M5S1 - Confidence Intervals

Professor Jarad Niemi

STAT 226 - Iowa State University

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Outline

- Confidence intervals
    - Relation to Central Limit Theorem
    - Based on the Empirical Rule
    - Finding z critical values
    - significance level
    - confidence level
    - margin of error

Sample mean as an estimator for the population mean

Recall that due to the CLT, $\overline{X} \dot{\sim} N(\mu, \sigma^2/n)$ where

- $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is the (random) sample mean,
- μ is the population mean,
- σ^2 is the population variance, and
- n is the sample size.

Suppose μ is unknown. Then $\overline{X}$ is an unbiased estimator for μ, since

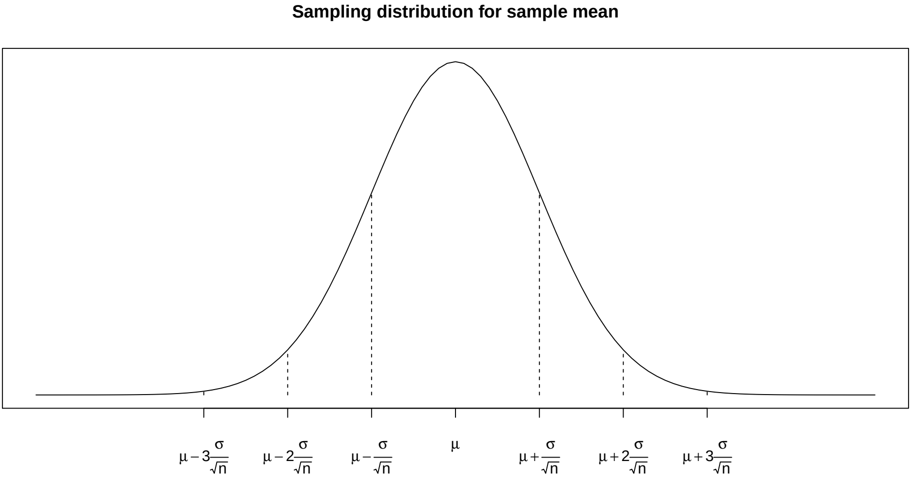
$$E[\overline{X}] = \mu,$$

and its variability decreases with increased sample size since

$$SD[\overline{X}] = \sqrt{Var[\overline{X}]} = \sigma/\sqrt{n}.$$

How can we use this knowledge to describe our uncertainty in μ?

How close is $\overline{X}$ to μ?



Empirical Rule Confidence Intervals

From the Central Limit Theorem, we can write

$$
\begin{aligned}
P\left(\mu - \frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + \frac{\sigma}{\sqrt{n}}\right) &\approx 0.68 \\
P\left(\mu - 2\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 2\frac{\sigma}{\sqrt{n}}\right) &\approx 0.95 \\
P\left(\mu - 3\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 3\frac{\sigma}{\sqrt{n}}\right) &\approx 0.997
\end{aligned}
$$

We can rewrite these inequalities by subtracting $\overline{X}$, subtracting μ, and multiplying by -1:

$$
\begin{aligned}
P\left(\overline{X} - \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + \frac{\sigma}{\sqrt{n}}\right) &\approx 0.68 \\
P\left(\overline{X} - 2\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 2\frac{\sigma}{\sqrt{n}}\right) &\approx 0.95 \\
P\left(\overline{X} - 3\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 3\frac{\sigma}{\sqrt{n}}\right) &\approx 0.997
\end{aligned}
$$

We will call these intervals, e.g. $\left(\overline{X} - \frac{\sigma}{\sqrt{n}}, \overline{X} + \frac{\sigma}{\sqrt{n}}\right)$, confidence intervals and their confidence level is the probability (usually written as a percentage).

Example

US Bank provides students with savings accounts having no monthly maintenance fee and a low minimum monthly transfer. US Bank is interested in knowing the mean monthly balance of all its student savings accounts. They know the standard deviation of balances is \$20. They take a random sample of 64 student savings accounts and record that at the end of the month the sample mean savings was \$105. Construct a 68% confidence interval for the mean monthly balance.

Let X_i be the end of the month balance for student i. Then $E[X_i] = \mu$, the mean monthly balance, is unknown, but $SD[X_i] = \sigma = \$20$ is known. We obtained a sample of size $n = 64$ with a sample mean $\overline{x} = \$105$. To obtain the 68% confidence interval for μ, we calculate

$$\begin{aligned} \overline{x} \pm \frac{\sigma}{\sqrt{n}} &= \left(\overline{x} - \frac{\sigma}{\sqrt{n}}, \overline{x} + \frac{\sigma}{\sqrt{n}}\right) \\ &= \left(\$105 - \frac{\$20}{\sqrt{64}}, \$105 + \frac{\$20}{\sqrt{64}}\right) \\ &= (\$102.5, \$107.5) \end{aligned}$$

Confidence Intervals for μ when σ is known

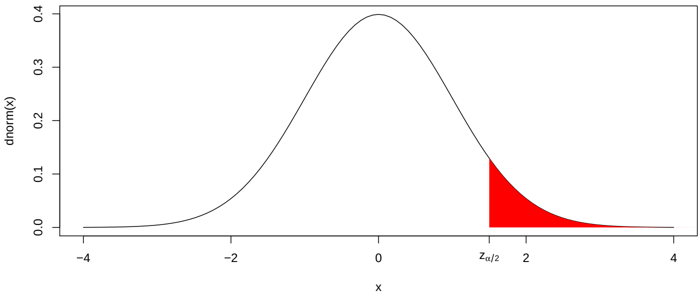
Definition

Let μ be the population mean and σ be the known population standard deviation. Choose a significance level α which you can convert to a confidence level $C = 100(1-\alpha)\%$ and a z critical value $z_{\alpha/2}$ where $P(Z > z_{\alpha/2}) = \alpha/2$. You obtain a random sample of observations from the population and calculate the sample mean $\overline{X}$. Then a $C = 100(1-\alpha)\%$ confidence interval for μ is

$$\overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

where $z_{\alpha/2} \cdot \sigma/\sqrt{n}$ is called the margin of error.

Finding z critical values



Recall that

$$P(Z > z_{\alpha/2}) = P(Z < -z_{\alpha/2}).$$

Check that

C	α	$\alpha/2$	$z_{\alpha/2}$
68%	0.32	0.16	≈ 1
95%	0.05	0.025	≈ 2
99.7%	0.003	0.0015	≈ 3

Example

US Bank provides students with savings accounts having no monthly maintenance fee and a low minimum monthly transfer. US Bank is interested in knowing the mean monthly balance of all its student savings accounts. They know the standard deviation of balances is \$20. They take a random sample of 64 student savings accounts and record that at the end of the month the sample mean savings was \$105. Construct a 80% confidence interval for the mean monthly balance.

Let X_i be the end of the month balance for student i. Then $E[X_i] = \mu$, the mean monthly balance, is unknown, but $SD[X_i] = \sigma = \$20$ is known. We obtained a sample of size $n = 64$ with a sample mean $\overline{x} = \$105$. For a confidence level of 80%, we have $\alpha = 0.2$, $\alpha/2 = 0.1$ and $z_{\alpha/2} \approx 1.28$. Then we calculate

$$\overline{x} \pm z_{alpha/2} \frac{\sigma}{\sqrt{n}} = \$105 \pm 1.28 \frac{\$20}{\sqrt{64}} = (\$101.8, \$108.2)$$

which is an 80% confidence interval for μ