Assignment 1 - COMP 2804

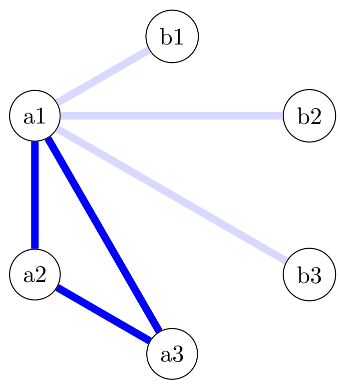
October 20, 2024

1. Extension of the Erdős–Rényi–Sós Friendship Theorem

Given that we know that a complete graph K_6 has at least one monochromatic cycle:

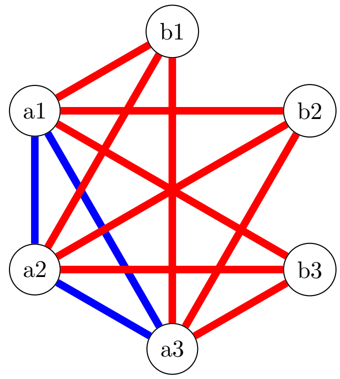
1. We know that a complete graph K_6 whose edges consist of 2 colors has at least one monochromatic cycle. Within that cycle, each of its vertices has at least 2 edges that are one color which forms the cycle, and 3 others of any color. Based on this, we can conclude the following:

   - If any of the remaining 3 edges connected to any of the vertices in the monochromatic cycle were the same color as the cycle itself, this would produce a monochromatic lollipop.

     Any connection to the cycle via a_1 creates a lollipop. We can observe that the same is true for a_2 and a_3.

   - Assume the monochromatic cycle is blue. If all of the 3 vertices in the monochromatic cycle had their remaining 3 edges be red, then in the cycle a_1, a_2, a_3, each vertex has a red edge connecting it to b_1, b_2 and b_3. We thus have red edges $a_i b_j$ for each $i, j \in \{1, 2, 3\}$ which fulfills the definition of a $K_{3,3}$ graph.



Red $K_{3,3}$

Thus we have proven that any complete graph K_6 colored with 2 colors, must contain either a monochromatic lollipop or monochromatic $K_{3,3}$. If we choose to make the monochromatic triangle red instead, the same result will occur with the colors inverted.

2. Proving a binomial identity

To create a bijective function $f : S_0 \to S_1$, we need to make a function that for every element of S_0, there will be a single unique element S_1. Consider the following function

> Take a random element x where $x \in S$
> Let $\{e_1, ..., e_n\}$ be the elements of S_0 and $\{o_1, ..., o_m\}$ be the elements of S_1
> For i in n:
>   If $x \in e_i$, remove x from e_i
>   If $x \notin e_i$, add x to e_i

If we take a set with an even number of elements and either add or remove one, we will always end up with a set of odd numbers. Since all elements in the subsets including x are in S, for each set in S_0 there will be a a set in S_1 that is mapped onto by the function. Note that if $e_i = S$, $x \in e_i$ so we will never get a set that is greater than S.

3. A summation game

Let's play a game. We start with a list of numbers $1, 2, 3, \ldots, 2024$. We take turns by picking two numbers from the list, say a and b, and removing them. Then, we insert exactly one of the numbers $|a - b|$ or $a + b$ into the list. Note that the length of the list decreases by one after each change. The game stops when only one number is left on the list. Prove that the final remaining number is even.
The summation of the sequence is:

$$\begin{aligned} \sum_{i=1}^{2024} i &= \frac{2024(2025)}{2} \\ &= 2049300 \end{aligned}$$

$\therefore$ The summation of the sequence is an even number.
When we remove 2 numbers a and b, and insert a + b, the sum remains the same since $\sum 0 = 0$
When we remove 2 numbers a and b, and insert $|a - b|$, the sum is reduced by either $2a$ or $2b$, depending on which is lesser. In this case, the sum will always remain even because $\sum 2n = 2\sum n$ where is is some integer.
$\therefore$ Every time we change the list, the sum remains even.
$\therefore$ When there is only one number left, that number must be even for the sum of the list to be even. $\square$

4. A predicate logic exercise

Prove that Resolution is valid using the laws of propositional logic and any of the other rules of inference besides Resolution.
To get to $b \lor c$ given the following arguments:

(1) $a \lor b$

(2) $\neg a \lor c$

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(3) $(a \implies c)$	Implication Relation (2)
(4) $(\neg a \implies b)$	Implication Relation (1)
(5) $(\neg b \implies \neg\neg a)$	Contrapositive (4)
(6) $(\neg b \implies a)$	Double Negation (5)
(7) $\neg b \implies c$	Hypothetical Syllogism (6, 3)
(8) $b \lor c$	Implication Relation (6)

$\square$

5. Proof by induction

To prove the closed form for the following recursive function:

$$f(n) = \begin{cases} 0 & \text{where } n = 0 \\ 1 & \text{where } n = 1 \\ 4 \cdot f(n-2) & \text{where } n \geq 2 \end{cases}$$

Base Cases: $f(0) = 2^{k-2}((-1)^{k+1} + 1) = 0$, $f(1) = 2^{k-2}((-1)^{k+1} + 1) = 1$
Inductive Hypothesis: $f(k) = 2^{k-2}((-1)^{k+1} + 1)$ for all $k \in \{0, ..., n-1\}$

$$\begin{aligned} f(n) &= 4 \times f(n-2) \\ &= 4 \times (2^{k-4}((-1)^{k-1} + 1) \\ &= 2^{k-2}((-1)^{k-1} + 1 \\ &= 2^{k-2}(\frac{-1 \times -1}{-1 \times -1}(-1)^{k-1} + 1) \\ &= 2^{k-2}\frac{(-1)^2(-1)^{k-1} + 1}{1} \\ &= 2^{k-2}((-1)^{k+1} + 1) \end{aligned} \tag{1}$$

This is equal to the inductive hypothesis so we have proven that it is equal to the recursive function.