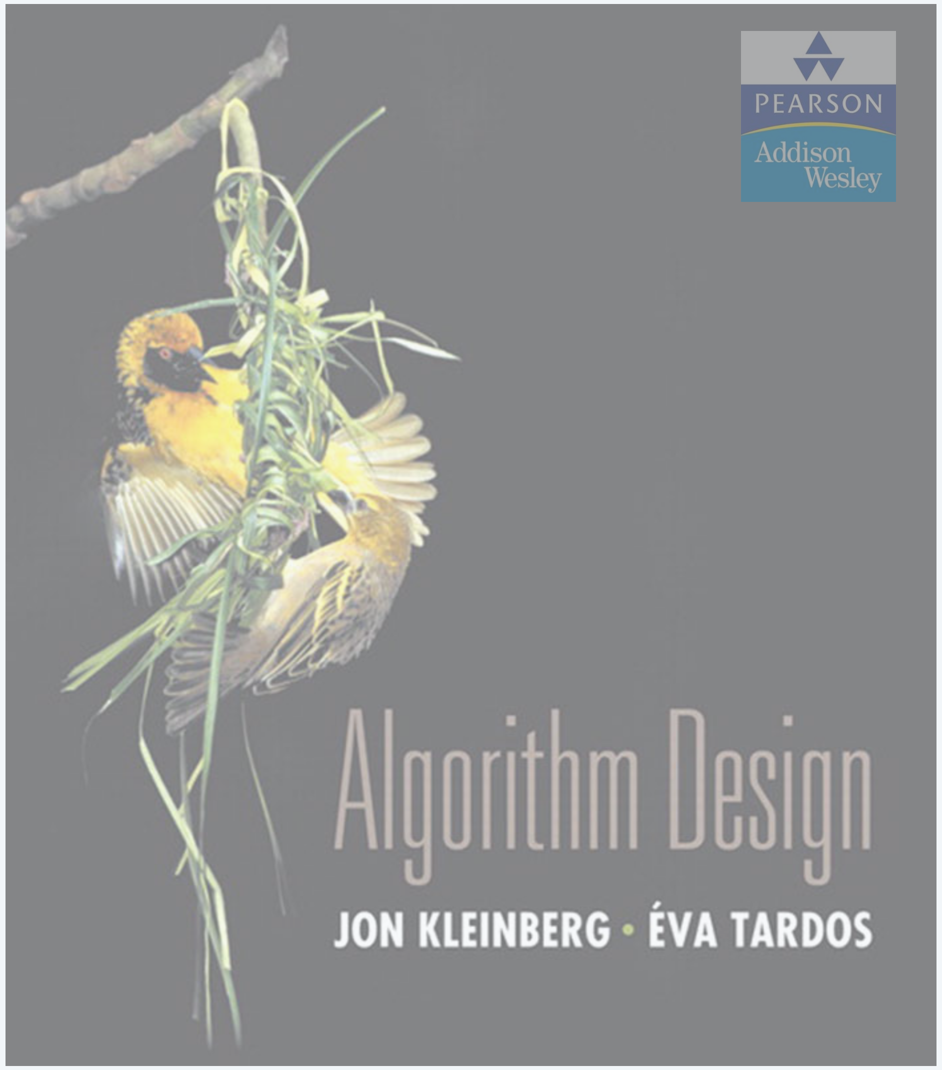


SECTION 11.7

11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Generalized load balancing

Input. Set of m machines M; set of n jobs J.

- Job $j \in J$ must run contiguously on an authorized machine in $M_j \subseteq M$.
- Job $j \in J$ has processing time t_j.
- Each machine can process at most one job at a time.

Def. Let J_i be the subset of jobs assigned to machine i.
The load of machine i is $L_i = \Sigma_{j \in J_i} t_j$.

Def. The makespan is the maximum load on any machine $= \max_i L_i$.

Generalized load balancing. Assign each job to an authorized machine to minimize makespan.

Generalized load balancing: integer linear program and relaxation

ILP formulation. x_{ij} = time machine i spends processing job j.

$$
\begin{array}{llllll}
(IP) & \min & L & & & \\
& \text{s. t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
& & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
& & x_{ij} & \in & \{0,\ t_j\} & \text{for all } j \in J \text{ and } i \in M_j \\
& & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

LP relaxation.

$$
\begin{array}{llllll}
(LP) & \min & L & & & \\
& \text{s. t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
& & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
& & x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
& & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

Generalized load balancing: lower bounds

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. Let L be optimal value to the LP. Then, optimal makespan $L^* \geq L$.

