SI231 - Matrix Computations, Fall 2020-21

Homework Set #1

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Acknowledgements:

1) Deadline: **2020-09-27 23:59:59**
2) No handwritten is accepted. You need to use LaTeX. (If you have difficulty in using LaTeX, you are allowed to use **Word** for the first and the second homework to accommodate yourself.)
3) Do use the given template.

I. Understanding rank, range space and null space

Problem 1. (4 points $\times$ 5)

1) For matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, prove that $\mathbb{R}^n = \mathcal{N}(\mathbf{A}) \oplus \mathcal{R}(\mathbf{A}^T)$ [1].

   Hint: $\mathsf{dim}(\mathcal{N}(\mathbf{A})) + \mathsf{dim}(\mathcal{R}(\mathbf{A}^T)) = n$.
2) For matrices $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{B} \in \mathbb{R}^{m\times n}$, prove that $\mathsf{rank}(\mathbf{A}+\mathbf{B}) \leq \mathsf{rank}(\mathbf{A}) + \mathsf{rank}(\mathbf{B})$.
3) For matrices $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{B} \in \mathbb{R}^{n\times p}$, prove that $\mathsf{rank}(\mathbf{AB}) \leq \min\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\}$ and $\mathsf{rank}(\mathbf{AB}) = n$ only when $\mathbf{A}$ has full-column rank and $\mathbf{B}$ has full-row rank.
4) For matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times p}$, prove that $\mathcal{R}(\mathbf{A}|\mathbf{B}) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B})$ [2]
5) For matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times p}$, prove that

$$\mathsf{rank}(\mathbf{A}|\mathbf{B}) = \mathsf{rank}(\mathbf{A}) + \mathsf{rank}(\mathbf{B}) - \mathsf{dim}(\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}))\,.$$

   Hint: Recall the result in 4).

[1] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subspaces of $\mathbb{R}^n$, if $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$ and $\mathcal{S}_1 + \mathcal{S}_2 = \mathbb{R}^n$, we define the **direct sum** $\mathbb{R}^n = \mathcal{S}_1 \oplus \mathcal{S}_2$.

[2] Here $\mathbf{A}|\mathbf{B}$ denotes a new matrix combined by $\mathbf{A}$ and $\mathbf{B}$. For example, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$, then $\mathbf{A}|\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \end{bmatrix}$.

II. UNDERSTANDING SPAN, SUBSPACE

Problem 1. (10 points) For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, prove that span($\mathcal{S}$) is the intersection of all subspaces that contain $\mathcal{S}$, i.e., prove that span$(\mathcal{S}) = \mathcal{M}$ where $\mathcal{M} := \cap_{s \subseteq \mathcal{V}} \mathcal{V}$ is the intersection of all subspaces that contain $\mathcal{S}$ and $\mathcal{V}$ denotes the subspace containing $\mathcal{S}$.

Hint: Prove that span$(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq$ span$(\mathcal{S})$.

III. Basis, dimension and projection

Problem 1. (2 points $\times$ 2) Determine the dimension of each of the following vector spaces:

1) The space of polynomials having degree n or less;
2) The space of $n \times n$ symmetric matrices.

Problem 2. Some Important linear transformations

1) **Rotations.** (6 points) A rotation matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix ($\mathbf{R}\mathbf{R}^T = \mathbf{I}$) such that $\det(\mathbf{R}) = 1$.
   - According to above definition, find all rotation matrix in $\mathbb{R}^{2\times 2}$.
   - Geometrically, if $\mathbf{R} \in \mathbb{R}^{2\times 2}$, then $\mathbf{R}\mathbf{x}$ means we rotate the vector $\mathbf{x} \in \mathbb{R}^2$ from some angle $\theta \in [0, 2\pi]$ in anti-clockwise direction. For $\mathbf{x} = [\cos(\pi/4), \sin(\pi/4)]^T$, compute $\mathbf{R}\mathbf{x}$, where $\mathbf{R}$ represents the matrix that rotating $\mathbf{x}$ by $7\pi/12$ in anti-clockwise direction.

