# SI 211: Numerical Analysis
# Homework 5

Prof. Boris Houska

Deadline: Nov 4, 2020

1. The goal of this exercise is to compare different methods for approximating the integral
$$\int_0^4 e^x \, \mathrm{d}x \; .$$
For this aim, we first write the integral in the form
$$\int_0^4 e^x \, \mathrm{d}x = \sum_{i=0}^{N-1} \left\{ \int_{i/N}^{4(i+1)/N} e^x \, \mathrm{d}x \right\} \; ,$$
then apply Simpson's rule on each of the integrals separately, and sum up the result.

   (a) Plot the actual error of the integral approximation versus $N$ for $N \in \{\cancel{0,} 1, 2, \ldots, 100\}$.

   (b) Derive a theoretical bound on the integral approximation in dependence on $N$ and plot this upper bound, too.

2. Implement and compare the results of the closed Newton-Cotes formulas for $n = 3$ and $n = 5$ when approximating the integral
$$\int_0^{\pi/4} \sin(x) \, \mathrm{d}x = 1 - \sqrt{2}/2 \; .$$

3. The exact value of the integral
$$I(\omega) = \int_0^{\frac{\pi}{4}} \cos(\omega x) dx$$

is given by $I(\omega) = \frac{1}{\omega}\sin(\omega x)$ for any $\omega > 0$ In the following, we test how accurate a Gauss-Quadrature of the form

$$I_1(\omega) = \sum_{i=0}^{1} \alpha_i \cos(\omega x_i)$$

can approximate this integral. Explain how to compute the approximation $I_1(\omega) \approx I(\omega)$. You may use that the second order Legendre polynomial of order 2 on the interval [-1,1] has roots at $\pm\sqrt{\frac{1}{3}}$. How large is the approximation error $|I(1) - I_1(1)|$? What happens for large $\omega$? Plot your result.

4. We would like to develop a new numerical integration formula by passing through the following steps:

   (a) compute coefficients $c_1, c_2, c_3$ such that

   $$\forall i \in \{1,2,3\}, \qquad f(x_i) = c_1 + c_2\sin(x_i) + c_3\cos(x_i)$$

   for $x_1 = a$, $x_2 = \frac{a+b}{2}$, and $x_3 = b$. You may assume $b > a$ as well as $b - a < \frac{\pi}{2}$.

   (b) Derive an integral approximation of the form

   $$\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b \left[\, c_1 + c_2\sin(x) + c_3\cos(x)\,\right]\,\mathrm{d}x$$

   by working out an explicit expression for the integral on the right side.

   (c) Combine the above two results to show that the final numerical integration formula can be written in the form

   $$\int_a^b f(x)\,\mathrm{d}x \approx \alpha_1 f(a) + \alpha_2 f\left(\frac{a+b}{2}\right) + \alpha_3 f(b)\ .$$

   What are the coefficients $\alpha_1, \alpha_2$, and $\alpha_3$ ?

   Compare the above integration formula with Simpson's formula for the integrals

   $$\int_0^{0.5} \sin\left(\frac{9}{10}x\right)\,\mathrm{d}x \qquad \int_0^1 x^3\,\mathrm{d}x \qquad \text{and} \qquad \int_0^1 \cos(x)\,\mathrm{d}x$$

   Which integration formula is better? Discuss advantages and disadvantages.