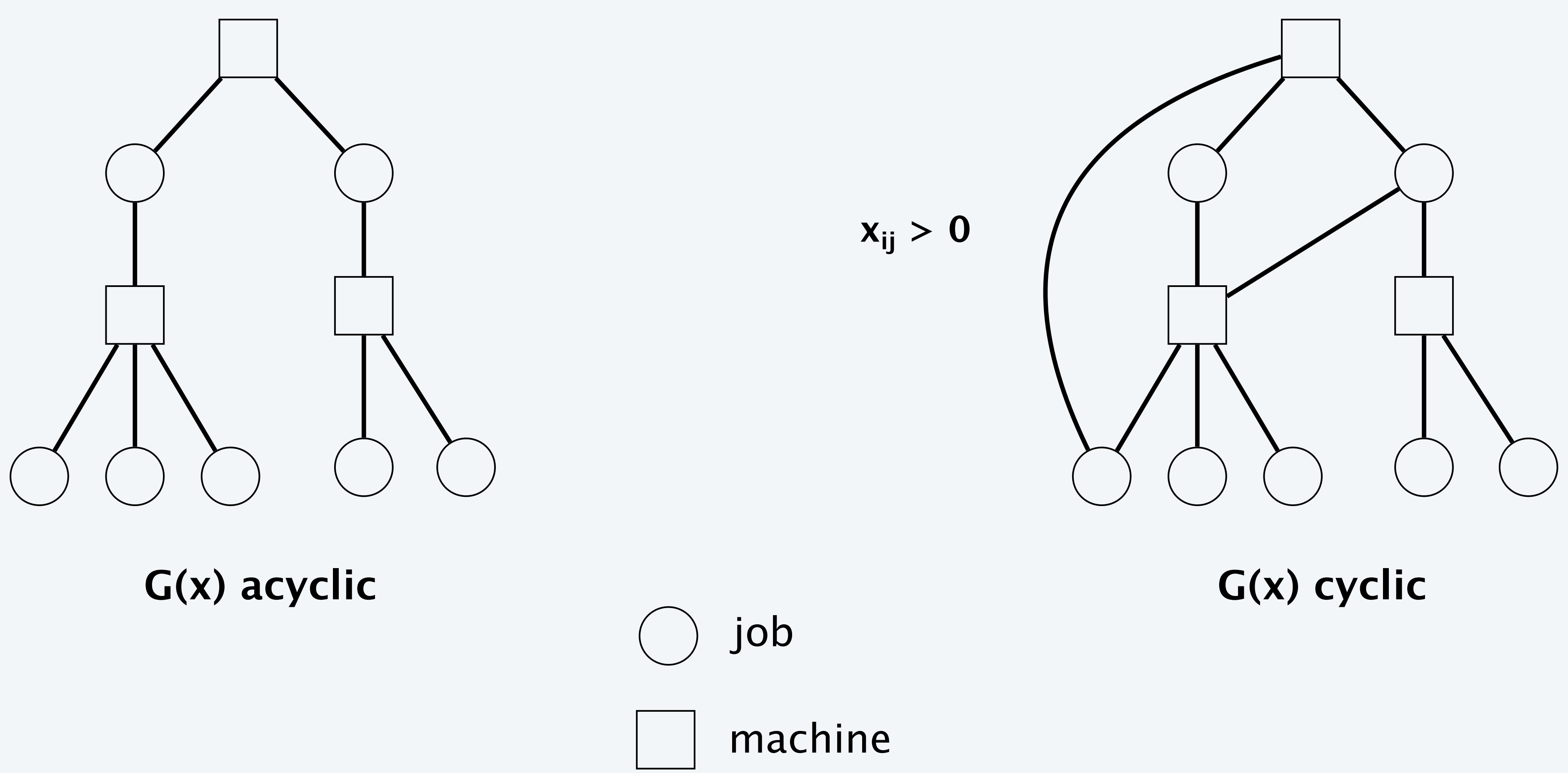
Pf. LP has fewer constraints than ILP formulation. ▪

