Linear System Report

Chengrui Zhang
School of Information Science and Technology
ShanghaiTech University, Shanghai, P. R. China
E-mail: {zhangchr}@shanghaitech.edu.cn

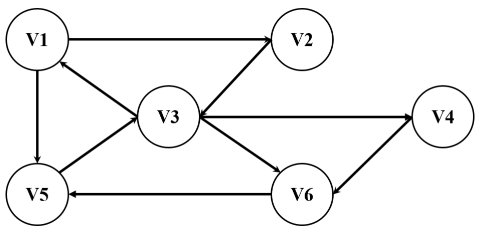


Fig. 1. Example: strongly directed network

I. INTRODUCTION

This report mainly focus on the interval consensus problem. In the standard consensus problem, the goal is to achieve an agreement among a group of agents in a distributed manner, for example, the formation control of mobile robots[1]. However, traditional consensus has limitation that an agent has no authority to veto certain values of consensus. Therefore, the constrained consensus problem has been proposed to ensure that every agent has the ability to control the consensus value. There are several approaches have been presented to fix this problem, like optimization algorithms, projection-based methods etc. to achieve consensus while respecting the positivity of the state variables.

In this report, we mainly focus on the constrained problem under the condition of deterministic network[2] and random network[3].

II. CONSTRAINED INTERVAL CONSENSUS OVER DETERMINISTIC NETWORKS

A. Overview

The goal of constrained consensus problem is to give each agent a possibility of limiting the interval of values in which a consensus value can be accepted. This paper[2] transforms this problem into interval consensus, and tries to analysis that under which conditions, the interval consensus can reach the goal.

To start with, they assume that the deterministic network is directed and strongly connected, which means that every two interaction points of this network have at least one directed path to reach each other, as shown in Fig. 1. In addition, every interaction point has an interval to constrain the transmitting value. Then, they model the constrained consensus problem into:

$$\dot{x} = -\Delta x + A\psi(x) \tag{1}$$

where A denotes the matrix abstracted from the strongly connected network, Δ denotes the input weight matrix, and $\psi(x) = [\psi_1(x_1)...\psi_n(x_n)]^T$, with:

$$\psi_i(x_i) = \begin{cases} p_i, \ if x_i < p_i \\ x_i, \ if p_i \leq x_i \leq q_i, i = 1, ..., n \\ q_i, \ if x_i > 1_i \end{cases} \tag{2}$$

After that, they propose a theorem that, if this system satisfy the assumption mentioned above, then the saturation $\psi_i(x_i)$ will force the consensus algorithm to take value in the intersection of the intervals $[p_i, q_i]$. Let $p_i = max_i \ p_i, q_i = max_i \ q_i$. Then, for any initial value x^0, there is $c^*(x^0) \in [p_*, q_*]$ and $lim_{t\to\infty} x_i(t) = c^*$ for all i. The proof is shown in II.B.

B. Derivation

The derivation can be divided into 4 steps, *Step 1–4*. *Step 1,2* utilize the *Lasalle* principle to prove that all those interaction points will converge to inverval $[p_*, q_*]$. *Step 3,4* prove the consensus that all the interaction points will converge to one value. From the above scheme, the interval consensus theorem has been proved and satisfy the goal.

Step 1. They introduce $H(x(t)) = max\{max_i \ x_i(t), q_*\}$. Clearly H is continuous and locally Lipschitz. If $H(x(t)) > q_*$, then $H(x(t)) = max_i \ x_i(t)$. Therefore:

