Set S03 - Random effects

STAT 401 (Engineering) - Iowa State University

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Regression models

For continuous Y_i, we have linear regression

$$Y_i \stackrel{ind}{\sim} N(\mu_i, \sigma^2), \quad \mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

For binary or count with an upper maximum Y_i, we have logistic regression

$$Y_i \stackrel{ind}{\sim} Bin(n_i, \theta_i), \quad \log\left(\frac{\theta_i}{1-\theta_i}\right) = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

For count data with no upper maximum, we have Poisson regression

$$Y_i \stackrel{ind}{\sim} Po(\lambda_i), \quad \log(\lambda_i) = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

But what if our observations cannot reasonably be assumed to be independent given these explanatory variables?

Random effect model

Suppose we have continuous observations Y_{ij} for individual i from group j. A random effects model (with a common variance) assumes

$$Y_{ij} = \mu + \alpha_j + \epsilon_{ij} \quad \epsilon_{ij} \stackrel{ind}{\sim} N(0, \sigma^2_\epsilon)$$

and, to make the α_i random effects, independent of ϵ_{ij}

$$\alpha_j \stackrel{ind}{\sim} N(0, \sigma^2_\alpha).$$

This makes observations within the group correlated since

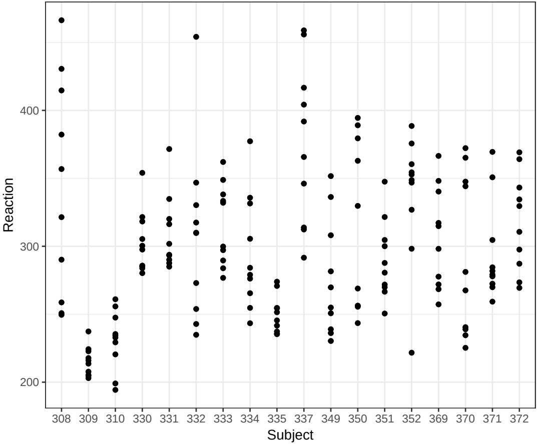
$$\begin{aligned} Cov[Y_{ij}, Y_{i'j}] &= Cov[\alpha_j + \epsilon_{ij}, \alpha_j + \epsilon_{i'j}] \\ &= Var[\alpha_j] = \sigma^2_\alpha \end{aligned}$$

and

$$Cor[Y_{ij}, Y_{i'j}] = \frac{Cov[Y_{ij}, Y_{i'j}]}{\sqrt{Var[Y_{ij}]Var[Y_{i'j}]}} = \frac{\sigma^2_\alpha}{\sigma^2_\alpha + \sigma^2_\epsilon}.$$

Sleep study example

```
ggplot(sleepstudy, aes(Subject, Reaction)) + geom_point() + theme_bw()
```



Sleep study example

```
summary(me <- lmer(Reaction ~ (1|Subject), sleepstudy))


Linear mixed model fit by REML [’lmerMod’]
Formula: Reaction ~ (1 | Subject)
   Data: sleepstudy

REML criterion at convergence: 1904.3

Scaled residuals:
    Min      1Q  Median      3Q     Max
-2.4983 -0.5501 -0.1476  0.5123  3.3446

Random effects:
 Groups   Name        Variance Std.Dev.
 Subject  (Intercept) 1278     35.75
 Residual             1959     44.26
Number of obs: 180, groups:  Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept)   298.51       9.05   32.98
```

Mixed effect model

Suppose we have continuous observations Y_{ij} for individual i from group j and an associated explanatory variable X_{ij}. A mixed effect model assumes

$$Y_{ij} = \beta_0 + \beta_1 X_{ij} + \alpha_j + \epsilon_{ij} \quad \epsilon_{ij} \overset{ind}{\sim} N(0, \sigma^2_\epsilon)$$

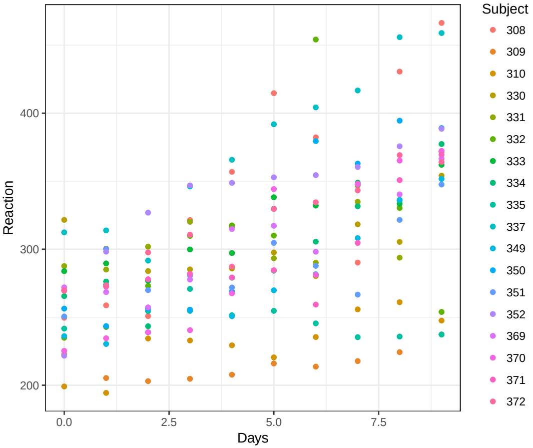
and, to make the α_i random effects, independent of ϵ_{ij}

$$\alpha_j \overset{ind}{\sim} N(0, \sigma^2_\alpha).$$

Again, this enforces a correlation between the observations within a group. This model is often referred to as a random intercept model because each group has its own intercept $(\beta_0 + \alpha_j)$ and these are *random* since α_j has a distribution. Thus this model is related to a model that includes a fixed effect for each subject. But here those subject specific effects are shrunk toward an overall mean (β_0).

