SI231 - Matrix Computations, Fall 2020-21

Solution of Homework Set #2

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I. General Linear System

Problem 1 (6 points + 9 points) This problem is graded by **Lin Zhu (zhulin@)**.

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -2 & 4 & -6 & 0 \\ 3 & 1 & 14 & -1 \\ -1 & 7 & -5 & 3 \end{bmatrix} \in \mathbb{R}^{4\times 4}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 1 & 1 \\ 2 & 2 & 2 & -1 \\ 5 & 5 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{4\times 4}$.

1) For $\mathbf{A}$ and $\mathbf{b} = (-1, 2, 5, 3)^T \in \mathbb{R}^4$, find $\mathcal{N}(\mathbf{A})$, $\mathcal{R}(\mathbf{A})$, then solve $\mathbf{Ax} = \mathbf{b}$.
2) For $\mathbf{B}$ and $\mathbf{b} = (1, 1, 1, 2)^T \in \mathbb{R}^4$, solve the linear equation system $\mathbf{Bx} = \mathbf{b}$ with Gauss Elimination, LU decomposition, and LU decomposition with partial pivoting, respectively. (Although not required, you are highly encouraged to write down your solution procedures in detail.)

Solution:

1) The zero space of A is $\mathcal{N}(\mathbf{A}) = \text{span}\{(8, 1, -2, -3)^T\}$ (2 points), the range space of A is $\mathcal{R}(\mathbf{A}) = \text{span}\{(1, -2, 3, 1)^T, (0, 4, 1, 7)^T, (2, 0, -1, 3)^T\}$ (2 points),
   the solution is $\mathbf{x} = (1, 1, 0, -1)^T + a(8, 1, -2, -3)^T, \forall a \in \mathbb{R}$ (2 points).
   Remarks: let $\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}]$, here we have $\mathbf{a_2} = -8\mathbf{a_1} + 2\mathbf{a_3} + 3\mathbf{a_4}$, so representation of $\mathcal{N}(\mathbf{A})$ is unique, but $\mathcal{R}(\mathbf{A})$ is not.
2) a) Gauss Elimination:

$$[\mathbf{B}|\mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & -1 & 1 \\ 2 & 3 & 1 & 1 & 1 \\ 2 & 2 & 2 & -1 & 1 \\ 5 & 5 & 2 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 1 \\ 0 & -1 & -5 & 3 & -1 \\ 0 & -2 & -4 & 1 & -1 \\ 0 & -5 & -13 & 8 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 1 \\ 0 & -1 & -5 & 3 & -1 \\ 0 & 0 & 6 & -5 & 1 \\ 0 & 0 & 12 & -7 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 1 \\ 0 & -1 & -5 & 3 & -1 \\ 0 & 0 & 6 & -5 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}. \text{(3 points)}$$

It follows that $\mathbf{x} = (1/6, 1/6, 1/6, 0)^T$.

b) The LU decomposition of $\mathbf{B}$ is

$$\mathbf{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 5 & 5 & 2 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -5 & 3 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{\mathbf{U}} .\text{(2 points)}$$

If $\mathbf{Bx} = \mathbf{b}$, we have $\mathbf{LUx} = \mathbf{b}$ and $\begin{cases} \mathbf{Lz} = \mathbf{b} \\ \mathbf{Ux} = \mathbf{z} \end{cases}$, i.e. $\mathbf{Ux} = \mathbf{L}^{-1}\mathbf{b}$,(1 point)

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -5 & 3 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & -1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$

then work out $\mathbf{x} = (1/6, 1/6, 1/6, 0)^T$.

c) The LU decomposition with partial pivoting of $\mathbf{B}$ is

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \cdot \mathbf{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/5 & 1 & 0 & 0 \\ 1/5 & 1 & 1 & 0 \\ 2/5 & 0 & 1/2 & 1 \end{bmatrix}}_{\mathbf{L}} \cdot \underbrace{\begin{bmatrix} 5 & 5 & 2 & 3 \\ 0 & 1 & 1/5 & -1/5 \\ 0 & 0 & 12/5 & -7/5 \\ 0 & 0 & 0 & -3/2 \end{bmatrix}}_{\mathbf{U}} .\text{(2 points)}$$

If $\mathbf{Bx} = \mathbf{b}$, we have $\mathbf{PBx} = \mathbf{LUx} = \mathbf{Pb}$ and $\begin{cases} \mathbf{Lz} = \mathbf{Pb} \\ \mathbf{Ux} = \mathbf{z} \end{cases}$, i.e. $\mathbf{Ux} = \mathbf{L}^{-1}\mathbf{Pb}$,(1 points)

$$\begin{bmatrix} 5 & 5 & 2 & 3 \\ 0 & 1 & 1/5 & -1/5 \\ 0 & 0 & 12/5 & -7/5 \\ 0 & 0 & 0 & -3/2 \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} 2 \\ 1/5 \\ 2/5 \\ 0 \end{bmatrix},$$

and work out $\mathbf{x} = (1/6, 1/6, 1/6, 0)^T$.

