SI 211: Numerical Analysis
Homework 4

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Deadline: Nov 4, 2020

1. Prove that for all $x \in \mathbb{R}^n$ the inequality

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty$$

holds.

2. Let $H, \langle \cdot, \cdot \rangle$ be a Hilbert space with norm $\|x\| = \sqrt{\langle x, x \rangle}$. Prove that $\langle x, y \rangle = 0$ if and only if we have $\|x + \alpha y\| = \|x - \alpha y\|$ for all scalars α.

3. Prove that the Legendre polynomials

$$P_n = \frac{1}{2^n n!} \frac{\partial^n}{\partial x^n} \left(x^2 - 1\right)^n$$

are orthogonal with respect to the L_2-scalar product on the interval [-1,1].

4. Solve the least-squares optimization problem

$$\min_{p \in P_2} \int_1^2 |f(x) - p(x)|^2 dx$$

for $f(x) = e^x$ by using Legendre polynomials. Here, P_2 denotes the set of polynomials of order 2.

5. Solve the least-squares optimization problem

$$\min_{p \in P_1} \int_0^\infty |f(x) - p(x)|^2 e^{-x} dx$$

for $f(x) = x^2$. Here, P_1 denotes the set of polynomials of order 1.