Generalized load balancing: structure of LP solution

Lemma 3. Let x be solution to LP. Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. Then $G(x)$ is acyclic.

can transform x into another LP solution where G(x) is acyclic if LP solver doesn't return such an x

Pf. (deferred)



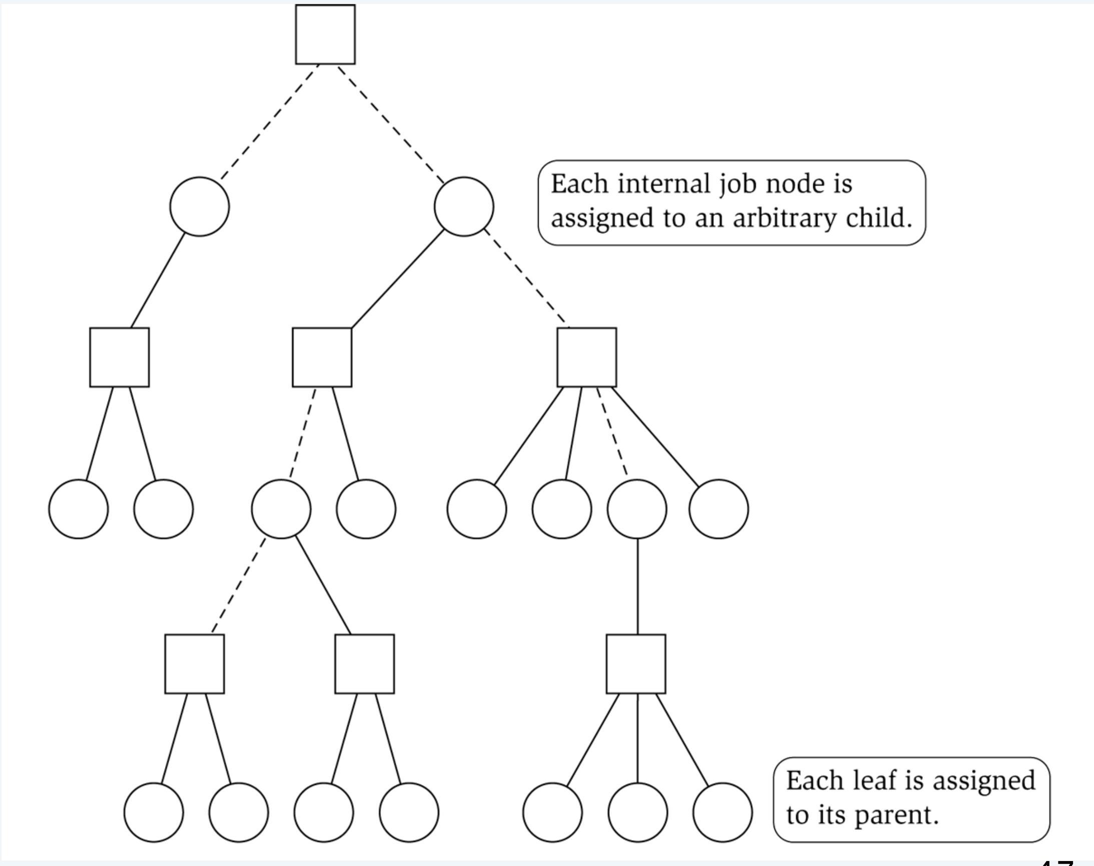
Generalized load balancing: rounding

Rounded solution. Find LP solution x where $G(x)$ is a forest. Root forest $G(x)$ at some arbitrary machine node r.

- If job j is a leaf node, assign j to its parent machine i.
- If job j is not a leaf node, assign j to any one of its children.

Lemma 4. Rounded solution only assigns jobs to authorized machines.

