Homework 10 Solution

Yikun Zhang[1]

Chapter 5. Ex.9 *If f is of moderate decrease, then*

$$\int_{-R}^{R}(1-\frac{|\xi|}{R})\hat{f}(\xi)e^{2\pi ix\xi}d\xi=(f*\mathcal{F}_R)(x), \tag{1}$$

where the Fejér kernel on the real line is defined by

$$\mathcal{F}_R(t)=\begin{cases} R(\frac{sin\,\pi tR}{\pi tR})^2 & if\ t\neq 0,\\ Rif\ t=0. \end{cases}$$

Show that $\{\mathcal{F}_R\}$ is a family of good kernels as $R\to\infty$, and therefore (1) tends uniformly to $f(x)$ as $R\to\infty$. This is the analogue of Fejér's theorem for Fourier series in the context of the Fourier transform.

Proof. We first derive the explicit formula of the Fejér kernel from (1).
From (1), we know that $(f*\mathcal{F}_R)(x)=\int_{-\infty}^{\infty}\int_{-R}^{R}(1-\frac{|\xi|}{R})f(y)e^{2\pi i(x-y)\xi}d\xi dy$, where we can change the order of integration because f is of moderate decrease. Thus, if $t\neq 0$, then

$$\begin{aligned}
\mathcal{F}_R(t)&=\int_{-R}^{R}(1-\frac{|\xi|}{R})e^{2\pi it\xi}d\xi\\
&=\int_{0}^{R}(1-\frac{\xi}{R})e^{2\pi it\xi}d\xi+\int_{-R}^{0}(1+\frac{\xi}{R})e^{2\pi it\xi}d\xi\\
&=\frac{e^{2\pi itR}-1}{2\pi it}-\frac{R}{2\pi itR}e^{2\pi itR}+\frac{e^{2\pi itR}-1}{(2\pi it)^2R}+\frac{1-e^{-2\pi itR}}{2\pi it}-\frac{-R}{2\pi itR}e^{-2\pi itR}-\frac{1-e^{-2\pi itR}}{(2\pi it)^2R}\\
&=\frac{e^{2\pi itR}+e^{-2\pi itR}-2}{(2\pi it)^2R}\\
&=R(\frac{\sin\pi tR}{\pi tR})^2.
\end{aligned} \tag{2}$$

If $t=0$, $\mathcal{F}_R(t)=\int_{-R}^{R}(1-\frac{|\xi|}{R})d\xi=R$.
Then, to prove that $\{\mathcal{F}_R\}$ is a family of good kernels as $R\to\infty$, we first show that

$$\int_{-\infty}^{\infty}\mathcal{F}_R(t)dt=\frac{2}{\pi}\int_{0}^{\infty}(\frac{\sin u}{u})^2du=-\frac{2}{\pi}\frac{sin^2u}{u}\Big|_0^\infty+\frac{2}{\pi}\int_0^\infty\frac{\sin 2u}{u}du=1,$$

where we change the variable by $u=\pi tR$ and use the fact that $\int_0^\infty\frac{\sin u}{u}du=\frac{\pi}{2}$.
Since $\mathcal{F}_R\geq 0$, $\int_{-\infty}^{\infty}|\mathcal{F}_R(t)|dt\leq M$ also holds.

[1]School of Mathematics, Sun Yat-sen University

For any $\delta > 0$, $\int_{|t|>\delta} |\mathcal{F}_R(t)|dt = \int_{|u|>\delta\pi R} \frac{1}{\pi}(\frac{\sin u}{u})^2 du \to 0$ as $R \to \infty$, since $\int_{-\infty}^{\infty}(\frac{\sin t}{t})^2 dt$ converges.
As a consequence, by the continuity of f, (1) tends uniformly to $f(x)$ as $R \to \infty$. $\square$

Chapter 5. Ex.10 *Below is an outline of a different proof of the Weierstrass approximation theorem.*
Define the ***Landau*** *kernels by*

$$L_n(x) = \begin{cases} \frac{(1-x^2)^n}{c_n} & if\ -1 \le x \le 1, \\ 0 if\ |x| \ge 1, \end{cases}$$

where c_n is chosen so that $\int_{-\infty}^{\infty} L_n(x)dx = 1$. Prove that $\{L_n\}_{n\ge0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-\frac{1}{2}, \frac{1}{2}]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-\frac{1}{2}, \frac{1}{2}]$ which converges uniformly to f.

Proof. By the choice of c_n, we immediately have $\int_{-\infty}^{\infty} L_n(x)dx = 1$.
Since $1 - x^2 \ge 1 - x \ge 0$ when $x \in [-1, 1]$, we obtain that

$$1 = \int_{-1}^{1} \frac{(1-x^2)^n}{c_n}dx = 2\int_0^1 \frac{(1-x^2)^n}{c_n}dx \ge 2\int_0^1 \frac{(1-x)^n}{c_n}dx = \frac{2}{(n+1)c_n},$$

yielding that $c_n \ge \frac{2}{n+1}$.
Thus $L_n(x) \ge 0$ and $\int_{-\infty}^{\infty} |L_n(x)|dx \le M$, where M is a constant.
Moreover, for any $\eta > 0$, $\int_{|x|\ge\eta} L_n(x)dx = 2\int_\eta^1 \frac{(1-x^2)^n}{c_n}dx \le (n+1)(1-\eta)(1-\eta^2)^n \to 0$, as $n \to \infty$.
As a result, $\{L_n\}$ is a family of good kernels and by Theorem 4.1 in Chapter 2, $(f * L_n)(x)$ converges uniformly to f on $[-\frac{1}{2}, \frac{1}{2}]$ if f is a continuous function supported in $[-\frac{1}{2}, \frac{1}{2}]$.
Since $L_n(x)$ indeed is a polynomial of x, $(f * L_n)(x) = \int_{-1/2}^{1/2} f(y)L_n(x-y)dy$ is a sequence of polynomials of x. $\square$

Chapter 5. Ex.12 *Show that the function defined by*

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x)$$

satisfied the heat equation for $t > 0$ and $\lim_{t\to0} u(x,t) = 0$ for every x, but u is <u>not</u> continuous at the origin.

Proof. We are just making some direct computations when verifying $u(x,t) = \frac{x}{2\sqrt{\pi}t^{\frac{3}{2}}}e^{-\frac{x^2}{4t}}$ satisfies the heat equation. We thus only write down the ultimate result

$$\frac{\partial^2 u}{\partial x^2} = (-\frac{3x}{4\sqrt{\pi}t^{\frac{5}{2}}} + \frac{x^3}{8\sqrt{\pi}t^{\frac{7}{2}}})e^{-\frac{x^2}{4t}} = \frac{\partial u}{\partial t}.$$

By L'Hospital Rule, $\lim_{t\to 0} u(x,t) = \lim_{t\to 0} \frac{x}{2\sqrt{\pi}t^{\frac{3}{2}}e^{\frac{x^2}{4t}}} = 0$.
However, $\lim_{x^2=4ct, x\to 0} u(x,t) = \lim_{x\to 0} \frac{4c^{\frac{3}{2}}}{\sqrt{\pi}x^2} e^{-c} = \infty \neq 0$.
Thus u is not continuous at the origin. $\square$