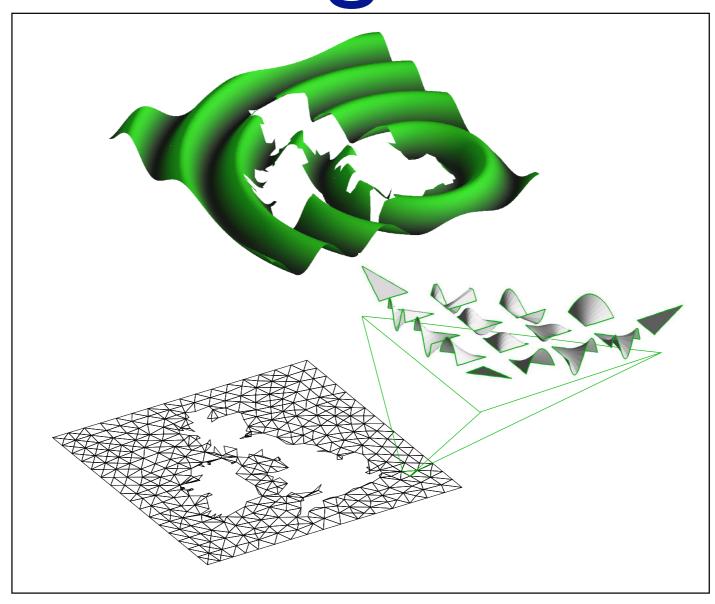
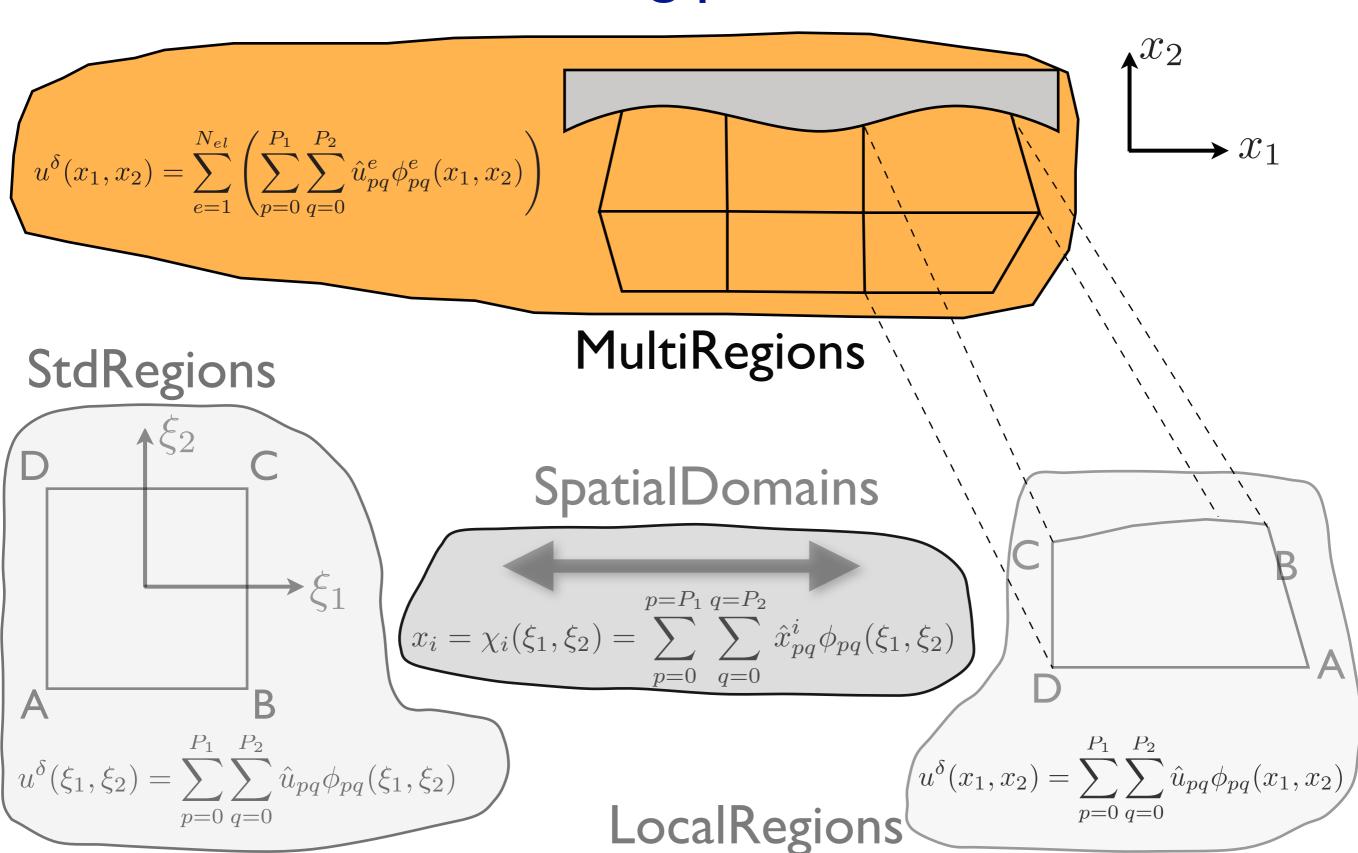
Expansions in Multiple Regions



