Bayesian Spatial Analysis

Dr. Jarad Niemi

STAT 615 - Iowa State University

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Spatial modeling

Three main types of spatial data:

- Point/geo-referenced
- Areal-referenced
- Point process/pattern

Point-referenced spatial data

Features

- Some spatial domain $\mathcal{D}$ is under study

Examples

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- Bird point counts

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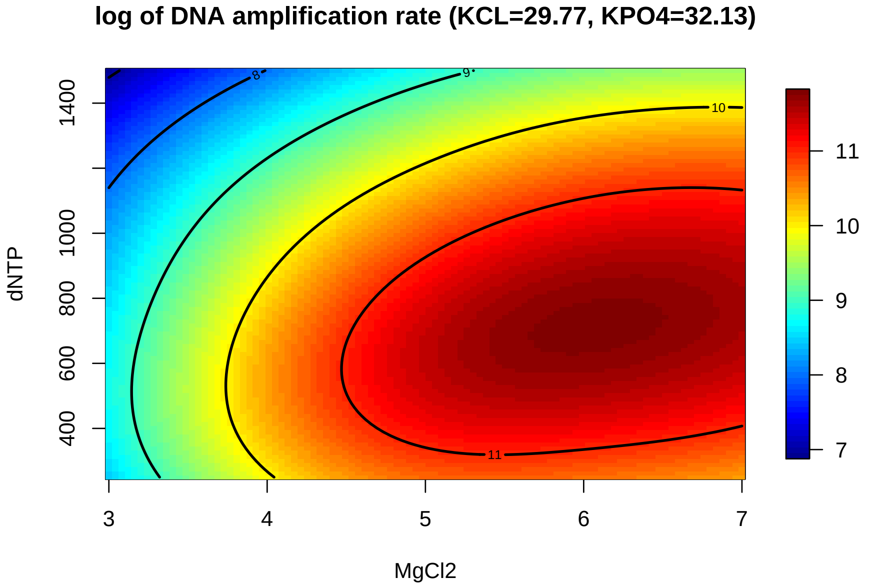
Example spatial process



Intrinsic stationarity

Definition

A process $Y(s)$ is intrinsically stationary if ($E[Y(s+h) - Y(s)] = 0$ and)

$$E[(Y(s+h) - Y(s))^2] = Var[Y(s+h) - Y(s)] = 2\gamma(h)$$

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Definition

A process $Y(s)$ is isotropic if the semivariogram function depends only on $||h||$, the length of the separation vector. Otherwise the process is anisotropic.

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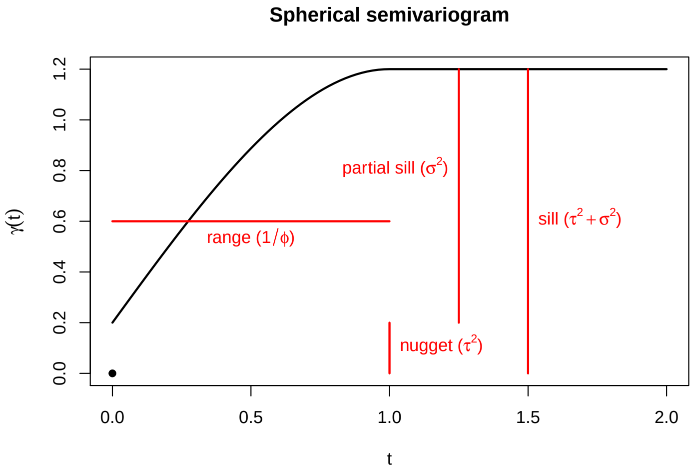
Thus, if the process is ergodic, an intrinsicly stationary process is also weakly stationary.

