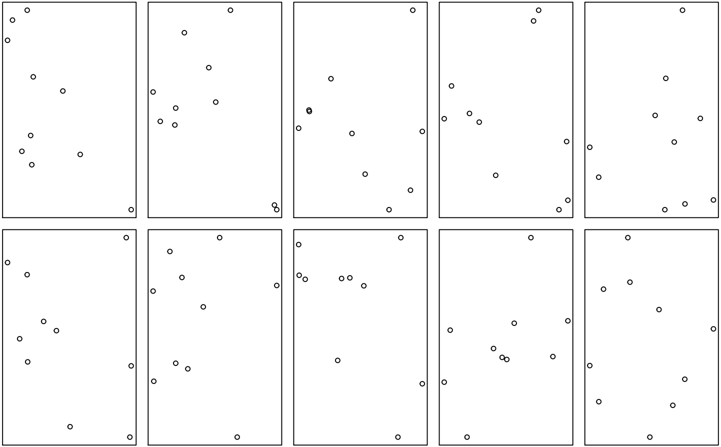
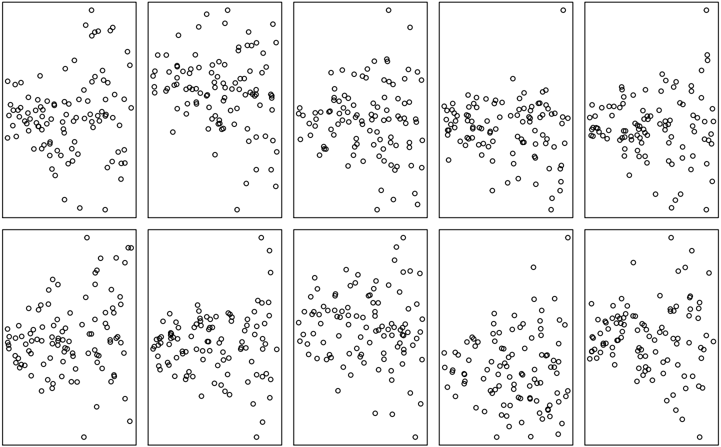


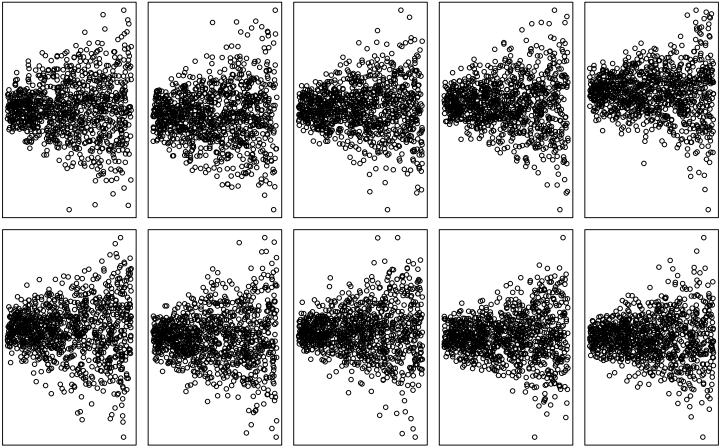
# Non-constant variance (n=10, $\sigma_2/\sigma_1 = 4$)



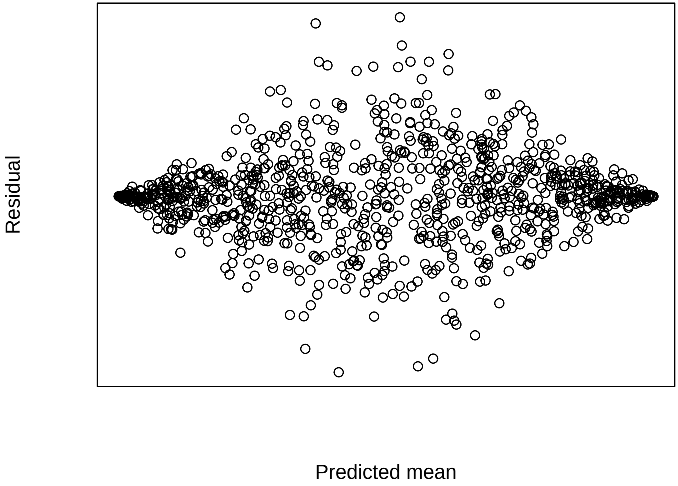
# Non-constant variance (n=100, $\sigma_2/\sigma_1 = 4$)