     Hint: draw a plot of $\mathbf{x}$ and $\mathbf{R}\mathbf{x}$.
2) **Reflections.** (8 points) Let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector, $\|\mathbf{u}\|_2 = 1$. For a given vector $\mathbf{x} \in \mathbb{R}^n$ and a hyperplane $\mathcal{H}_u = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{u}^T\mathbf{x} = 0\}$. Let $\mathbf{Q} = \mathbf{I} - \mathbf{u}\mathbf{u}^T$. Then a vector $\mathbf{y} \in \mathbb{R}^n$ is said to be a *reflection* of $\mathbf{x}$ with respect to $\mathcal{H}$ if their projections onto the hyperplane $\mathcal{H}$ (denoted as $\mathbf{Q}\mathbf{x}$ and $\mathbf{Q}\mathbf{y}$ respectively) satisfy

$$\mathbf{Q}\mathbf{x} = \mathbf{Q}\mathbf{y}\,, \quad \|\mathbf{x} - \mathbf{Q}\mathbf{x}\|_2 = \|\mathbf{y} - \mathbf{Q}\mathbf{y}\|_2\,.$$

   See Figure.1 for visualization.

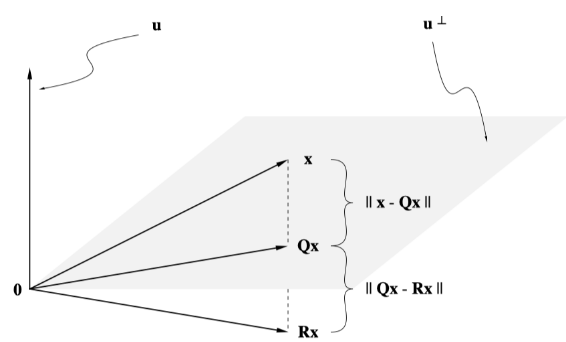

Figure 1. Reflection of $\mathbf{x}$

   A Householder matrix has the form $\mathbf{H} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$. Prove that $\mathbf{H}\mathbf{x}$ is a reflection of $\mathbf{x}$ with respect to $\mathcal{H}_u$.

IV. DIRECT SUM

Problem 1. (10 points) Let $\mathcal{V}$ be a vector space, and $\mathcal{B}$ be a basis for $\mathcal{V}$. Suppose that there exist subsets $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{B}$, such that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Then show that $\mathcal{V} = \text{span}(\mathcal{B}_1) \oplus \text{span}(\mathcal{B}_2)$.

Problem 2. (10 points) Let $\mathcal{V}$ be a real vector space of dimension n. Let $\mathcal{S}$ be a subspace of $\mathcal{V}$ of dimension $d \leq n$. Prove that there exists a subspace $\mathcal{T}$ of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$.

V. Understanding the Matrix Norm

Problem 1. (7 points $\times$ 2) Matrix norm is induced by vector norm,

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|\mathbf{A}\mathbf{x}\|_p\,, \quad \mathbf{A}\in\mathbb{R}^{m\times n}, \mathbf{x}\in\mathbb{R}^{n\times 1}\,,$$

prove that

1) the matrix 1-norm

$$\begin{aligned}\|\mathbf{A}\|_1 = \max_{\|\mathbf{x}\|_1=1} \|\mathbf{A}\mathbf{x}\|_1 &= \max_j \sum_i^m |a_{ij}| \\ &= \text{the largest absolute column sum.}\end{aligned}$$

2) the matrix ∞-norm

$$\begin{aligned}\|\mathbf{A}\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|\mathbf{A}\mathbf{x}\|_\infty &= \max_i \sum_j^n |a_{ij}| \\ &= \text{the largest absolute row sum.}\end{aligned}$$

VI. UNDERSTANDING THE HÖLDER INEQUALITY

Problem 1. (6 points $\times$ 3) Hölder inequality:

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q \,,$$

for any p, q such that $1/p + 1/q = 1$, $p \geq 1$. Derive this inequality by exexcuting the following steps:

1) Consider the function $f(t) = (1-\lambda) + \lambda t - t^\lambda$ for $0 < \lambda < 1$, establish the inequality

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta \,,$$

for nonnegative real numbers α and β.

2) Let $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|_p$ and $\hat{\mathbf{y}} = \mathbf{y}/\|\mathbf{y}|_q$, and apply the inequality of part 1) to obtain

$$\sum_{i=1}^{n} |\hat{x}_i \hat{y}_i| \leq \frac{1}{p}\sum_{i=1}^{n} |\hat{x}_i|^p + \frac{1}{q}\sum_{i=1}^{n} |\hat{y}_i|^q = 1 \,.$$

3) Deduce the Hölder inequality with the above results.

4) (Bouns question: 10 points) Prove the general form of triangle inequality

$$\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p \,.$$

Hint: For $p > 1$, let q be the number such that $1/q = 1 - 1/p$. Verify that for scalars α and β,

$$|\alpha+\beta|^p = |\alpha+\beta||\alpha+\beta|^{p/q} \leq |\alpha||\alpha+\beta|^{p/q} + |\beta||\alpha+\beta|^{p/q}$$

and make use of Hölder's inequality.