MATH 516 Theorem 3.18 (Extended)

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Theorem 1 (Affine envelope representation)**.** *A proper function* $f : \mathbf{E} \to \bar{\mathbf{R}}$ *admits an affine minorant if and only if* $(\bar{co}\, f)$ *is proper. Under these two equivalent conditions, the equality holds:*

$$(\bar{co}\, f)(x) = \sup\{g(x) : g : \mathbf{E} \to \bar{\mathbf{R}} \textit{ is an affine minorant of } f\}. \tag{1}$$

Proof. Define the set $Q := \mathrm{cl}(\mathrm{conv}(\mathrm{epi}\, f))$. Clearly if f admits an affine minorant, then $\bar{\mathrm{co}}\, f$ never takes the value $-\infty$ and is therefore proper. (Note that an affine minorant, which is finite at any given point in $\mathbf{E}$, serves as an lower bound for f.) Henceforth, we assume that $(\bar{\mathrm{co}}\, f)$ is proper. We will show that f admits at least one affine minorant and that (1) holds, thereby completing the proof. Applying Theorem 2.21 in the Lecture Note, we deduce that Q can be written as an intersection of halfspaces in $\mathbf{E} \times \mathbf{R}$. (That is,

$$Q = \bigcap_{(a,\eta,b)\in\mathcal{F}} \{(x,r) \in \mathbf{E} \times \mathbf{R} : \langle (a,\eta),(x,r)\rangle \le b\},$$

where $\mathcal{F} = \{(a,\eta,b) \in \mathbf{E} \times \mathbf{R} \times \mathbf{R} : \langle (a,\eta),(x,r)\rangle \le b \text{ for all } (x,r) \in \mathrm{epi}\, f\}$.) Observe that one of the halfspaces in this representation must be nonvertical ("Nonvertical" here means that the last coordinate of (a,η) in the intersecting representation of Q, which is exactly η in this case, is nonzero. The terminology "vertical" comes from the fact that when $\eta = 0$, the last coordinate of (x,r), which is exactly r in this case, can take any value in $(-\infty,\infty)$. The hyperplane that defines the halfspace, $\{(x,r) \in \mathbf{E} \times \mathbf{R} : \langle (a,0),(x,r)\rangle = b\}$, is parallel to the last coordinate axis and vertical to the spaces $\{(x,r) : r = \text{constant}\}$.); otherwise, Q would be a union of vertical lines (or an intersection of vertical halfspaces), thereby contradicting that $\bar{\mathrm{co}}\, f$ is proper.
Let us write this nonvertical halfspace as the epigraph of an affine minorant g_1 of f; we will use this function shortly.

Let $h(\cdot)$ be the function defined on the right-hand side of (1). We will show that $(\bar{\mathrm{co}}\, f)$ and h have the same epigraphs. Since h is a pointwise supremum, we may write $\mathrm{epi}\, h$ as an intersection of halfspaces:

$$\mathrm{epi}\, h = \bigcap\{\mathrm{epi}\, g : g : \mathbf{E} \to \bar{\mathbf{R}} \text{ is an affine minorant of } f\}. \tag{2}$$

In particular, the inclusion $Q \subset \mathrm{epi}\, h$ clearly holds. (This is because Q can be written as an intersection of halfspaces that contain Q itself, as mentioned earlier.) Suppose now for the sake of contradiction that there exists a point $(\bar{x},\bar{r}) \in \mathrm{epi}\, h$ that is not in Q. The separation theorem (Theorem 2.19 in the Lecture Notes) yields $(a,\mu) \in \mathbf{E} \times \mathbf{R}$ and $b \in \mathbf{R}$ such that the halfspace

$$H = \{(x,r) : \langle (a,\mu),(x,r)\rangle \le b\}$$

contains Q and does not contain $(\bar{x},\bar{r})$. By the nature of epigraphs, the inequality $\mu \le 0$ holds. (Note that $(x,+\infty) \in \operatorname{epi} f$. If $\mu > 0$, then halfspace representation implies that $\langle (a,\mu),(x,+\infty)\rangle = \langle a,x\rangle + \infty \le b$, i.e., $\langle a,x\rangle \le -\infty$ for this specific x. This contradicts the fact that we can take x to be a point in $\mathbf{E}$ with a finite value.) If $\mu < 0$, then H is nonvertical, thereby contradicting the definition of h. (The contradiction lies in the fact that when H is a nonvertical halfspace that contains Q, it naturally defines an affine minorant of f. Hence, by (2), $\operatorname{epi} h \subset H$ and thus $(\bar{x},\bar{r}) \in \operatorname{epi} h \subset H$, contradicting to the separation property of H.) Thus, we may assume $\mu = 0$. The strategy now is to perturb H by using g_1 in order to make it nonvertical. To this end, define the function $g_2(x) := \langle a,x\rangle - b$ and observe $H = \{(x,r) : g_2(x) \le 0\}$. In particular, every point $x \in \operatorname{dom} f$ satisfies $g_2(x) \le 0$. We therefore deduce

$$\lambda g_2(x) + g_1(x) \le f(x)$$

for all points $x \in \mathbf{E}$ and any $\lambda > 0$. (Recall that g_1 is an affine minorant of f.) Thus the function $g_3(x) := \lambda g_2(x) + g_1(x)$ is another affine minorant of f. Taking into account $g_2(\bar{x}) > 0$ (Separation property of H: $(\bar{x},\bar{r}) \notin H$.), we arrive at the contradiction

$$h(\bar{x}) \ge g_3(\bar{x}) = \lambda g_2(\bar{x}) + g_1(\bar{x}),$$

when $\lambda > 0$ is sufficiently large. (Notice that as long as $f(\bar{x})$ is finite, $h(\bar{x}) \le f(\bar{x})$ is also finite. We may take λ to be sufficiently large to induce the contradiction here.) $\square$