SI 211: Numerical Analysis Homework 1

Prof. Boris Houska

Due on Step 30, 2020, 23:59

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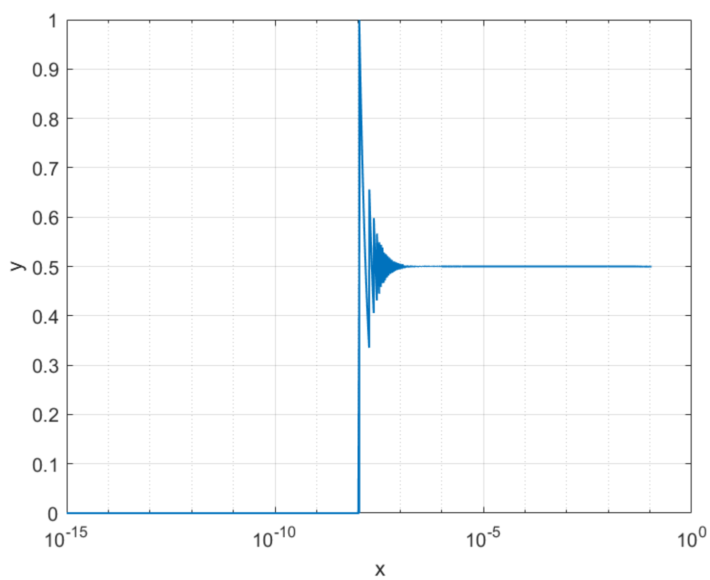
1. Floating point numbers. What is the bit representations of the floating point number 5.25 using the IEEE standard for double precision numbers? Solve this problem with pen and paper first. Use a computer program to verify your result.
   Ans:
   $\because 5.25 = 1.3125 * 4 = +(1+0.3125) * 2^2 = +(1+0.25+0.0625) * 2^2$
   $\therefore m = 0.3125, e = 2$
   $\therefore bits(5.25) =' 0100000000010101000000000000000000000000000000000000000000000000'$
   This string can be obtained by '**bitstring(5.25)**' in Julia.

2. Numerical evaluation error. Evaluate the function

$$f(x) = \frac{1-\cos(x)}{x^2}$$

   with a compute program of your choice using the standard IEEE double precision floating point format. Plot the numerical result on a logarithmic scale for $x \in [10^{-15}, 10^{-1}]$.
   Ans: The code is in '**HW1_2.m**', and the figure is shown below.



3. Taylor expansion. Consider again the function

$$f(x) = \frac{1 - \cos(x)}{x^2}$$

from the first homework problem. Can you approximate f by using a Taylor approximation? Does this help you to evaluate f with higher accuracy?

Ans:

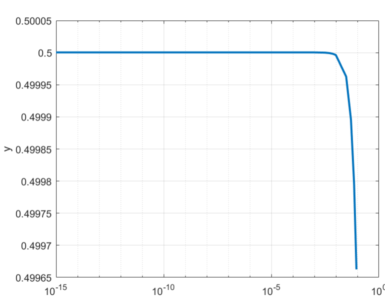
The derivatives of f:

$$f'(x) = \frac{x^2 sin(x) - 2x + 2x cos(x)}{x^4}$$
$$f''(x) = \frac{-4x sin(x) + x^2 cos(x) - 6 cos(x) + 6}{x^4}$$

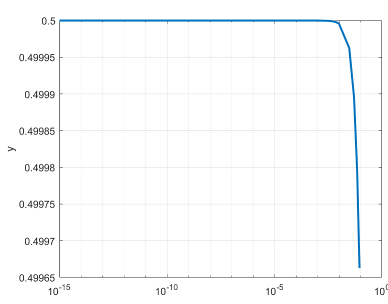
Therefore, we let the Taylor approximation be:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x)(x - x_0)^2$$

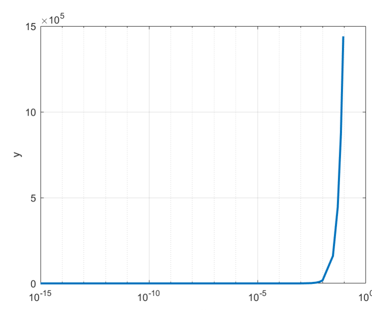
The code is in '**HW1_3.m**'. After using Taylor approximation, we find that the stability in small x is better than its in Problem 2. However, the accuracy of f highly depends on the choice of the base point x_0 in Taylor expansion. We've plot some figures of f with different x_0 shown below.



