Fourier Analysis and its Applications

2016 Yat-sen Class

Sun Yat-sen University

Department of Mathematics

September 8, 2017

Introduction to the Course

Class Time: Thursday
14:20–16:00

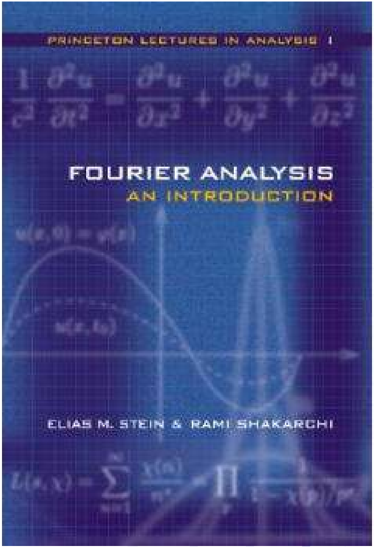
Location: Room 416

Instructor: Prof.Lixin Yan
(颜立新老师)

Contact: 020-84110123

Office: Room 702

Email: mcsylx@mail.sysu.edu.cn



TA: Yikun Zhang (张奕堃)

Email: yikunzhang@foxmail.com

Course Website:
`https://zhangyk8.github.io/teaching/fourier`

Office Hour: Wednesday 16:00-17:30

Location: 逸仙学院负一层

Support Materials:

- Trigonometric Series (三角级数) Third Edition (Written by Antoni Zygmund)
- Stanford Open Course
  `http://open.163.com/special/opencourse/fouriertransforms.html`

Author of the Textbook

Elias M. Stein

Professor Emeritus of Mathematics, Princeton University

Main Awards: Wolf Prize (1999)

R.Shakarchi: PhD student of C.Feffermen

Stein's Doctoral Students

Figure: Charles Fefferman (1978 Fields, 2017 Wolf)

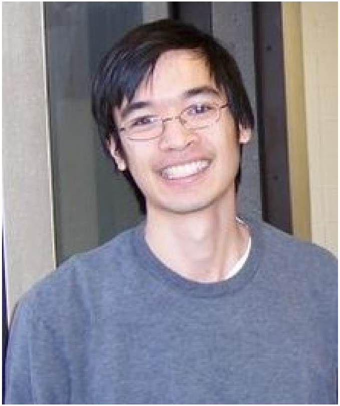


Figure: Terence Tao (2006 Fields)

Who is Fourier?

Biography of J.Fourier

Joseph Fourier(1768–1830)

- Profession: Mathematician, Egyptologist and Administrator
- Other Achievement: Discover the **Greenhouse Effect**

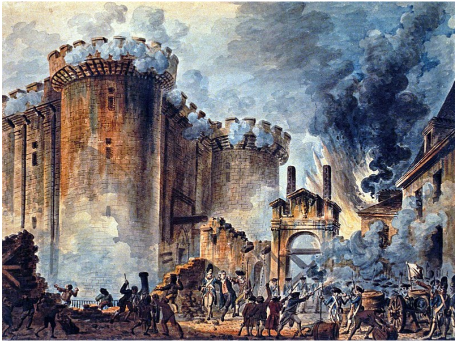
Early Life

- The son of a tailor and a housewife
- Talent in Latin, French, and Literature
- Real interest: **Mathematics**

Yesterday was my 21st birthday, at that age Newton and Pascal had already acquired many claims to immortality.

Politics vs Mathematics

- **French Revolution**
- Egyptian Expedition



Mathematics Career

1. Mathematics teacher at the **École Polytechnique**
2. The chair of Analysis and Mechanics

ÉCOLE POLYTECHNIQUE ParisTech

Academic Advisors of Fourier

Teacher's Name	His Advisor	The age when he first met Fourier
Pierre-Simon Laplace	Jean d'Alembert	45
Gaspard Monge	NA	49
Joseph-Louis Lagrange	Leonhard Eulern	59

Fourier gave charming descriptions of these famous mathematicians.