$$D^+ H(x(t)) = D^+ max_i \ x_i(t) = max_i \dot{x}_i(t) \tag{3}$$

$$= max_i[-\sum_j a_{ij}(x_i(t) - \psi_i(x_j(t)))] \tag{4}$$

Then we can find that: (i). $\psi_j(x_j(t)) \leq x_j(t) \ if x_j(t) > q_*$; (ii). $\psi_j(x_j(t)) \leq q_* \ if x_j(t) \leq q_*$. Combine the two cases, they find that $x_i(t) - \psi_j(x_j(t)) \geq 0$ since $x_i(t) > q_*$. Therefore $D^+ H(x(t)) \leq 0 \ if H(x(t)) > q_*$, which further assures that if $H(x(t_*)) = q_*$, then $H(x(t)) = q_*$ for all $t \geq t_*$. $H(x(t))$ is non-increasing function for all t. Similarly, define $h(x(t)) = min\{min_i \ x_i(t), p_*\}$, and $h(x(t))$ is a non-decreasing function for all t. Let $V(x(t)) = H(x(t)) - h(x(t))$, then $D^+ V(x(t)) \leq 0$.

Step 2. Denote $Z = \{x : D^+ V(x) = 0\}$, this step shows that $Z \in [p_*, q_*]$ when Graph G_A is strongly connected.

They use a contradiction argument. Let $x^* = (x_1^*...x_n^*)^T \in Z$ with $x_i^* \notin [p_*, q_*]$. Then there must exist a $x_i^* \notin [p_*, q_*]$. By symmetry they assume $x_i^* > q_*$, and consider a solution $x(t)$ of (1) with $x(0) = x^*$. Then, since the G_A is strongly connected, x_i^* will be attracted by other nodes which values is strictly smaller than x_i^*. Therefore, there is $\epsilon > 0$ such that

$x_j(\epsilon) < x_i^*$ for all j. This is to say,$H(x(\epsilon)) < H(x(0))$ and therefore the trajectory cannot be within Z. We have proved $Z \in [p_*, q_*]$.

Now, by the *Lassale* principle, there holds:

$$x(t) \to [p_*, q_*] \tag{5}$$

as $t \to \infty$.

Step 3. By (5), for any $\delta > 0$, there is $T(\delta) > 0$ such that along (1), there holds $|x_i(t) - \psi_i(x_i(t))| \leq \delta$ for all $t \geq T$. Therefore, they rewrite (1) as:

$$\dot{x}_i = -\sum_j a_{ij}(x_i - x_j) + w_i(t) \tag{6}$$

where $w_i(t) = a_{ij}(\psi(x_j) - x_j)$, and conclude that $|w_i(t)| \leq \delta$ for $t \geq T(\delta)$. Then, they invoking Lemma 4 shown in Fig. 2, Therefore, they get that:

Lemma 4 *Consider the following consensus dynamics:*

$$\dot{x}_i = \sum_j a_{ij}(x_j - x_i) + w_i(t),\ i = 1, \ldots, n.$$

Let the induced graph $\mathcal{G}_A$ of $A = [a_{ij}]$ contain a directed spanning tree. Then for any $\epsilon > 0$, there exist $\delta > 0$ such that

$$\limsup_{t \to +\infty} \max_{i,j} |x_i(t) - x_j(t)| \leq \epsilon$$

for all initial value $x(t_0)$ when $\|w(t)\|_\infty \leq \delta$, where $\|w(t)\|_\infty := \max_{i \in \mathrm{V}} \sup_{t \in [t_0, \infty)} |w_i(t)|$.

Fig. 2. Lemma 4 in [2]

$$\lim_{t \to \infty} sup \max_{i,j} |x_i(t) - x_j(t)| = 0 \tag{7}$$

Step 4. In this step, they finally show that each $x_i(t)$ admits a finite limit. Let c^* be a limit point of $x_j(t)$ for a fixed j. Based on the fact that $Z \in [p_*, q_*]$, there must hold $c^* \in [p_*, q_*]$. If $p_* = q_*$, the result already holds. they assume $p_* < q_*$ in the following. According to (7), for any $\epsilon > 0$, there exists $t_* > 0$ such that $|x_i(t_*) - c^*| \leq \epsilon$ for all i. The proof is complete.

