# STAT 401A - Statistical Methods for Research Workers

## Simple linear regression

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last updated: October 20, 2014

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Recall the one-way ANOVA model:

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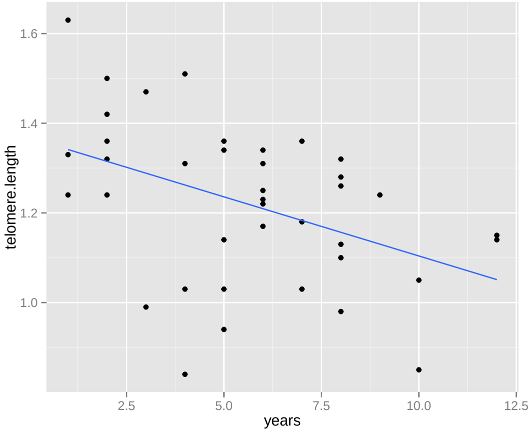
Terminology (all of these are equivalent):

| | |
|---|---|
| response | explanatory |
| outcome | covariate |
| dependent | independent |
| endogenous | exogenous |


1.6
1.4
1.2
1.0
telomere.length
2.5
5.0
7.5
10.0
12.5
years

# Telomere length

http://www.pnas.org/content/101/49/17312

*People who are stressed over long periods tend to look haggard, and it is commonly thought that psychological stress leads to premature aging and the earlier onset of diseases of aging.*

*...*

*This design allowed us to examine the importance of perceived stress and measures of objective stress (caregiving status and chronicity of caregiving stress based on the number of years since a child's diagnosis).*

*...*

*Telomere length values were measured from DNA by a quantitative PCR assay that determines the relative ratio of telomere repeat copy number to single-copy gene copy number (T/S ratio) in experimental samples as compared with a reference DNA sample.*

# Parameter interpretation

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  $\beta_1$ is the expected increase in the response for each unit increase in the explanatory variable.

- $\sigma$ is the standard deviation of the response for a fixed value of the explanatory variable.

Remove the mean:

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$$\begin{aligned}
\hat{\beta}_1 &= SXY/SXX \\
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\hat{\sigma}^2 &= SSE/(n-2) \qquad \text{df} = n-2
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$$\begin{array}{rl} SXY & = \sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) \\ SXX & = \sum_{i=1}^n (X_i - \overline{X})(X_i - \overline{X}) = \sum_{i=1}^n (X_i - \overline{X})^2 \\ SSE & = \sum_{i=1}^n r_i^2 \end{array}$$

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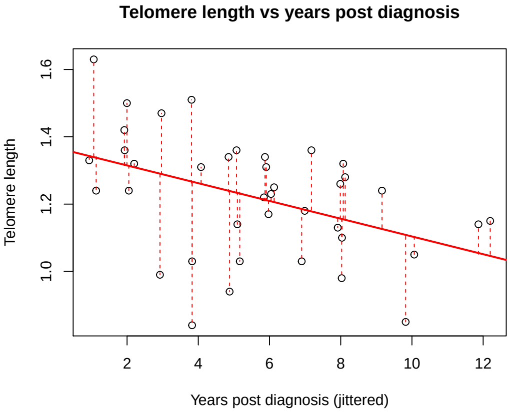
The coefficient of determination ($R^2$) is the proportion of the total response variation explained by the explanatory variable(s).


Telomere length vs years post diagnosis
Telomere length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
Years post diagnosis (jittered)

# Pvalues and confidence interval

We can compute two-sided pvalues via

$$2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_0}{SE(\beta_0)}\right|\right) \qquad \text{and} \qquad 2P\left(t_{n-2} < -\left|\frac{\hat{\beta}_1}{SE(\beta_1)}\right|\right)$$

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These test the null hypothesis that the corresponding parameter is zero.

We can construct $100(1-\alpha)\%$ two-sided confidence intervals via

$$\hat{\beta}_0 \pm t_{n-2}(1-\alpha/2)SE(\beta_0) \qquad \text{and} \qquad \hat{\beta}_1 \pm t_{n-2}(1-\alpha/2)SE(\beta_1)$$

These provide ranges of the parameters consistent with the data.