Laplace and Monge

To Laplace:

His voice is quiet but clear, and he speaks precisely, though not very fluently; his appearance is pleasant, and he dresses very simply. His teaching of mathematics is in no way remarkable and he covers the material ***very rapidly****...*

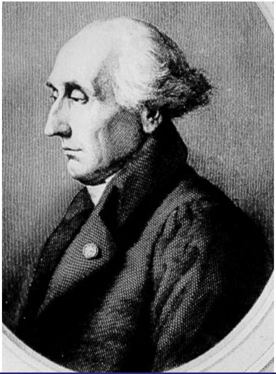
To Monge:

Monge has a loud voice and he is energetic, ingenious and very learned. The subject that he teaches is a fascinating one, and he describes it with the greatest possible clarity. He is even considered to be ***too clear, or, rather to deal with his material too slowly****...*

J.Lagrange

To Lagrange:

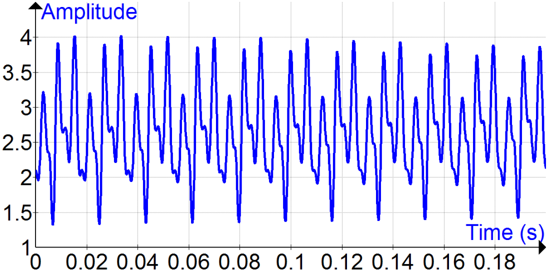
Lagrange, the foremost scholar of Europe, appears to be between 50 and 60 years old, though he is in fact younger; he dresses very quietly, in black or brown. He speaks only in discussion, and some of what he says excites ridicule. The other day he said ***"There are a lot of important things to be said on this subject, but I shall not say them".***



What is Fourier Analysis?

Background

- **Trigonometric functions and its convergence property**
- **Fourier Transform**
- **...**



Is Fourier Analysis Useful?

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

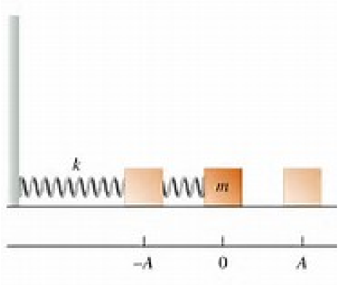
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + \frac{(-1)^{n-1}}{n^2} + \cdots$$

邓东皋　尹小玲
数学分析简明教程
高等教育出版社（第二版），2006年12月

Simple Harmonic Motion

$$-ky(t) = my''(t)$$



Solution

$$y(t) = acos(ct) + bsin(ct)$$

See Chapter 1 for details

Two Problems

d'Alembert, Euler, Taylor, Bernoulli,...

$\Longrightarrow$ Solution for the oscillator equation

Problem 1

Given a function f on $[0, \pi]$ (with $f(0) = f(\pi) = 0$), can we find coefficients A_m so that $f(x) = \sum_{m=1}^{\infty} A_m sin(mx)$?

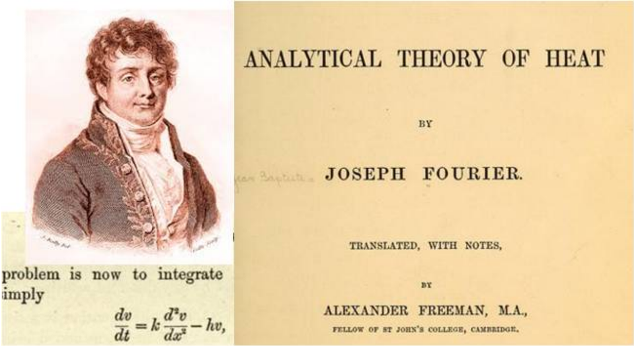
Problem 2

Given any reasonable function F on $[-\pi, \pi]$, is it true that

$$F(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}? \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{-inx}dx$$

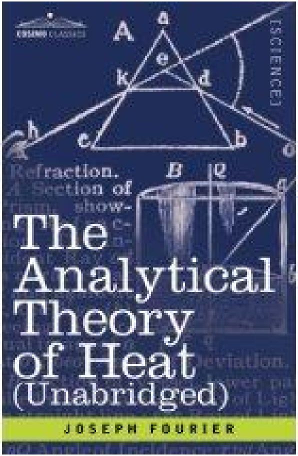
The Analytical Theory of Heat (1822)

Idea: Any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable.



- No rigorous proof

- Still unprecedented



Trigonometric Series: Dirichlet's Works

Dirichlet

- Poisson summation formula
- Modernization of the concept of function

Theorem A

For any given value of x, the sum of the Fourier series is $f(x)$ if $f(x)$ is continuous at that point x, and is $\frac{1}{2}[f(x-0)+f(x+0)]$ if $f(x)$ is discontinuous at that point.

See Chapter 3 for details.