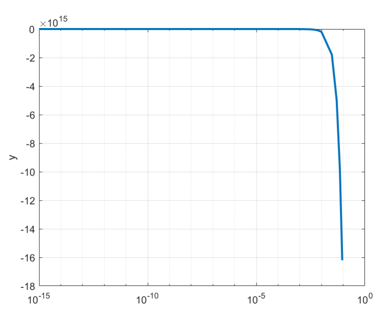
(a) $x0 = 1e^{-1}$



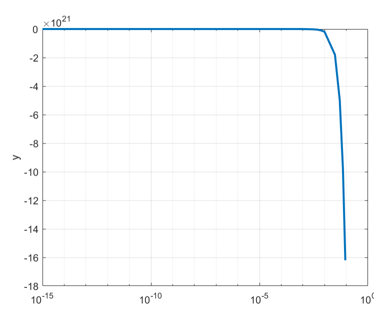
(b) $x0 = 1e^{-3}$



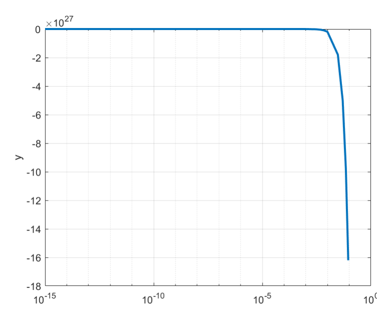
(c) $x0 = 1e^{-6}$



(d) $x0 = 1e^{-9}$



(e) $x0 = 1e^{-12}$



(f) $x0 = 1e^{-15}$

4. Numerical differentiation. Consider the five-point differentiation formula

$$f'(x) \approx \frac{1}{12h}\left[-25f\left(x_0\right)+48f\left(x_0+h\right)-36f\left(x_0+2h\right)+16f\left(x_0+3h\right)-3f\left(x_0+4h\right)\right]$$

(a) What is the mathematical approximation error of this formula?
Ans: By the 4^{th} order Taylor expansion, we can find:

$$\begin{aligned}
f(x_0) &= f(x_0) \\
f(x_0+h) &= f(x_0)+f'(x_0)h+\frac{1}{2}f''(x_0)h^2+\frac{1}{6}f'''(x_0)h^3+\frac{1}{24}f^{(4)}(x_0)h^4+O(h^4) \\
f(x_0+2h) &= f(x_0)+2f'(x_0)h+2f''(x_0)h^2+\frac{4}{3}f'''(x_0)h^3+\frac{2}{3}f^{(4)}(x_0)h^4+O(h^4) \\
f(x_0+3h) &= f(x_0)+3f'(x_0)h+\frac{9}{2}f''(x_0)h^2+\frac{9}{2}f'''(x_0)h^3+\frac{27}{8}f^{(4)}(x_0)h^4+O(h^4) \\
f(x_0+4h) &= f(x_0)+4f'(x_0)h+8f''(x_0)h^2+\frac{32}{3}f'''(x_0)h^3+\frac{32}{3}f^{(4)}(x_0)h^4+O(h^4)
\end{aligned}$$

Therefore, put those equations above into the right side of five-point differentiation formula, we can find that the right side is equal to:

$$right\ side = f'(x)+O(h^4)$$

The mathematical approximation error is $O(h^4)$.

(b) For which values of h would you expect that this formula leads to a minimum approximation error taking both the mathematical as well as the numerical approximation error into account?
Ans: The approximation error is:

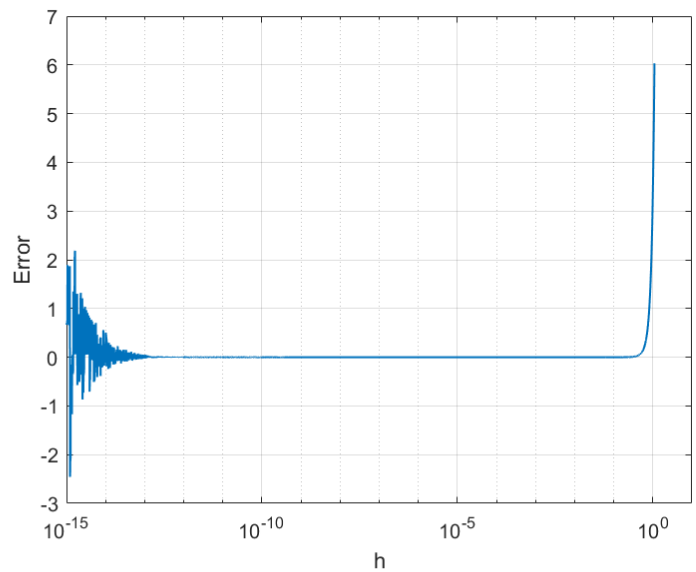
$$approximation\ error = O(h^4+\frac{eps}{h})$$

Therefore, in order to minimum the approximation error, we need to:

$$(h^4+\frac{eps}{h})' = 4h^3-\frac{eps}{h^2} = 0$$
$$\therefore h = \sqrt[5]{eps}$$

(c) Implement the above differentiation formula in a compute program of your choice and use it to find the derivative of the test function $f(x) = e^x$ at $x = 1$. Plot the total derivative evaluation error as a function of h and interpret your results.

Ans: The code is in '**HW1_4.m**'. From figures below, we find that the minimum error can be

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The value of h with the minimum approximation error:     7.580000e-04
The value of h derived by us:    7.400960e-04
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obtained around our derived value. When h is too small, the numerical error will be large. When h is too small, the h will not suitable for approximate derivative.