Model checking

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STAT 544 - Iowa State University

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Outline

We assume $p(y|\theta)$ and $p(\theta)$, so it would be prudent to determine if these assumptions are reasonable.

- (Prior) sensitivity analysis
- Posterior predictive checks
  - Graphical checks
  - Posterior predictive pvalues

Prior sensitivity analysis

Since a prior specifies our prior belief, we may want to check to determine whether our conclusions would change if we held different prior beliefs. Suppose a particular scientific question can be boiled down to

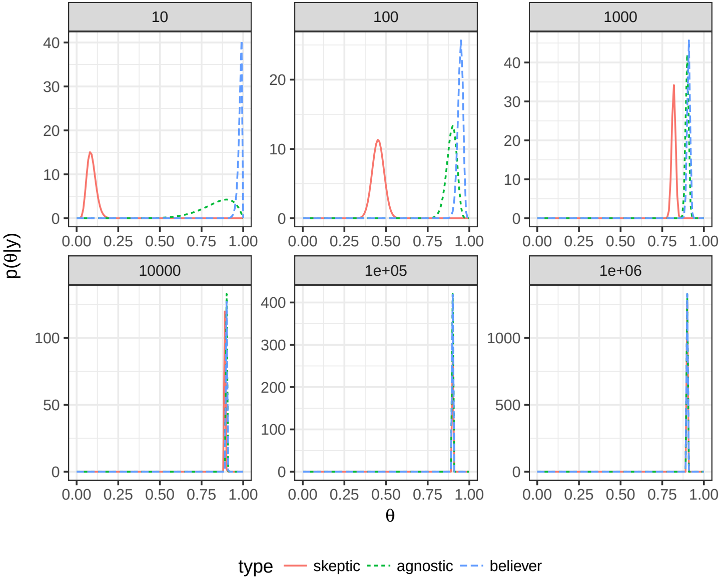
$$Y_i \stackrel{ind}{\sim} Ber(\theta)$$

and that there is wide disagreement about the value for θ such that the following might reasonably characterize different individual beliefs before the experiment is run:

- Skeptic: $\theta \sim Be(1, 100)$
- Agnostic: $\theta \sim Be(1, 1)$
- Believer: $\theta \sim Be(100, 1)$

An experiment is run and the posterior under these different priors are compared.

Posteriors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p(θ|y)
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Hierarchical variance prior

Recall the normal hierarchical model

$$y_i \stackrel{ind}{\sim} N(\theta_i, s_i^2), \quad \theta_i \stackrel{ind}{\sim} N(\mu, \tau^2)$$

which results in the posterior distribution for τ of

$$p(\tau|y) \propto p(\tau) V_\mu^{1/2} \prod_{i=1}^{\mathrm{I}} (s_i^2 + \tau^2)^{-1/2} \exp\left(-\frac{(y_i - \hat{\mu})^2}{2(s_i^2 + \tau^2)} \right)$$

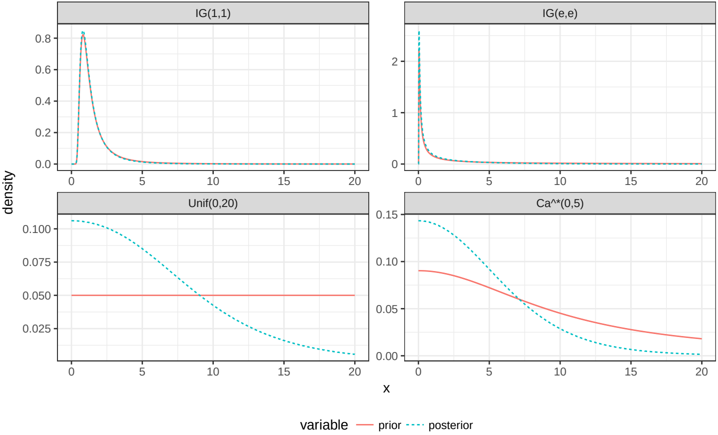
As an attempt to be non-informative, consider an $IG(\epsilon, \epsilon)$ prior for τ^2. As an alternative, consider $\tau \sim Unif(0, C)$ or $\tau \sim Ca^+(0, C)$ where C is problem specific, but is chosen to be relatively large for the particular problem.