C. Experiment

The experiment results are shown in Fig. 3, which is simulated by a 4 interaction points contained deterministic network. The initial state of each points is $4, 0.1, -1, -4$ respectively, and the constrained interval of each points is $[-0.5, 0.5], [-1, 1.8], [-1, 1], [-0.5, 0.2]$ respectively. This network satisfies the assumption mentioned above. In Fig.3, a) represents the condition that the common interval belongs to $[-0.5, 0.2]$; b) represents the condition that common interval equals to 0.5; c) represents the condition that there is no common interval. The results match our theoretical analysis results. The consensus value of a) lies in $[-0.5, 2]$, b) equals to 0.5 and c) stay unstable.

III. Constrained interval consensus over random networks

A. Overview

After the analysis in deterministic network, Weiming Fu et al. expand the interval consensus problem into random network[3]. The goal of interval consensus problem remains unchanged, while the network structure will change over time. As shown in Fig. 4, G is the origin undirected graph, and S is the sub-graph generated by G. At each time step, the network will randomly select a structure from S. This paper figures out that if the common interval is not $\emptyset$ and the weight of each edge satisfy $2\varepsilon \sum_{j \in N_i} a_{ij} < 1$, where ε denotes the step size, then the interval consensus can reach the goal under any initial state.

B. Modeling

To start with, they consider a network of n nodes indexed in the set $V = \{1, ...n\}$. The underlying interaction structure of the network is described by an undirected graph $G = (V, E)$, which is assumed to be connected. Here $E \in V \times V$ is an unordered edge set and $(i, j) \in E$ means that there are communication links between node i and node j. Denoted by $N_i = \{j \in V : (i, j) \in E\}$, the set of neighbours of node i in the ubderlying grapg G. The set of all subgraphs of G is denoted as $S = \{G_k = (V, E_k)\}_{k \in \{1,...,c\}}$, with $c > 0$ as a finite integer and $U_{k=1}^c E_k = E$. Let μ be a finite probability measure over S satisfying $\mu(G_k) \geq \alpha, k \in \{1, ...c\}$ for some $\alpha > 0$.

Then, they denote the communication among the nodes at time $t = 0, 1, 2...$ is drawn from S by the probability measure μ independently with other time and node states, and the resulting random graph process is $g_t = G_{\sigma(t)} \in S, t = 0, 1, 2...$ with $\sigma : N \to \{1...c\}$ being the random variable that determines the network topology. The neighbor set of node i at time t is denoted as $N_i(t) = \{j : (i, j) \in E_{\sigma(t)}$. In addition, they let F_t be the σ-algebra generated by $g_0...g_t$ for $t \geq 0$.

Each node i holds a state $x_i(t)$ at time $t \in N$. Furthermore, node i also holds an interval $I_i = [p_i, q_i]$ determining the range of the values node i can transmit. The nodes send $\psi_i(x_i(t))$ to its neighbors instead of $x_i(t)$, shown in (2). With the above notations, the interval consensus problem of node i is described by:

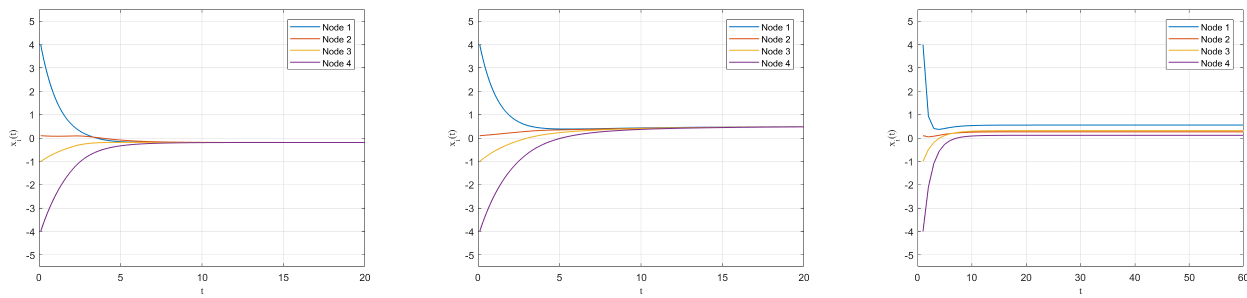
$$x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i(t)} a_{ij}(\psi_j(x_j(t)) - x_i(t)) \tag{8}$$