Remarks: In 2), you will lose 1 point if wrong or no solution to $\mathbf{Bx} = \mathbf{b}$ is given, but no more points for the right solution.

II. Understanding Various Matrix Decompositions

Problem 2 (10 points) This problem is graded by **Sihang Xu (xush@)**.

Consider the following symmetric matrix $\mathbf{A} \in \mathbb{R}^{4\times 4}$

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Decide under which conditions $\mathbf{A}$ has the LU decomposition and then give the LU decomposition of $\mathbf{A}$.

Solution: The LU decomposition of $\mathbf{A}$ is given by

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

The four conditions are $a \neq 0, b \neq a, c \neq b, d \neq c$.

Problem 3 (5 points + 10 points) This problem is graded by **Zhicheng Wang (wangzhch1@)**.

1) Consider a 3×3 matrix
$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 5 & 1 \\ 1 & 1 & 8 \end{bmatrix},$$
find the LDM (also called LDU) decomposition of $\mathbf{A}$, i.e., factor $\mathbf{A}$ as $\mathbf{A} = \mathbf{LDM}^T$ (or $\mathbf{A} = \mathbf{LDU}$), where $\mathbf{L} \in \mathbb{R}^{3\times 3}$ is a lower triangular with unit diagonal entries, $\mathbf{D} \in \mathbb{R}^{3\times 3}$ is a diagonal matrix, and $\mathbf{M} \in \mathbb{R}^{3\times 3}$ is a lower triangular with unit diagonal entries ($\mathbf{U} \in \mathbb{R}^{3\times 3}$ is an upper triangular with unit diagonal entries).
2) Consider a 3×3 matrix
$$\mathbf{B} = \begin{bmatrix} 8 & 1 & 1 \\ 1 & 5 & 1 \\ 4 & 2 & 2 \end{bmatrix},$$
find the UL decomposition of $\mathbf{B}$, i.e., factor $\mathbf{B}$ as $\mathbf{B} = \mathbf{UL}$, where $\mathbf{U} \in \mathbb{R}^{3\times 3}$ is upper triangular with unit diagonal entries and $\mathbf{L} \in \mathbb{R}^{3\times 3}$ is lower triangular.

   Hint: $\mathbf{B} = \mathbf{PAP}$, where $\mathbf{P}$ is a unit anti-diagonal matrix [1].

Solution:

1) LU decomposition of $\mathbf{A}$ can be given in
$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 2 & 4 \\ 0 & 4 & -1 \\ 0 & 0 & 6 \end{bmatrix}}_{\mathbf{U}}, \text{(3 points)}$$
then by the definition of LDM decomposition, we have that,
$$\mathbf{D} = \mathrm{diag}(\mathbf{U}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \text{(1 point)} \quad \mathbf{M} = \mathbf{U}^T\mathbf{D}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{4} & 1 \end{bmatrix}. \text{(1 point)}$$
Rewrite $\mathbf{A}$ in the LDM decomposition form,
$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{M^T} \text{ or } \mathbf{U}'}.$$

[1] **Anti-diagonal matrix:** Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a permutation matrix, then an anti-diagonal matrix is a square matrix where all the entries are zero except those on the diagonal going from the lower left corner to the upper right corner, known as the anti-diagonal. For example,
$$\mathrm{adiag}(a_1, \dots, a_n) = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \cdots & \cdots & 0 \end{bmatrix},$$
and consequently, unit anti-diagonal matrix means $\mathrm{adiag}(1, \dots, 1)$, also known as the **exchange matrix** or **permutation matrix**.

2) Notice that $\mathbf{B} = \mathbf{PAP}$, where $\mathbf{P}$ is a unit anti-diagonal matrix. We can verify that $\mathbf{P} = \mathbf{P}^T = \mathbf{P}^{-1}$. If $\mathbf{A}$ has LU decomposition $\mathbf{A} = \mathbf{LU}$, then

$$\mathbf{B} = \mathbf{PAP} = \mathbf{PLUP} = \mathbf{PL}(\mathbf{PP}^{-1})\mathbf{UP} = \underbrace{(\mathbf{PLP})}_{\mathbf{U}'}\underbrace{(\mathbf{PUP})}_{\mathbf{L}'}, \text{(5 points)}$$

by the results of 1), we have that

$$\mathbf{U}' = \mathbf{PLP} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \text{(2 points)} \qquad \mathbf{L}' = \mathbf{PUP} = \begin{bmatrix} 6 & 0 & 0 \\ -1 & 4 & 0 \\ 4 & 2 & 2 \end{bmatrix}. \text{(3 points)}$$

Problem 4 (7 points + 6 points + 7 points + 5 points) This problem is graded by **Yijia Chang (changyj@)**.

Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, suppose that the LDM (LDU) decomposition of $\mathbf{A}$ exists, prove that

1) the LDM (LDU) decomposition of $\mathbf{A}$ is *uniquely* determined;
2) if $\mathbf{A}$ is a symmetric matrix, then its LDM (LDU) decomposition must be $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, which is called LDL decomposition in this case;
3) $\mathbf{A}$ is a symmetric and positive definite matrix if and only if its Cholesky decomposition exists (i.e., there exists a matrix $\mathbf{G} \in \mathbb{R}^{n\times n}$ such that $\mathbf{A} = \mathbf{G}\mathbf{G}^T$, where $\mathbf{G}$ is lower triangular with *positive* diagonal entries);
4) if $\mathbf{A}$ is a symmetric and positive definite matrix, then its Cholesky decomposition is *uniquely* determined.

Hint: You can directly utilize the following lemmas,

- the inverse (if it exists) of a lower (resp. upper) triangular matrix is also lower (resp. upper) triangular;
- the product of two lower (resp. upper) triangular matrices is lower (resp. upper) triangular;
- also, if such two lower (resp. upper) triangular matrices have unit diagonal entries, then their product also has unit diagonal entries.

Solution:

1) Assume that $\mathbf{A}$ has two **LDM** decompositions as $\mathbf{A} = \mathbf{L}_1\mathbf{D}_1\mathbf{M}_1^T = \mathbf{L}_2\mathbf{D}_2\mathbf{M}_2^T$, we expect to prove that $\mathbf{L}_1 = \mathbf{L}_2$, $\mathbf{D}_1 = \mathbf{D}_2$, and $\mathbf{M}_1 = \mathbf{M}_2$.

   First, note that the existence of the LDM decomposition implies that $\mathbf{A}$ is nonsingular. Hence, the determinant of $\mathbf{A}$ satisfies $|\mathbf{A}| \neq 0$. Besides, since $|\mathbf{A}| = |\mathbf{L}_1| \times |\mathbf{D}_1| \times |\mathbf{M}_1| = |\mathbf{L}_2| \times |\mathbf{D}_2| \times |\mathbf{M}_2|$, we have that $\mathbf{L}_1$, $\mathbf{D}_1$, $\mathbf{M}_1$, $\mathbf{L}_2$, $\mathbf{D}_2$, and $\mathbf{M}_2$ are all nonsingular (*i.e.*, invertible).

   Second, since $\mathbf{L}_1\mathbf{D}_1\mathbf{M}_1^T = \mathbf{L}_2\mathbf{D}_2\mathbf{M}_2^T$, we have

   $$\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{D}_2\mathbf{M}_2^T\left(\mathbf{M}_1^T\right)^{-1}\mathbf{D_1}^{-1}.$$

   We can check that the left hand side $\mathbf{L}_2^{-1}\mathbf{L}_1$ is lower triangular, while the right hand side $\mathbf{D}_2\mathbf{M}_2^T\left(\mathbf{M}_1^T\right)^{-1}\mathbf{D_1}^{-1}$ is upper triangular. Hence, the left hand side and the right hand side are both diagonal matrices.

   Third, the diagonal entries of the left hand side $\mathbf{L}_2^{-1}\mathbf{L_1}$ must be one, which implies $\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{I}$ and accordingly $\mathbf{L}_1 = \mathbf{L}_2$. Similarly, we can derive that $\mathbf{M}_2^T\left(\mathbf{M}_1^T\right)^{-1} = \mathbf{D}_2^{-1}\mathbf{L}_2^{-1}\mathbf{L}_1\mathbf{D}_1$, which can further deduce that $\mathbf{M}_1 = \mathbf{M}_2$. Finally, we have $\mathbf{D}_1 = \mathbf{L}_1^{-1}\mathbf{A}\left(\mathbf{M}_1^T\right)^{-1} = \mathbf{L}_2^{-1}\mathbf{A}\left(\mathbf{M}_2^T\right)^{-1} = \mathbf{D}_2$, which concludes this proof.

   Note: If you utilize the uniqueness of LU decomposition to prove the uniqueness of LDU decomposition, you need to first prove the uniqueness of LU decomposition, then describe the relationships between such two decompositions.

2) Since $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{M}^T$ and $\mathbf{A}$ is symmetric, we have $\mathbf{L}\mathbf{D}\mathbf{M}^T = \mathbf{A} = \mathbf{A}^T = \mathbf{M}\mathbf{D}\mathbf{L}^T$. From the proof in 1), we learn that $\mathbf{L}$ and $\mathbf{M}^T$ are both invertible. Then we can derive that

   $$\mathbf{D}\mathbf{M}^T\left(\mathbf{L}^T\right)^{-1} = \mathbf{L}^{-1}\mathbf{M}\mathbf{D}.$$

   Note that $\left(\mathbf{L}^T\right)^{-1}$ and $\mathbf{M}^T$ are both upper triangular, while $\mathbf{L}^{-1}$ and $\mathbf{M}$ are both lower triangular. We can check that the left hand side $\mathbf{D}\mathbf{M}^T\left(\mathbf{L}^T\right)^{-1}$ is upper triangular and the right hand side $\mathbf{L}^{-1}\mathbf{M}\mathbf{D}$ is lower triangular. Hence, the left hand side and the right hand side are both diagonal matrices. Since $\mathbf{D}$ is a diagonal matrix, $\mathbf{L}^{-1}\mathbf{M}$ is also a diagonal matrix. Moreover, the diagonal entries of $\mathbf{L}^{-1}\mathbf{M}$ must be one, which implies $\mathbf{L}^{-1}\mathbf{M} = \mathbf{I}$ and accordingly $\mathbf{L} = \mathbf{M}$.

