Homework 3 Solution

Yikun Zhang[1]

Chapter 2. Problem 1 *One can construct Riemann integrable functions on* $[0,1]$ *that have a dense set of discontinuities as follows.*
(a) Let $f(x)=0$ *when* $x<0$*, and* $f(x)=1$ *if* $x\geq 0$*. Choose a countable dense sequence* $\{r_n\}$ *in* $[0,1]$*. Then, show that the function*

$$F(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}f(x-r_n)$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$*.*
(b) Consider next

$$F(x)=\sum_{n=1}^{\infty}3^{-n}g(x-r_n),$$

where $g(x)=sin\,1/x$ *when* $x\neq 0$*, and* $g(0)=0$*. Then* F *is integrable, discontinuous at each* $x=r_n$*, and fails to be monotonic in any subinterval of* $[0,1]$*.*
(c) The original example of Riemann is the function

$$F(x)=\sum_{n=1}^{\infty}\frac{(nx)}{n^2},$$

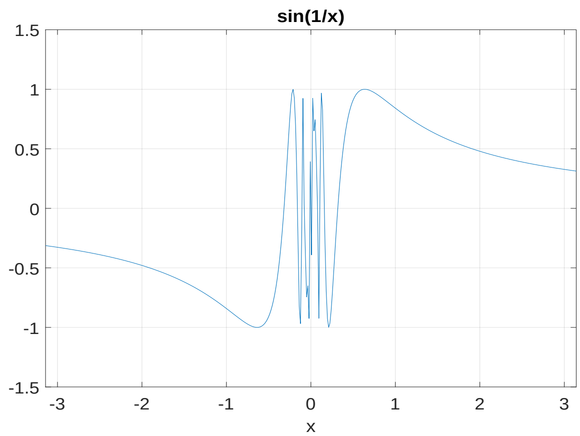
where $(x)=x$ *for* $x\in(-1/2,1/2]$ *and* (x) *is continued to* $\mathbb{R}$ *by periodicity, that is,* $(x+1)=(x)$*. It can be shown that* F *is discontinuous whenever* $x=m/2n$*, where* $m,n\in\mathbb{Z}$ *with* m *odd and* $n\neq 0$*.*

Proof. (a) First we have $|F(x)|\leq\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$, yielding that F is bounded.
Then $F(x)$ is nondecreasing because $f(x)$ is nondecreasing.
Thus, by the Proposition 1.3 in the Appendix, we know that $F(x)$ is integrable.
For any countable dense set $\{r_n\}$ in $[0,1]$, like $\mathbb{Q}\cap[0,1]$, it suffices to prove that F is discontinuous at an arbitrary point r_k selected from the reordered set $\{r_n\}$, where $r_k<r_{k+1}$, $k=1,2,...$
The left limit of F at r_k is $\lim\limits_{x\to r_k^-}F(x)=\lim\limits_{x\to r_k^-}\sum\limits_{n=1}^{\infty}\frac{1}{n^2}f(x-r_n)=\sum\limits_{n=1}^{k-1}\frac{1}{n^2}$, while the right limit of F at the same point is $\lim\limits_{x\to r_k^+}F(x)=\lim\limits_{x\to r_k^+}\sum\limits_{n=1}^{\infty}\frac{1}{n^2}f(x-r_n)=\sum\limits_{n=1}^{k}\frac{1}{n^2}\neq\lim\limits_{x\to r_k^-}F(x)$.
Therefore, F has discontinuities as all points of the sequence $\{r_n\}$.

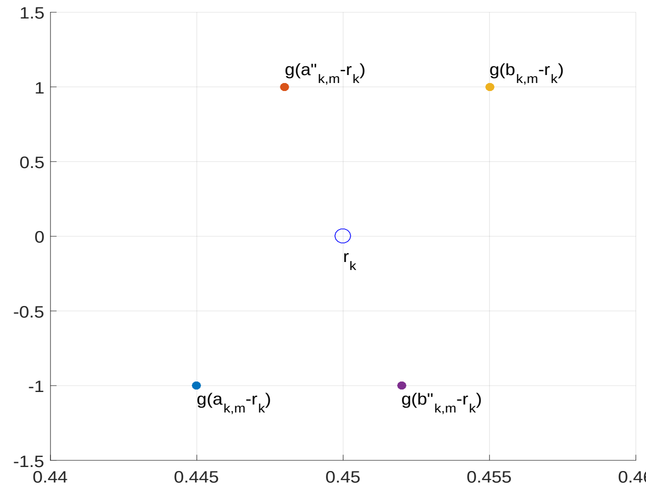
(b) Similarly, we have $|F(x)|\leq\sum\limits_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}$, showing that F is bounded.
Moreover, since $g(x)=sin\frac{1}{x}$ is discontinuous only at the point $x=0$, for each $n\in\mathbb{Z}^+$,

