Matrix computation
Shanghaitech University
Fall 2020

Homework 1

Name: Chengrui Zhang, Student Number: 2019233183

This homework answers the problem set sequentially.

1 Part I

1.1 1)

In order to prove this, we need to prove that $S_1 \cap S_2 = \{\mathbf{0}\}$ and $S_1 + S_2 = \mathbf{R}^n$. For the former one, we find that for $x_1 \in N(A)$ and $x_2 \in R(A^T)$

$$x_1^T x_2 = x_1^T A^T y = (Ax_1)^T y = 0$$

Therefore, $N(A) \perp R(A^T), N(A) \cap R(A^T) = \{\mathbf{0}\}$
Then, for the latter one,

$$N(A) + R(A^T) = \{x + y | x \in N(A), y \in R(A^T)\}$$

Let $B_1 = \{p_1, p_2 \dots p_r\}, B_2 = \{p_{r+1}, p_{r+2} \dots p_n\}$ be the basis of $N(A), R(A^T)$, respectively, then $B_1 \cap B_2 = \mathbf{0}$.
Let $v = x + y$, then v is a linear combination of B_1, B_2, $rank(v) = r + n - r = n$.
Therefore, $N(A) + R(A^T) = R^n$.

1.2 2)

Suppose that $P = \{p_1, p_2 \dots p_r\}, Q = \{q_1, q_2 \dots q_k\}$ are the basis of A, B, respectively, then $A + B$ can be represented by (A, B).
Therefore, $rank(A + B) \leq rank(A, B) \leq rank(A) + rank(B)$

1.3 3)

$$AB = (a_1, a_2 \dots a_n) \begin{Bmatrix} b_{11} & b_{12} & \cdots & b_{n1} \\ b_{21} & \ddots & & \vdots \\ \vdots & & & \\ b_{n1} & \cdots & & b_{nn} \end{Bmatrix}$$

where a_i is a $m \times 1$ vector. Therefore, we find that every element in AB can be represented by A, therefore, $rank(AB) \leq rank(A)$.
Similarly, $rank(AB) \leq rank(B)$. In general, $rank(AB) \leq min(rank(A), rank(B))$.
In addition,

$$\because dimN(A) \cap R(B) \subseteq N(A)$$
$$\therefore dimN(A) \cap R(B) \leq dimN(A) = n - rank(A)$$

In addition, we know that

$$rank(AB) = rank(B) - dimN(A) \cap R(B) \geq rank(A + rank(B) - n$$

Therefore, we know that $rank(AB) = n$ if and only if $rank(A) = rank(B) = n$

1.4 4)

By the definition of range space, we find that,

$$R(A|B) = [A, B] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax_1 + Bx_2$$
$$R(A) + R(B) = Ay_1 + By_2$$

Therefore, the subspace $R(A|B)$ and $R(A) + R(B)$ can be the linear combination of each other easily when $x_1 = y_1, x_2 = y_2$, which means that $R(A|B) \subseteq R(A) + R(B)$ and $R(A) + R(B) \subseteq R(A|B)$.
Therefore, $R(A|B) = R(A) + R(B)$

1.5 5)

$$\begin{aligned} rank(A|B) &= dim(R(A|B)), for\ the\ conclusion\ from\ 4) \\ &= dim(R(A) + R(B)) \\ &= dim(R(A)) + dim(R(B)) - dim(R(A) \cap R(B)) \\ &= rank(A) + rank(B) - dim(R(A) \cap R(B)) \end{aligned}$$

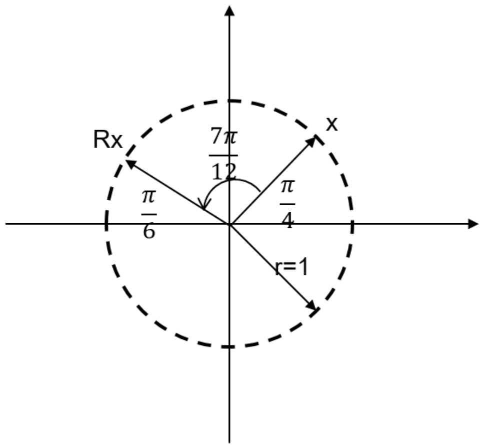
2 Part II

To prove that $span(S) \subseteq M$, let $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.
$\because S \subseteq V$, $\therefore$ all V contain x.
$\therefore M := \cap_{S \subseteq V} V$ contains x, $\therefore span(S) \subseteq M$.
To prove that $M \subseteq span(S)$, we consider the situation that, $span(S)$ is a special subspace that contains S, therefore, $\because M := \cap_{S \subseteq V} V$, $\therefore M \subseteq span(S)$. In general, $M = span(S)$.
Therefore, the subspace $span(S)$ is the intersection of all subspaces that contain S

3 Part III

3.1 Problem 1

1) $n + 1$;
2)$\frac{n^2+n}{2n}$;



3.2 Problem 2

3.2.1

Suppose that,

$$R = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$
$$\therefore RR^T = \begin{bmatrix} x_1^2 + x_2^2 & x_1x_3 + x_2x_4 \\ x_1x_3 + x_2x_4 & x_1^2 + x_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$det(R) = x_1x_4 - x_2x_3 = 1$$
$$\therefore we\ can\ get, x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1$$

From the above equations we can get:

$$x_1 = x_4, x_2 = -x_3, x_1^2 + x_2^2 = 1 \tag{1}$$

All the R that satisfy Eq.(1) can be a rotation matrix.
From the figure shown above, we can easily get the Rx. $Rx = [cos(\frac{5\pi}{6}), sin(\frac{5\pi}{6})]$.