The big picture



Outline

- 1D formulation of Helmholtz Problem
 - Elemental formulation
 - Global assembly
- 2D Formulation of Helmholtz problem
 - Global assembly ——

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Helmholtz problem

Poisson Equation: $\nabla^2 u - \lambda u = f$

$$\nabla^2 u - \lambda u = f$$

ID:
$$\mathbb{L}(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0, \qquad u(0) = g_{\mathcal{D}}, \qquad \frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}.$$

$$u(0) = g_{\mathcal{D}}, \qquad \frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}$$

Integral formulation (MWR):
$$\int_0^l v \frac{\partial^2 u}{\partial x^2} - \int_0^l \lambda v u \ dx + \int_0^l v f \ dx = 0.$$

Integrate by parts:
$$\int_0^l v \frac{\partial^2 u}{\partial x^2} dx = \left[v \frac{\partial u}{\partial x} \right]_0^l - \int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx$$

Weak formulation:
$$\int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \int_0^l \lambda v u \ dx = \int_0^l v f \ dx + \left[v \frac{\partial u}{\partial x} \right]_0^l$$

Enforcing Neumann BC's:

$$\begin{bmatrix} v(l) \frac{\partial u}{\partial x} \Big|_{l} - v(0) \frac{\partial u}{\partial x} \Big|_{0} \end{bmatrix}$$

Helmholtz problem:

Imposing Dirichlet boundary conditions

ID:
$$\mathbb{L}(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0,$$
 $u(0) = g_{\mathcal{D}},$ $\frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}.$

$$u(0) = g_{\mathcal{D}}, \qquad \frac{\partial}{\partial t}$$

$$\frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}$$

Weak formulation:
$$\int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \int_0^l \lambda v u \ dx = \int_0^l v f \ dx + \left[v \frac{\partial u}{\partial x} \right]_0^l$$

$$u^{\delta} = u^{\mathcal{D}} + u^{\mathcal{H}}$$

Lift BC:
$$u^{\delta} = u^{\mathcal{D}} + u^{\mathcal{H}}$$
 $u^{\mathcal{D}}(0) = g_{\mathcal{D}}$ $u^{\mathcal{H}}(0) = 0$

$$u^{\mathcal{H}}(0) = 0$$

(Homogenize problem)

$$\int_{0}^{l} \frac{\partial v^{\mathcal{H}}}{\partial x} \frac{\partial u^{\mathcal{H}}}{\partial x} dx + \lambda \int_{0}^{l} v^{\mathcal{H}} u^{\mathcal{H}} dx = \int v f^{*} dx$$
$$f^{*} = f - v(l)g_{\mathcal{N}} - \frac{\partial v^{\mathcal{D}}}{\partial x} \frac{\partial u^{\mathcal{D}}}{\partial x} + \lambda v^{\mathcal{H}} u^{\mathcal{H}}$$