# Calculations by hand

```
   n Xbar Ybar   s_X    s_Y    r_XY
1 39 5.59 1.22 2.935 0.1798 -0.4307
```

$$
\begin{aligned}
SXX &= (n-1)s_X^2 = (39-1) \times 2.935427^2 = 327.4358 \\
SYY &= (n-1)s_Y^2 = (39-1) \times 0.1797731^2 = 1.228098 \\
SXY &= (n-1)s_X s_Y r_{XY} = (39-1) \times 2.935427 \times 0.1797731 \times -0.4306534 = -8.635897 \\
\hat\beta_1 &= SXY/SXX = -8.635897/327.4358 = -0.02637432 \\
\hat\beta_0 &= \overline{Y} - \hat\beta_1 \overline{X} = 1.220256 - (-0.02637432) \times 5.589744 = 1.367682 \\
R^2 &= r_{XY}^2 = (-0.4306534)^2 = 0.1854624 \\
SSE &= SYY(1-R^2) = 1.228098(1-0.1854624) = 1.000332 \\
\hat\sigma^2 &= SSE/(n-2) = 1.000332/(39-2) = 0.027036 \\
\hat\sigma &= \sqrt{\hat\sigma^2} = \sqrt{0.027036} = 0.1644263 \\
SE(\hat\beta_0) &= \hat\sigma \sqrt{\frac{1}{n} + \frac{\overline{X}^2}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{39} + \frac{5.589744^2}{327.4358}} = 0.05721115 \\
SE(\hat\beta_1) &= \hat\sigma \sqrt{\frac{1}{(n-1)s_X^2}} = 0.1644263\sqrt{\frac{1}{327.4358}} = 0.009086742 \\
p_{H_0:\beta_0=0} &= 2P\left(t_{n-2} < -\left|\frac{1.367682}{0.05721115}\right|\right) = 2P(t_{37} < -23.90586) < 0.0001 \\
p_{H_0:\beta_1=0} &= 2P\left(t_{n-2} < -\left|\frac{-0.02637432}{0.009086742}\right|\right) = 2P(t_{37} < -2.902506) < 0.0062 \\
CI_{95\%\ \beta_0} &= \hat\beta_0 \pm t_{n-2}(1-\alpha/2)SE(\hat\beta_0) = 1.367682 \pm 2.026192 \times 0.05721115 = (1.251761, 1.483603) \\
CI_{95\%\ \beta_1} &= \hat\beta_1 \pm t_{n-2}(1-\alpha/2)SE(\hat\beta_1) = -0.02637432 \pm 2.026192 \times 0.009086742 = (-0.044785804 - 0.007962836)
\end{aligned}
$$

```
DATA t;
  INFILE 'telomeres.csv' DSD FIRSTOBS=2;
  INPUT years length;

PROC CORR DATA=t;
  VAR length;
  WITH years;
  RUN;
```

```
                                        The CORR Procedure

                                    1 With Variables:    years
                                    1      Variables:    length


                                        Simple Statistics

   Variable           N          Mean       Std Dev           Sum       Minimum       Maximum

   years             39       5.58974       2.93543     218.00000       1.00000      12.00000
   length            39       1.22026       0.17977      47.59000       0.84000       1.63000


                              Pearson Correlation Coefficients, N = 39
                                     Prob > |r| under H0: Rho=0

                                                   length

                                        years      -0.43065
                                                     0.0062
```

```
PROC GLM DATA=t;
  MODEL length = years / CLPARM;
  RUN;

                                 The GLM Procedure
                        Number of Observations Read         39
                        Number of Observations Used         39

Dependent Variable: length

                                             Sum of
      Source                      DF        Squares     Mean Square    F Value    Pr > F
      Model                        1     0.22776588      0.22776588       8.42    0.0062
      Error                       37     1.00033156      0.02703599
      Corrected Total             38     1.22809744

                      R-Square     Coeff Var      Root MSE    length Mean
                      0.185462      13.47473      0.164426       1.220256

      Source                      DF      Type I SS     Mean Square    F Value    Pr > F
      years                        1     0.22776588      0.22776588       8.42    0.0062

      Source                      DF    Type III SS     Mean Square    F Value    Pr > F
      years                        1     0.22776588      0.22776588       8.42    0.0062

                                     Standard
 Parameter          Estimate            Error    t Value    Pr > |t|      95% Confidence Limits
 Intercept       1.367682067       0.05721112      23.91      <.0001     1.251761335   1.483602799
 years          -0.026374315       0.00908674      -2.90      0.0062    -0.044785794  -0.007962836
```