Covariance functions for isotropic models

Model	Covariance function, $C(t)$	Semivariogram, $\gamma(t)$
Linear	$C(t)$ does not exist	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$
Spherical	$C(t) = \begin{cases} 0 \\ \sigma^2 \left[1 - \frac{3}{2}\phi t + \frac{1}{2}(\phi t)^3\right] \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 & \text{if} \\ \tau^2 + \sigma^2 \left[\frac{3}{2}\phi t - \frac{1}{2}(\phi t)^3\right] & 0 \\ 0 & \text{o} \end{cases}$
Exponential	$C(t) = \begin{cases} \sigma^2 \exp(-\phi t) \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - \exp(-\phi t)] & t > \\ 0 & \text{oth} \end{cases}$
Powered exponential	$C(t) = \begin{cases} \sigma^2 \exp(-\lvert\phi t\rvert^p) \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - \exp(-\lvert\phi t\rvert^p)] \\ 0 \end{cases}$
Gaussian	$C(t) = \begin{cases} \sigma^2 \exp(-\phi^2 t^2) \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[1 - \exp(-\phi^2 t^2)\right] \\ 0 \end{cases}$
Rational quadratic	$C(t) = \begin{cases} \sigma^2 \left(1 - \frac{t^2}{(1+\phi^2)}\right) \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \frac{t^2}{(1+\phi^2)} & t > 0 \\ 0 & \text{otherwise} \end{cases}$
Wave	$C(t) = \begin{cases} \sigma^2 \frac{\sin(\phi t)}{\phi t} \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[1 - \frac{\sin(\phi t)}{\phi t}\right] & t > \\ 0 & \text{other} \end{cases}$
Power law	$C(t)$ does not exist	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t^\lambda & t > 0 \\ 0 & \text{otherwise} \end{cases}$
Matérn	$C(t) = \begin{cases} \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} (2\sqrt{v} t\phi)^\nu K_\nu(2\sqrt{\nu} t\phi) \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[1 - \frac{(2\sqrt{\nu} t\phi)^\nu}{2^{\nu-1}\Gamma(\nu)} (2\sqrt{v} \right. \\ 0 \end{cases}$
Matérn ($\nu = 3/2$)	$C(t) = \begin{cases} \sigma^2 (1 + \phi t) \exp(-\phi t) \\ \tau^2 + \sigma^2 \end{cases}$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - (1 + \phi t) \exp(-\phi \\ 0 \end{cases}$

Spherical semivariogram



Matérn

Perhaps the most important isotropic process is the Matérn process with covariance

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Special cases are the exponential ($\nu = 1/2$) and Gaussian ($\nu \to \infty$).

Strong stationarity

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A process $Y(s)$ is strongly (or strictly) stationary if, for any set of $n \geq 1$ sites $\{s_1, \ldots, s_n\}$ and any $h \in \mathbb{R}^d$,

$$(Y(s_1), \ldots, Y(s_n))^\top \stackrel{d}{=} (Y(s_1 + h), \ldots, Y(s_n + h))^\top$$

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The reverse is not necessarily true.

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For a Gaussian process, weak stationarity and strong stationarity are equivalent.

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where Σ is constructed from the parameters ν, ϕ, τ^2, and σ^2 and the distances between locations, e.g. $||s_1 - s_2||$.

Consider point-referenced data at spatial locations $s_1, \ldots, s_n$, model this data as

$$Y(s) = \mu(s) + w(s) + \epsilon(s)$$

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where H is a correlation matrix with $H_{ij} = \rho(s_i - s_j; \phi)$ and ρ is a valid isotropic correlation function on $\mathbb{R}^r$, i.e. Matérn:

$$\rho(u; \nu, \phi) = \frac{(u/\phi)^\nu K_\nu(u/\phi)}{2^{\nu-1}\Gamma(\nu)}$$

as defined in `geoR:matern`.

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$$\mu(s) = x(s)^\top \beta.$$

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As a general rule, non-informative priors can be chosen for β, e.g. $p(\beta) \propto 1$.