Discrete Approximation

Global approximation - C^0

$$u \Rightarrow u^{\delta} = \sum \hat{u}_i \Phi_i(x)$$

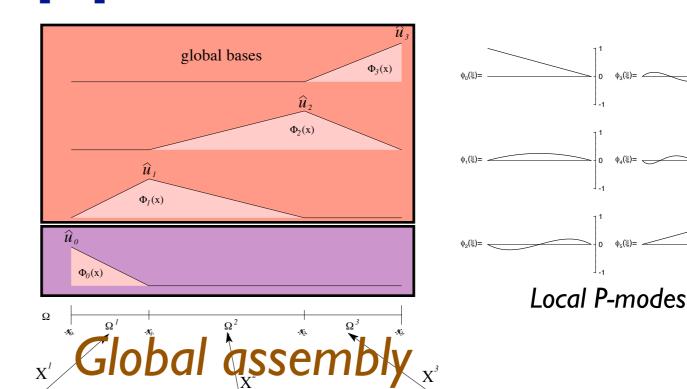
$$v \Rightarrow v^{\delta} = \sum_{i}^{t} \hat{v}_{i} \Phi_{i}(x)$$

$$u^{\mathcal{D}}(0) = g_{\mathcal{D}}$$

$$u^{\mathcal{D}}(0) = g_{\mathcal{D}} \qquad u^{\mathcal{H}}(0) = 0$$

Local approximation

$$u^{\delta} = \sum_{i} \hat{u}_{i} \Phi_{i}(x) = \sum_{e}^{net} \sum_{p} \hat{u}_{p}^{e} \phi_{p}(x)$$



local bases

$$\int_{0}^{l} \frac{\partial v^{\mathcal{H}}}{\partial x} \frac{\partial u^{\mathcal{H}}}{\partial x} dx + \lambda \int_{0}^{l} v^{\mathcal{H}} u^{\mathcal{H}} dx = \int v f^{*} dx$$
$$f^{*} = f - v(l) g_{\mathcal{N}} - \frac{\partial v^{\mathcal{D}}}{\partial x} \frac{\partial u^{\mathcal{D}}}{\partial x} + \lambda v^{\mathcal{H}} u^{\mathcal{H}}$$

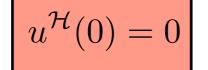
Discrete spaces

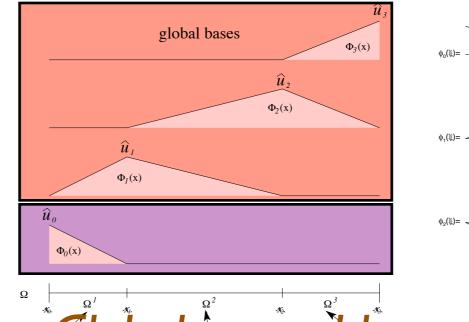
Global approximation - C^0

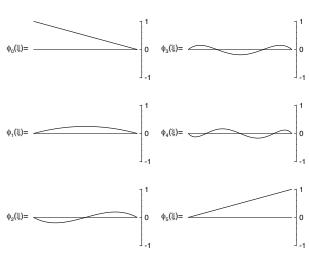
$$u \Rightarrow u^{\delta} = \sum \hat{u}_i \Phi_i(x)$$

$$v \Rightarrow v^{\delta} = \sum_{i}^{\delta} \hat{v}_{i} \Phi_{i}(x)$$

$$u^{\mathcal{D}}(0) = g_{\mathcal{D}}$$



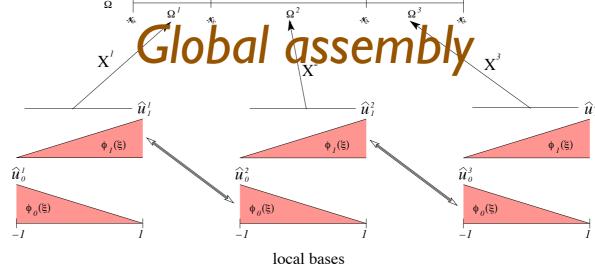




Local P-modes

Local approximation

$$u^{\delta} = \sum_{i} \hat{u}_{i} \Phi_{i}(x) = \sum_{e}^{net} \sum_{p} \hat{u}_{p}^{e} \phi_{p}(x) \tilde{u}_{p}^{e}$$