Pf. If job j is assigned to machine i, then $x_{ij} > 0$. LP solution can only assign positive value to authorized machines. ▪



Generalized load balancing: analysis

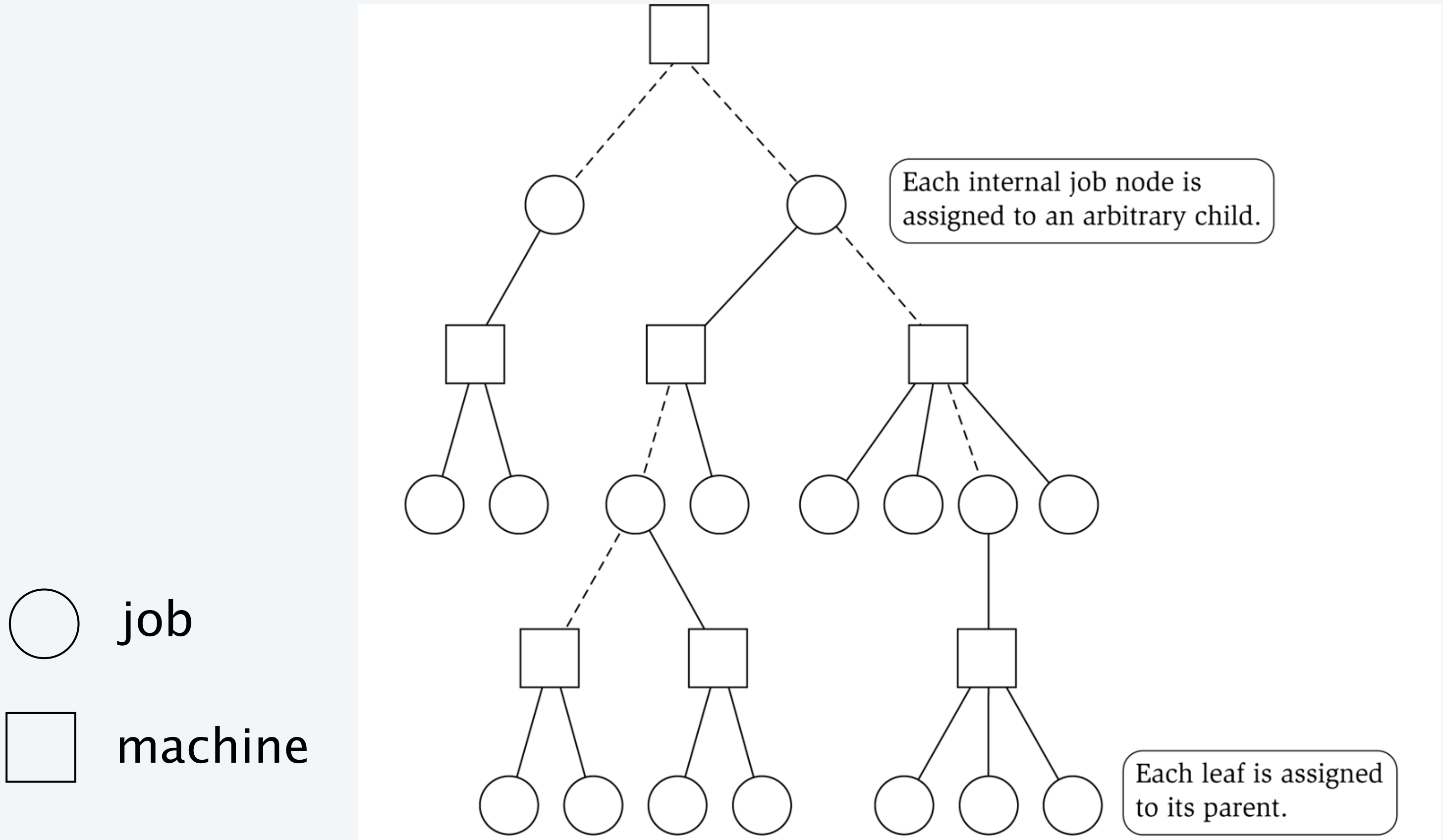
Lemma 5. If job j is a leaf node and machine $i = parent(j)$, then $x_{ij} = t_j$.

Pf.

- Since i is a leaf, $x_{ij} = 0$ for all $j \neq parent(i)$.
- LP constraint guarantees $\Sigma_i \, x_{ij} = t_j$. ▪

Lemma 6. At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to machine i is $parent(i)$. ▪



Generalized load balancing: analysis

Theorem. Rounded solution is a 2-approximation.

Pf.