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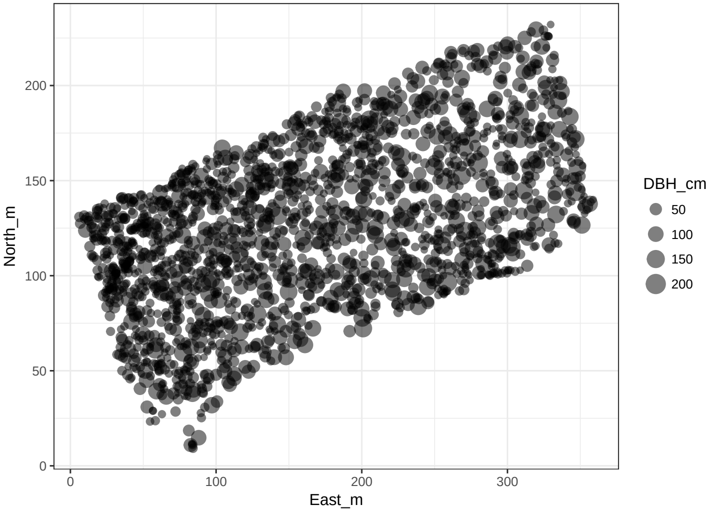
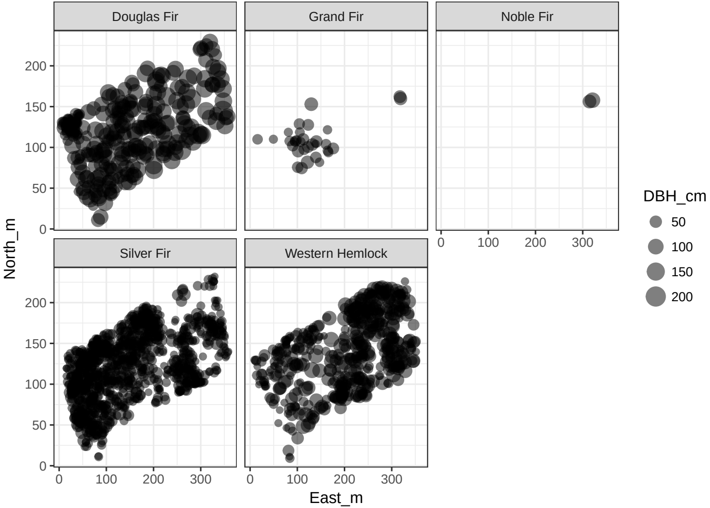
Typically, independent priors are chosen so that

$$p(\theta) = p(\beta)p(\sigma^2)p(\tau^2)p(\phi).$$

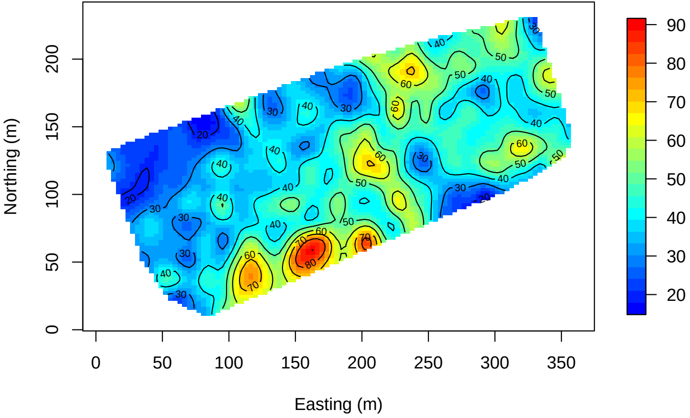
As a general rule, non-informative priors can be chosen for β, e.g. $p(\beta) \propto 1$. However, improper (or vague proper) priors for the variance parameters can lead to improper (or computationally improper) posteriors.

Diameter at breast height (DBH) for an experimental forest




Douglas Fir
Grand Fir
Noble Fir
Silver Fir
Western Hemlock
North_m
East_m
0
50
100
150
200
0
100
200
300
DBH_cm
50
100
150
200

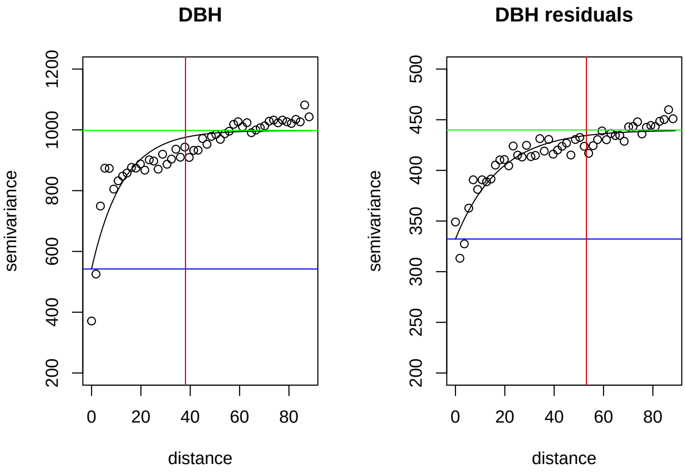
Interpolation of mean DBH (ignoring species)