3) (4 points) $\mathbf{A}$ is a symmetric and positive definite matrix $\Rightarrow$ its Cholesky decomposition exists)

Note that a positive definite matrix must be nonsingular. According to the conclusion in 2), we have $\mathbf{A} = \mathbf{LDL}^T$. For any vector $\mathbf{x} \in \mathbb{R}^n$, there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} = \mathbf{L}^T\mathbf{x}$. Since A is a positive definite matrix, we can derive that

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{x}^T\mathbf{LDL}^T\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} > 0.$$

Hence, the diagonal entries of $\mathbf{D}$ are all positive. Let $\mathbf{D}^{'}$ denote a diagonal matrix, where $\mathbf{D}^{'}_{ii} = \sqrt{\mathbf{D}_{ii}}$ for $i = 1, 2, 3, ..., n$. Then we have

$$\mathbf{A} = \mathbf{LDL}^T = \mathbf{LD}^{'}(\mathbf{D}^{'})^T\mathbf{L}^T = \mathbf{LD}^{'}(\mathbf{LD}^{'})^T.$$

Let $\mathbf{G} = \mathbf{LD}^{'}$, then $\mathbf{A} = \mathbf{GG}^T$, where $\mathbf{G}$ is lower triangular with positive diagonal elements.

($\mathbf{A}$ is a symmetric and positive definite matrix $\Leftarrow$ its Cholesky decomposition exists)

(1 point) *First*, since $\mathbf{A}^T = (\mathbf{GG}^T)^T = \mathbf{GG}^T = \mathbf{A}$, we can learn that $\mathbf{A}$ is a symmetric matrix.

(2 points) *Second*, for any non-zero vector $\mathbf{x} \in \mathbb{R}^n$, we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{GG}^T\mathbf{x} = (\mathbf{G}^T\mathbf{x})^T(\mathbf{G}^T\mathbf{x}) > 0,$$

which implies that A is a positive definite matrix.

4) Assume that $\mathbf{A}$ has two Cholesky decompositions as $\mathbf{A} = \mathbf{G}_1\mathbf{G}_1^T = \mathbf{G}_2\mathbf{G}_2^T$, we expect to prove that $\mathbf{G}_1 = \mathbf{G}_2$.

First, since $\mathbf{G}_1$ and $\mathbf{G}_2$ are both lower triangular with positive diagonal entries, $\mathbf{G}_1$ and $\mathbf{G}_2$ must be invertible.

Second, since $\mathbf{A} = \mathbf{G}_1\mathbf{G}_1^T = \mathbf{G}_2\mathbf{G}_2^T$, we have

$$\mathbf{G}_1^{-1}\mathbf{G}_2 = (\mathbf{G}_1^{-1}\mathbf{G}_2)^T.$$

We can check that the left hand side $\mathbf{G}_1^{-1}\mathbf{G}_2$ is lower triangular, while the right hand side $(\mathbf{G}_1^{-1}\mathbf{G}_2)^T$ is upper triangular. Hence, the left hand side and the right hand side are both diagonal matrices. Let $\mathbf{G}_0 = \mathbf{G}_1^{-1}\mathbf{G}_2$, then $\mathbf{G}_0$ is a diagonal matrix.

Third, by $\mathbf{G}_1\mathbf{G}_1^T = \mathbf{G}_2\mathbf{G}_2^T$, we can also derive that

$$\mathbf{I} = \mathbf{G}_1^{-1}\mathbf{G}_2\mathbf{G}_2^T(\mathbf{G}_1^T)^{-1} = \mathbf{G}_0\mathbf{G}_0^T.$$

Hence, the diagonal entries of $\mathbf{G}_0$ must be 1 or -1. Note that $\mathbf{G}_2 = \mathbf{G}_1\mathbf{G}_0$. Considering the diagonal entries of $\mathbf{G}_2$ and $\mathbf{G}_1$, we have $(\mathbf{G}_2)_{ii} = (\mathbf{G}_1)_{ii} \times (\mathbf{G}_0)_{ii}$ for $i = 1, 2, ..., n$. Since the diagonal entries of $\mathbf{G}_2$ and $\mathbf{G}_1$ are required to be positive, $(\mathbf{G}_0)_{ii}$ can only be 1 for $i = 1, 2, ..., n$, *i.e.*, $\mathbf{G}_0 = \mathbf{I}$. Accordingly, $\mathbf{G}_1 = \mathbf{G}_2$, which concludes this proof.

Problem 5 (10 points + 5 points) This problem is graded by **Zhihang Xu (xuzhh@)**.

Consider matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ in the following form,

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix},$$

where a_j, b_j, and c_j are non-zero entries. The matrix in such form is known as a **Tridiagonal Matrix** in the sense that it contains three diagonals.