# Regression in R

```
m = lm(telomere.length~years, Telomeres)
with(Telomeres, cor(telomere.length,years))
```

```
[1] -0.4307
```

```
anova(m)
```

```
Analysis of Variance Table

Response: telomere.length
          Df Sum Sq Mean Sq F value Pr(>F)
years      1  0.228   0.228    8.42 0.0062 **
Residuals 37  1.000   0.027
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Regression in R

```
m = lm(telomere.length~years, Telomeres)
summary(m)



Call:
lm(formula = telomere.length ~ years, data = Telomeres)

Residuals:
    Min      1Q  Median      3Q     Max 
-0.4222 -0.0854  0.0206  0.1074  0.2887 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.36768    0.05721    23.9   <2e-16 ***
years       -0.02637    0.00909    -2.9   0.0062 ** 
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.164 on 37 degrees of freedom
Multiple R-squared:  0.185, Adjusted R-squared:  0.163 
F-statistic: 8.42 on 1 and 37 DF,  p-value: 0.0062


confint(m)


               2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

# Conclusion

Telomere length at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% confidence interval of (1.25, 1.48).

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Telomere length at the time of diagnosis of a child's chronic illness is estimated to be 1.37 with a 95% confidence interval of (1.25, 1.48). For each year since diagnosis, the telomere length decreases by 0.026 with a 95% confidence interval of (0.008, 0.045) on average. The proportional of variability in telomere length described by years since diagnosis is 18.5%.

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http://www.pnas.org/content/101/49/17312

> *The zero-order correlation between chronicity of caregiving [years] and mean telomere length, r,is* $-0.445$ *(P <0.01).*
> *[*$R^2 = 0.198$ *was shown in the plot.]*

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> *The zero-order correlation between chronicity of caregiving [years] and mean telomere length, r,is* $-0.445$ *(P <0.01). [*$R^2 = 0.198$ *was shown in the plot.]*

**Remark** I'm guessing our analysis and that reported in the paper don't match exactly due to a discrepancy in the data.

# Summary

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$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

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  - At a constant value of the explanatory variable, $\sigma^2$ is the variance of the responses.
  - The coefficient of determination ($R^2$) is the percentage of the total response variation explained by the explanatory variable(s).

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We know $\beta_0 = E[Y|X = 0]$, but what about $X = x$?

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but there is uncertainty in both $\beta_0$ and $\beta_1$. So the standard error of $E[Y|X=x]$ is

$$SE(E[Y|X=x]) = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(\overline{X}-x)^2}{(n-1)s_X^2}}$$

and a $100(1-\alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1-\alpha/2)SE(E[Y|X=x])$$

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$$Pred\{Y|X = x\} = \hat{\beta}_0 + \hat{\beta}_1 x$$

but the uncertainty includes the variability due to $\sigma^2$. So the standard error of $Pred\{Y|X = x\}$ is

$$SE(Pred\{Y|X = x\}) = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(\overline{X} - x)^2}{(n-1)s_X^2}}$$

and a $100(1 - \alpha)\%$ confidence interval is

$$\hat{\beta}_0 + \hat{\beta}_1 x \pm t_{n-2}(1 - \alpha/2) SE(Pred\{Y|X = x\}).$$

```
DATA tnew;
  INPUT years;
  DATALINES;
  4
  ;

DATA combined;
  SET t tnew;
  RUN;

PROC PRINT DATA=combined;
  WHERE years=4;
  RUN;
```

| Obs | years | length |
|---|---|---|
| 10 | 4 | 1.51 |
| 11 | 4 | 1.31 |
| 15 | 4 | 1.03 |
| 16 | 4 | 0.84 |
| 40 | 4 | . |

```
PROC GLM DATA=combined;
 MODEL length = years;
 OUTPUT OUT=combinedreg PREDICTED=predicted  LCLM=lclm UCLM=uclm LCL=lcl UCL=ucl;
 RUN;

PROC PRINT DATA=combinedreg;
  WHERE length=.;             /* . is missing data in SAS */
  RUN;
```

