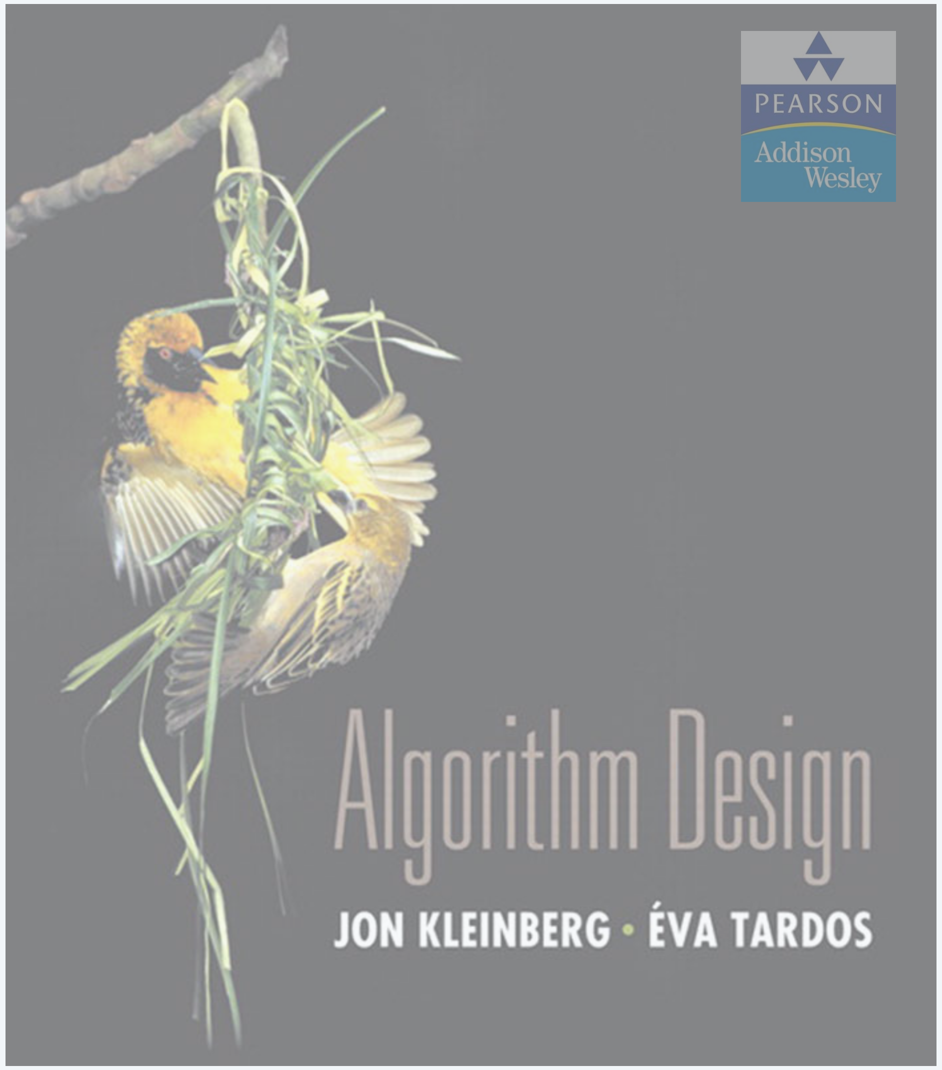


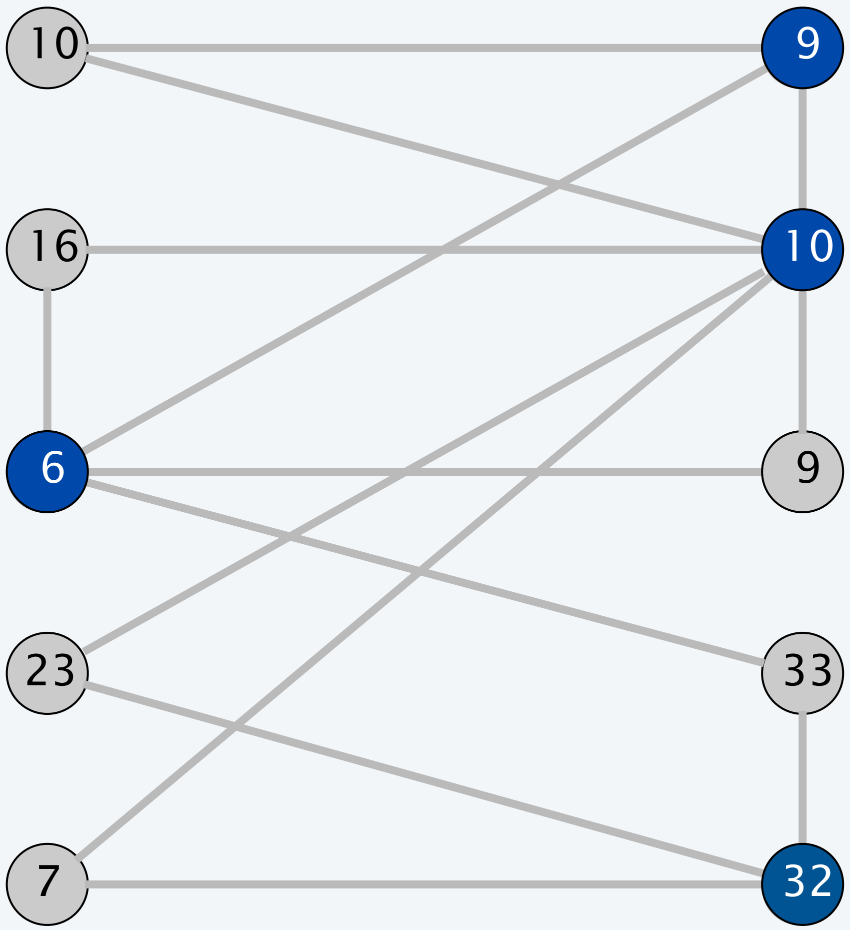
SECTION 11.6

11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Weighted vertex cover

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.



total weight = 6 + 9 + 10 + 32 = 57

Weighted vertex cover: ILP formulation

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.

Integer linear programming formulation.

- Model inclusion of each vertex i using a 0/1 variable x_i.

$$x_i = \begin{cases} 0 & \text{if vertex } i \text{ is not in vertex cover} \\ 1 & \text{if vertex } i \text{ is in vertex cover} \end{cases}$$

  Vertex covers in 1–1 correspondence with 0/1 assignments:
  $S = \{ i \in V : x_i = 1\}$.

- Objective function: minimize $\Sigma_i \, w_i \, x_i$.

- For every edge (i, j), must take either vertex i or j (or both): $x_i + x_j \geq 1$.

Weighted vertex cover: ILP formulation

Weighted vertex cover. Integer linear programming formulation.

$$
\begin{array}{llrcll}
(ILP) & \min & \displaystyle\sum_{i \in V} w_i \, x_i & & & \\
& \text{s.t.} & x_i + x_j & \geq & 1 & (i, j) \in E \\
& & x_i & \in & \{0,\ 1\} & i \in V
\end{array}
$$

Observation. If x^* is optimal solution to ILP, then $S = \{ i \in V : x_i^* = 1\}$ is a min-weight vertex cover.

Integer linear programming

Given integers a_{ij}, b_i, and c_j, find integers x_j that satisfy:

$$\begin{array}{rrcl} \min & c^{\mathsf{T}} x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \\ & x & & \text{integral} \end{array}$$

$$\begin{array}{rrcll} \min & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \\ & x_j & & \text{integral} & 1 \leq j \leq n \end{array}$$

Observation. Vertex cover formulation proves that INTEGER-PROGRAMMING is an **NP**-hard optimization problem.