1) LU decomposition is particularly efficient in the case of tridiagonal matrices. Find the LU decomposition of $\mathbf{A}$ (derivation is expected) and try to complete the Algorithm 1.

Algorithm 1: LU decomposition for tridiagonal matrices

Input : Tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Output: LU decomposition of $\mathbf{A}$.

1 *Complete the algorithm here...*

2) Consider symmetric tridiagonal matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and } \mathbf{B} = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix},$$

and give the LU decompositions and the LDL^T decompositions of $\mathbf{A}$ and $\mathbf{B}$ respectively.

Solution:

1) Solution #1: Following the standard procedure of LU decomposition, we have that

$$\mathbf{M}_1 = \begin{bmatrix} 1 & & & \\ -\frac{a_2}{b_1} & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_1\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ 0 & b_2^{(1)} & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix}$$

with $b_2^{(1)} = -\frac{a_2}{b_1}c_1 + b_2$,

$$\mathbf{M}_2 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & -\frac{a_3}{b_2^{(1)}} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_2\mathbf{M}_1\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ 0 & b_2^{(1)} & c_2 & 0 & 0 & 0 \\ 0 & 0 & b_3^{(2)} & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix}$$

with $b_3^{(2)} = b_3 - \frac{a_3}{b_2^{(1)}}c_2$. After $n-1$ steps, we can obtain that

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & & & & & \\ \frac{a_2}{b_1} & 1 & & & & \\ & \frac{a_3}{b_2^{(1)}} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & \frac{a_n}{b_{n-1}^{(n-2)}} & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ 0 & b_2^{(1)} & c_2 & 0 & 0 & 0 \\ 0 & 0 & b_3^{(2)} & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & & b_{n-1}^{(n-2)} & c_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & b_n^{(n-1)} \end{bmatrix}}_{\mathbf{U}}$$

$$= \underbrace{\begin{bmatrix} 1 & & & & & \\ \alpha_2 & 1 & & & & \\ & \alpha_3 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & \alpha_n & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \beta_1 & c_1 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & c_2 & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & & \beta_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & \beta_n \end{bmatrix}}_{\mathbf{U}}. \tag{1}$$

By writing in an iterative form, we have that

$$\begin{aligned} \beta_1 &= b_1\,, \\ \alpha_k &= \frac{a_k}{\beta_{k-1}}\,, \quad k = 2, \ldots, n \\ \beta_k &= b_k - \alpha_k c_{k-1}\,, \quad k = 2, \ldots, n. \end{aligned}$$

Solution # 2: By the special structure of tridiagonal matrix, the LU decomposition of $\mathbf{A}$ must have the following form (you can verify it by hand),

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & & & & \\ \alpha_2 & 1 & & & \\ & \alpha_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha_n & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \beta_1 & c_1 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & c_2 & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & c_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & & \beta_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & \beta_n \end{bmatrix}}_{\mathbf{U}},$$

then by simple matrix multiplication we can obtain the same recursion relations. The complete algorithm is provided as follows.

Algorithm 2: LU decomposition for tridiagonal matrices

Input : Tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Output: LU decomposition of $\mathbf{A}$.

Initilization: $\beta_1 = b_1$.

for $k = 2, \ldots, n$ **do**

  if $\beta_k == 0$ **then**

    Stop the program.

  else

    $\alpha_k = a_k/\beta_{k-1}$,

    $\beta_k = b_k - \alpha_k c_{k-1}$.

  end

end

Give $\mathbf{L}$ and $\mathbf{U}$ in the form of (1).

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2) Use the conclusion from 1), the LU decompositions of $\mathbf{A}$ and $\mathbf{B}$ can be directly given as follows:

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{U}}, \quad \mathbf{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}}_{\mathbf{U}}.$$

By the definition of LDL decomposition, we have that $\mathbf{D} = \mathrm{Diag}(u_{11}, u_{22}, u_{33})$. Then the LDL decompositions of $\mathbf{A}$ and $\mathbf{B}$ are given as follows:

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{L}^T}, \quad \mathbf{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{L}^T}.$$

Remarks:

1) In sub-problem 1), the derivation of finding the LU decomposition of tridiagonal matrices takes 5 points (it means that, even if the results you give are not correct, you still can get 5 points), and the results takes 2 points. According to the conclusion, completing algorithm takes 3 points. Here we give two means of deviation, the first one is following the standard LU decomposition steps and the second one can be obtained by observing the structure of triangonal matrices, either way is accepted. The statement *By the special structure of tridiagonal matrix, the LU decomposition of* $\mathbf{A}$ *must have the following form ...* is essential. You cannot just give the LU decomposition of $\mathbf{A}$, in that case, you lose 5 points.
2) In the process of writing pseudo algorithm, there are some common mistakes.
   - **Input** and **Output** should be clearly identified. For example, in this problem, the output is matrices $\mathbf{L}$ and $\mathbf{U}$, then the relation between α_k, β_k and $\mathbf{L}, \mathbf{U}$ should be pointed clearly.
   - **Initialization** is needed in many cases.
   - Some **stopping criterion** should be considered for the completeness of the algorithm.
3) The intention of sub-problem 2) is that, according to the conclusion of sub-problem 1), you can directly give the LU decomposition of $\mathbf{A}$ and $\mathbf{B}$, and consequently, you can also quickly give the LDL decomposition simply by the definition.