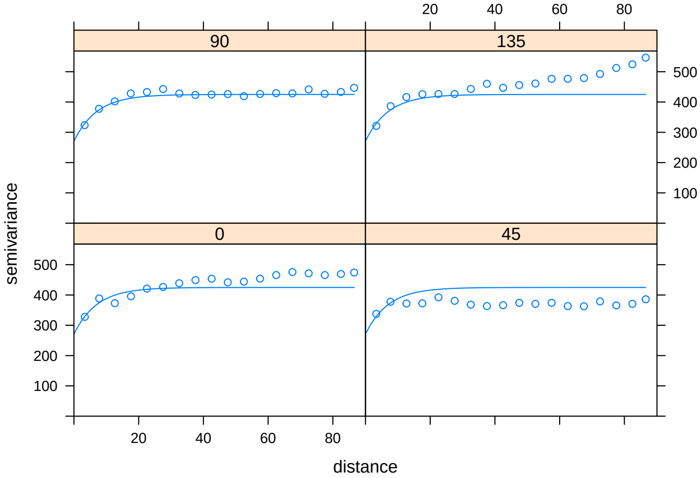
Regression

```
## variog: computing omnidirectional variogram
## variofit: covariance model used is exponential
## variofit: weights used: equal
## variofit: minimisation function used: nls
##
## Call:
## lm(formula = DBH_cm ~ Species, data = d)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -78.423  -9.969  -3.561  10.924 118.277
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)              89.423      1.303  68.629   <2e-16 ***
## SpeciesGrand Fir        -51.598      4.133 -12.483   <2e-16 ***
## SpeciesNoble Fir         -5.873     15.744  -0.373    0.709
## SpeciesSilver Fir       -68.347      1.461 -46.784   <2e-16 ***
## SpeciesWestern Hemlock  -48.062      1.636 -29.377   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 22.19 on 1950 degrees of freedom
## Multiple R-squared:  0.5332,Adjusted R-squared:  0.5323
## F-statistic: 556.9 on 4 and 1950 DF,  p-value: < 2.2e-16
```

Variogram (exponential model)



Isotropy?



90
135
0
45
20
40
60
80
100
200
300
400
500
semivariance
distance

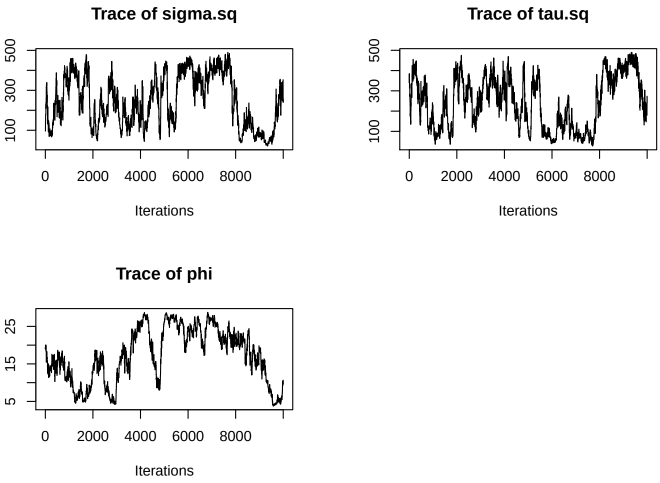
spBayes

```
p = nlevels(d$Species)
r = spLM(DBH_cm ~ Species,
        data = d,
        coords = as.matrix(d[c('East_m','North_m')]),
        knots = c(6,6,.1), # for spatial prediction
        cov.model = 'exponential',

        starting = list(tau.sq   = fit.DBH.resid$nugget,
                        sigma.sq = fit.DBH.resid$cov.pars[1],
                        phi      = fit.DBH.resid$cov.pars[2]),

        tuning = list(tau.sq   = 0.015,
                      sigma.sq = 0.015,
                      phi      = 0.015),

        priors = list(beta.Norm   = list(rep(0,p), diag(1000,p)),
                      phi.Unif    = c(3/1,3/0.1),
                      sigma.sq.IG = c(2,200),
                      tau.sq.IG   = c(3,300)),

        n.samples = 10000,
        n.report = 200,
        verbose=TRUE)
```

