SI 211: Numerical Analysis Homework 4

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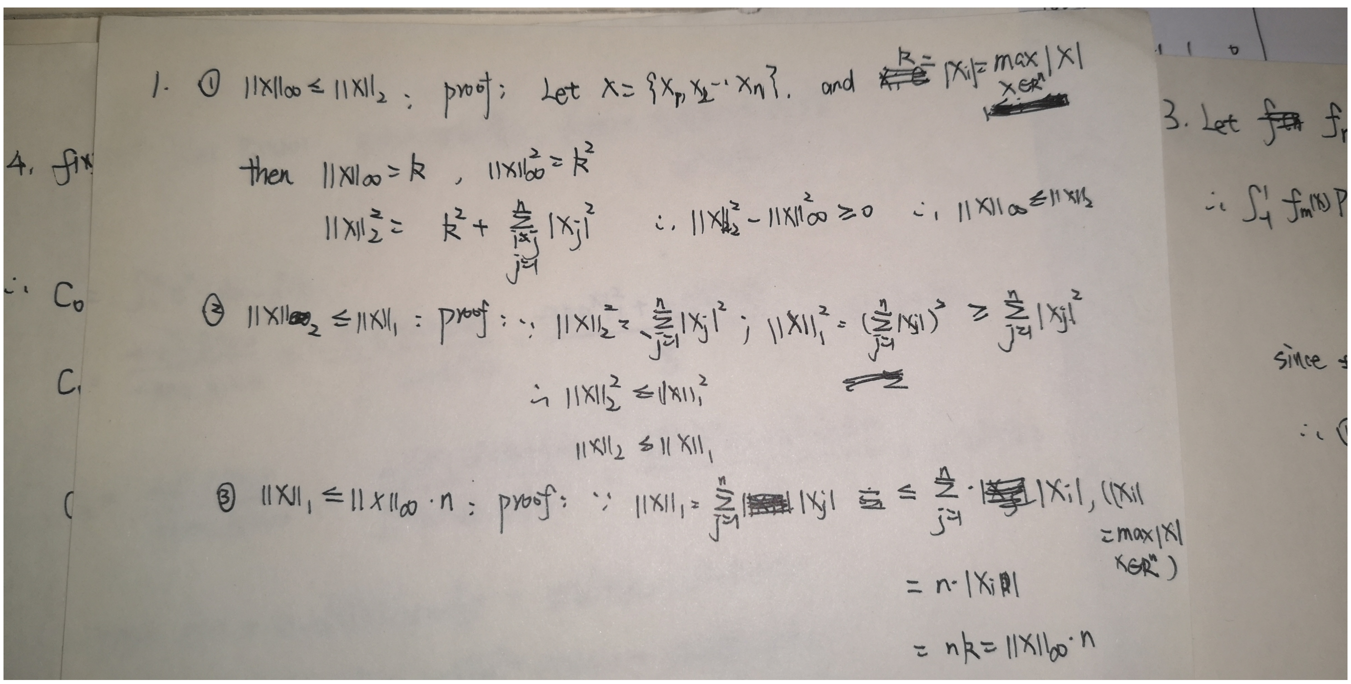
1. P1
   Ans:



1. ① $\|x\|_\infty \le \|x\|_2$: proof: Let $x = \{x_1, x_2, \cdots x_n\}$. and $k = |x_i| = \max_{x\in\mathbb{R}^n} |x|$

then $\|x\|_\infty = k$, $\|x\|_\infty^2 = k^2$

$\|x\|_2^2 = k^2 + \sum_{j=1, j\ne i}^{n} |x_j|^2$ ∴ $\|x\|_2^2 - \|x\|_\infty^2 \ge 0$ ∴ $\|x\|_\infty \le \|x\|_2$

② $\|x\|_2 \le \|x\|_1$: proof: ∵ $\|x\|_2^2 = \sum_{j=1}^{n} |x_j|^2$; $\|x\|_1^2 = (\sum_{j=1}^{n} |x_j|)^2 \ge \sum_{j=1}^{n} |x_j|^2$

∴ $\|x\|_2^2 \le \|x\|_1^2$

$\|x\|_2 \le \|x\|_1$

③ $\|x\|_1 \le \|x\|_\infty \cdot n$: proof: ∵ $\|x\|_1 = \sum_{j=1}^{n} |x_j| \le \sum_{j=1}^{n} |x_i|$, ($|x_i| = \max_{x\in\mathbb{R}^n} |x|$)

$= n \cdot |x_i|$

$= nk = \|x\|_\infty \cdot n$

2. P2

Ans:

I.

2. if $\langle x, y\rangle = 0$, then $\langle x, \alpha y\rangle = 0$, $\langle x, -\alpha y\rangle = 0$

$\therefore$ and $\|x+\alpha y\|^2 = \langle x+\alpha y, x+\alpha y\rangle = \langle x+\alpha y, x\rangle + \langle x+\alpha y, \alpha y\rangle$

$$= \langle x, x\rangle + \alpha^2\langle y, y\rangle + 2\alpha\langle x, y\rangle$$

$$= \langle x, x\rangle + \alpha^2\langle y, y\rangle$$

similarly, $\|x-\alpha y\|^2 = \langle x, x\rangle + (-\alpha)^2\langle y, y\rangle = \langle x, x\rangle + \alpha^2\langle y, y\rangle = \|x+\alpha y\|^2$

$\therefore \|x-\alpha y\| = \|x+\alpha y\|$

if $\|x+\alpha y\| = \|x-\alpha y\|$ then $\|x+\alpha y\|^2 = \|x-\alpha y\|^2$

$\therefore \langle x, x\rangle + 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle = \langle x, x\rangle + (-2\alpha)\cdot\langle x, y\rangle + \alpha^2\langle y, y\rangle$