III. PROGRAMMING

Problem 6 (5 points + 10 points) This problem is graded by **Song Mao (maosong@)**.

In this problem, we explore the efficiency of the LU method together with the classical linear system solvers we have learnt in linear algebra.

1) Derive the complexity of the LU decomposition. Particularly, how many flops does the LU decomposition require? The corresponding pseudo code (in Matlab) is provided as follows:

```
function [L,U]= Naive_lu(A)
    n = size(A,1)
    L = eye(n)
    U = A
    for k=1:n-1
        for j=k+1:n
            L(j,k)=U(j,k)/U(k,k)
            U(j,k:n)=U(j,k:n)-L(j,k)*U(k,k:n)
        end
    end
    for k=2:n
        U(k,1:k-1)=0
    end
end
```

2) Randomly generate a non-singular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^{n\times 1}$, then program the following methods to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$:

- **The inverse method:** Use the inverse of $\mathbf{A}$ to solve the problem, which can be written as,
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\,.$$
- **Cramer rule:** Suppose $\mathbf{x} = [x_1, \ldots, x_n]^T$, and we denote $\mathbf{A}_{-i}(\mathbf{b})$ the matrix that we replace i-th column of $\mathbf{A}$ with $\mathbf{b}$. Then we have
$$x_i = \frac{\det(\mathbf{A}_{-i}(\mathbf{b}))}{\det(\mathbf{A})}, i = 1, \ldots, n\,.$$
auss
- **Gauss Elimination:** We perform row operations on the augmented matrix $[\mathbf{A}|\mathbf{b}]$, and use back substitution to obtain the solution $\mathbf{x}$.
- **LU decomposition.** We first find the LU decomposition of $\mathbf{A}$, then we solve $\mathbf{L}\mathbf{y} = \mathbf{b}$ and $\mathbf{U}\mathbf{x} = \mathbf{y}$.

In your homework, you are required to submit the time-consuming plot (**one figure**) of given methods against the size of matrix $\mathbf{A}$ (i.e., n), where $n = 100, 150, \ldots, 1000$ (you can try larger n and see what will happen, be careful with memory use of your PC!).

Remarks:

- Do not use built-in Matlab function A/b, inv(A) or lu(A). Otherwise, your results will contradict the complexity analysis, and your score will be discounted. You can implement the simplest version of these methods by yourself.
- In Matlab, to randomly generate a matrix or a vector, you can use randn function to generate normally distributed random numbers.

Solution:

1) According to the pseudo code, the number of required flops F can be represented by
$$F = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} [1 + 2(n-k+1)] = \sum_{k=1}^{n-1} (n-k)[3 + 2(n-k)].$$
Let $t = n - k$. By $\sum_{t=1}^{n-1} t = \frac{1}{2}n(n-1)$ and $\sum_{t=1}^{n-1} t^2 = \frac{1}{6}n(n-1)(2n-1)$, we can derive that
$$F = \sum_{t=1}^{n-1} (2t^2 + t) = \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n.$$
Hence, the LU decomposition algorithm requires $\mathcal{O}(\frac{2}{3}n^3)$ flops.
2) The figure is shown in Figure 1, which is consistent with the complexity analysis. That is, LU decomposition is the most efficient one as the size of matrix grows; Inverse method and Cramer rule become slow and untractable.

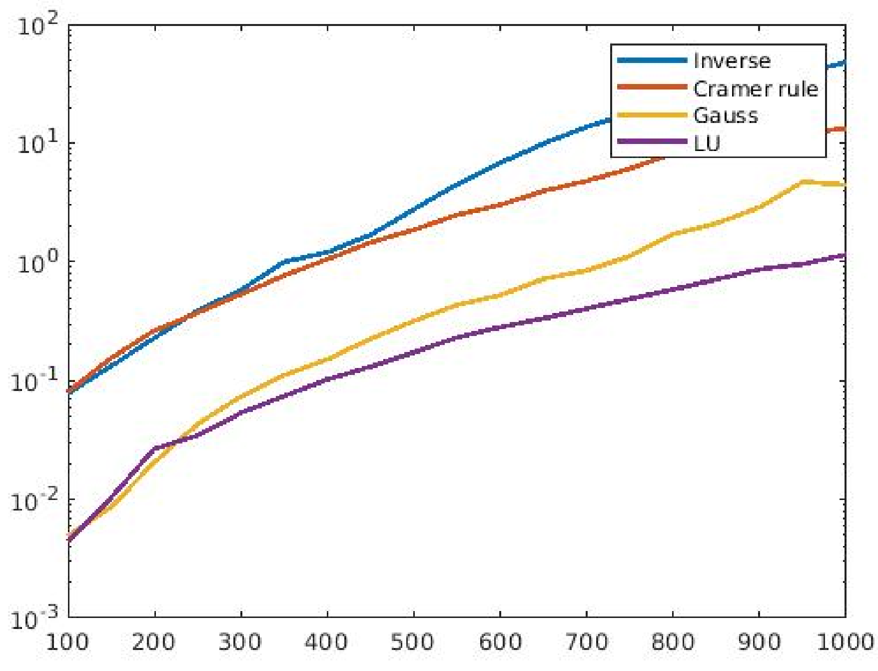

Figure 1: One example solution to problem III.2)