The 8-schools example has the following data:

	1	2	3	4	5	6	7	8
y	28	8	-3	7	-1	1	18	12
s	15	10	16	11	9	11	10	18

Posterior for 8-schools example

Reproduction of Gelman 2006:



Summary

For a default prior on a variance (σ^2) or standard deviation (σ), use

1. Easy to remember.
   - Half-Cauchy on the standard deviation ($\sigma \sim Ca^+(0, C)$).
2. Harder to remember
   - Data-level variance
     - Use default prior ($p(\sigma^2) \propto 1/\sigma^2$)
   - Hierarchical standard deviation
     - Use uniform prior (Unif$(0, C)$) if there are enough reps (5 or more) of that parameter.
     - Use half-Cauchy prior ($Ca^+(0, C)$) otherwise.

When assigning the values for C

- For a uniform prior (Unif$(0, C)$) make sure C is large enough to capture any reasonable value for the standard deviation.
- For a half-Cauchy prior ($Ca^+(0, C)$) err on the side of making C too small since the heavy tails will let the data tell you if the standard deviation needs to be larger whereas a value of C that is too large will put too much weight toward large values of the standard deviation and make the prior more informative.

Posterior predictive checks

Let y^{rep} be a replication of y, then

$$p(y^{rep}|y) = \int p(y^{rep}|\theta, y)p(\theta|y)d\theta = \int p(y^{rep}|\theta)p(\theta|y)d\theta.$$

where y is the observed data and θ are the model parameters.

To simulate a full replication:

1. Simulate $\theta^{(j)} \sim p(\theta|y)$ and
2. Simulate $y^{rep,j} \sim p(y|\theta^{(j)})$.

To assess model adequacy:

- Compare plots of replicated data to the observed data.
- Calculate posterior predictive pvalues.

Airline accident data

Let

- y_i be the number of fatal accidents in year i
- x_i be the number of 100 million miles flown in year i

Consider the model

$$Y_i \stackrel{ind}{\sim} Po(x_i\lambda) \qquad p(\lambda) \propto 1/\sqrt{\lambda}.$$

	year	fatal_accidents	passenger_deaths	death_rate	miles_flown
1	1976	24	734	0	3863
2	1977	25	516	0	4300
3	1978	31	754	0	5027
4	1979	31	877	0	5481
5	1980	22	814	0	5814
6	1981	21	362	0	6033
7	1982	26	764	0	5877
8	1983	20	809	0	6223
9	1984	16	223	0	7433
10	1985	22	1066	0	7107