$$= (1 - \varepsilon \sum_{j \in N_i(t)} a_{ij}) x_j(t) + \varepsilon \sum_{j \in N_i(t)} \psi_j(x_j(t)) \tag{9}$$

where $a_{ij} = a_{ji} > 0$ is the weight of edge (i, j), and $\varepsilon > 0$ is a step size.

From the modeling above, they propose a theorem that:

Theorem 1. Suppose $\bigcap_{m=1}^n I_m \neq \emptyset$ and $2\varepsilon \sum_{j \in N_i(t)} a_{ij} < 1$ for all $i \in V$. Then for any initial value $\mathbf{x}(0)$, there is a ran-



(a) Common interval $\in \{-0.5 \sim 0.2\}$ (b) Common interval $= 0.5$ (c) Common interval $\in \varnothing$

Fig. 3. Simulation result in deterministic network

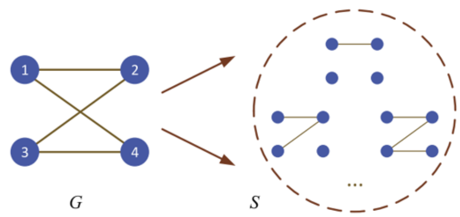


Fig. 4. Example: random network

dom variable $y_*(\mathbf{x}(0) \in \bigcap_{m=1}^{n} I_m)$, such that $P(\underset{t\to\infty}{lim}\, x_i(t) = y_*, i \in V) = 1, q_* = min_{i\in V}\ q_i, p_* = max_{i\in V}\ p_i$.

C. Derivation

Similar to the derivation in II.B, the proof of Theorem 1 also has two parts, the convergence and consensus. *Part 1* use *Lemma 1,2*, which is shown in Fig. 5, to prove the consensus. *Part 2*, unlike *Step 1,2* in II.B, doesn't use *Lasalle* principle but prove the convergence directly.

Lemma 1. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, and $A \in \mathbb{R}^{n\times n}$. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$ and let the eigenvalues of A be $\lambda, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of $A + \mathbf{x}\mathbf{y}^T$ are $\lambda + \mathbf{y}^T\mathbf{x}, \lambda_2, \ldots, \lambda_n$.*

Lemma 2. *Let $\{v_t\}_{t\in\mathbb{N}}$ be a sequence of non-negative random variables with $\mathbb{E}\{v_0\} < \infty$. Assume that for any $t \in \mathbb{N}$, $\mathbb{E}\{v_{t+1}|v_0, \ldots, v_t\} \le (1-\xi_t)v_t + \theta_t$, where $\{\xi_t\}_{t\in\mathbb{N}}$ and $\{\theta_t\}_{t\in\mathbb{N}}$ are two sequences of non-negative numbers satisfying $\forall t \in \mathbb{N}$, $0 \le \xi_t \le 1$, $\sum_{t=0}^{\infty}\xi_t = \infty$, $\sum_{t=0}^{\infty}\theta_t < \infty$, and $\lim_{t\to\infty}\theta_t/\xi_t = 0$. Then, $\lim_{t\to\infty} v_t = 0$ almost surely.*

Fig. 5. Lemma 1,2 in [3]

Part 1. In this part, they first generalize the notion of robust consensus for deterministic network to the random case.