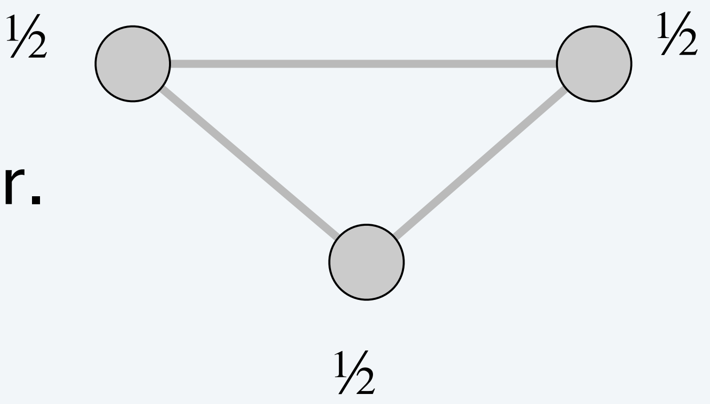
Weighted vertex cover: LP relaxation

Linear programming relaxation.

$$
\begin{array}{llrcll}
(LP) & \min & \displaystyle\sum_{i \in V} w_i \, x_i & & & \\
& \text{s.t.} & x_i + x_j & \geq & 1 & (i, j) \in E \\
& & x_i & \geq & 0 & i \in V
\end{array}
$$

Observation. Optimal value of *LP* is ≤ optimal value of *ILP*.

Pf. *LP* has fewer constraints.



Note. *LP* solution x^* may not correspond to a vertex cover.
(even if all weights are 1)

Q. How can solving *LP* help us find a low-weight vertex cover?

A. Solve *LP* and round fractional values in x^*.

Weighted vertex cover: LP rounding algorithm

Lemma. If x^* is optimal solution to LP, then $S = \{ i \in V : x_i^* \geq ½ \}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

- Consider an edge $(i, j) \in E$.
- Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq ½$ or $x_j^* \geq ½$ (or both) $\Rightarrow$ (i, j) covered.

Pf. [S has desired weight]

- Let S^* be optimal vertex cover. Then

$$\sum_{i \in S^*} w_i \underset{\text{LP is a relaxation}}{\geq} \sum_{i \in S} w_i \, x_i^* \underset{x_i^* \geq ½}{\geq} \frac{1}{2} \sum_{i \in S} w_i$$

Theorem. The rounding algorithm is a 2-approximation algorithm.

Pf. Lemma + fact that LP can be solved in poly-time.

Weighted vertex cover inapproximability

Theorem. [Dinur–Safra 2004] If **P** $\neq$ **NP**, then no ρ-approximation algorithm for WEIGHTED-VERTEX-COVER for any $\rho < 1.3606$ (even if all weights are 1).

On the Hardness of Approximating Minimum Vertex Cover

Irit Dinur* Samuel Safra†

May 26, 2004

Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.

Open research problem. Close the gap.

Weighted vertex cover inapproximability

Theorem. [Kohot–Regev 2008] If Unique Games Conjecture is true, then no $2 - \varepsilon$ approximation algorithm for WEIGHTED-VERTEX-COVER for any $\varepsilon > 0$.

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JOURNAL OF COMPUTER AND SYSTEM SCIENCES

Journal of Computer and System Sciences 74 (2008) 335–349

Vertex cover might be hard to approximate to within $2 - \varepsilon$

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Received 28 May 2003; received in revised form 25 April 2006

Available online 13 June 2007

Abstract

Based on a conjecture regarding the power of unique 2-prover-1-round games presented in [S. Khot, On the power of unique 2-Prover 1-Round games, in: Proc. 34th ACM Symp. on Theory of Computing, STOC, May 2002, pp. 767–775], we show that vertex cover is hard to approximate within any constant factor better than 2. We actually show a stronger result, namely, based on the same conjecture, vertex cover on k-uniform hypergraphs is hard to approximate within any constant factor better than k.

Keywords: Hardness of approximation; Vertex cover; Unique games conjecture

Open research problem. Prove the Unique Games Conjecture.