- Let $J(i)$ be the jobs assigned to machine i.
- By LEMMA 6, the load L_i on machine i has two components:
  - leaf nodes:

$$\sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} t_j \overset{\text{Lemma 5}}{=} \sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} x_{ij} \le \sum_{j \in J} x_{ij} \overset{\text{LP}}{\le} \underbrace{L}_{\text{optimal value of LP}} \overset{\text{Lemma 2 (LP is a relaxation)}}{\le} L^*$$

  - parent:

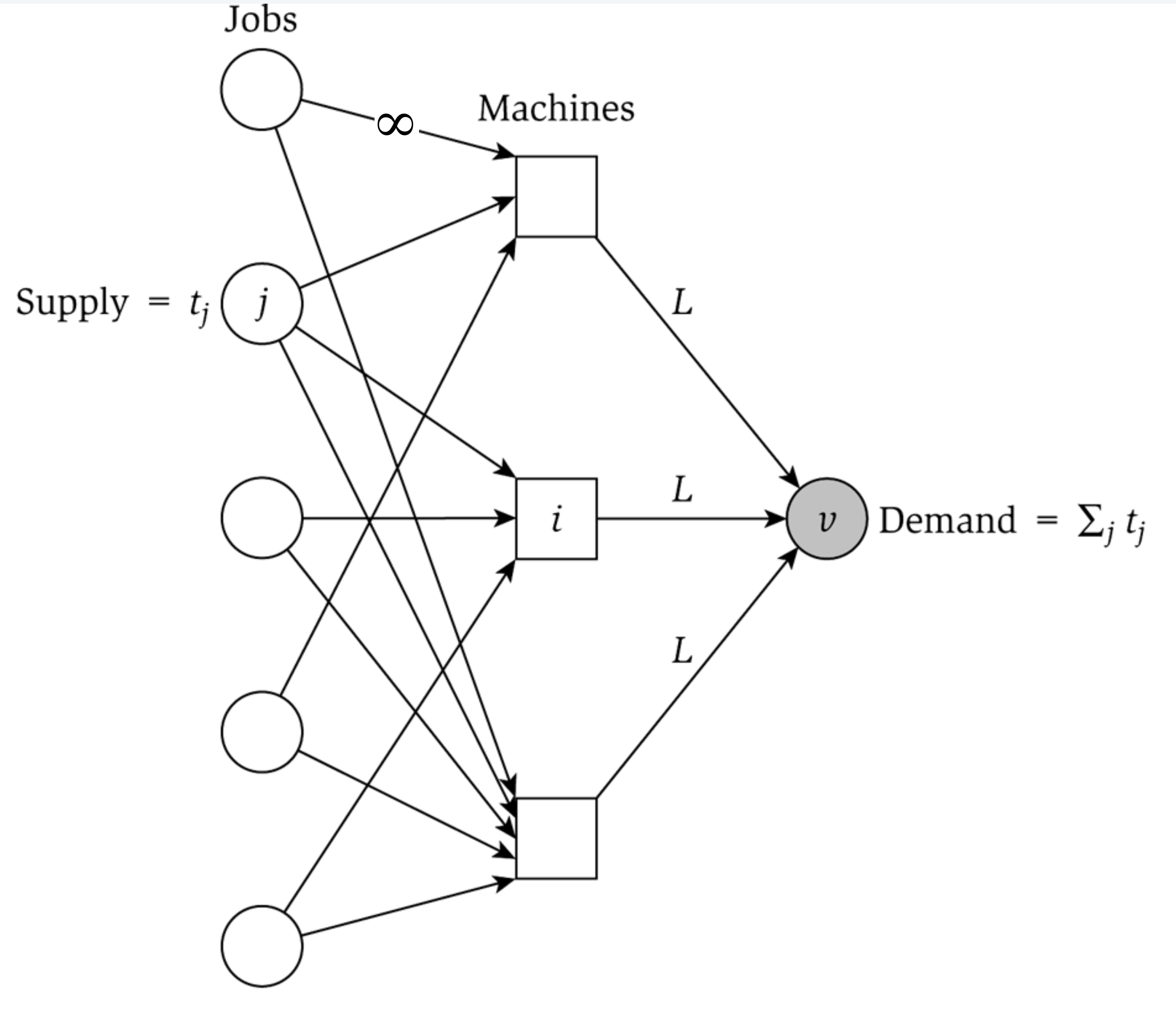
$$t_{\text{parent}(i)} \overset{\text{Lemma 1}}{\le} L^*$$