Proposition 1. Consider the network dynamics below.

$$x_i(t+1) = x_i(t) + \varepsilon \sum_{j\in N_i(t)} a_{ij}(x_j(t) - x_i(t)) + w_i(t)$$
$$i \in V, t \in N, \tag{10}$$

where $2\varepsilon \sum_{j\in N_i(t)} a_{ij} < 1$. If $w_i(t)$ is F_t measurable with $|w_i(t)| \le w^*$ for $w^* > 0$ and $P(\underset{t\to\infty}{lim}\, w_i(t) = 0) = 1$, then for any initial states, there holds that $P(\underset{t\to\infty i,j\in V}{lim\ max}|x_i(t)-x_j(t)| = 0) = 1$.

In order to prove Proposition 1, they rewrite (10) to

$$x(t+1) = (I - \varepsilon L_t)x(t) + w(t) \tag{11}$$

where L_t is the Laplacian of the random graph g_t and $w(t) = (w_1(t), \ldots w_n(t))^T$. Denote $J = \frac{1}{n}11^T$, where 1 is the n-dimensional all-one vector. Then, they introduce $y(t) = (I - J)x(t)$. Therefore, the relationship between $y(t)$ and $y(t+1)$ is $y(t+1) = (I-J)x(t+1) = M_t y(t) + (I-J)w(t)$, where $M_t = 1 - \varepsilon L_t - J$.

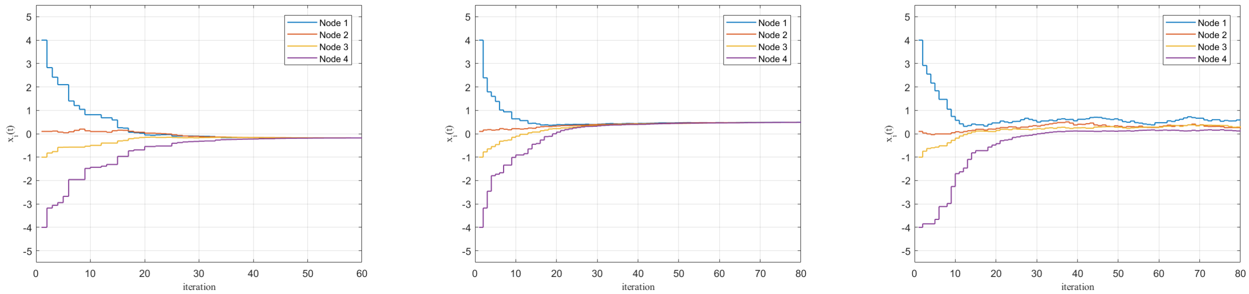
After that, they utilize the characteristic of *Laplacian Matrix* of graph G, and find that $E\{M_t^2\} \le E\{M_t\}$. $E\{M_t\} = 1 - \varepsilon E\{L_t\} - J = I - \varepsilon L^*(G) - J$, where $L^*(G)$ is a weighted *Laplacian Matrix* of graph G. After using Lemma 1, they prove the Lemma:

Lemma 3: Suppose $2\varepsilon \sum_{j\in N_i(t)} a_{ij} < 1$ for all $i \in V$. Then $0 < \lambda_max(E\{M_t^2\}) < 1$.

Then, by applying Cauchy–Schwarz inequality and Lemma 3, they find that $E\{|y(t+1)|^2 \mid |y(t)|^2, \ldots |y(0)|^2\} \le [1 - (\lambda^* - h_1(t))]|y(t)|^2 + h_2(t), t \ge T$, where $(\lambda^* - h_1(t))$ and $h_2(t)$ satisfy $0 \le (\lambda^* - h_1(t)) \ge 1$, $\sum_{t=T}^{\infty}(\lambda^* - h_1(t)) = \infty$, $\sum_{t=T}^{\infty} h_2(t) < \infty$, and $lim_{t\to\infty}\ h_2(t)/(\lambda^* - h_1(t)) = 0$ for all $t \ge T$.

