# Homework 6 Solution

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**Chapter 3. Ex.2** *Prove that the vector space $\ell^2(\mathbb{Z})$ is complete.*

**Proof.** Suppose that $A_k = \{a_{k,n}\}_{n\in\mathbb{Z}}$ with $k = 1, 2, ...$ is a Cauchy sequence. Then for any $\epsilon > 0$, there exists an $N > 0$ such that

$$|a_{k,n} - a_{k',n}| \le ||A_k - A_{k'}|| < \epsilon/2, \text{ whenever } k, k' > N.$$

Thus, for each $n \in \mathbb{Z}$, $\{a_{k,n}\}_{k=1}^{\infty}$ is a Cauchy sequence of complex numbers, therefore it converges to a limit, say $b_n$. Let $B = (..., b_{-1}, b_0, b_1, ...)$ and $A_{k,N}, B_N$ denote the truncated element

$$A_{k,N} = (..., 0, a_{k,-N}, ..., a_{k,-1}, a_{k,0}, a_{k,1}, ..., a_{k,N}, 0, ...), B_N = (..., 0, b_{-N}, ..., b_{-1}, b_0, b_1, ..., b_N, 0, ...),$$

respectively.

By taking partial sums of $||A_k - A_{k'}||$, we have

$$||A_{k,N} - A_{k',N}|| \le ||A_k - A_{k'}|| < \epsilon/2.$$

Letting $k' \to \infty$, $||A_{k,N} - B_N|| \le \epsilon/2$. Letting $N \to \infty$, we obtain that $||A_k - B|| \le \epsilon/2 < \epsilon$, yielding that $||A_k - B|| \to 0$ as $k \to \infty$.

Finally, we are left to prove that $B \in \ell^2(\mathbb{Z})$.

Since $||A_k - B|| \to 0$ as $k \to \infty$ and $A_k \in \ell^2(\mathbb{Z})$ for each $k$, $||A_k|| < \infty$ and thus

$$||B|| \le ||B - A_k|| + ||A_k|| < \epsilon + ||A_k|| < \infty, \text{ when } k \text{ is large.} \quad \square$$

**Chapter 3. Ex.5** *Let*

$$f(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ log(1/\theta) & \text{for } 0 < \theta \le 2\pi, \end{cases}$$

*and define a sequence of functions in $\mathcal{R}$ by*

$$f_n(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ f(\theta) & \text{for } 1/n < \theta \le 2\pi. \end{cases}$$

*Prove that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$. However, $f$ does not belong to $\mathcal{R}$.*

**Proof.** By L'Hospital's Rule, it is easy to prove that $\lim\limits_{\theta\to 0} \theta(log\theta)^2 = 0$ and $\lim\limits_{\theta\to 0} \theta(log\theta) = 0$. Therefore, we have $\int_a^b (log\theta)^2 d\theta \to 0$ if $0 < a < b$ and $b \to 0$, where we use the fact that $\int (log\theta)^2 d\theta = \theta(log\theta)^2 - 2\theta(log\theta) + 2\theta + C$ and $C$ is a constant.

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Thus, $\forall \epsilon > 0, \exists N > 0$, for $n > m > N$,

$$||f_n(\theta) - f_m(\theta)|| = (\frac{1}{2\pi}\int_{\frac{1}{n}}^{\frac{1}{m}} [log(1/\theta)]^2 d\theta)^{\frac{1}{2}} = (\frac{1}{2\pi}\int_{\frac{1}{n}}^{\frac{1}{m}} (log\theta)^2 d\theta)^{\frac{1}{2}} < \epsilon,$$

showing that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$ and $\lim\limits_{n\to\infty} f_n = f$. However, $f \notin \mathbb{R}$, since it is not bounded. $\square$

**Chapter 3. Ex.7** *Show that the trigonometric series*

$$\sum_{n\geq 2} \frac{1}{log\, n} sin\, nx$$

*converges for every $x$, yet it is not the Fourier series of a Riemann integrable function.*

**Proof.** First we have $|\sum\limits_{n=2}^{N} sin(nx)| \leq \frac{1}{sin(\delta/2)}$ when $|x| \geq \delta > 0$ and $\lim\limits_{x\to 0}\sum\limits_{n=2}^{N} sin(nx) = 0$.

Thus, $\sum\limits_{n=2}^{N} sin(nx) = 0$ is bounded while $\frac{1}{log\, n}$ is monotonic and tends to 0 as $n \to \infty$. By Dirichlet's test, $\sum\limits_{n\geq 2} \frac{1}{log\, n} sin\, nx$ converges for every $x$.

If $\frac{1}{log\, n}$ is the Fourier coefficient of a Riemann integrable functions, by Parseval's identity, one must have $\sum\limits_{n\geq 2} \frac{1}{2}|\frac{1}{log\, n}|^2 = ||f||^2 < \infty$. However, $\sum_{n\geq 2} \frac{1}{log\, n}$ diverges, which leads to a contradiction. $\square$

**Remark:** Likewise, $\sum\limits_{n=1}^{\infty} \frac{1}{n^{2\alpha}}$ is also a divergent series when $0 < \alpha \leq \frac{1}{2}$. Hence the same is true for $\sum \frac{sin\, nx}{n^{\alpha}}$ when $0 < \alpha \leq \frac{1}{2}$.