SI 211: Numerical Analysis
Homework 7

Prof. Boris Houska

Deadline: Dec 2, 2020

1. Provide short answers to the following questions
   (a) What is the main idea of Newtons method?
   (b) What is the main idea of Gauss-Newton methods?
   (c) What is the local convergence rate of the exact Newton method?
   (d) Under which conditions converges Newtons method in one step?
   (e) What is the Armijo line search condition and what is it good for?

2. Consider the function $f(x) = x^3 - 2x + 2$. What happens if you apply Newton's method to the equation $f(x) = 0$ starting with $x_0 = 0$? Work out the iterates of Newton's method explicitly.

3. The goal of this exercise is to implement and exact Newton methods with Gauss-Newton methods by minimizing the nonlinear function

$$f(x,y) = \frac{1}{2}(x-1)^2 + \frac{1}{2}\left(10\left(y-x^2\right)\right)^2 + \frac{1}{2}y^2 \qquad (1)$$

   (a) Derive the gradient and Hessian matrix of the function in (1). Then, re-write it in the form $f(x,y) = \frac{1}{2}\|R(x,y)\|_2^2$, where $R : \mathbb{R}^2 \to \mathbb{R}^3$ is the residual function. Derive the Gauss-Newton Hessian approximation and compare it with the exact one. When do the two matrices coincide?
   (b) Implement your own Newton's Method with exact Hessian information and full steps. Start from the initial point $(x_0, y_0) = (-1,-1)$ and use as termination condition $\|\nabla f(x_k, y_k)\|_2 \leq 10^{-3}$. Keep track of the iterates (x_k, y_k) and plot the results.

(c) Update your code to use the Gauss-Newton Hessian approximation instead. Plot the difference between exact and approximate Hessian as a function of the iterations.

(d) Compare the performance of the implemented methods. Consider the iteration path (x_k, y_k), the number of iterations and the run time.