- Thus, the overall load $L_i \le 2L^*$. ▪

Generalized load balancing: flow formulation

Flow formulation of LP.

$$\begin{array}{llll}
\sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
\sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}$$



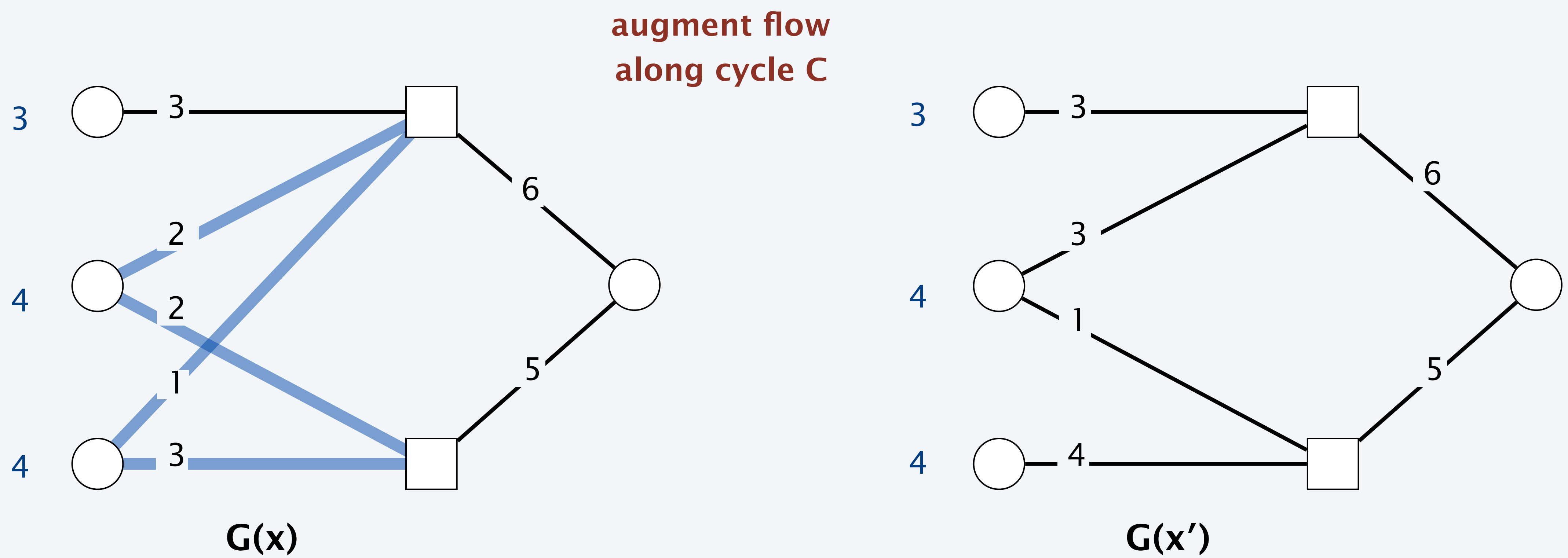
Observation. Solution to feasible flow problem with value L are in 1-to-1 correspondence with LP solutions of value L.

Generalized load balancing: structure of solution

Lemma 3. Let (x, L) be solution to LP. Let $G(x)$ be the graph with an edge from machine i to job j if $x_{ij} > 0$. We can find another solution (x', L) such that $G(x')$ is acyclic.

Pf. Let C be a cycle in $G(x)$.

- Augment flow along the cycle C. ← flow conservation maintained
- At least one edge from C is removed (and none are added).
- Repeat until $G(x')$ is acyclic. ▪



Conclusions

Running time. The bottleneck operation in our 2-approximation is solving one LP with $mn+1$ variables.

Remark. Can solve LP using flow techniques on a graph with $m+n+1$ nodes: given L, find feasible flow if it exists. Binary search to find L^*.

Extensions: unrelated parallel machines. [Lenstra–Shmoys–Tardos 1990]

- Job j takes t_{ij} time if processed on machine i.
- 2-approximation algorithm via LP rounding.
- If **P** ≠ **NP**, then no no ρ-approximation exists for any $\rho < 3/2$.

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APPROXIMATION ALGORITHMS FOR SCHEDULING UNRELATED PARALLEL MACHINES

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