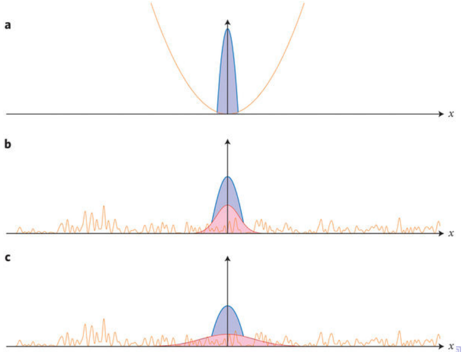
Riemann's Contribution

1. Riemann integral
2. Riemann localization principle

Riemann Localization Principle

Theorem B

For a bounded integrable function $f(x)$, the convergence of its Fourier series at a point x in $[-\pi, \pi]$ depends only on the behaviour of $f(x)$ in an arbitrarily small neighborhood of that point x.



Further Development: Before 20^{th} Century

Figure: R.Lipschitz

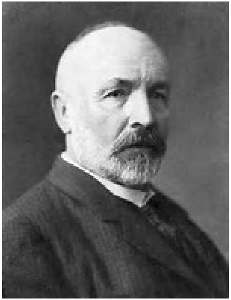


Figure: Georg Cantor

Figure: Karl Weierstrass

Weierstrass Approximation Theorem

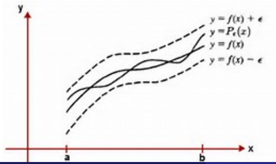
Theorem C

Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.

Theorem D (Weierstrass)

Let f be a continuous function on the closed and bounded interval $[a, b] \subset \mathbb{R}$. Then for any $\epsilon > 0$, there exists a polynomial P such that

$$\sup_{x \in [a,b]} |f(x) - P(x)| < \epsilon.$$



Chapter 2 Exercise 16 (Homework)

Riemann's Guess

Continuous but nowhere differentiable function?

$$R(x) = \sum_{n=1}^{\infty} \frac{sin(n^2 x)}{n^2}$$

In 1969 Gerver successfully proved that

The function R is actually differentiable at all the rational multiples of π of the form $\frac{\pi p}{q}$ with p and q odd integers.

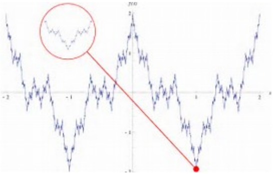
Weierstrass's Example

Theorem E

If $ab > 1 + \frac{3\pi}{2}$, $0 < b < 1$, $a > 1$,$a \in 2\mathbb{Z} + 1$ *then the function*

$$W(x) = \sum_{n=1}^{\infty} b^n cos(a^n x)$$

is nowhere differentiable.



See Chapter 4 for details.

Remark: This kind of functions is uncountable.

Drawback of Riemann Integral:

- Newton-Leibniz Formula
- Convergence of Fourier series

Lebesgue's Theory

Harmonic Analysis

Figure: A.N.Kolmogorov

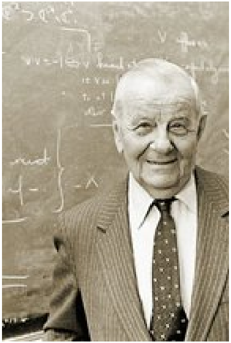


Figure: A.Zygmund

Figure: Elias Stein

Connections and Influence

Modern Fourier Analysis (I)

Luzin's Conjecture

If $f \in L^2[0, 2\pi]$, then its Fourier series converges almost everywhere.

Theorem E (L.Carleson and Hunt)

The Fourier series of any integrable function $f(x)$ i.e., $f \in L^p[0, 2\pi]\,, 1 < p < \infty$, converges almost everywhere to $f(x)$.

Luzin's Conjecture in high dimensions: **Still Unsolved**

Kolmogorov's Counterexample (1926)

The preceding theorem fails when $p = 1$, that is, the Fourier series of L^1 functions diverges everywhere.

Example

Let $\lambda_1, ..., \lambda_n$ be increasing odd integers to be defined.

- $m_1 = n \,, 2m_k + 1 = \lambda_k(2n+1)$
- $A_k = k\frac{4\pi}{2n+1} \,, 1 \le k \le n$
- $\phi(x) = \frac{m_k^2}{n}$ for $x \in \Delta_k := [A_k - m_k^{-2}, A_k + m_k^{-2}]$ and 0 elsewhere.

Consider the following candidate:

$$\phi(x) = \sum_{k=1}^{\infty} \frac{\phi_{n_k}(x)}{\sqrt{M_{n_k}}}$$

Then the fourier series of ϕ diverges almost elsewhere on $[0, 2\pi]$.

Partial Differential Equation (II)

Wave equation: $\Delta u = u_{tt}$

Heat equation: $cu_t = \Delta u$

Laplace's equation: $\Delta u = 0$

$Schrödinger's\, equation$: $iu_t + \Delta u = 0$

Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} = -\nabla p$$

$$div\mathbf{u} = 0$$

$$p : Pressure$$

....