$$\sum_{i} \hat{v}_{j} \left\{ \sum_{j} \int_{0}^{l} \left[\frac{\partial \Phi_{i}^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_{j}^{\mathcal{H}}}{\partial x} + \lambda \Phi_{i}^{\mathcal{H}} \Phi_{j}^{\mathcal{H}} \right] \hat{u}_{j} dx = \int \Phi_{i}^{\mathcal{H}} f^{*} dx \right\}$$

$$\mathbf{L}[i][j] \qquad \mathbf{M}[i][j] \qquad \mathbf{f}[i]$$

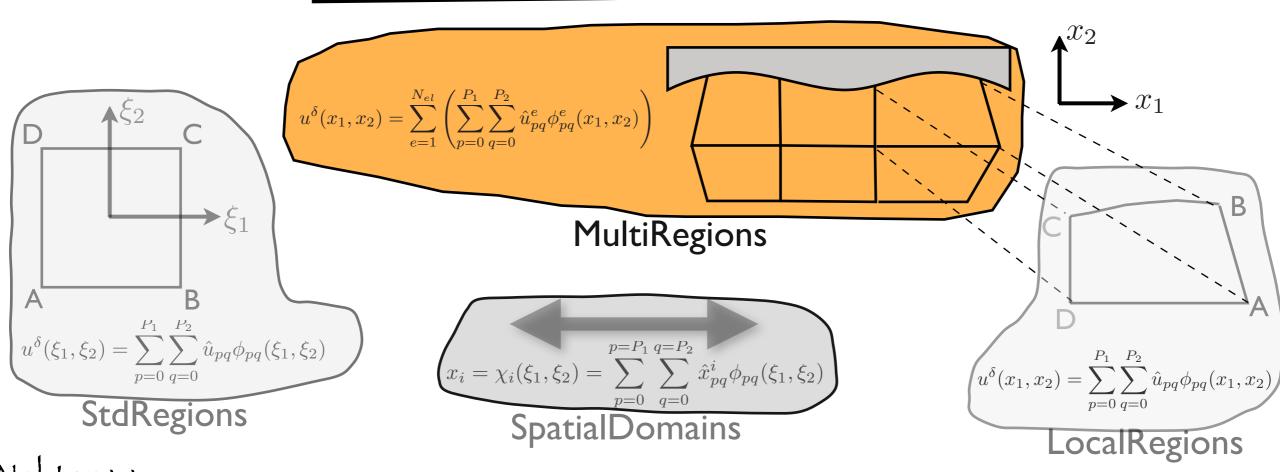
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 - Elemental formulation
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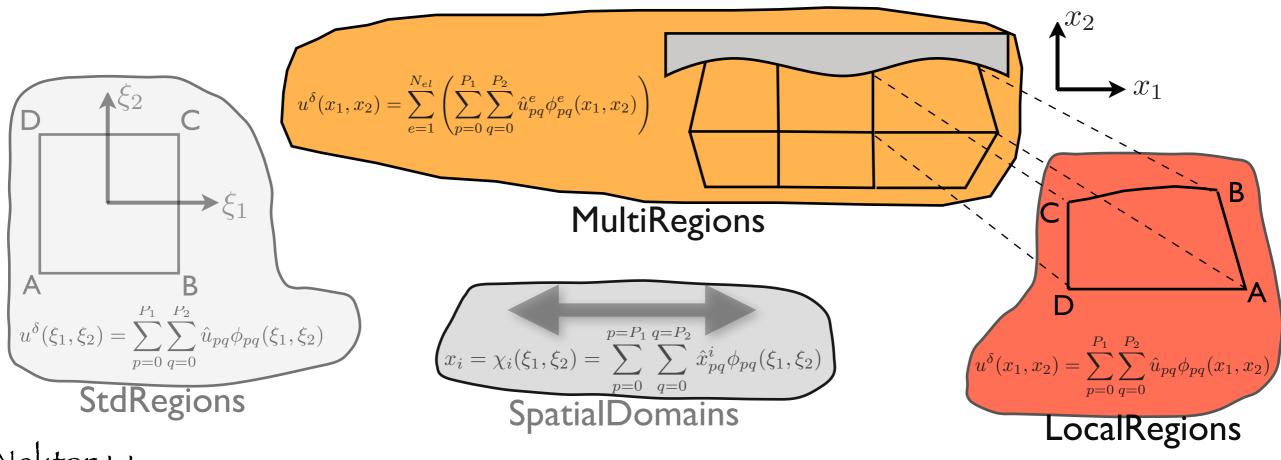
$$\sum_{i} \hat{v}_{j} \left\{ \sum_{j} \int_{0}^{l} \left[\frac{\partial \Phi_{i}^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_{j}^{\mathcal{H}}}{\partial x} + \lambda \Phi_{i}^{\mathcal{H}} \Phi_{j}^{\mathcal{H}} \right] \hat{u}_{j} dx = \int \Phi_{i}^{\mathcal{H}} f^{*} dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx$$