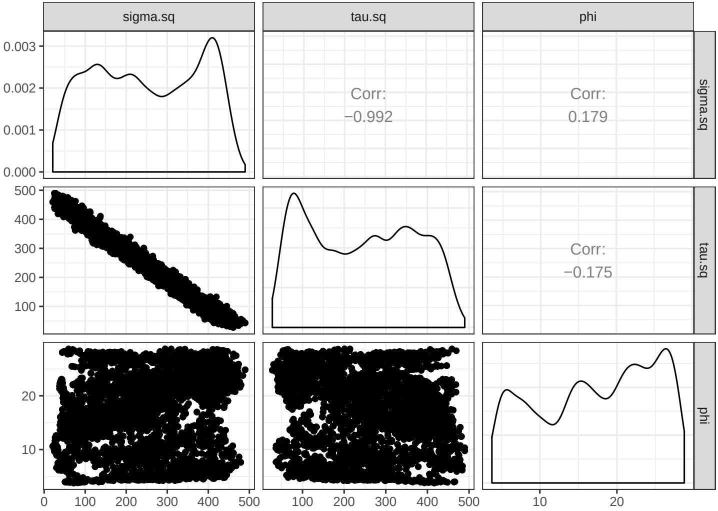
```
## ----------------------------------------
##  General model description
## ----------------------------------------
## Model fit with 1955 observations.
##
## Number of covariates 5 (including intercept if specified).
##
## Using the exponential spatial correlation model.
##
## Using modified predictive process with 36 knots.
##
## Number of MCMC samples 10000.
##
## Priors and hyperpriors:
##  beta normal:
##  mu: 0.000 0.000 0.000 0.000 0.000
##  cov:
##  1000.000 0.000 0.000 0.000 0.000
##  0.000 1000.000 0.000 0.000 0.000
##  0.000 0.000 1000.000 0.000 0.000
##  0.000 0.000 0.000 1000.000 0.000
##  0.000 0.000 0.000 0.000 1000.000
##
##  sigma.sq IG hyperpriors shape=2.00000 and scale=200.00000
##  tau.sq IG hyperpriors shape=3.00000 and scale=300.00000
##  phi Unif hyperpriors a=3.00000 and b=30.00000
## -------------------------------------------------
##  Sampling
## -------------------------------------------------
## Sampled: 200 of 10000, 2.00%
## Report interval Metrop. Acceptance rate: 36.50%
## Overall Metrop. Acceptance rate: 36.50%
## -------------------------------------------------
## Sampled: 400 of 10000, 4.00%
## Report interval Metrop. Acceptance rate: 36.50%
```

Traceplots

```
plot(r$p.theta.samples, density=FALSE)
```

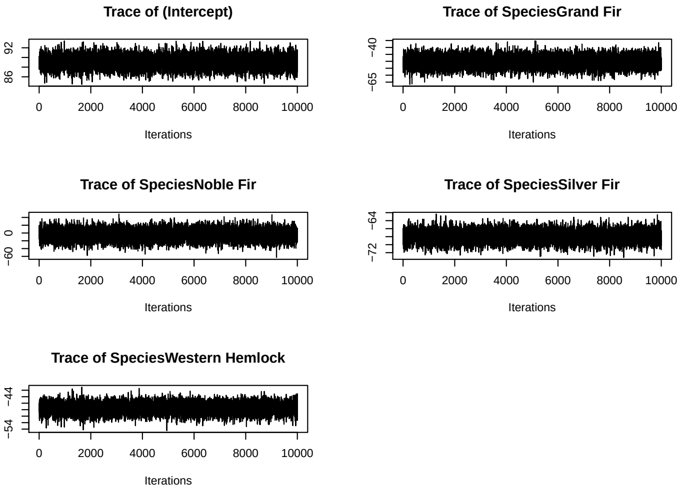


```
r$p.theta.samples[burnin:nreps,] %>%
  as.data.frame %>%
  GGally::ggpairs() +
  theme_bw()
```



