Homework 15 Solution

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Chapter 7. Ex.12 *Suppose that G is a finite abelian group and $e : G \to \mathbb{C}$ is a function that satisfies $e(x \cdot y) = e(x)e(y)$ for all $x, y \in G$. Prove that either e is a identically 0, or e never vanishes. In the second case, show that for each x, $e(x) = e^{2\pi i r}$ for some $r \in \mathbb{Q}$ of the form $r = \frac{p}{q}$, where $q = |G|$.*

Proof. Let 0_G be the identity of the finite abelian group G. By the definition of e, we know that $e(0_G + 0_G) = e(0_G) \cdot e(0_G) = e(0_G)$, yielding that $e(0_G) = 1$ or $e(0_G) = 0$.
If $e(0_G) = 0$, then for every $a \in G$, $e(a) = e(a + 0_G) = e(a) \cdot e(0_G) = 0$, showing that e is identically 0.
If $e(0_G) = 1$, letting $q = |G|$, for each $x \in G$, $[e(x)]^q = e(q \cdot x) = e(0_G) = 1$.
Hence $e(x) = e^{\frac{2\pi i p}{q}}$ for some $p = 0, 1, ..., q-1$. $\square$

Chapter 7. Ex.13 *In analogy with ordinary Fourier series, one may interpret finite Fourier expansions using convolutions as follows. Suppose G is a finite abelian group, 1_G its unit, and V the vector space of complex-valued functions on G.*
(a) The convolution of two functions f and g in V is defined for each $a \in G$ by

$$(f * g)(a) = \frac{1}{|G|} \sum_{b \in G} f(b) g(a \cdot b^{-1}).$$

Show that for all $e \in \hat{G}$ one has $\widehat{(f * g)}(e) = \hat{f}(e)\hat{g}(e)$.
(b) Use Theorem 2.5 to show that if e is a character on G, then

$$\sum_{e \in \hat{G}} e(c) = 0 \text{ whenever } c \in G \text{ and } c \neq 1_G.$$

(c) As a result of (b), show that the Fourier series $Sf(a) = \sum_{e \in \hat{G}} \hat{f}(e)e(a)$ of a function $f \in V$ takes the form

$$Sf = f * D,$$

where D is defined by

$$D(c) = \sum_{e \in \hat{G}} e(c) = \begin{cases} |G| & \text{if } c = 1_G, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Since $f * D = f$, we recover the fact that $Sf = f$. Loosely speaking, D corresponds to a "Dirac delta function"; it has unit mass

$$\frac{1}{|G|} \sum_{c \in G} D(c) = 1,$$

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and (1) says that this mass is concentrated as the unit element in G. Thus D has the same interpretation as the "limit" of a family of good kernels.

Proof. (a) For all $e \in \hat{G}$, we prove the result by some algebra,

$$\begin{aligned}
\widehat{(f * g)}(e) &= (f * g, e) \\
&= \frac{1}{|G|}\sum_{a\in G}\frac{1}{|G|}\sum_{b\in G} f(b)g(a\cdot b^{-1})\overline{e(a)} \\
&= \frac{1}{|G|}\sum_{b\in G}\frac{1}{|G|}f(b)\sum_{a\in G} g(a\cdot b^{-1})\overline{e(a)} \\
&= \frac{1}{|G|}\sum_{b\in G}\frac{1}{|G|}f(b)\sum_{y\in G} g(y)\overline{e(b\cdot y)} \\
&= (\frac{1}{|G|}\sum_{b\in G} f(b)\overline{e(b)})\cdot(\frac{1}{|G|}\sum_{y\in G} g(y)\overline{e(y)}) \\
&= \hat{f}(e)\hat{g}(e).
\end{aligned} \tag{2}$$

(b) If $c \neq 1_G$, then there exists an $e' \in \hat{G}$ such that $e'(c) \neq 1$.
In fact, suppose that $e(c) = 1$ for all $e \in \hat{G}$. Let H be the cyclic group generated by c. Since $c \neq 1_G$, we have $|H| > 1$ and thus the factor group G/H, which is formed by the cosets of H, has the order $|G/H| < |G|$. Each $e \in \hat{G}$ induces a character in G/H, and different e's induce different characters. But it is impossible because there are exactly $|G/H|$ characters on G/H.

Therefore, $e'(c)\sum_{e\in\hat{G}} e(C) = \sum_{e\in\hat{G}}(e'\cdot e)(c) = \sum_{e\in\hat{G}} e(c)$, and since $e'(c) \neq 1$, we obtain that $\sum_{e\in\hat{G}} e(c) = 0$.

(c) By some direct computations, we have

$$\begin{aligned}
Sf(a) &= \sum_{e\in\hat{G}} \hat{f}(e)e(a) \\
&= \sum_{e\in\hat{G}}\frac{1}{|G|}\sum_{x\in G} f(x)\overline{e(x)}e(a) \\
&= \frac{1}{|G|}\sum_{x\in G} f(x)\sum_{e\in\hat{G}} e(a\cdot x^{-1}) \\
&= (f * D)(a),
\end{aligned} \tag{3}$$

where

$$D(c) = \sum_{e\in\hat{G}} e(c) = \begin{cases} |G| & \text{if } c = 1_G, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

as the result of (b). Hence $Sf = f * D = f$. $\square$