Sleep study example

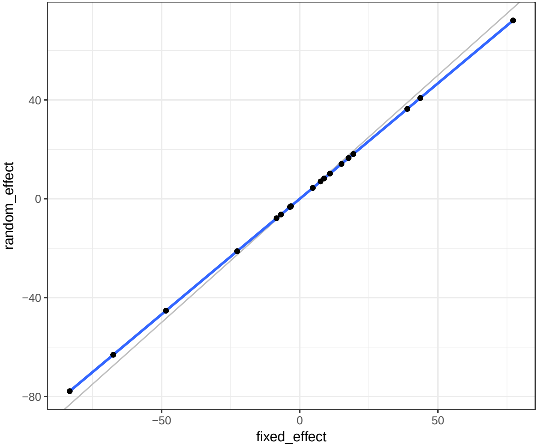
```
ggplot(sleepstudy, aes(Days, Reaction, color = Subject)) +
  geom_point() + theme_bw()
```



Sleep study example

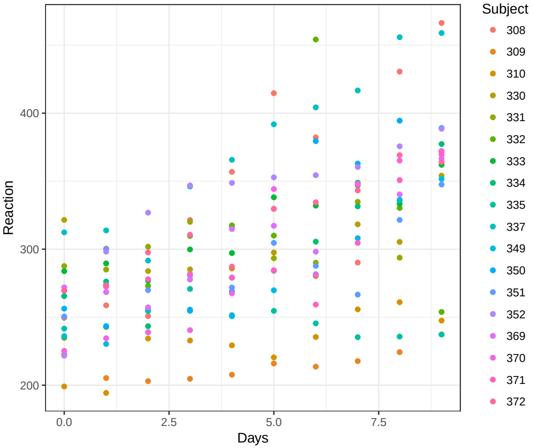
```
summary(me <- lmer(Reaction ˜ Days + (1|Subject), sleepstudy))


Linear mixed model fit by REML ['lmerMod']
Formula: Reaction ˜ Days + (1 | Subject)
   Data: sleepstudy

REML criterion at convergence: 1786.5

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.2257 -0.5529  0.0109  0.5188  4.2506

Random effects:
 Groups   Name        Variance Std.Dev.
 Subject  (Intercept) 1378.2   37.12
 Residual              960.5   30.99
Number of obs: 180, groups:  Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept) 251.4051     9.7467   25.79
Days         10.4673     0.8042   13.02

Correlation of Fixed Effects:
     (Intr)
Days -0.371
```

Shrinkage



Sleep study example

```
ggplot(sleepstudy, aes(Days, Reaction, color = Subject)) +
  geom_point() + theme_bw()
```



Random slope model

Suppose we have continuous observations Y_{ij} for individual i from group j. A mixed effect model with group specific slopes assumes

$$Y_{ij} = \beta + 0 + \beta_1 X_{ij} + \alpha_{0j} + \alpha_{1j} X_{ij} + \epsilon_{ij} \quad \epsilon_{ij} \overset{ind}{\sim} N(0, \sigma^2_\epsilon)$$

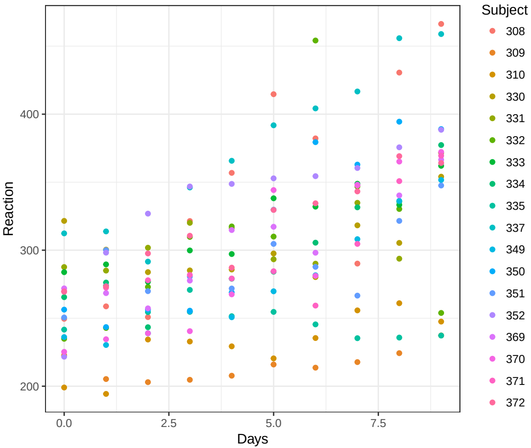
and, independent of ϵ_{ij},

$$\left(\begin{array}{c} \alpha_{0j} \\ \alpha_{1j} \end{array} \right) \overset{ind}{\sim} N(0, \Sigma_\alpha)$$

$N(0, \Sigma_\alpha)$ represents a bivariate normal with mean 0 and covariance matrix Σ_α. This model is often referred to as a random slope model because each group has its own slope $(\beta_1 + \alpha_{1j})$ and these are *random* since α_{1j} has a distribution. Thus this model is related to a model that includes an interaction between the group and the explanatory variable, but here those subject specific slopes are shrunk toward an overall slope (β_1).