# Non-constant variance (n=1000, $\sigma_2/\sigma_1 = 4$)



# Extreme non-constant variance (football)



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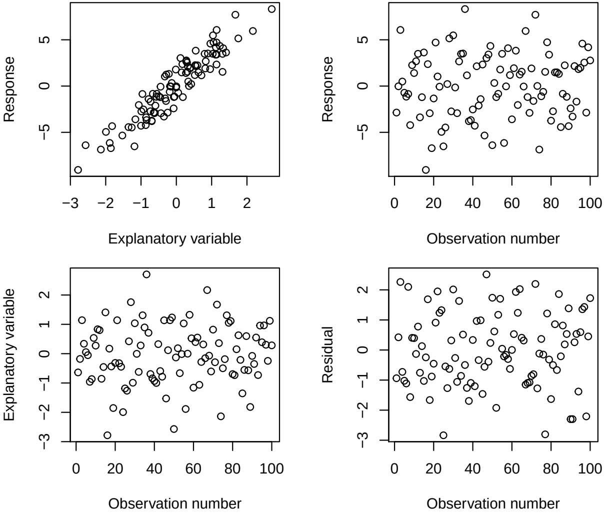
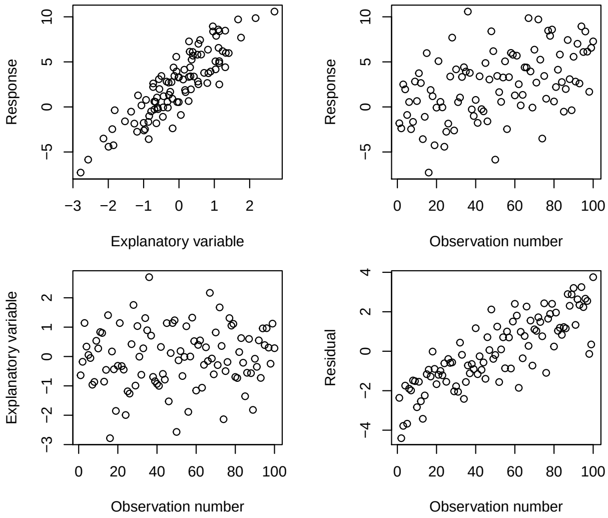
# No evidence for lack of independence

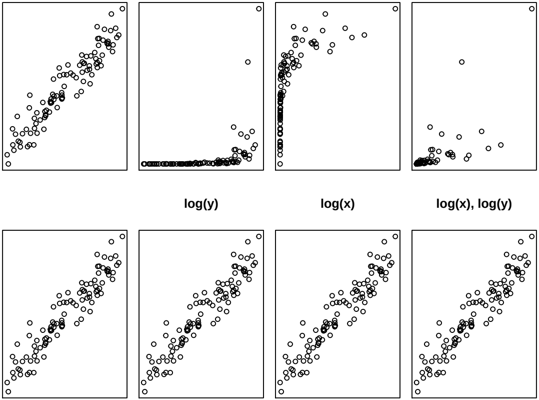


# Evidence for lack of independence



# Linearity

Assess using scatterplots of (transformed) response vs (transformed) explanatory variable:



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Do the following

1. Calculate extra sum of squares.
2. Calculate extra degrees of freedom
3. Calculate

$$\text{F-statistic} = \frac{\text{Extra sum of squares / Extra degrees of freedom}}{\hat{\sigma}^2_{full}}$$

4. Compare this to an F-distribution with
   - numerator degrees of freedom = extra degrees of freedom
   - denominator degrees of freedom = degrees of freedom in estimating $\hat{\sigma}^2_{full}$

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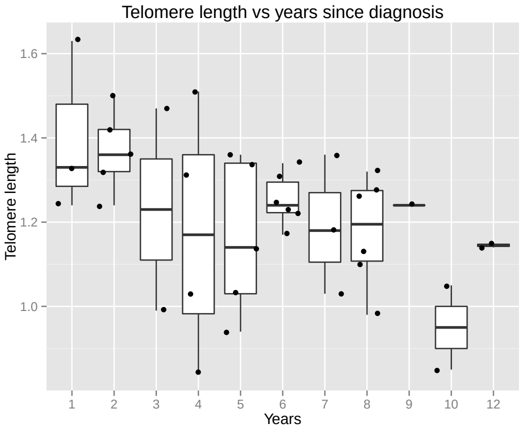
- Lack-of-fit F-test requires multiple observations at a few $X_j$ values!

