Hitting Distribution of 2-dimensional Brownian Motions on a Wedge

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Problem 1. *Suppose that a 2-dimensional Brownian motion starts inside a wedge with angle α. We want to derive the explicit formula for the hitting distribution of the Brownian motion on the wedge. In other words, if the distance between the hitting point and the vertex is R, we are supposed to derive the density of the random variable R. See Figure 1 for details.*

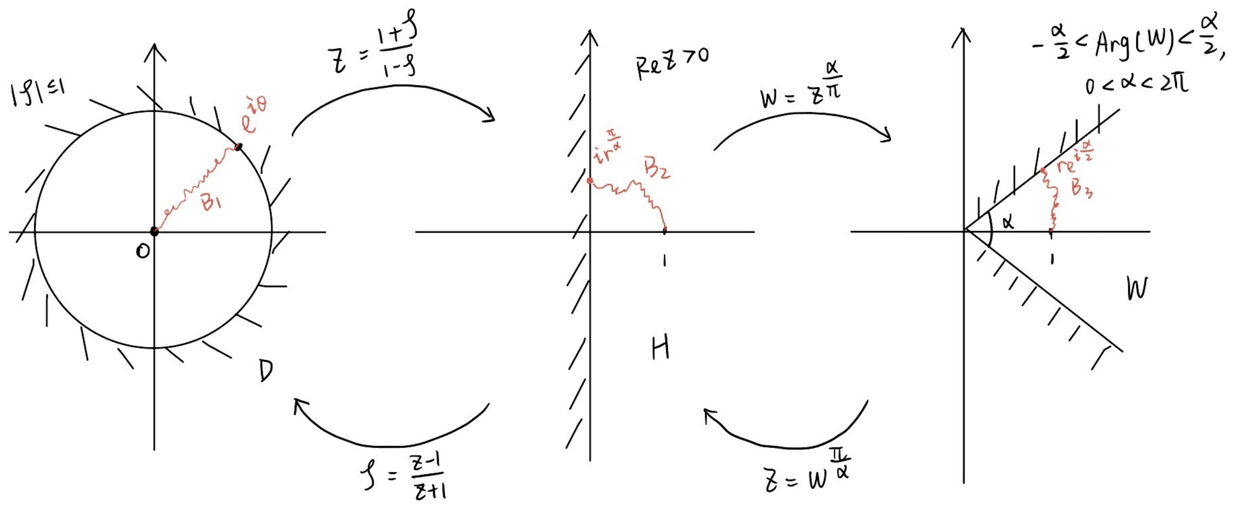

Figure 1: Conformal Mappings and 2-dim Brownian Motions

Proof. We need a conformal mapping between a unit disc and the wedge with angle α.
Let $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, and $W = \{\omega \in \mathbb{C} : -\frac{\alpha}{2} < \mathrm{Arg}(w) < \frac{\alpha}{2}\}$. The conformal mapping $f : D \to H$ is $z = f(\zeta) = \frac{1+\zeta}{1-\zeta}$, while the conformal mapping $g : H \to W$ is $\omega = g(z) = z^{\alpha/\pi}$, where $0 < \alpha < 2\pi$.
Thus, the conformal mapping from D to W is

$$\omega = g \circ f(\zeta) = \left(\frac{1+\zeta}{1-\zeta}\right)^{\alpha/\pi},$$

and its inverse is $\zeta = f^{-1} \circ g^{-1}(\omega) = \frac{\omega^{\pi/\alpha}-1}{\omega^{\pi/\alpha}+1}$.
Let $B_1(t), B_2(t), B_3(t)$ be Brownian motions in D, H, W, respectively. Then $B_1(0) = 0, B_2(0) = 1, B_3(0) = 1$. Let $\tau_D = \inf\{t \geq 0 : B_1(t) \notin D\}$, $\tau_H = \inf\{t \geq 0 : B_2(t) \notin H\}$, and $\tau_W = \inf\{t \geq 0 : B_3(t) \notin W\}$. We know that for any point $e^{i\theta}$ on ∂D, it will be mapped to

$$z = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = \frac{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}{\sin\frac{\theta}{2}(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2})} = i\cot\frac{\theta}{2}$$

on ∂H by f. Moreover, $i\cot\frac{\theta}{2}$ will be mapped to $(\cot\frac{\theta}{2})^{\alpha/\pi} \cdot e^{i\alpha/2}$ on ∂W by g. Hence

$$\begin{aligned} P(|B_3(\tau_W)| \leq r) &= P(-r^{\pi/\alpha} \leq \operatorname{Im} B_2(\tau_H) \leq r^{\pi/\alpha}) \\ &= P(2\text{arccot}\, r^{\pi/\alpha} \leq \operatorname{Arg} B_1(\tau_D) \leq 2\pi - 2\text{arccot}\, r^{\pi/\alpha}) \\ &= 1 - \frac{2\text{arccot}\, r^{\pi/\alpha}}{\pi} \end{aligned}$$

by the symmetry of the 2-dimensional Brownian motion B_1. On the other hand, if the density of R is $\phi(x)$, then

$$P(|B_3(\tau_W)| \leq r) = \int_{-r}^{r} \phi(x)dx,$$

and $\phi(x)$ is an even function by the symmetry of B_3. Therefore,

$$\phi(r) + \phi(-r) = \frac{\partial}{\partial r} P(|B_3(\tau_W)| \leq r) = \frac{2r^{\pi/\alpha-1}}{\alpha(1+r^{2\pi/\alpha})},$$

that is,

$$\phi(r) = \frac{r^{\pi/\alpha-1}}{\alpha(1+r^{2\pi/\alpha})}.$$

Additionally, we can discuss the cases when the β^{th} moment of R exists. Note that

$$E|R|^\beta = 2\int_0^\infty \frac{r^{\beta+\frac{\pi}{\alpha}-1}}{\alpha(1+r^{2\pi/\alpha})} dr.$$

The β^{th} moment of R exists if and only if the integrand is at least in the order of $O(\frac{1}{r})$ as $r \to \infty$, that is, there exists a constant c such that

$$\lim_{r\to\infty} \frac{r^{\beta+\frac{\pi}{\alpha}-1}}{\alpha(1+r^{2\pi/\alpha})} / (\frac{1}{r}) = c.$$

Therefore, $\beta + \frac{\pi}{\alpha} < \frac{2\pi}{\alpha} \iff \alpha\beta < \pi$ in order for the existence of the β^{th} moment of R. □