Example: Wave Equation

If this equation is subject to the initial conditions

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x),$$

this is called the **Cauchy problem** for the wave equation.

Theorem F

A solution of the Cauchy problem for the wave equation is

$$u(x,t) = \int_{\mathbb{R}^d} [\hat{f}(\xi) cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{sin(2\pi|\xi|t)}{2\pi|\xi|}] e^{2\pi i x\cdot\xi} d\xi,$$

where $\hat{f}$ is the inverse Fourier transform of f.

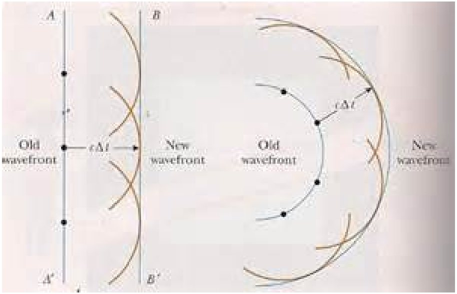
See Chapter 6 for details.

Interpretation in Physics – Huygens Principle

The solutions of the wave equation in one dimension is

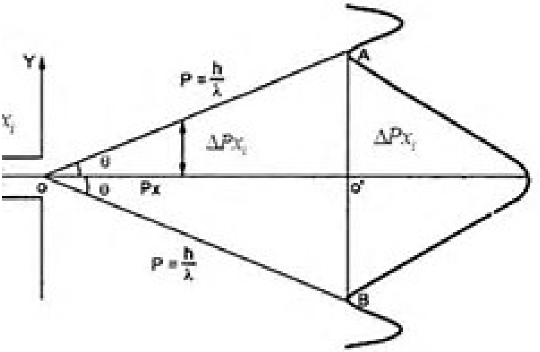
$$u(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy,$$

known as **d'Alembert's formula**.



The Heisenberg Uncertainty Principle, 1927

$(Uncertainty\ of\ position) \times (Uncertainty\ of\ momentum) \leq \frac{h}{16\pi^2}$,
where h is Planck's constant.



The Heisenberg Uncertainty Principle (1927)

Theorem

Suppose ψ *is a function in* $\mathbf{S}(\mathbb{R})$ *which satisfies the normalizing condition* $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. *Then*

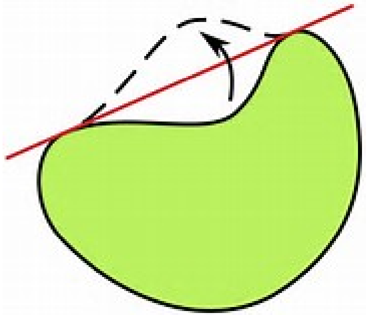
$$(\int_{-\infty}^{\infty} x^2|\psi(x)|^2 dx)(\int_{-\infty}^{\infty} \xi^2|\hat{\psi}(\xi)|^2 d\xi) \geq \frac{1}{16\pi^2},$$

and equality holds if and only if $\psi(x) = Ae^{-Bx^2}$ *where* $B > 0$ *and* $A^2 = \sqrt{\frac{2B}{\pi}}$.

See Chapter 5 for details.

Geometry (IV)

Among all simple closed curves of length l in the plane $\mathbb{R}^2$, which one encloses the largest area?



Fourier analysis can give a rigorous proof.

Statement of the Theorem

Theorem G (Isoperimetric inequality)

Suppose that Γ is a simple closed curve in $\mathbb{R}^2$ of length l and let $\mathcal{A}$ denote the area of the region enclosed by this curve. Then

$$\mathcal{A} \leq \frac{l^2}{4\pi}$$

with equality if and only if Γ is a circle.

The proof depends mainly on **Parseval's identity**.

Ergodicity (V)

Theorem H (Weyl's equidistribution)

If γ is irrational, then the sequence of fractional parts $\langle\gamma\rangle, \langle 2\gamma\rangle, \langle 3\gamma\rangle, \ldots$ is equidistributed in $[0,1)$.

See visualization and geometric interpretation on the course website.

See Chapter 4 for details.

