Homework 1 Solution

Yikun Zhang[1]

Chapter 2. Ex.16 *The Weierstrass approximation theorem states: Let f be a continuous function on the closed and bounded interval $[a,b] \subset \mathbb{R}$. Then, for any $\epsilon > 0$, there exists a polynomial P such that*

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

Prove this by applying Corollary 5.4 of Fejér's theorem and using the fact that the exponential function e^{ix} can be approximated by polynomials uniformly on any interval.

Proof. With loss of generosity, we assume that $a > 0$. Define

$$F(x) = \begin{cases} f(x), & x \in [a,b] \\ f(2b-x), & x \in (b, 2b-a]. \end{cases}$$

Then $F(x)$ is continuous on $[a, 2b-a]$ with $F(a) = F(2b-a)$. We can extend F to be a continuous $(2b-2a)$-periodic function on the real line $\mathbb{R}$.

Letting $x = (\frac{b-a}{\pi})t + b$, $\phi(t) = F[(\frac{b-a}{\pi})t + b]$ would be a 2π-periodic function with $\phi(-\pi) = \phi(\pi)$ on $\mathbb{R}$.

By Corollary 5.4 of Fejér's theorem, for any $\epsilon > 0$, there exists a trigonometric polynomial Q such that $|\phi(t) - Q(t)| < \frac{\epsilon}{2}$ for all $-\pi \le t \le \pi$.

Thus, by letting $t = \frac{\pi(x-b)}{b-a}$, we obtain that $|F(x) - Q[\frac{\pi(x-b)}{b-a}]| < \frac{\epsilon}{2}$ for all $a < x < 2b-a$.

Denote $Q[\frac{\pi(x-b)}{b-a}]$ by $Q_1(x)$ and it can be written as $Q_1(x) = \sum_{n=M}^{N} a_n e^{\frac{n\pi i x}{b-a}}$, where $N, M \in \mathbb{Z}$.

By Taylor's Expansion, we know that $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \cdots + \frac{(ix)^n}{n!} + \frac{e^{\theta x}}{(n+1)!}(ix)^{n+1}$, where $x \in [a, 2b-a], \theta \in (0,1)$.

Therefore, with $|\frac{e^{\theta x}}{(n+1)!}(ix)^{n+1}| < \frac{e^{2b-a}(2b-a)^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$, there exists a polynomial $P_n(x)$ such that $|a_n e^{\frac{n\pi x}{b-a}} - P_n(x)| < \frac{\epsilon}{2(N-M+1)}$ for each n, $M \le n \le N$.

Let $P(x)$ be $\sum_{n=M}^{N} P_n(x)$. Therefore,

$$\begin{aligned} |F(x) - P(x)| &\le |F(x) - Q_1(x)| + |Q_1(x) - P(x)| \\ &\le \frac{\epsilon}{2} + (N-M+1) \cdot \frac{\epsilon}{2(N-M+1)} \qquad (1) \\ &< \epsilon \end{aligned}$$

Restricting $P(x)$ to $[a,b]$, we obtain the desired polynomial. ☐

[1]School of Mathematics, Sun Yat-sen University