Homework 16 Solution

Yikun Zhang[1]

Chapter 6. Ex.2 *Suppose that $R : \mathbb{R}^3 \to \mathbb{R}$ is a proper rotation.*
(a) Show that $p(t) = \det(R - tI)$ is a polynomial of degree 3, and prove that there exists $\gamma \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma) = \gamma.$$

(b) If $\mathcal{P}$ denotes the plane perpendicular to γ and passing through the origin, show that

$$R : \mathcal{P} \to \mathcal{P},$$

and that this linear map is a rotation.

Proof. (a) Since any proper rotation matrix can be expressed as a 3×3 matrix, we have immediately that

$$p(t) = \det(R - tI) = \begin{vmatrix} r_{11} - t & r_{12} & r_{13} \\ r_{21} & r_{22} - t & r_{23} \\ r_{31} & r_{32} & r_{33} - t \end{vmatrix} \tag{1}$$

is a polynomial of degree 3 and the coefficient of t^3 is (-1).
Moreover, using the fact that R is a proper rotation, we know that $p(0) = \det(R) = 1 > 0$ and $\lim_{t \to +\infty} p(t) = -\infty$. Since a polynomial is necessarily continuous on $\mathbb{R}$, there exists a $\lambda > 0$ with $p(\lambda) = 0$. Then $R - \lambda I$ is singular, so its kernel is nontrivial. We choose a nonzero vector in its kernel and normalize it to obtain $\gamma \in S^2$. The corresponding eigenvalue must be 1 because R, a proper rotation, preserves the inner product.

(b) For any $x \in \mathcal{P}$, we have that $\gamma \cdot R(x) = R(\gamma) \cdot x = \gamma \cdot x = 0$, showing that $R(x)$ is also perpendicular to γ and thus R maps $\mathcal{P}$ to $\mathcal{P}$.
Meanwhile, R retains linearity and preservation of the inner product as it did in $\mathbb{R}$. Hence, it is still a rotation on $\mathcal{P}$. $\square$

Chapter 6. Ex.4 *Let A_d and V_d denote the area and volume of the unit sphere and unit ball in $\mathbb{R}^d$, respectively.*
(a) Prove the formula

$$A_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

so that $A_2 = 2\pi, A_3 = 4\pi, A_4 = 2\pi^2, \ldots.$ Here $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the Gamma function.

[1]School of Mathematics, Sun Yat-sen University

Proof. (a)(*Method 1*) We derive the formula via the equality $\int_{\mathbb{R}^d} e^{-\pi|x|^2}dx = 1$ and polar coordinates.

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^d} e^{-\pi|x|^2}dx \\
&= \int_{S^{d-1}} \int_0^\infty e^{-\pi r^2} r^{d-1} dr d\sigma(\gamma) \\
&= A_d \int_0^\infty e^{-\pi r^2} r^{d-1} dr \\
&\overset{u=\pi r^2}{=} \frac{A_d}{2\pi^{\frac{d}{2}}} \int_0^\infty u^{\frac{d}{2}-1} e^{-u} du \\
&= \frac{A_d}{2\pi^{\frac{d}{2}}} \Gamma(\frac{d}{2}),
\end{aligned} \tag{2}
$$

which in turn shows that $A_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

(*Method 2 (From Jian Yao)*) We begin with the definition of A_d and derive the formula directly.

$$
\begin{aligned}
A_d &= \int_{S^{d-1}} d\sigma(\gamma) \\
&= \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{d-2}\theta_1 \cdots \sin\theta_{d-2} d\theta_{d-1} \cdots d\theta_1 \\
&= 2\pi \prod_{i=1}^{d-2} \int_0^\pi \sin^{d-i-1}\theta_i d\theta_i \\
&= 2\pi \prod_{i=1}^{d-2} 2\int_0^{\frac{\pi}{2}} \sin^{2(\frac{d-i}{2})-1}\theta_i \cos^{2(\frac{1}{2})-1}\theta_i d\theta_i \\
&= 2\pi \prod_{i=1}^{d-2} Beta(\frac{d-i}{2}, \frac{1}{2}) \\
&= 2\pi \prod_{i=1}^{d-2} \frac{\Gamma(\frac{d-i}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d-i+1}{2})} \\
&= 2\pi \frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} \cdot \frac{\Gamma(\frac{d-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{d-1}{2})} \cdots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.
\end{aligned} \tag{3}
$$

(b) Let $A_d(R)$ and $V_d(R)$ denote the area and volume of the sphere and ball centered at the origin with radius R, respectively.

Then we can construct the volume $V_d(R)$ by adding infinitely thin spherical shells of radius

$0 \le r \le R$. In equation form, it becomes

$$V_d(R) = \int_0^R A_d(r)dr.$$

Therefore, $dV_d(r) = A_d(r)dr$ and by $A_d(r) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}r^{d-1}$, where the factor r^{d-1} is a consequence of dimensional analysis, we conclude that $V_d = \int_0^1 A_d(r)dr = \int_0^1 r^{d-1}dr = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$. $\square$

Chapter 6. Ex.5 *Let A be a $d \times d$ positive definite symmetric matrix with real coefficients. Show that*

$$\int_{\mathbb{R}^d} e^{-\pi(x,A(x))}dx = (det(A))^{-\frac{1}{2}}.$$

This generalizes the fact that $\int_{\mathbb{R}^d} e^{-\pi|x|^2}dx = 1$, which corresponds to the case where A is the identity.

Proof. Applying the spectral theorem to write $A = RDR^{-1}$, where R is a rotation, $R^{-1} = R^T$, and D is diagonal with entries $\lambda_1, ..., \lambda_d$, we can transform the integral as

$$\int_{\mathbb{R}^d} e^{-\pi(x,A(x))}dx = \int_{\mathbb{R}^d} e^{-\pi x^T Ax}dx = \int_{\mathbb{R}^d} e^{-\pi x^T RDR^T x}dx = \int_{\mathbb{R}^d} e^{-\pi y^T Dy}|\det(A)|^{-1}dy = \int_{\mathbb{R}^d} e^{-\pi y^T Dy}dy,$$

where $y = R^T x$ and $|\det(A)|^{-1} = 1$. Since A is a positive definite symmetric matrix with real coefficients, we know that all the eigenvalues are strictly greater than 0 and thus

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-\pi y^T Dy}dy &= \int_{\mathbb{R}^d} e^{-\pi|z|^2}|\det(D^{-\frac{1}{2}})|dz \\
&= |\det(D^{-\frac{1}{2}})| \\
&= \prod_{i=1}^{d} \lambda_i^{-\frac{1}{2}} \\
&= (\det(A))^{-\frac{1}{2}}. \qquad \square
\end{aligned} \tag{4}$$