[1]School of Mathematics, Sun Yat-sen University

$3^{-n}g(x-r_n)$ has a unique discontinuity at r_n. Together with Lemma 1.6 in the Appendix and the fact that $\{r_n\}$ is countable, we conclude that the discontinuities of F has measure 0 and thus F is integrable.
Next we want to prove that F is discontinuous at each $x = r_n$ and fails to be monotonic in any subinterval of $[0,1]$, by leveraging the fluctuation of $sin\frac{1}{x}$ around 0. See Figure (a) for a visualization.



(a) Plotting for $sin\,\frac{1}{x}$



(b) Sample Configuration of these Four Points

Since $\{r_n\}$ is dense in $[0,1]$, any subinterval of $[0,1]$ contains at least one element in $\{r_n\}$, says r_k.
Consider four numerical sequences $a_{k,m} = r_k - \frac{1}{\frac{\pi}{2}+2m\pi}$, $a'_{k,m} = r_k - \frac{1}{\frac{3\pi}{2}+2m\pi}$, $b_{k,m} = r_k + \frac{1}{\frac{\pi}{2}+2m\pi}$, and $b'_{k,m} = r_k + \frac{1}{\frac{3\pi}{2}+2m\pi}$. See Figure (b) for a visualization.
Then $g(a_{k,m}-r_k) = g(b'_{k,m}-r_k) = -1$, and $g(a'_{k,m}-r_k) = g(b_{k,m}-r_k) = 1$.
We respectively divide the summation in $F(a_{k,m})$ and $F(b_{k,m})$ into three parts as follows,

$$F(a_{k,m}) = \sum_{n=1}^{k-1} 3^{-n}g(a_{k,m}-r_n) \ + \ 3^{-k}g(a_{k,m}-r_k) \ + \sum_{n=k+1}^{\infty} 3^{-n}g(a_{k,m}-r_n) = I_1^a \ + II_2^a \ + III_3^a$$

$$F(b_{k,m}) = \sum_{n=1}^{k-1} 3^{-n}g(b_{k,m}-r_n) \ + \ 3^{-k}g(b_{k,m}-r_k) \ + \sum_{n=k+1}^{\infty} 3^{-n}g(b_{k,m}-r_n) = I_1^b \ + II_2^b \ + III_3^b.$$

Since $\lim_{m\to\infty} a_{k,m} = \lim_{m\to\infty} b_{k,m} = r_k$, we obtain that $I_1^a = I_1^b$ when m is sufficiently large.
Using the fact that $3^{-k} > \sum_{n>k} 3^{-n}$ and $g(a_{k,m}-r_k) = -1$, we know that

$$II_2^a \ + III_3^a \leq -3^{-k} + \Big|\sum_{n=k+1}^{\infty} 3^{-n}g(a_{k,m}-r_n)\Big| < -3^{-k} + \sum_{n>k} 3^{-n} < 0.$$

On the other hand, we also have

$$II_2^b + III_3^b \geq 3^{-k} - |\sum_{n=k+1}^{\infty} 3^{-n} g(a_{k,m} - r_n)| > 3^{-k} - \sum_{n>k} 3^{-n} > 0.$$

Therefore, we obtain that $F(a_{k,m}) < F(b_{k,m})$.
Likewise, we can prove that $F(a_{k,m}) < F(a'_{k,m})$ and $F(a'_{k,m}) > F(b'_{k,m})$ with the same argument. Since $a_{k,m} < a'_{k,m} < r_k < b'_{k,m} < b_{k,m}$ for each m and $|a_{k,m} - b_{k,m}| \to 0$ as $m \to 0$, F is discontinuous at r_k and fails to be monotonic in $[a_{k,m}, b_{k,m}$ and thus in any subinterval of $[0, 1]$.

(c) Let $x_m = \frac{m}{2n}$, where $m, n \in \mathbb{Z}$ with m odd and $n \neq 0$.
Then $|F(x_m + 0) - F(x_m - 0)| = \frac{1}{2m^2} > 0$. Thus F is discontinuous at x_m. $\square$