Fast Fourier Transform (VI)

- **Cooley-Tukey algorithm** (Most common)
- Factorizing the Discrete Fourier Transform matrix into a product of sparse factors

Discrete Fourier Transform

If we denote by $a_k^N(F)$ the k^{th} Fourier coefficient of F on $\mathbb{Z}(N)$, then it is defined by

k^{th} Fourier coefficient

$a_k^N(F) = \frac{1}{N}\sum_{r=0}^{N-1} F(r)w_N^{kr}$, *where* $w_N = e^{-\frac{2\pi i}{N}}$

Theorem I

If F is a function on $\mathbb{Z}(N)$, then

$$F(q) = \sum_{k=0}^{N-1} a_k^N(F)e^{2\pi ikq/N}$$

Fast Fourier Transform

FFT can improve the bound $O(N^2)$.

Theorem J

Given $w_N = e^{-\frac{2\pi i}{N}}$ with $N = 2^n$, it is possible to calculate the Fourier coefficients of a function on $\mathbb{Z}(N)$ with at most

$$4 \cdot 2^n n = 4Nlog_2(N) = O(NlogN)$$

operations.

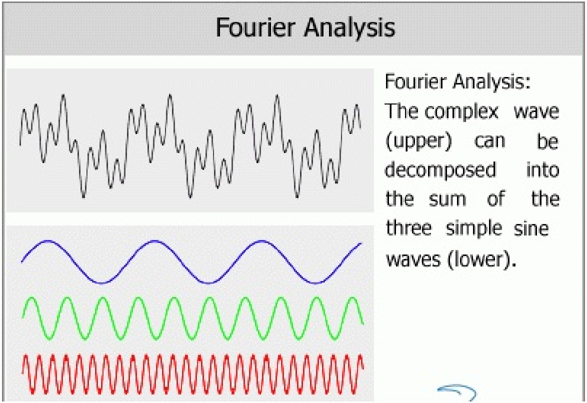
See Chapter 7 for details.

Applications of FFT

- Signal processing
- Cryptography
- Filtering algorithms (Time Series)
- Quantum mechanics

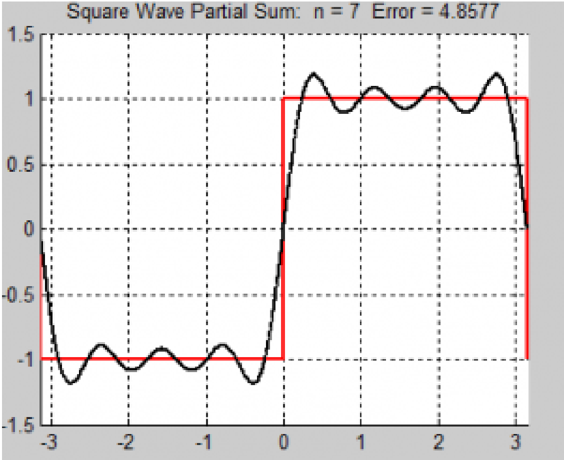
Example 1: Representation of Complicated signals

- A complex periodic signal can be decomposed into the sum of several simple Fourier series.



Example 2: Approximation of Impulses and Complex Functions

- An impulse signal can be approximated by trigonometric series.



Example 3:

Express the signal function f into trigonometric series

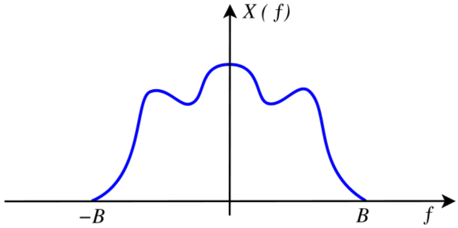
$$f(t) = a_0 + \sum_k (a_k cos(kt) + b_k sin(kt))$$

- *Signal(Data) Compression*
- *Noise Reduction*

Example 4: Nyquist-Shannon Sampling Theorem

Theorem K

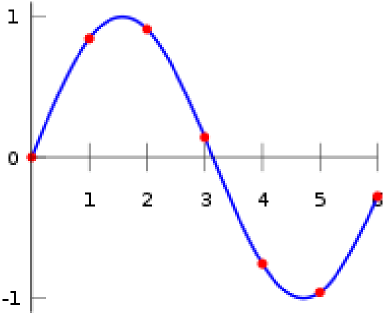
If a function $x(t)$ contains frequencies no higher than B hertz, it is completely determined by giving its ordinates at a series of points space $\frac{1}{2B}$ seconds part.



1. Continuous function

2. Discrete sequence

3. Continuous function



Prestige of Fourier Analysis

Applications:

- Physics
- Signal Processing
- Statistics
- Biology
- Acoustics
- Oceanography
- ...

Wikipedia
Fourier Analysis
https://en.wikipedia.org/wiki/Fourier_analysis, Retrieved on September 6, 2017

Discussion