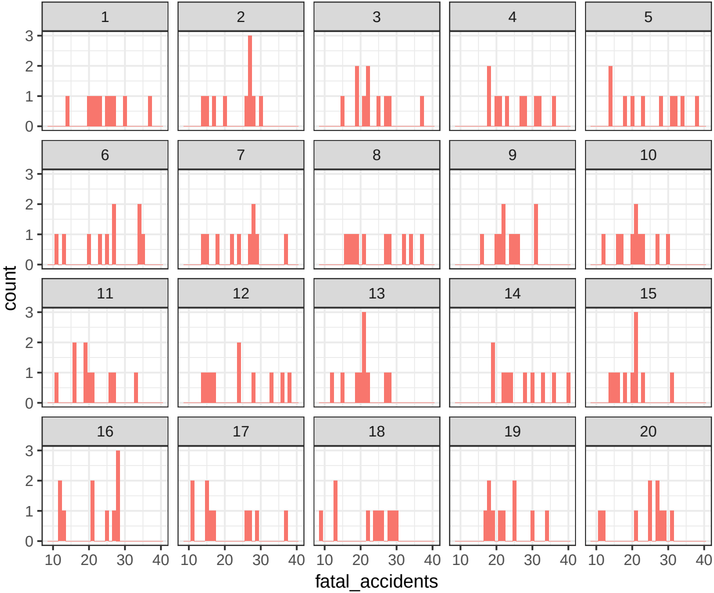
Posterior replications of the data

Under Jeffreys prior, the posterior is

$$\lambda|y \sim Ga(0.5 + n\overline{y}, n\overline{x}).$$

So to obtain a replication of the data, do the following

1. $\lambda^{(j)} \sim Ga(0.5 + n\overline{y}, n\overline{x})$ and
2. $y_i^{rep,j} \stackrel{ind}{\sim} Po(x_i \lambda^{(j)})$ for $i = 1, \ldots, n$.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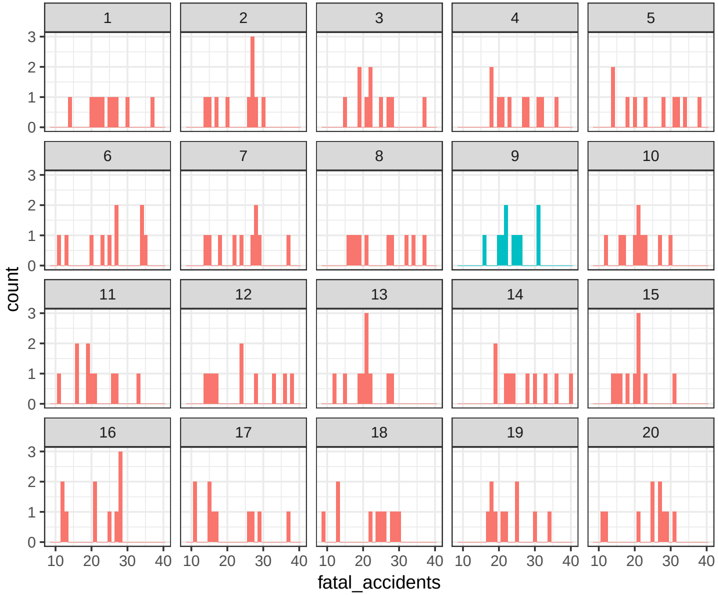
fatal_accidents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How might this model not accurately represent the data?

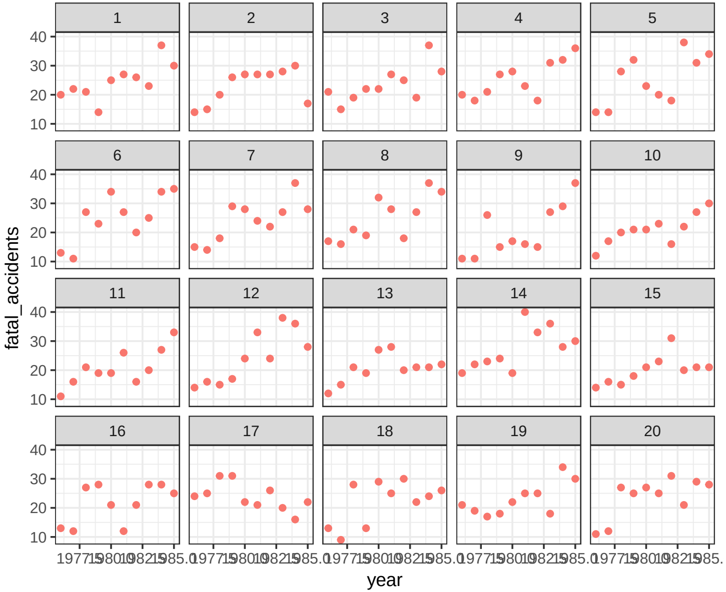
Let

- y_i be the number of fatal accidents in year i
- x_i be the number of 100 million miles flown in year i

Consider the model

$$Y_i \stackrel{ind}{\sim} Po(x_i\lambda) \qquad p(\lambda) \propto 1/\sqrt{\lambda}.$$