Traceplot 2s

```
plot(r$p.beta.samples, density=FALSE)
```



Summary statistics

```r
summary(r$p.theta.samples)
```

```
##
## Iterations = 1:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##            Mean      SD Naive SE Time-series SE
## sigma.sq 246.59 127.324  1.27324         31.224
## tau.sq   244.94 126.725  1.26725         30.490
## phi       17.24   7.204  0.07204          2.743
##
## 2. Quantiles for each variable:
##
##            2.5%    25%    50%    75% 97.5%
## sigma.sq 43.200 133.07 238.13 366.61 446.7
## tau.sq   51.208 124.40 252.81 355.24 449.0
## phi       4.593  11.46  17.55  23.58  27.9
```

Summary statistics 2

```
summary(r$p.beta.samples)


##
## Iterations = 1:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##                            Mean     SD Naive SE Time-series SE
## (Intercept)              89.011  1.291  0.01291        0.01291
## SpeciesGrand Fir        -50.361  4.125  0.04125        0.04125
## SpeciesNoble Fir         -4.406 14.034  0.14034        0.14034
## SpeciesSilver Fir       -67.923  1.458  0.01458        0.01458
## SpeciesWestern Hemlock -47.605  1.631  0.01631        0.01631
##
## 2. Quantiles for each variable:
##
##                           2.5%    25%     50%     75%  97.5%
## (Intercept)              86.48  88.13  89.006  89.873  91.53
## SpeciesGrand Fir        -58.49 -53.11 -50.306 -47.617 -42.21
## SpeciesNoble Fir        -32.44 -13.91  -4.382   5.187  22.83
## SpeciesSilver Fir       -70.79 -68.90 -67.917 -66.944 -65.09
## SpeciesWestern Hemlock -50.79 -48.69 -47.615 -46.506 -44.44
```

Spatial surface

If interest resides in w, draws can be obtained using the following relationship

$$p(w|y) = \int p(w|\sigma^2, \phi, y) p(\sigma^2, \phi|y) d\sigma^2 d\phi$$

which suggests the following strategy:

Spatial surface

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$$p(w|y) = \int p(w|\sigma^2, \phi, y) p(\sigma^2, \phi|y) d\sigma^2 d\phi$$

which suggests the following strategy:

1. Run the MCMC sampler to obtain draws $(\sigma^2, \phi)^{(g)} \sim p(\sigma^2, \phi|y)$
2. After burn-in and for $g = 1, \ldots, G$, sample $w^{(g)} \sim p(w|(\sigma^2, \phi)^{(g)}, y)$.

Prediction

For prediction at points $s_{01}, \ldots, s_{0m}$ and denoting $Y_0 = (Y(s_{01}), \ldots, Y(s_{0m}))^\top$ and design matrix X_0 having rows $x(s_{0j})^\top$,

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$$p(y_0|y, X, X_0) = \int p(y_0|y, \theta, X_0)p(\theta|y, X)d\theta \approx \frac{1}{G}\sum_{g=1}^{G} p(y_0|y, \theta^{(g)}, X_0).$$

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It is more common to take draws $y_0^{(g)} \sim p(y_0|y, \theta^{(g)}, X_0)$ and estimate the predictive distribution using

$$p(y_0|y, X, X_0) \approx \frac{1}{G}\sum_{g=1}^{G} \delta_{y_0^{(g)}}$$

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where $p(y_0|y, \theta, X_0)$ has a conditional normal distribution.

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$$\left(\begin{array}{c} y \\ y_0 \end{array}\right) \sim N\left(\left[\begin{array}{c} X\beta \\ X_0\beta \end{array}\right], \left[\begin{array}{cc} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{array}\right]\right)$$

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where

$$\begin{array}{ll} \Omega_{11} & = \sigma^2 H(\phi) + \tau^2 \mathrm{I} \\ \Omega_{22} & = \sigma^2 + \tau^2 \\ \Omega_{12}^\top & = \sigma^2 [\rho(d_{01};\phi), \ldots, \rho(d_{0n};\phi)] \end{array}$$

and $d_{ij} = ||s_i - s_j||$.

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and $d_{ij} = ||s_i - s_j||$.