$\therefore 4\alpha\langle x, y\rangle = 0$ $\quad\therefore \langle x, y\rangle = 0$ for $\alpha \neq 0$

↳ note that α must be non-zero because if $\alpha = 0$, we get $\|x\| = \|x\|$ is an invalid condition

3. P3

   Ans:

3. Let $f_m(x) = a_0 + a_1 x + \cdots + a_m x^m$, $m<n$ then $f(m) = \sum_{j=0}^{m} \beta_j P_j(x)$

$\therefore \int_{-1}^{1} f_m(x) P_n(x)\, dx = \frac{1}{2^n n!} \cdot \int_{-1}^{1} f_m(x) \frac{d^n}{dx^n}(x^2-1)^n dx$

$= \frac{1}{2^n \cdot n!} \cdot f_m(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \Big|_{-1}^{1} - \frac{1}{2^n n!} \cdot \int_{-1}^{1} f_m'(x) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$ --- ①

since ± 1 are the zeros of $\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n$

$\therefore$ ① $= -\frac{1}{2^n n!} \cdot \int_{-1}^{1} f_m'(x) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$ ~ ②

similarly,

② $= (-1)^m \cdot \frac{a_m \cdot m!}{2^n \cdot n!} \cdot \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n dx$

$= (-1)^m \cdot \frac{a_m \cdot m!}{2^n \cdot n!} \cdot \frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)\Big|_{-1}^{1} = 0$

$\therefore$ for $\forall f_m(x)$, $m<n$, we have $\langle f_m(x), P_n(x)\rangle = 0$

$\therefore$ $\langle P_0, P_n\rangle = 0$, $\langle P_1, P_n\rangle = 0$ ... $\langle P_{n-1}, P_n\rangle = 0 \Rightarrow$ q.e.d.

4. P4

Ans:

4. $f(x) = e^x$, Let $P_0(x) = 1$, $P_1(x) = 2(x - \frac{3}{2})$, $P_2(x) = \frac{1}{2} \cdot [3 \cdot (2x-3)^2 - 1]$

$= 6x^2 - 18x + 13$

$\therefore C_0 = \int_1^2 e^x \cdot 1\, dx = e^2 - e$

$$C_1 = \frac{\langle e^x, P_1(x) \rangle}{\langle P_1(x), P_1(x) \rangle} = \frac{\int_1^2 e^x (2x-3)\, dx}{\int_1^2 (2x-3)^2\, dx} = \frac{-3e^x|_1^2 + 2\int_1^2 e^x \cdot x\, dx}{\frac{1}{3}} = 3(-e^2 + 3e)$$

$$C_2 = \frac{\langle e^x, P_2(x) \rangle}{\langle P_2(x), P_2(x) \rangle} = \frac{\int_1^2 e^x (6x^2 - 18x + 13)\, dx}{\int_1^2 (6x^2 - 18x + 13)^2\, dx} = \frac{6\int_1^2 x^2 e^x\, dx - 18e^2 + 13(e^2 - e)}{\frac{1}{5}} = 5(7e^2 - 19e)$$

$$\therefore p(x) = e^2 - e + 3(-e^2 + 3e) \cdot 2(x - \tfrac{3}{2}) + 5(7e^2 - 19e) \cdot (6x^2 - 18x + 13)$$

$$= 30(7e^2 - 19e) \cdot x^2 + (-636e^2 + 1728e)x + \cancel{(1466e^2 - 1264e)}$$

$$+ 1465e^2 - 1263e$$

5. P5

Ans:

5. Let $I = \left| f(x) e^{-\frac{1}{2}x} - p(x) e^{-\frac{1}{2}x} g \right|$, define $\langle f, p \rangle = \int_0^\infty f(x) p(x) e^{-x} dx$

let $\cancel{\tilde{g}_0 = a_0 = 1,\ a_1 = x}$ $a_0(x) = 1$, $a_1(x) = x$

$\tilde{g}_0 = a_0 \quad \therefore \langle \tilde{g}_0, \tilde{g}_0 \rangle = \int_0^\infty 1^2 \cdot e^{-x} dx = -e^{-x}\Big|_0^\infty = 1 \quad \therefore g_0 = \frac{\tilde{g}_0}{\|\tilde{g}_0\|} = 1$

$\tilde{g}_1 = a_1 - \langle a_1, g_0 \rangle g_0 = x - \int_0^\infty x \cdot e^{-x} dx \cdot 1 = x + x e^{-x}\Big|_0^\infty - \int_0^\infty e^{-x} dx = x - 1$

$\langle \tilde{g}_1, \tilde{g}_1 \rangle = \int_0^\infty (x-1)^2 e^{-x} dx = \int_0^\infty (x^2 - 2x + 1) e^{-x} dx = -2 + 1 \cdot \int_0^\infty x^2 e^{-x} dx = -2 + 1 + 2 = 1$

$\therefore g_1 = \frac{\tilde{g}_1}{\|\tilde{g}_1\|} = x - 1$

$\therefore c_0 = \langle g_0, f \rangle = \int_0^\infty x^2 e^{-x} dx = 2 \; ; \; c_1 = \langle g_1, f \rangle = \int_0^\infty x^2 (x-1) e^{-x} dx$

$= \int_0^\infty x^3 e^{-x} dx - 2$

$= 0 - \int_0^\infty x^3 \, de^{-x} - 2$

$= -x^3 e^{-x}\Big|_0^\infty + \int_0^\infty 3x^2 \cdot e^{-x} dx - 2$

$= 4$

$\therefore p = c_0 g_0 + c_1 g_1 = 4x - 2$