Remarks:

1) The definition of *flop* is: **The float operations of float numbers**. So the division($/$), multiplication($\times$), addition($+$) and subtraction($-$) should be taken into consideration. However, the assignment ($=$) is not an operation on float numbers by convention.
2) The aim of problem 2) is to: understand **why we need to learn matrix computation?** The reason is that the classic linear system solvers (Inverse method and Cramer Rule method) is untractable when the size of coefficient matrix becomes large.
3) The performances of Inverse method and Cramer Rule depend on implementation. For example, if you use the adjoint matrix $\mathbf{A}^*$ to obtain $\mathbf{A}^{-1}$, it will be far more complicated than Cramer Rule. However, if you solve the matrix equation $\mathbf{AB} = \mathbf{I}$, it will be faster than Cramer Rule. Both results are accepted.
4) As you can see, *the classic methods are slower than decomposition methods*. So if your result is in-consistent with this observation, you will lose points. You may wonder why Cramer Rule method is as good as Gauss Elimination method, this is because the *det()* function in MATLAB is optimized, you can implement a simple version of *det()* and the observation will become clear.

Grade Policy

1) If you didn't select pages for these two problems or say nothing about your solution to Problem 2), you will get 0 point. Please **specify pages related your solution when you submit Regrade Request**.
2) If you haven't provided any details for Problem 1) and your answer is wrong, you got 0 point. (**The answer is wrong**)
3) Technically speaking, LU method is a "matrix form" of Gauss Elimination method, their performances should be similar to each other, this is the reason why we add the Gauss Elimination method into this problem. Regardless of implementation, you will lose 7 points for this vital error.
4) The main goal of Problem 2) is to **compare** four methods, so if you didn't put the result of four methods into one figure, you will get 5 points deducted. (It's OK if you use two or more figures to explain your ambiguous results)
5) If you didn't put your result (one figure) in your PDF or in your code files, you will lose 5 points.
6) Please note that it's wrong to write $2/3\mathcal{O}(n^3)$, since $\mathcal{O}(\cdot)$ means *it's an order of* $\cdot$, Please use the correct form.

Problem 7 (Bonus Problem: 10 points + 8 points + 2 points) This problem is graded by **Xinyue Zhang (zhangxy11@)**.

Given a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, consider the roundoff error in the process of solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ by Gaussian elimination in three stages:

1. Decompose $\mathbf{A}$ into $\mathbf{L}\mathbf{U}$, in a machine with roundoff error $\mathbf{E}$, $\bar{\mathbf{L}}$ and $\bar{\mathbf{U}}$ are computed instead, i.e.,
$$\mathbf{A} + \mathbf{E} = \bar{\mathbf{L}}\bar{\mathbf{U}}\,.$$
2. Solving $\mathbf{L}\mathbf{y} = \mathbf{b}$, numerically with roundoff error $\delta\bar{\mathbf{L}}$, $\hat{\mathbf{y}} = \mathbf{y} + \delta\mathbf{y}$ are computed instead.
$$(\bar{\mathbf{L}} + \delta\bar{\mathbf{L}})(\mathbf{y} + \delta\mathbf{y}) = \mathbf{b}\,.$$
3. Solving $\mathbf{U}\mathbf{x} = \mathbf{y}$, numerically with roundoff error $\delta\bar{\mathbf{U}}$, $\hat{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ are computed instead.
$$(\bar{\mathbf{U}} + \delta\bar{\mathbf{U}})(\mathbf{x} + \delta\mathbf{x}) = \hat{\mathbf{y}}\,.$$

Finally, we can get the computed solution $\hat{\mathbf{x}}$ and
$$\begin{aligned}\mathbf{b} =&(\bar{\mathbf{L}} + \delta\bar{\mathbf{L}})(\bar{\mathbf{U}} + \delta\bar{\mathbf{U}})(\mathbf{x} + \delta\mathbf{x})\\ =&(\mathbf{A} + \delta\mathbf{A})(\mathbf{x} + \delta\mathbf{x})\,.\end{aligned}$$

1) Prove the relative error of $\mathbf{x}$ has an upper bound as follows
$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{1}{1 - \kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|},$$
where $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ denotes the condition number of matrix $\mathbf{A}$ (Suppose $\mathbf{A}$ and $\mathbf{A}+\delta\mathbf{A}$ are nonsingular and $\|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\| < 1$).