Thus $y_0|y,\theta,X,X_0$ is normal with

$$\begin{array}{rl} E[Y(s_0)|y,\theta,X,X_0] & = x_0^\top\beta + \Omega_{12}^\top\Omega_{22}^{-1}(y - X\beta) \\ Var[Y(s_0)|y,\theta,X,X_0] & = \sigma^2 + \tau^2 - \Omega_{12}^\top\Omega_{22}^{-1}\Omega_{12} \end{array}$$

Generalized linear spatial modeling

Let $Y(s)$ be the response of interest with

$$E[Y(s)] = g^{-1}(x(s)^\top \beta + w(s))$$

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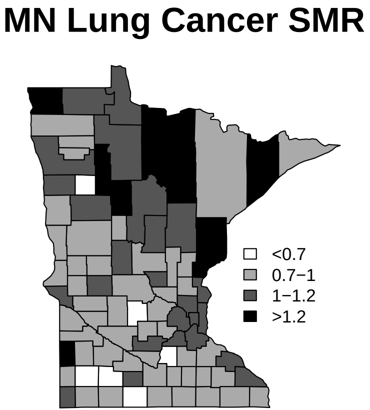
For GLMs (other than linear models), $w(s)$ cannot be integrated out and therefore a common MCMC strategy is

1. Sample $\beta| \ldots$.
2. Sample $w| \ldots$.
3. Sample $\theta| \ldots$ (the spatial parameters [no nugget]).

Choropleth



Modeling areal units

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This defines a *Markov Random Field*.

Brook's Lemma

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Definition

Brook's Lemma states that

$$\frac{p(y_1,\ldots,y_n)}{p(y_1',\ldots,y_n')} = \frac{p(y_1|y_2,\ldots,y_n)}{p(y_1'|y_2,\ldots,y_n)} \cdot \frac{p(y_2|y_1',y_3,\ldots,y_n)}{p(y_2'|y_1',y_3,\ldots,y_n)} \cdots \frac{p(y_n|y_1',\ldots,y_{n-1}')}{p(y_n'|y_1',\ldots,y_{n-1}')}$$

for all $(y_1',\ldots,y_n')$.

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for all $(y_1',\ldots,y_n')$.

If

$$p(y_1',\ldots,y_n') = \int \frac{p(y_1|y_2,\ldots,y_n)}{p(y_1'|y_2,\ldots,y_n)} \cdot \frac{p(y_2|y_1',y_3,\ldots,y_n)}{p(y_2'|y_1',y_3,\ldots,y_n)} \cdots \frac{p(y_n|y_1',\ldots,y_{n-1}')}{p(y_n'|y_1',\ldots,y_{n-1}')} dy_1,\ldots,dy_n < \infty$$

then $p(y_1,\ldots,y_n)$ is a proper joint distribution.

Conditionally autoregressive models

More generally, we can consider

$$Y_i|y_{-i} \sim N\left(\sum_{j\neq i} b_{ij}y_j, \tau_i^2\right)$$

Through Brook's Lemma, we have

$$p(y_1,\ldots,y_n) \propto \exp\left(-\frac{1}{2}y^\top D^{-1}[\mathrm{I}-B]y\right)$$

where

- B has elements b_{ij}
- D is diagonal with elements τ_i^2

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In order for $D^{-1}[\mathrm{I}-B]$ to be symmetric, we need $\frac{b_{ij}}{\tau_i^2} = \frac{b_{ji}}{\tau_j^2}$ for all i,j.

Proximity matrix

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  - neighbors defined by those who are within distance δ
  - K-nearest neighbors

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- "distance"
  - inverse intercentroidal distance
  - inverse minimum distance plus c

Intrinsicially autoregressive model

Recall

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This is called the *intrinsically autoregressive* model.

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a prior for ρ that induces a reasonable amount of spatial association should put most of its mass near 1.

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- If $\rho = 0$, then we have conditional independence. But the variance decreases with the number of neighbors which is perplexing.
- ρ needs to be very close to 1 to obtain a consequential amount of spatial association.

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- Choose $\rho = 1$ and estimate a mean (remove mean from the fixed effect)
- Let $\rho \sim Be(18, 2)$ (Banerjee pg 164) and estimate it.

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This distribution is proper so long as $0 \le \rho < 1$. Lee (2011) argued that this CAR should be preferred for a variety of reasons.