Therefore, they invoke Lemma 2 to conclude that $P(lim_{t\to\infty}\ y(t) = 0) = 1$, i.e., $P(lim_{t\to\infty} max_{i,j\in V}|x_j(t) - x_j(t)| = 0) = 1$. The consensus has been proved. *Part 2*. They design $H(x(t)) = max\ \{max_i\ x_i(t),\ q_*\}$ and $h(x(t)) = min\{min_i\ x_i(t), p_*\}$. Then, define $I_a^+(z)$with $I_a^+(z) = z$ if $z > a$ and $I_a^+(z) = a$ otherwise, therefore:

$$\begin{aligned} H(x(t+1)) &= I_{q*}^+(\underset{i\in V}{max}\ x_i(t+1)) \\ &= I_{q*}^+(\underset{i\in V}{max}((1 - \varepsilon \sum_{j\in N_i(t)} a_{ij})x_i(t) \\ &+ \varepsilon \sum_{j\in N_i(t)} a_{ij}\psi_j(x_j(t)))) \end{aligned}$$



(a) Common interval $\in \{-0.5 \sim 0.2\}$ (b) Common interval $= 0.5$ (c) Common interval $\in \emptyset$

Fig. 6. Simulation result in random network

$$
\begin{aligned}
&\le I_{q*}^{+}(\underset{i\in V}{max}((1-\varepsilon\sum_{j\in N_i(t)} a_{ij})I_{q*}^{+}(\underset{j\in V}{max} x_j(t)) \\
&+\varepsilon\sum_{j\in N_i(t)} a_{ij} I_{q*}^{+}(\underset{j\in V}{max}\ x_j(t)))) \\
&= H(x(t))
\end{aligned}
$$

regardless of the choice of g_t (therefore of $N_i(t)$ as well). This proves that $H(x(t+1)) \le H(x(t))$ is a sure event. Similarly, $h(x(t+1)) \ge h(x(t))$. Therefore, Lemma 4, shown below, has been proved.

Lemma 4: $H(x(t+1)) \le H(x(t))$ and $h(x(t+1)) \ge h(x(t))$ are sure events if $\varepsilon\sum_{j\in N_i(t)} a_{ij} < 1$.

Using Lemma 4, they prove the Lemma 5:

Lemma 5: Let $\bigcap_{m=1}^{n} I_m \neq \emptyset$ and $\varepsilon\sum_{j\in N_i(t)} a_{ij} < 1$, there holds $P(\underset{t\to\infty}{lim} H(x(t) = q_*) = 1$ and $P(\underset{t\to\infty}{lim} h(x(t) = p_*) = 1$.

From the above scheme, they find that if they define the measure $w_i(t)$ as $\varepsilon\sum_{j\in N_i(t)} a_{ij}(\psi_j(x_j(t)) - x_i(t))$ the (8) can be written as:

$$
x_i(t+1) = x_i(t) + \varepsilon\sum_{j\in N_i(t)} a_{ij}(\psi_j(x_j(t)) - x_i(t)) + w_i(t) \tag{12}
$$

Now from Lemma 4, one can know that there is a constant D_0 such that $|w_i(t)| \le D_0$ is a sure event for all i and t. From the form of its definition, $w_i(t)$ is F_t-measurable. Moreover, Lemma 5 ensures $P(lim_{t\to\infty} w_i(t) = 0) = 1$. Thus, we can obtain by applying Proposition 1 that

$$
P(\underset{t\to\infty i,j\in V}{lim\ max}|x_i(t) - x_j(t)| = 0) = 1. \tag{13}
$$

Therefore, the interval consensus in random network has been proved.

D. Experiment

Similar with experiment in II.C. The experiment results are shown in Fig. 6, which is simulated by a 4 interaction points contained random network. The initial state of each points is $4, 0.1, -1, -4$ respectively, and the constrained interval of each points is $[-0.5, 0.5], [-1, 1.8], [-1, 1], [-0.5, 0.2]$ respectively. This network satisfies the assumption mentioned above. In Fig.3, a) represents the condition that the common interval belongs to $[-0.5, 0.2]$; b) represents the condition that common interval equals to 0.5; c) represents the condition that there is no common interval. The results match our theoretical analysis results. The consensus value of a) lies in $[-0.5, 2]$, b) equals to 0.5 and c) stay unstable.

