Homework 4 Solution

Yikun Zhang[1]

Chapter 2. Ex.15 *Prove that the Fejér kernel is given by*

$$F_N(x) = \frac{1}{N}\frac{sin^2(Nx/2)}{sin^2(x/2)}.$$

Proof. First we note that $D_N(x) = \sum\limits_{n=0}^{N} \omega^n + \sum\limits_{n=1}^{N} \omega^{-n} = \frac{1-\omega^{N+1}}{1-\omega} + \frac{\omega^{-N}-1}{1-\omega} = \frac{\omega^{-N}-\omega^{N+1}}{1-\omega}$, where $\omega = e^{ix}$.
Therefore,
$NF_N(x) = \sum\limits_{n=0}^{N-1} \frac{\omega^{-n}-\omega^{n+1}}{1-\omega} = \frac{1}{1-\omega}(\frac{1-\omega^{-N}}{1-\omega^{-1}} - \frac{\omega-\omega^{N+1}}{1-\omega}) = \frac{\omega^{1-N}-2\omega+\omega^{N+1}}{(1-\omega)^2} = \frac{(\omega^{-\frac{N}{2}}-\omega^{\frac{N}{2}})^2}{(\omega^{-\frac{1}{2}}-\omega^{\frac{1}{2}})^2} = \frac{sin^2(\frac{Nx}{2})}{sin^2(\frac{x}{2})}$. □

Chapter 2. Problem 2 *Let D_N denote the Dirichlet kernel*

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{sin((N+1/2)\theta)}{sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)| d\theta.$$

(a) Prove that

$$L_N \geq c\, log\, N$$

for some constant $c > 0$. A more careful estimate gives

$$L_N = \frac{4}{\pi^2} log\, N + O(1).$$

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function f_n such that $|f_n| \leq 1$ and $|S_n(f_n)(0) \geq c' \, log\, n$.

Proof. (a) We directly prove the more precise estimate $L_N = \frac{4}{\pi^2} log\, N + O(1)$ and the result $L_N \geq c\, log\, N$ follows.
First we know that $|sin\frac{\theta}{2}| \leq |\frac{\theta}{2}|$ for all $\theta \in \mathbb{R}$. Thus $|D_N(\theta)| \geq \frac{2|sin(N+\frac{1}{2})\theta|}{|\theta|}$.

[1]School of Mathematics, Sun Yat-sen University

Therefore,

$$
\begin{aligned}
L_N &\geq \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{|sin(N+\frac{1}{2})\theta|}{|\theta|}d\theta = \frac{2}{\pi}\int_{0}^{\pi}\frac{|sin(N+\frac{1}{2})\theta|}{|\theta|}d\theta \\
&\overset{t\,=\,(N+\frac{1}{2})\theta}{=} \frac{2}{\pi}\int_{0}^{(N+\frac{1}{2})\pi}\frac{|sin\,t|}{|t|}dt \\
&= \frac{2}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi}\frac{|sin\,t|}{|t|}dt + \frac{2}{\pi}\int_{N\pi}^{(N+\frac{1}{2})\pi}\frac{|sin\,t|}{|t|}dt \\
&\geq \frac{2}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi}\frac{|sin\,t|}{k\pi}dt + \frac{2}{\pi}\int_{N\pi}^{(N+\frac{1}{2})\pi}\frac{|sin\,t|}{(N+\frac{1}{2})\pi}dt \qquad (1) \\
&= \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} + \frac{2}{(N+\frac{1}{2})\pi^2} \\
&\geq \frac{4}{\pi^2}\sum_{k=1}^{N}log(1+\frac{1}{k}) + O(1) \\
&\geq \frac{4}{\pi^2}log\,N + O(1),
\end{aligned}
$$

where we use the fact that $\int_{(k-1)\pi}^{k\pi}|sin\,t|dt = 2$ and $\int_{N\pi}^{(N+\frac{1}{2})\pi}|sin\,t|dt = 1$. □

(b) First we construct the function g_n as follows,

$$
g_n(x) = \begin{cases} 1 & \text{when } D_n(x) \geq 0, \\ -1 & \text{when } D_n(x) < 0. \end{cases}
$$

Then by Lemma 3.2, we can approximate g_n by continuous functions $\{h_k\}_{k=1}^{\infty}$ satisfying $|h_k| \leq 1$ and $\int_{-\pi}^{\pi}|g_n(x) - h_k(x)|dx < \pi\epsilon^2$ for any $\epsilon > 0$ when $k \geq K$ and K is sufficiently large.
Let $f_n = h_K$ and thus $\int_{-\pi}^{\pi}|g_n(x) - f_n(x)|dx < \pi\epsilon^2$. By (a) and Cauchy's Inequality, we have

$$
\begin{aligned}
|S_N(f_n)(0)| &= |\frac{1}{2\pi}\int_{-\pi}^{\pi}f_n(y)D_N(y)dy| \\
&= |\frac{1}{2\pi}\int_{-\pi}^{\pi}g_n(y)D_N(y)dy + \frac{1}{2\pi}\int_{-\pi}^{\pi}(f_n(y)-g_n(y))D_N(y)dy| \\
&\geq c\,log\,N - |\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_n(y)-g_n(y))D_N(y)dy| \\
&\geq c\,log\,N - (\frac{1}{2\pi}\int_{-\pi}^{\pi}|f_n(y)-g_n(y)|^2dy)^{\frac{1}{2}}(\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(y)|^2dy)^{\frac{1}{2}} \qquad (2) \\
&\geq c\,log\,N - (\frac{1}{\pi}\int_{-\pi}^{\pi}|f_n(y)-g_n(y)|dy)^{\frac{1}{2}}(2n+1) \\
&\geq c\,log\,N - \epsilon(2n+1),
\end{aligned}
$$

where we use the fact that $|D_n(x)| \leq 2n+1$ and $|f_n(y) - g_n(y)| \leq |f_n(y)| + |g_n(y)| \leq 2$. Therefore, for each fixed $n \leq 1$, by letting $\epsilon \to 0$ and modifying c to c', we obtain that f_n satisfying $|f_n| \leq 1$ and $|S_n(f_n)(0) \geq c' \, log \, n$. $\square$

Chapter 1. Ex.10 *Show that the expression of the Laplacian*

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also, prove that

$$|\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2 = |\frac{\partial u}{\partial r}|^2 + \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2.$$

Proof. In polar coordinates, $u(x,y) = u(rcos\,\theta, rsin\,\theta)$. Thus,

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} cos\,\theta + \frac{\partial u}{\partial y} sin\,\theta, & (1) \\ \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-rsin\,\theta) + \frac{\partial u}{\partial y} rcos\,\theta. & (2) \end{cases}$$

Furthermore,

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2} cos^2\theta + (\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}) sin\,\theta cos\,\theta + \frac{\partial^2 u}{\partial^2 y} sin^2\theta, & (3) \\ \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} r^2 sin^2\theta - (\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}) r^2 sin\,\theta cos\,\theta + \frac{\partial^2 u}{\partial^2 y} r^2 cos^2\theta - r(\frac{\partial u}{\partial x} cos\,\theta + \frac{\partial u}{\partial y} sin\,\theta). & (4) \end{cases}$$

Letting $\frac{1}{r^2}(4) + (3)$, we obtain that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - r\frac{\partial u}{\partial r}$.
Moreover, with $(1)^2 + \frac{1}{r^2}(2)$, we also have $|\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2 = |\frac{\partial u}{\partial r}|^2 + \frac{1}{r^2}|\frac{\partial u}{\partial \theta}|^2$. $\square$