CAR as a model for random effects

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- Y_i represent the (continuous) response for observation i

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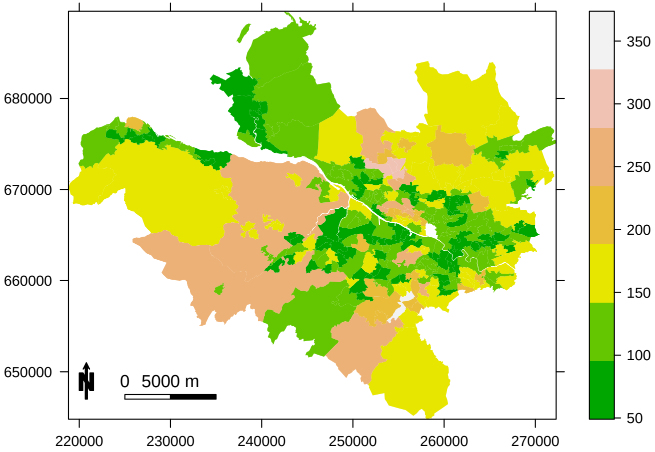
$$p(\omega_1, \ldots, \omega_S) \propto \exp\left(-\frac{1}{2\tau^2}\omega^\top [D_w - \rho W]\omega\right)$$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Housing price model

Let

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Housing price model

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- Y_i be the logarithm of the median home price in each Intermediate Geography (IG) to the north of the river Clude in the Greater Glasgow and Clyde health board,
- use explanatory variables
  - crime: crime rate (number of crimes per 10,000 people) in each IG (logged),
  - rooms: median number of rooms in a property in each IG,
  - type: predominant property type in each IG with levels: detached, flat, semi, terrace,
  - sales: percentage of properties that sold in each IG in a year, and
  - driveshop: average time taken to drive to a shopping centre in minutes (logged).

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where

- β are the regression parameters and
- ω_i are assumed to come from an intrinsic CAR model with proximity matrix indicating those regions that share a border

```
propertydata.spatial@data$logprice <- log(propertydata.spatial@data$price)
propertydata.spatial@data$logdriveshop <- log(propertydata.spatial@data$driveshop)


###################################################
### code chunk number 9: CARBayes.Rnw:495-498
###################################################
library(splines)
form <- logprice~ns(crime,3)+rooms+sales+factor(type) + logdriveshop
model <- lm(formula=form, data=propertydata.spatial@data)


###################################################
### code chunk number 10: CARBayes.Rnw:505-510
###################################################
library(spdep)
W.nb <- poly2nb(propertydata.spatial, row.names = rownames(propertydata.spatial@data))
W.list <- nb2listw(W.nb, style="B")
resid.model <- residuals(model)
moran.mc(x=resid.model, listw=W.list, nsim=1000)


##
##  Monte-Carlo simulation of Moran I
##
## data:  resid.model
## weights: W.list
## number of simulations + 1: 1001
##
## statistic = 0.2733, observed rank = 1001, p-value = 0.000999
## alternative hypothesis: greater
```

```
##
## Call:
## lm(formula = form, data = propertydata.spatial@data)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.91319 -0.15992  0.00136  0.15647  0.81675
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)          4.436135   0.157971  28.082  < 2e-16 ***
## ns(crime, 3)1       -0.358967   0.089006  -4.033 7.24e-05 ***
## ns(crime, 3)2       -0.617084   0.165152  -3.736 0.000229 ***
## ns(crime, 3)3       -0.299454   0.126516  -2.367 0.018670 *
## rooms                0.193827   0.029268   6.623 2.02e-10 ***
## sales                0.002034   0.000362   5.619 4.93e-08 ***
## factor(type)flat    -0.215967   0.066412  -3.252 0.001298 **
## factor(type)semi    -0.153610   0.057750  -2.660 0.008301 **
## factor(type)terrace -0.280023   0.072634  -3.855 0.000146 ***
## logdriveshop        -0.089084   0.025588  -3.482 0.000585 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2243 on 260 degrees of freedom
## Multiple R-squared:  0.6206,Adjusted R-squared:  0.6075
## F-statistic: 47.26 on 9 and 260 DF,  p-value: < 2.2e-16
```

Bayesian analysis using Leroux CAR

```
W <- nb2mat(W.nb, style = "B")
model.spatial <- S.CARleroux(formula = form,
                             data = propertydata.spatial@data,
                             family = "gaussian",
                             W = W,
                             burnin = 20000,
                             n.sample = 120000, thin = 10)

## Setting up the model

## Error:  the formula inputted contains an error, e.g the variables may be different lengths.
```