# Telomere length



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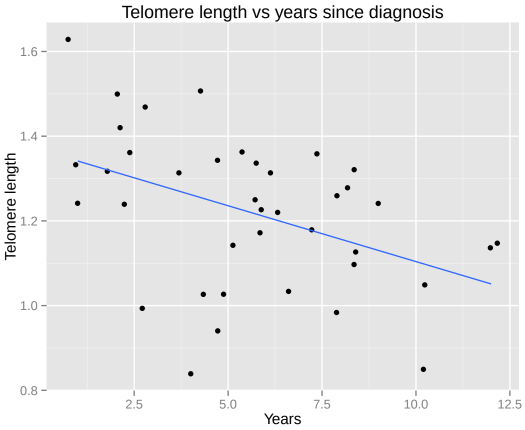

# SAS code

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC REG DATA=t;
  MODEL length = years / CLB LACKFIT;
  RUN;
```

```
                              The REG Procedure
                                Model: MODEL1
                          Dependent Variable: length

                  Number of Observations Read          39
                  Number of Observations Used          39


                            Analysis of Variance

                                     Sum of           Mean
        Source               DF     Squares         Square    F Value    Pr > F

        Model                 1     0.22777        0.22777       8.42    0.0062
        Error                37     1.00033        0.02704
          Lack of Fit         9     0.18223        0.02025       0.69    0.7093
          Pure Error         28     0.81810        0.02922
        Corrected Total      38     1.22810
```

```
# Use as.factor to turn a continuous variable into a categorical variable
m_anova = lm(telomere.length ~ as.factor(years), Telomeres)
m_reg   = lm(telomere.length ~           years , Telomeres)
anova(m_reg, m_anova)


Analysis of Variance Table

Model 1: telomere.length ~ years
Model 2: telomere.length ~ as.factor(years)
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2     28 0.818  9     0.182 0.69   0.71
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No evidence of a lack of fit.

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  - Transform explanatory variable
  - Add other explanatory variable(s)

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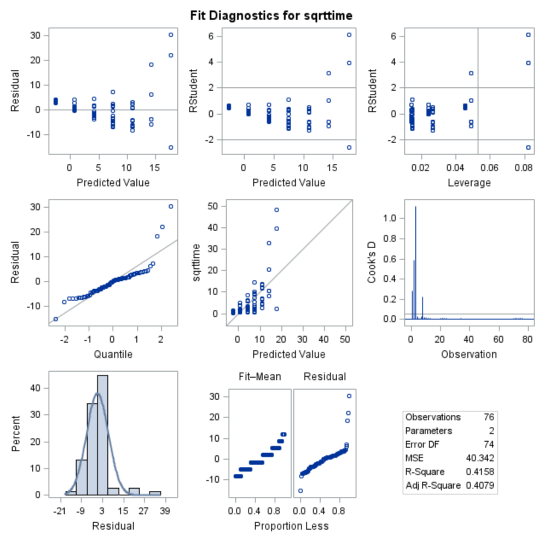
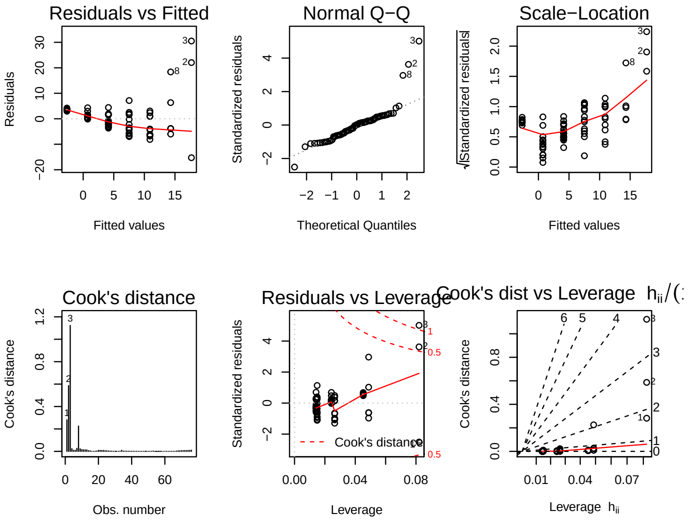
  - Response vs explanatory variable should be linear
  - Lack-of-fit F-test should not be significant

# Default diagnostics in SAS




| | |
|---|---|
| Observations | 76 |
| Parameters | 2 |
| Error DF | 74 |
| MSE | 40.342 |
| R-Square | 0.4158 |
| Adj R-Square | 0.4079 |

# Default diagnostics in R



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To construct confidence intervals for $e^\beta$, find a confidence interval for $\beta$ and exponentiate the endpoints, i.e. if $(L, U)$ is a confidence interval for $\beta$, then $(e^L, e^U)$ is a confidence interval for $e^\beta$.