

## I03 - Bayesian parameter estimation

STAT 587 (Engineering)  
Iowa State University

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# Outline

- Bayesian parameter estimation
  - Condition on what is known
  - Describe **belief** using probability
  - Terminology
    - Prior  $\rightarrow$  posterior
    - Posterior expectation
    - Credible intervals
  - Binomial example
    - Beta distribution

# A Bayesian statistician

Let

- $y$  be the data we will collect from an experiment,
- $K$  be everything we know for certain about the world (aside from  $y$ ), and
- $\theta$  be anything we don't know for certain.

My definition of a Bayesian statistician is an individual who makes decisions based on the probability distribution of those things we don't know conditional on what we know, i.e.

$$p(\theta|y, K).$$

Typically, the  $K$  is dropped from the notation.

# Bayes' Rule

Bayes' Rule applied to a partition  $P = \{A_1, A_2, \dots\}$ ,

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{\infty} P(B|A_i)P(A_i)}$$

Bayes' Rule also applies to probability density (or mass) functions, e.g.

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

where the integral plays the role of the sum in the previous statement.

# Parameter estimation

Let  $y$  be data from some model with unknown parameter (vector)  $\theta$ . Then

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

and we use the following terminology

Terminology	Notation
Posterior	$p(\theta y)$
Prior	$p(\theta)$
Model (likelihood)	$p(y \theta)$
Prior predictive (marginal likelihood)	$p(y)$

Bayesian parameter estimation involves updating your prior **belief** about  $\theta$ ,  $p(\theta)$ , into a posterior **belief** about  $\theta$ ,  $p(\theta|y)$ , based on the data observed.

## Bayesian notation

We now have two distributions for our parameter  $\theta$ : prior and posterior. To distinguish these two, we will have no conditioning in the prior and we will condition on  $y$  in the posterior. For example,

	Prior	Posterior
Density	$p(\theta)$	$p(\theta y)$
Expectation	$E[\theta]$	$E[\theta y]$
Variance	$Var[\theta]$	$Var[\theta y]$
Probabilities	$P(\theta < c)$	$P(\theta < c y)$

## Binomial model

Suppose  $Y \sim \text{Bin}(n, \theta)$ , then

$$p(y|\theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}.$$

A reasonable default prior is the uniform distribution on the interval  $(0, 1)$

$$p(\theta) = \text{I}(0 < \theta < 1).$$

Using Bayes Rule, you can find

$$\theta|y \sim \text{Be}(1 + y, 1 + n - y).$$

# Beta distribution

The **beta distribution** defines a distribution for a probability, i.e. a number on the interval  $(0,1)$ . The probability density function is

$$p(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{Beta(a,b)} \mathbf{I}(0 < \theta < 1)$$

where  $a, b > 0$  and  $Beta$  is the beta function, i.e.

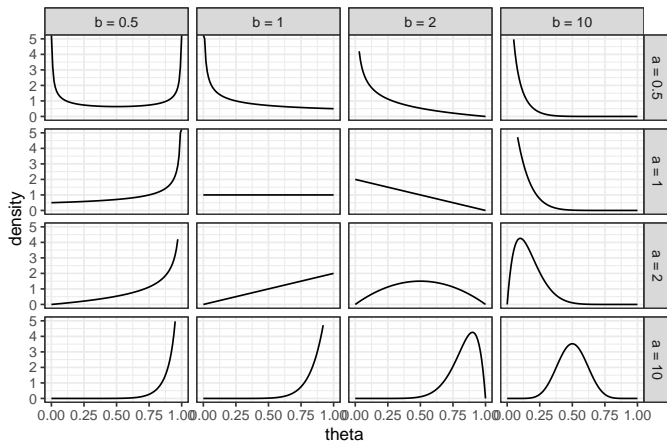
$$Beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{and} \quad \Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx.$$

The beta distribution has the following properties:

- $E[\theta] = \frac{a}{a+b},$
- $Var[\theta] = \frac{ab}{(a+b)^2(a+b+1)},$  and
- $Be(1,1) \stackrel{d}{=} Unif(0,1).$