Sleep study example

```
ggplot(sleepstudy, aes(Days, Reaction, color = Subject)) +
  geom_point() + theme_bw()
```

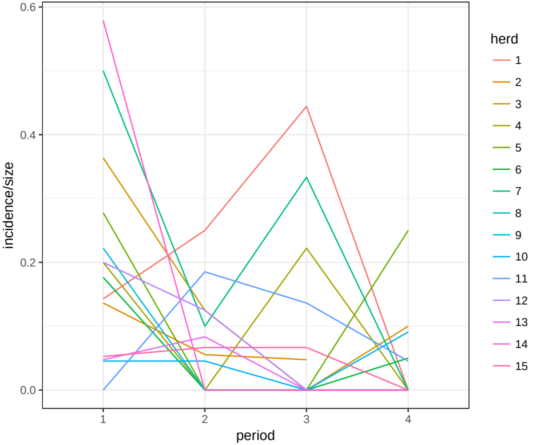


Sleep study example

```
summary(me <- lmer(Reaction ~ Days + (Days|Subject), sleepstudy))


Linear mixed model fit by REML ['lmerMod']
Formula: Reaction ~ Days + (Days | Subject)
   Data: sleepstudy

REML criterion at convergence: 1743.6

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.9536 -0.4634  0.0231  0.4634  5.1793

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 Subject  (Intercept) 612.09   24.740
          Days         35.07    5.922   0.07
 Residual             654.94   25.592
Number of obs: 180, groups:  Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept)  251.405      6.825   36.84
Days          10.467      1.546    6.77

Correlation of Fixed Effects:
     (Intr)
Days -0.138
```

Contagious bovine pleuropneumonia (CBPP)

```
ggplot(cbpp, aes(period, incidence/size, color=herd, group=herd)) +
  geom_line() + theme_bw()
```



Generalized linear mixed effect models

The same idea can be utilized in generalized linear models, e.g. logistic and Poisson regression.

A mixed effect logistic regression model for CBPP count is

$$\begin{aligned} Y_{ph} &\overset{ind}{\sim} Bin(n_{ph}, \theta_{ph}) \\ \text{logit}\,(\theta_{ph}) &= \beta_0 + \beta_1 \mathrm{I}(p=2) + \beta_2 \mathrm{I}(p=3) + \beta_3 \mathrm{I}(p=4) + \alpha_h \\ \alpha_h &\overset{ind}{\sim} N(0, \sigma^2_\alpha) \end{aligned}$$

where $p = 1, 2, 3, 4$ stands for the period and $h = 1, \ldots, 15$ stands for the herd.

When used in GLMs, these models are called generalized linear mixed models (GLMMs).

GLMMs in R

```
me <- glmer(cbind(incidence, size - incidence) ~ period + (1 | herd),
            data = cbpp, family = binomial)
summary(me)


Generalized linear mixed model fit by maximum likelihood (Laplace Approximation) [’glmerMod’]
 Family: binomial  ( logit )
Formula: cbind(incidence, size - incidence) ~ period + (1 | herd)
   Data: cbpp

     AIC      BIC   logLik deviance df.resid
   194.1    204.2    -92.0    184.1       51

Scaled residuals:
    Min      1Q  Median      3Q     Max
-2.3816 -0.7889 -0.2026  0.5142  2.8791

Random effects:
 Groups Name        Variance Std.Dev.
 herd   (Intercept) 0.4123   0.6421
Number of obs: 56, groups:  herd, 15

Fixed effects:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -1.3983     0.2312  -6.048 1.47e-09 ***
period2      -0.9919     0.3032  -3.272 0.001068 **
period3      -1.1282     0.3228  -3.495 0.000474 ***
period4      -1.5797     0.4220  -3.743 0.000182 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

Contrasts

Is there a linear trend in logit(θ_{ph}) by period?

```
ls <- lsmeans(me, ~ period, type="response") # for intrepretability
ls


 period       prob         SE df  asymp.LCL asymp.UCL
 1      0.19807921 0.03672693 NA 0.13569523 0.2798569
 2      0.08391784 0.02363110 NA 0.04775454 0.1433443
 3      0.07401714 0.02241761 NA 0.04040242 0.1317591
 4      0.04842565 0.01959184 NA 0.02163870 0.1048199

Confidence level used: 0.95
Intervals are back-transformed from the logit scale


co <- contrast(ls, list(‘linear trend‘ = c(-1.5, -0.5, 0.5, 1.5)))
confint(co)


 contrast     odds.ratio         SE df  asymp.LCL asymp.UCL
 linear trend 0.08735598 0.05765302 NA 0.02396174 0.3184688

Confidence level used: 0.95
Intervals are back-transformed from the log odds ratio scale
```

Summary

There are a variety of opinions about when to use fixed effects and when to use random effects, e.g. https://stats.stackexchange.com/questions/4700/what-is-the-difference-between-fixed-effect-random-effect-and-mixed-effect-mode.

I am in favor of using random effects whenever we have enough levels (~ 5) of the effect to estimate the variance and we can consider the levels *exchangeable*.

For example, in the CBPP data set,

- period only has 4 levels and their are not exchangeable because they are ordered
- herd has 15 levels and the herds are exchangeable

thus I would treat period as a fixed effect and herd as random effect.