	year	fatal_accidents	passenger_deaths	death_rate	miles_flown	.n
1	1976	24	734	0	3863	9
2	1977	25	516	0	4300	9
3	1978	31	754	0	5027	9
4	1979	31	877	0	5481	9
5	1980	22	814	0	5814	9
6	1981	21	362	0	6033	9
7	1982	26	764	0	5877	9
8	1983	20	809	0	6223	9
9	1984	16	223	0	7433	9
10	1985	22	1066	0	7107	9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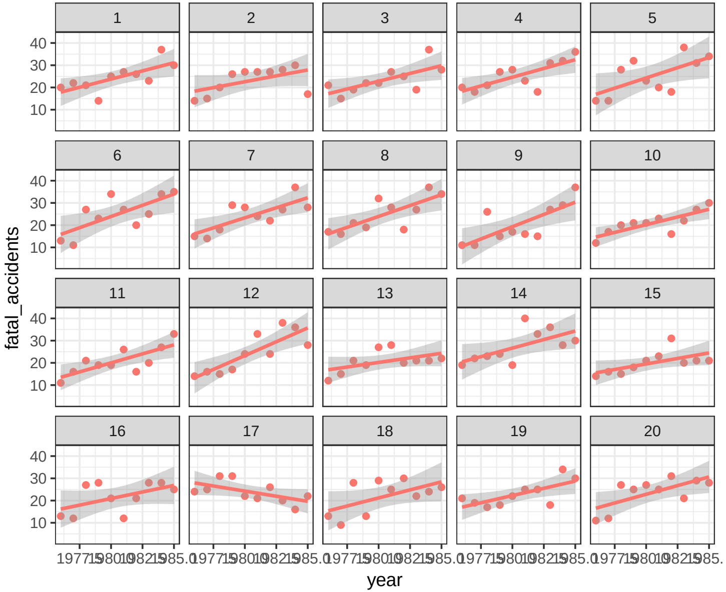
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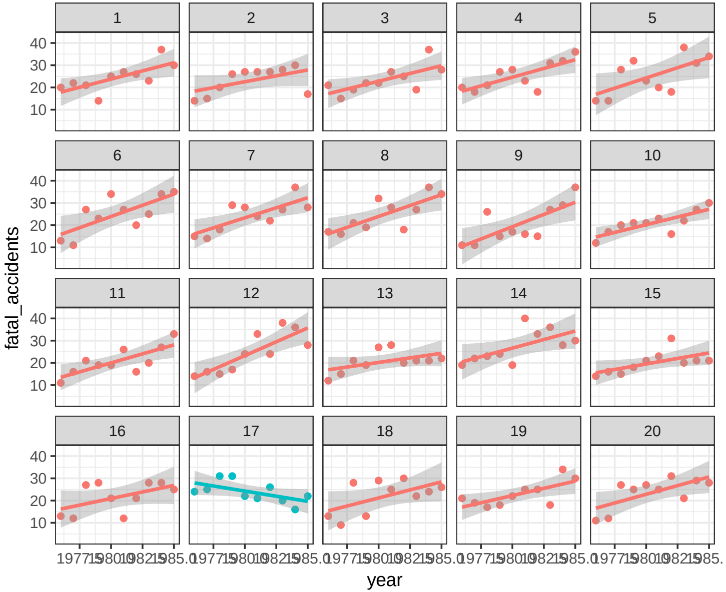
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1980.0
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1985.0

Posterior predictive pvalues

To quantify the discrepancy between observed and replicated data:

1. Define a test statistic $T(y, \theta)$.
2. Define the posterior predictive pvalue

   $$p_B = P(T(y^{rep}, \theta) \geq T(y, \theta)|y)$$

   where y^{rep} and θ are random. Typically this pvalue is calculated via simulation, i.e.

   $$\begin{aligned} p_B &= E_{y^{rep},\theta}[\mathrm{I}(T(y^{rep}, \theta) \geq T(y, \theta))|y] \\ &= \int \int \mathrm{I}(T(y^{rep}, \theta) \geq T(y, \theta))p(y^{rep}|\theta)p(\theta|y)dy^{rep}d\theta \\ &\approx \frac{1}{J}\sum_{j=1}^{J} \mathrm{I}(T(y^{rep,j}, \theta^{(j)}) \geq T(y, \theta^{(j)})) \end{aligned}$$

   where $\theta^{(j)} \sim p(\theta|y)$ and $y^{rep,j} \sim p(y|\theta^{(j)})$.

Small or large pvalues are (possible) cause for concern.

Posterior predictive pvalue for slope

Let

$$Y_i^{obs} = \beta_0^{obs} + \beta_1^{obs}\, i$$

where

- Y_i^{obs} is the observed number of fatal accidents in year i and
- β_1^{obs} be the test statistic.