3.2.2

$$\because QH = I - 2uu^T - uu^T + 2uu^Tuu^T$$
$$while\ uu^Tuu^T = uu^T\ can\ be\ easily\ proved.$$
$$\therefore QH = I - uu^T = Q$$
$$\therefore Qx = QHx = Qy$$

In addition,

$$x - Qx = x - x + uu^Tx = x - u(u^Tx)$$
$$\therefore ||y - Qy||_2 = ||Hx - QHx||_2 = ||x - 2uu^Tx - x + uu^Tx||_2 = ||uu^Tx||_2$$
$$and\ ||x - Qx||_2 = ||x - x + uu^Tx||_2 = ||uu^Tx||_2$$
$$\therefore ||x - Qx||_2 = ||y - Qy||_2$$

In general, we find that Hx is a reflection of x with respect to H_u

4 Part IV

4.1 Problem 1

Obviously, we find that $B_1 \subseteq B$ and $B_2 \subseteq B$. Since B is a basis for V, our goal is equal to the $span(B) = span(B_1) \bigoplus span(B_2)$.
In order to satisfy $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$, we suppose that $span(B_1) = span(b_1, b_2, \ldots, b_r)$, $span(B_2) = span(b_{r+1}, b_{r+2}, \ldots, b_n)$ and $span(B) = span(b_1, b_2, \ldots, b_n)$. Then, $\because B_1 \cap B_2 = \varnothing, \therefore span(B_1) \cap span(B_2) = \{\mathbf{0}\}$.
By the definition of direct sum, we can find that; $span(B_1) \bigoplus span(B_2) = \{u + v | u \in span(B_a)\ and\ v \in span(B_2)\}$.
$\therefore z = u + v$ is a linear expression of B_1, B_2, and can also be a linear expression of B. Similarly, B can be a linear expression of z. Therefore, $span(B) = span(B_1) \bigoplus span(B_2)$, which means that $V = span(B_1) \bigoplus span(B_2)$.

4.2 Problem 2

Obviously, $dim(V) = n$. Let B, B_1 is a basis for V, S, respectively. Therefore $dim(S) = d$. Suppose that $span(B_1) = span(b_1, b_2, \ldots, b_d), span(B) = span(b_1, b_2, \ldots, b_n)$.
Let Γ is the subspace of V, which satisfy $dim(\Gamma) = n - d < n$, and suppose the basis for Γ, called B_2. In order to satisfy that $B_1 \cup B_2 = B, B_1 \cap B_2 = \varnothing$.
$\therefore B_2 = \{b_{n-d+1}, b_{n-d+2}, \ldots, b_n\}, span(B_2) = span(b_{n-d+1}, b_{n-d+2}, \ldots, b_n)$.
From the conclusion in Problem 1, we can find that $span(B) = span(B_1) \bigoplus span(B_2)$, which is equal to $V = S \bigoplus \Gamma$.

5 Part V

5.1 Problem 1

Let $A = [a_1, a_2, \ldots, a_n]^T, a_i \in R^{1\times n}$.

$$\begin{aligned}
||Ax||_1 &= ||[a_1x, a_2x, \ldots, a_nx]^T]||_1 \\
&= |a_1x| + |a_2x| + \cdots + |a_nx| \\
&= \sum_i \sum_j |a_{ij}x_j| \\
&\leq \sum_i \sum_j |a_{ij}||x_j| \\
&= \sum_j |x_j| \sum_i |a_{ij}| \\
&\leq (\sum_j |x_j|)(max_j \sum_i |a_{ij}|) \\
&= max_j \sum_i |a_{ij}|
\end{aligned}$$

$\therefore ||Ax||_1 \leq max_j \sum_i |a_{ij}|$. In order to make the equal sign true, we find that when $x = e_k$, where A_{*k} is the column with the largest absolute sum, then $||Ax||_1 = max_j \sum_i |a_{ij}|$.

$\therefore ||Ax||_1$ has a clear upper bound and there exists a $||x||_1 = 1$ that make the equal sign true. Therefore, $\underset{||x||_1=1}{max}||Ax||_1 = max_j \sum_i^m |a_{ij}|$ is true.