# Beta densities

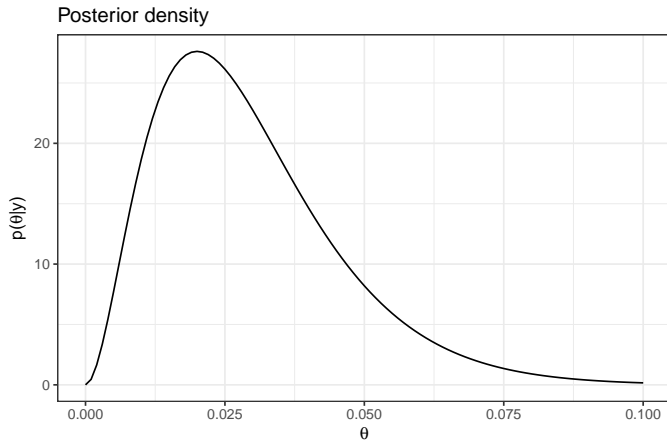


## Beta posterior

Suppose we have made 100 sensors according to a particular protocol and 2 have a sensitivity below a pre-determined threshold. Let  $Y$  be the number below the threshold. Assume  $Y \sim \text{Bin}(n, \theta)$  with  $n = 100$  and  $\theta \sim \text{Be}(1, 1)$ , then

$$\theta|y \sim \text{Be}(1 + y, 1 + n - y) \stackrel{d}{=} \text{Be}(3, 99).$$

# Posterior density



## Posterior expectation

Often times it is inconvenient to provide a full posterior and so we often summarize using a point estimate from the posterior. For a point estimate, we can use the posterior expectation:

$$\hat{\theta}_{Bayes} = E[\theta|y] = \frac{1+y}{(1+y) + (1+n-y)} = \frac{1+y}{2+n}$$

```
(1+y)/(2+n)
```

```
[1] 0.02941176
```

Note that this is close, but not exactly equal to  $\hat{\theta}_{MLE} = y/n$ . Since the MLE is unbiased, this posterior expectation will generally be biased but it is still consistent since  $\hat{\theta}_{Bayes} \rightarrow \hat{\theta}_{MLE}$ .

# Credible intervals

A  $100(1 - a)\%$  **credible interval** is any interval  $(L, U)$  such that

$$1 - a = \int_L^U p(\theta|y)d\theta.$$

An **equal-tail**  $100(1 - a)\%$  **credible interval** is the interval  $L, U)$  such that

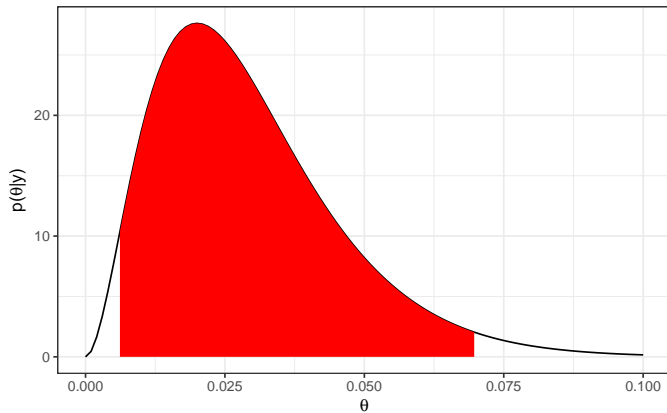
$$a/2 = \int_{-\infty}^L p(\theta|y)d\theta = \int_U^{\infty} p(\theta|y)d\theta.$$

```
# 95% credible interval is  
ci = qbeta(c(.025, .975), 1+y, 1+n-y)  
round(ci, 3)
```

```
[1] 0.006 0.070
```

# Equal-tail 95% credible interval

Posterior density with 95% area shaded



# Summary

Bayesian parameter estimation involves

1. Specifying a model  $p(y|\theta)$  for your data.
2. Specifying a prior  $p(\theta)$  for the parameter.
3. Deriving the posterior

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto p(y|\theta)p(\theta).$$

This equation updates your prior **belief**,  $p(\theta)$ , about the unknown parameter  $\theta$  into your posterior **belief**,  $p(\theta|y)$ , about  $\theta$ .

4. Calculating quantities of interest, e.g.
  - Posterior expectation,  $E[\theta|y]$
  - Credible interval

# Bayesian analysis for binomial model summary

Let  $Y \sim \text{Bin}(n, \theta)$  and assume  $\theta \sim \text{Be}(a, b)$ . Then

$$\theta|y \sim \text{Be}(a + y, b + n - y).$$

A default prior is  $\theta \sim \text{Be}(1, 1) \stackrel{d}{=} \text{Unif}(0, 1)$ .

R code for binomial analysis:

```
a <- 1; b <- 1           # default uniform prior
y <- 3; n <- 10          # data

curve(dbeta(x, a+y, b+n-y)) # posterior (pdf)
(a+y)/(a+b+n)              # posterior mean
qbeta(.5, a+y, b+n-y)      # posterior median
qbeta(c(.025, .975), a+y, b+n-y) # 95% equal tail credible interval

# Probabilities
pbeta(0.5, a+y, b+n-y)     # P(theta < 0.5 | y)

# Special cases
qbeta(c(0, .95), a+y, b+n-y) # if y=0, use a lower one-sided CI
qbeta(c(.05, 1), a+y, b+n-y) # if y=n, use a upper one-sided CI
```