Now, generate replicate data y^{rep} and fit

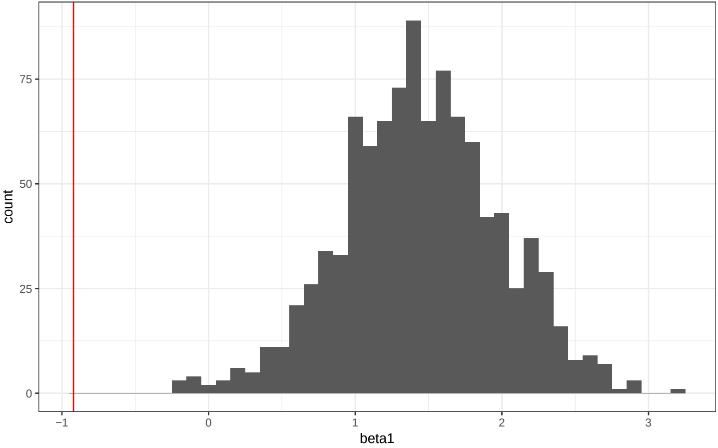
$$Y_i^{rep} = \beta_0^{rep} + \beta_1^{rep} i.$$

Now compare β_1^{obs} to the distribution of β_1^{rep}.

```r
mean(rep$beta1>observed_slope)
```

```
[1] 1
```

```r
ggplot(rep, aes(x=beta1)) + geom_histogram(binwidth=0.1) +
  geom_vline(xintercept=observed_slope, color="red") +
  theme_bw()
```



Consider a linear model for the λ_i

Consider the model

$$\begin{aligned} Y_i &\stackrel{ind}{\sim} Po(x_i\lambda_i) \\ \log(\lambda_i) &= \beta_0 + \beta_1 i \end{aligned}$$

where

- Y_i is the number of fatal accidents in year i
- x_i is the number of 100 million miles flown in year i
- λ_i is the accident rate in year i

Here the λ_i are a deterministic function of year, but (possibly) different each year.

Stan linear model for accident rate

```
model = "
data {
  int<lower=0> n;
  int<lower=0> y[n];
  vector<lower=0>[n] x;
}
transformed data {
  vector[n] log_x;
  log_x = log(x); # both x and logx need to be vectors
}
parameters {
  real beta[2];
}
transformed parameters {
  vector[n] log_lambda;
  for (i in 1:n) log_lambda[i] = beta[1] + beta[2]*i;
}
model {
  y ~ poisson_log(log_x + log_lambda); # _log indicates mean on log scale
}
"

m = stan_model(model_code = model)
r = sampling(m, list(n=nrow(d), y=d$fatal_accidents, x=d$miles_flown))
```

```
Inference for Stan model: c5fd0896f3ea971d9c6ac34c2a2fbedd.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

                 mean se_mean   sd   2.5%    25%    50%    75%  97.5% n_eff Rhat
beta[1]         -4.90    0.00 0.13  -5.16  -4.98  -4.90  -4.81  -4.63  1215    1
beta[2]         -0.10    0.00 0.02  -0.15  -0.12  -0.10  -0.09  -0.06  1220    1
log_lambda[1]   -5.00    0.00 0.11  -5.23  -5.08  -5.00  -4.93  -4.77  1267    1
log_lambda[2]   -5.11    0.00 0.10  -5.30  -5.17  -5.11  -5.04  -4.92  1371    1
log_lambda[3]   -5.21    0.00 0.08  -5.37  -5.26  -5.21  -5.16  -5.06  1604    1
log_lambda[4]   -5.32    0.00 0.07  -5.46  -5.36  -5.31  -5.27  -5.18  2170    1
log_lambda[5]   -5.42    0.00 0.07  -5.55  -5.46  -5.42  -5.38  -5.30  3461    1
log_lambda[6]   -5.53    0.00 0.07  -5.66  -5.57  -5.52  -5.48  -5.40  4000    1
log_lambda[7]   -5.63    0.00 0.08  -5.79  -5.68  -5.63  -5.58  -5.49  3561    1
log_lambda[8]   -5.74    0.00 0.09  -5.92  -5.79  -5.73  -5.67  -5.56  2721    1
log_lambda[9]   -5.84    0.00 0.11  -6.06  -5.91  -5.84  -5.77  -5.64  2183    1
log_lambda[10]  -5.94    0.00 0.13  -6.20  -6.03  -5.94  -5.86  -5.71  1895    1
lp__           516.88    0.03 1.02 514.03 516.53 517.21 517.58 517.82  1120    1

Samples were drawn using NUTS(diag_e) at Tue Feb 14 09:31:45 2017.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

Posterior predictive pvalue: slope

```
# Posterior predictive pvalue: slope
mean(rep_slopes>observed_slope)

[1] 0.49925
```