   Hint: The following equation might be useful,
$$\|(\mathbf{I} - \mathbf{B})^{-1}\| = \|\sum_{k=0}^{\infty}\mathbf{B}^k\| \leq \sum_{k=0}^{\infty}\|\mathbf{B}\|^k \leq \frac{1}{1 - \|\mathbf{B}\|}\,.$$
where $\mathbf{I} - \mathbf{B}$ is nonsingular and $\lim_{n\to\infty}\mathbf{B}^n = \mathbf{0}$.

2) Consider a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where
$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 10^{-10} & 10^{-10} \\ 1 & 10^{-10} & 10^{-10} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2(1 + 10^{-10}) \\ -10^{-10} \\ 10^{-10} \end{bmatrix}$$
find the solution $\mathbf{x}$, and calculate the condition number of $\mathbf{A}$ with the matrix infinite norm[2], i.e. $\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty\|\mathbf{A}^{-1}\|_\infty$. Suppose $|\delta\mathbf{A}| < 10^{-18}|\mathbf{A}|$[3], use $\kappa_\infty(\mathbf{A})$ to verify that
$$\|\delta\mathbf{x}\| < 10^{-7}\|\mathbf{x}\|.$$

[2]If $\mathbf{A} \in \mathbb{R}^{n\times n}$, then the matrix infinite norm is $\|\mathbf{A}\|_\infty = \max_{1<i<n}\sum_{j=1}^{n}|a_{i,j}|$.
[3]$|\mathbf{A}| \leq |\mathbf{B}|$ means each element in $\mathbf{A}$ is relative smaller to the corresponding element of $\mathbf{A}$

3) Discuss what you have observed from the previous 2 questions. What are the main factors that influence the relative error of the computed solution? Does the ill-conditioned matrix (i.e. the condition number is large) always lead to a large error of the solution?

Solution:

1) (10 points) The exact solution $\mathbf{x}$ can be represent as $\mathbf{A}^{-1}\mathbf{b}$, and the inexact solution $\hat{\mathbf{x}}$ can be represent as $(\mathbf{A}+\delta\mathbf{A})^{-1}\mathbf{b}$, then

$$\begin{aligned}
\frac{\|\hat{\mathbf{x}}-\mathbf{x}\|}{\|\mathbf{x}\|} &= \frac{\|(\mathbf{A}+\delta\mathbf{A})^{-1}\mathbf{b}-\mathbf{A}^{-1}\mathbf{b}\|}{\|\mathbf{A}^{-1}\mathbf{b}\|} \\
&= \frac{\|(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})^{-1}\mathbf{A}^{-1}\mathbf{b}-\mathbf{A}^{-1}\mathbf{b}\|}{\|\mathbf{A}^{-1}\mathbf{b}\|} \\
&= \frac{\|(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})^{-1}(\mathbf{A}^{-1}-(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})\mathbf{A}^{-1})\mathbf{b}\|}{\|\mathbf{A}^{-1}\mathbf{b}\|} \\
&= \frac{\|(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})^{-1}(-\mathbf{A}^{-1}\,\delta\mathbf{A}\,\mathbf{A}^{-1})\mathbf{b}\|}{\|\mathbf{A}^{-1}\mathbf{b}\|} \\
&\le \|(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})^{-1}\|\frac{\|\mathbf{A}^{-1}\,\delta\mathbf{A}\|\|\mathbf{A}^{-1}\mathbf{b}\|}{\|\mathbf{A}^{-1}\mathbf{b}\|} \\
&= \|(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})^{-1}\|\|\mathbf{A}^{-1}\,\delta\mathbf{A}\| \\
&\overset{\text{Neumann Series}}{\le} \frac{1}{1-\|\mathbf{A}^{-1}\delta\mathbf{A}\|}\|\mathbf{A}^{-1}\,\delta\mathbf{A}\| \\
&\le \frac{1}{1-\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}.
\end{aligned}$$

2) The solution $\mathbf{x}$ and the answer of condition number are as follows

$$\mathbf{x} = [10^{-10}, -1, 1]^T, \quad \text{(3points)}$$

$$\kappa_\infty(\mathbf{A}) = 2(10^{10}+1) \approx 2\times 10^{10}. \quad \text{(3 points)}$$

Since $|\delta\mathbf{A}| < 10^{-18}|A|$, then $\|\delta\mathbf{A}\|_\infty < 10^{-18}\|A\|_\infty$.

$$\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} < 10^{-18}\kappa(\mathbf{A}) = 2\times 10^{-8}.$$

Use the inequality in 1) we can verify that

$$\|\delta\mathbf{x}\| < \frac{1}{\frac{1}{\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}-1} \approx 2\times 10^{-8}\|\mathbf{x}\| < 10^{-7}\|\mathbf{x}\|. \quad \text{(2 points)}$$

3)
- (1 points) The main factors in the accuracy of the computed solution are condition number of $\mathbf{A}$ and the perturbation in $\mathbf{A}$ because of roundoff error.
- (1 points) Even though the matrix $\mathbf{A}$ is ill-conditioned (condition number is very large), a small perturbation of $\mathbf{A}$ might not cause a large change of the solution.