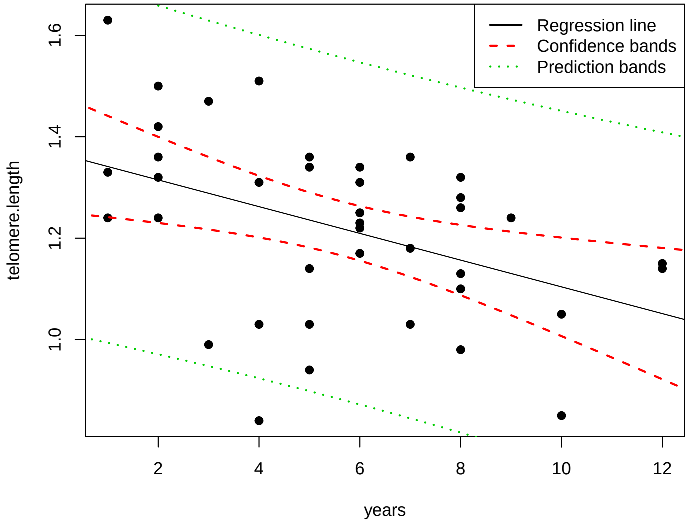
| Obs | years | length | predicted | lclm | uclm | lcl | ucl |
|---|---|---|---|---|---|---|---|
| 40 | 4 | . | 1.26218 | 1.20133 | 1.32303 | 0.92351 | 1.60086 |

```
m = lm(telomere.length~years, Telomeres)
new = data.frame(years=4)
predict(m, new, interval="confidence")
```

```
    fit   lwr   upr
1 1.262 1.201 1.323
```

```
predict(m, new, interval="prediction")
```

```
    fit    lwr   upr
1 1.262 0.9235 1.601
```


Regression line
Confidence bands
Prediction bands
telomere.length
1.0
1.2
1.4
1.6
2
4
6
8
10
12
years

# Shifting the intercept

The intercept ($\beta_0$) is the expected response when the explanatory variable is zero.

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Let $x$ be number of years since diagnosis, then

$$E[Y|X = x] = \tilde{\beta}_0 + \tilde{\beta}_1(x - 4) = (\beta_0 - 4\beta_1) + \beta_1 x$$

so our new parameters for the mean are

- intercept $\tilde{\beta}_0 = (\beta_0 - 4\beta_1)$

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Let $x$ be number of years since diagnosis, then

$$E[Y|X=x] = \tilde{\beta}_0 + \tilde{\beta}_1(x-4) = (\beta_0 - 4\beta_1) + \beta_1 x$$

so our new parameters for the mean are

- intercept $\tilde{\beta}_0 = (\beta_0 - 4\beta_1)$and
- slope $\tilde{\beta}_1 = \beta_1$ (unchanged).

```
DATA t;
  INFILE "telomeres.csv" DSD FIRSTOBS=2;
  INPUT years length;
  years4 = years-4;

PROC GLM DATA=combined;
 MODEL length = years;
 RUN;

PROC GLM DATA=combined;
 MODEL length = years4;
 RUN;
```

```
                                      Standard
 Parameter          Estimate             Error    t Value    Pr > |t|      95% Confidence Limits

 Intercept       1.367682067        0.05721112      23.91      <.0001      1.251761335   1.483602799
 years          -0.026374315        0.00908674      -2.90      0.0062     -0.044785794 -0.007962836

 Intercept       1.262184808        0.03003174      42.03      <.0001      1.201334726   1.323034890
 years4         -0.026374315        0.00908674      -2.90      0.0062     -0.044785794 -0.007962836
```

```r
m0 = lm(telomere.length ~   years   , Telomeres)
m4 = lm(telomere.length ~ I(years-4), Telomeres)

coef(m0)
```

```
(Intercept)       years
    1.36768    -0.02637
```

```r
coef(m4)
```

```
 (Intercept) I(years - 4)
     1.26218     -0.02637
```

```r
confint(m0)
```

```
                 2.5 %    97.5 %
(Intercept)  1.25176  1.483603
years       -0.04479 -0.007963
```

```r
confint(m4)
```

```
                  2.5 %    97.5 %
(Intercept)   1.20133  1.323035
I(years - 4) -0.04479 -0.007963
```