SI 211: Numerical Analysis Homework 2

Prof. Boris Houska

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1. Jacobi matrix
   Ans:
   The general form of Jacobi matrix is

$$\triangledown f(x) = \begin{pmatrix} 2x_1x_2 + x_2^2 & x_1^2 + 2x_1x_2 \\ 2x_1 + x_2 & x_1 \end{pmatrix}$$

   Therefore, when $x = (1,2)^T$,

$$\triangledown f(x) = \begin{pmatrix} 8 & 5 \\ 4 & 1 \end{pmatrix}$$

2. Polynomial interpolation
   Ans:
   Put these three points into the polynomial function, we can get:

$$\begin{cases} a_0 - a_1 - a_2 & = 6 \\ a_0 + 2a_1 + 8a_2 & = 12 \\ a_0 + 4a_1 + 64a_2 & = 66 \end{cases}$$

$$\therefore \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 8 \\ 1 & 4 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 66 \end{bmatrix}$$

   From the linear equation shown above, we can find that $a_0 = 6,\ a_1 = -1,\ a_2 = 1$

3. Interpolation with rational functions
Ans:
We find that, this polynomial can be rewrote as:

$$xP(x) = a_{-1} + a_0 x + a_1 x^2 + a_2 x^3$$

Then, we can use Newton polynomials to represent this equation:

$$xP(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Therefore, we can create the Divided Differences table shown below to find the parameters. The code is shown in '**HW2_3.m**' Finally, we find that:

$$\begin{array}{c|c|c|c|c} x_0 & x_0 y_0 & d_{01} & d_{02} & d_{03} \\ x_1 & x_1 y_1 & d_{12} & d_{13} & \\ x_2 & x_2 y_2 & d_{23} & & \\ x_3 & x_3 y_3 & & & \end{array}$$

$$a_{-1} = 0,\ a_0 = \frac{2}{3},\ a_1 = 4,\ a_2 = \frac{1}{3}$$

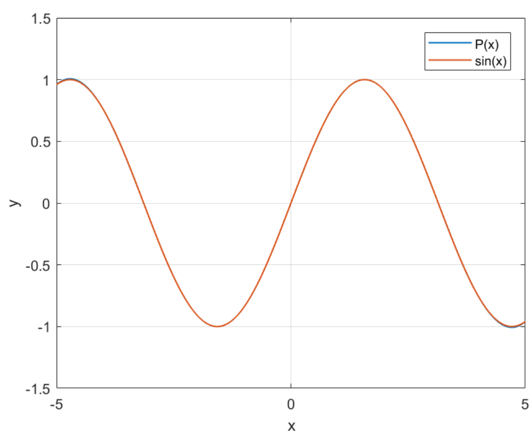
4. Hermite interpolation
Ans:
In this problem, we use Hermite interpolation to solve it. The divided difference table is shown below. The code is shown in '**HW2_4.m**'.

$$\begin{array}{c|c|cccc} a & f(a) & f'(a) & \frac{f(b)-f(a)-f'(a)(b-a)}{(b-a)^2} & \frac{2(f(a)-f(b))+(f'(a)+f'(b))(b-a)}{(b-a)^3} \\ a & f(a) & \frac{f(b)-f(a)}{b-a} & \frac{f(a)-f(b)+f'(b)(b-a)}{(b-a)^2} & \\ b & f(b) & f'(b) & & \\ b & f(b) & & & \end{array}$$

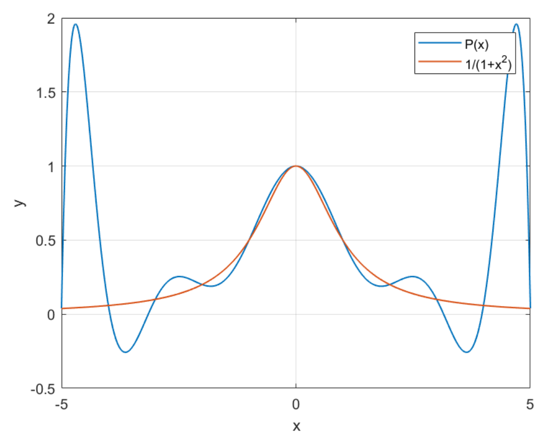
After solving this, we can get the parameters:

$$a_0 = -10,\ a_1 = 24,\ a_2 = -17,\ a_3 = 4$$
$$\therefore P(x) = 10 + 24x - 17x^2 + 4x^3$$

5. Polynomial approximation error
   Ans:
   The code is shown in '**HW2_5.m**'.
   For $sin(x)$, $|f(x)-p(x)| \leq \frac{1}{(n+1)!}\prod_{j=1}^{n}(x-x_j) \leq \frac{1}{(n+1)!}[\overline{x}-\underline{x}]^n$, therefore, in this case, $|f(x)-p(x)| \leq \frac{1}{(11)!}[5-(-5)]^{10} = 250.5211$, which is a large number. Comparing this value with the result obtained form simulation, which is shown below, we find that the error is very small in real simulation. The reasons that cause this difference is that, 1) the value is an upper bound of error, in this case, the error is much smaller than this value, because $\prod_{j=1}^{n}(x-x_j) \leq [\overline{x}-\underline{x}]^n$ is an excessive sufficient condition. 2) the upper bound error comes to 0 only when $n \to \infty$.



$sin(x)$



$\frac{1}{1+x^2}$

For $\frac{1}{1+x^2}$, we know that $|f^{(n)}(x)| \approx 2^n n! O(|x|^{-2-n})$. Since the maximum error can be represented as:

$$max|f(x)-p(x)| = \frac{1}{(n+1)!}\max_{\xi_x \in [\underline{x},\overline{x}]}\frac{\partial^{n+1}f(\xi_x)}{\partial x^{n+1}}\max_x \prod_{j=0}^{n}(x-x_j)$$
$$\approx 2^{n+1}O(|x|^{-3-n})\max_x \prod_{j=0}^{n}(x-x_j)$$

Therefore, the maximum error of $\frac{1}{1+x^2}$ will increase with the increase number of x. The polynomial figure will have large fluctuations.