IV. Comments

A. Difference between random network and deterministic network

From those two papers, we find that the derivation of convergence and consensus in random network and deterministic network are different. In deterministic case, the interval consensus problem can be modeled to a continuous model, however in the random case, this problem only can be modeled to discrete model.

In the derivation of convergence, deterministic problem utilizes the *Lasalle* principle. They design proper function $H(x(t))$ and $h(x(t))$ to construct the $V(x(t))$ and make the derivative of $V(x(t))$ always less than or equal to 0. Then, a contradiction argument is used to verify that the $Z : D^{+}V(x(t)) = 0$ always lies in the common interval. Those two steps prove the convergence of deterministic problem. However, random problem does not use *Lasalle* principle but use the same strategy. They first prove that the state of each points is bounded under some conditions, then they prove that the bounded set is equal to the common interval.

In the derivation of consensus, those two cases are similar, deterministic and random cases rely on the existing Lemma shown in Fig.2, Fig.5, to prove the consensus. In addition, it is worth noting that due to the random characteristic, the analysis in random case is all performed by the possibility or the expectation.

B. Drawbacks and advantages

Those two papers are complementary to discuss the interval consensus problem. One of the advantage is that, their result/theorem provide a theoretical guidance for us to design a network with constrained consensus. For the prevalent top-down design style, it is convenience for us to set the constraint of agents to satisfy our design demand.

Another advantage is the robustness, as shown in Fig. 7,8. We add some disturbance in deterministic and random networks at time=5 and iteration=50, respectively. From the figure we can find that, the disturbance will not affect the convergence and consensus characteristic in both networks.

However, even those networks are robust, there still have some small drawback. If the length of interaction of intervals larger than 0, then the disturbance will change the consensus value while the consensus value still lies in the common interval set. This phenomenon shows that this system is asymptotically stable outside the intersection of the allowed consensus intervals, because of the saturation, but marginally stable inside it.

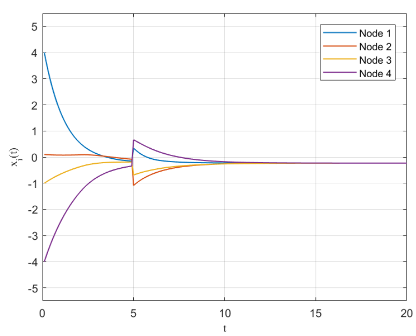


Fig. 7. Deterministic network with disturbance

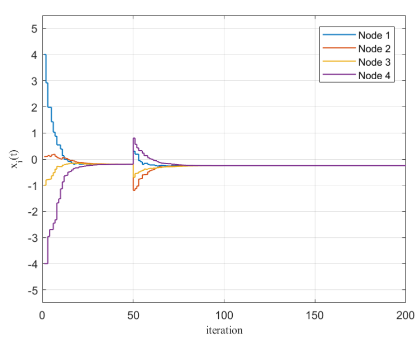


Fig. 8. Random network with disturbance

C. Future work

As presented in IV.B, those consensus problems are suitable for designing a system. Therefore, I think one of the future work is apply this theorem to neural network. Nowadays, it is hard for scientists to adjust the parameter manually, especially for the complex network like convolutional neural network(CNN), recursive neural network(RNN) etc.. From this theorem, we can find that we can adjust the constrain of each agent to make the result consensus, which may help us to adjust the parameters in neural network.

In addition, the neural network also has the form that each agent transmits values to their neighbors, especially in the forward and backward flow, which adjust the output value of each agents and adjust weight respectively. Furthermore, the strategy used in neural network like dropout, can be modeled as a random network. If we can prove the partial consensus in both deterministic and random networks, we can reduce the difficulty of parameters adjustment.

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