5.2 Problem 2

$$\begin{aligned}
||Ax||_\infty &= max_i|\sum_j a_{ij}x_j| \\
&\le max_i \sum_j |a_{ij}||x_j|, \because ||x||_\infty = 1, \therefore ||x_j|| \le 1
\end{aligned}$$

$$\therefore max_i|\sum_j a_{ij}x_j| \le max_i \sum_j |a_{ij}| \tag{2}$$

In addition,

$$|\sum_j a_{ij}x_j| = |a_{i1}x_1| + |a_{i2}x_2| + \cdots + |a_{in}x_n| \tag{3}$$

Similar to Problem 1, we need to find a x that satisfy $||x||_\infty = 1$ and make the equal sign true, which means that we need to find a x to make Eq.(2) equal to Eq.(3).
Therefore, we find that if A_{k*} is the row with the largest absolute value, and $x_j = \begin{cases} 1, a_{kj} \ge 0 \\ -1, a_{kj} < 0 \end{cases}$, then this x can make Eq.(2) equal to Eq.(3). Similar to Problem 1, the proof has been down.

6 Part VI

6.1 1)

$$\begin{gathered}
f'(t) = \lambda - \lambda t^{\lambda-1} \\
\therefore f'(t) < 0 \ when \ t < 1, \ f'(t) \ge 0 \ when \ t \ge 1. \\
\therefore f(t) \ge f(1) = 0
\end{gathered}$$

If we let $t = \frac{\alpha}{\beta}$, then,

$$\begin{gathered}
1 - \lambda + \frac{\alpha}{\beta}\lambda - (\frac{\alpha}{\beta})^\lambda \ge 0 \\
\therefore \beta^\lambda - \beta^\lambda\lambda + \alpha\lambda\beta^{\lambda-1} \ge \alpha^\lambda
\end{gathered}$$

$$\therefore \beta(1-\lambda) + \alpha\lambda \ge \alpha^\lambda\beta^{1-\lambda} \tag{4}$$

6.2 2)

We need to prove that

$$\sum_{i=1}^n |\hat{x}_i\hat{y}_i| \le \frac{1}{p}\sum_{i=1}^n |\hat{x}_i|^p + \frac{1}{q}\sum_{i=1}^n |\hat{y}_i|^q = 1 \tag{5}$$

For the equal sign, it is easy to verify that if we put the definition of *p-norm* into Eq.(5), the $\sum_{i=1}^{n}|\hat{x_i}|^p, \sum_{i=1}^{n}|\hat{y_i}|^q$ are all equal to 1, which means that we need to prove $\frac{1}{p}+\frac{1}{q}=1$. It is obvious.
For the inequality sign, we find that the sufficient condition of Eq.(5) is the following equation,

$$|\hat{x_i}\hat{y_i}| \leq \frac{1}{p}|\hat{x_i}|^p + \frac{1}{q}|\hat{y_i}|^q, for\ i \in [1,2,\ldots,n]$$

Since here $\hat{x_i}, \hat{y_i}$ are all constant, therefore this equation can be rewrite as,

$$|\hat{x_i}||\hat{y_i}| \leq \frac{1}{p}|\hat{x_i}|^p + \frac{1}{q}|\hat{y_i}|^q, for\ i \in [1,2,\ldots,n] \tag{6}$$

For Eq.(6), which is similar to Eq.(4), we find that if we let $\lambda = \frac{1}{p},\ 1-\lambda = \frac{1}{q},\ \alpha = |\hat{x_i}|^p,\ and\ \beta = |\hat{y_i}|^q$. Then we can get Eq.(6) by Eq.(4), then the Eq.(5) will be proved.

6.3 3)

$$\begin{aligned}
\because |x^T y| &= |\sum_{i=1}^{n} x_i y_i| \\
&\leq \sum_{i=1}^{n} |x_i y_i| \\
&= \sum_{i=1}^{n} |\hat{x_i}\hat{y_i}| * ||x||_p ||y||_q \\
&\leq (\frac{1}{p}|\hat{x_i}|^p + \frac{1}{q}|\hat{y_i}|^q) * ||x||_p ||y||_q \\
&= 1 * ||x||_p ||y||_q = ||x||_p ||y||_q
\end{aligned}$$

Therefore, the proof has been down.

6.4 4)

For the case that $p=1$, it is quite clear that the absolute value of sum is smaller than the sum of absolute value, which prove this equation.
For the case that $p>1$, we find that if we let $\frac{1}{q} = 1 - \frac{1}{p}$, then $p = \frac{p}{q}+1$
Therefore, let $\alpha = x_i, \beta = y_i$, then,

$$\begin{aligned}
||x+y||_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\
&= \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{\frac{p}{q}} \\
&\leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{\frac{p}{q}} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{\frac{p}{q}}
\end{aligned}$$

Then, we use the conclusion from 1),2),3),

$$\begin{aligned}\sum_{i=1}^{n} |x_i||x_i + y_i|^{\frac{p}{q}} &\leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |x_i + y_i|^{q * \frac{p}{q}})^{\frac{1}{q}} \\ &= (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |x_i + y_i|^p)^{\frac{p-1}{p}} \\ &= ||x||_p ||x + y||_p^{p-1}\end{aligned}$$

Similarly, we can find that $\sum_{i=1}^{n} |y_i||x_i + y_i|^{\frac{p}{q}} \leq ||y||_p ||x + y||_p^{p-1}$

$$\therefore ||x + y||_p^p \leq (||x||_p + ||y||_p)||x + y||_p^{p-1}$$
$$\therefore ||x + y||_p \leq ||x||_p + ||y||_p$$