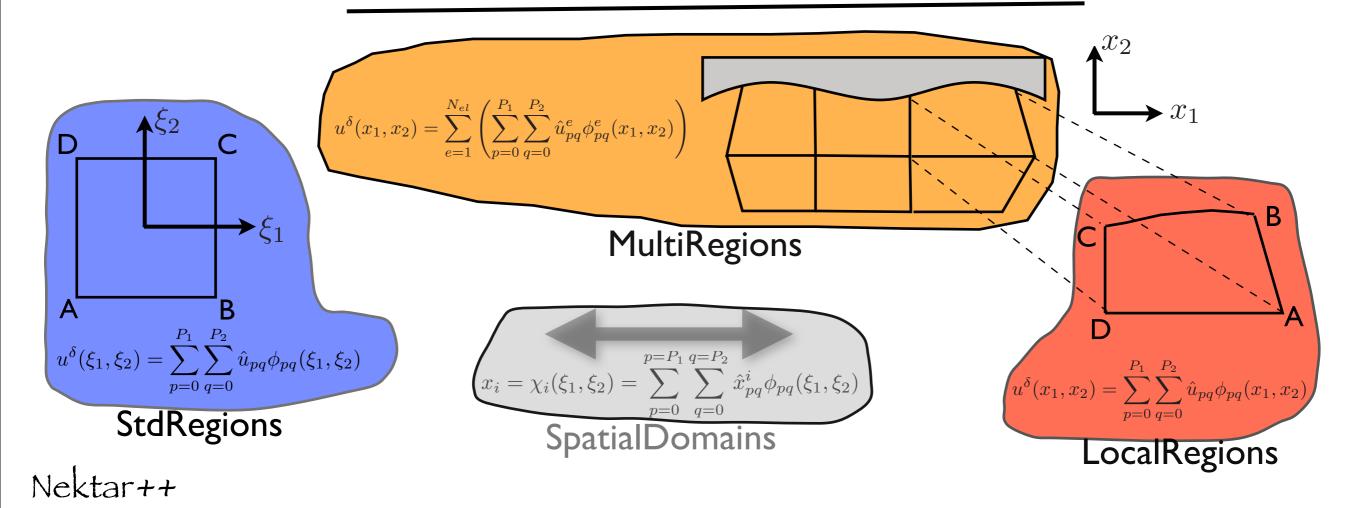
$$\sum_{i} \hat{v}_{j} \left\{ \sum_{j} \int_{0}^{l} \left[\frac{\partial \Phi_{i}^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_{j}^{\mathcal{H}}}{\partial x} + \lambda \Phi_{i}^{\mathcal{H}} \Phi_{j}^{\mathcal{H}} \right] \hat{u}_{j} dx = \int \Phi_{i}^{\mathcal{H}} f^{*} dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega_e} \phi_p(x) f^* dx = \sum_e^{nel} \sum_p \int_{\Omega_e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$



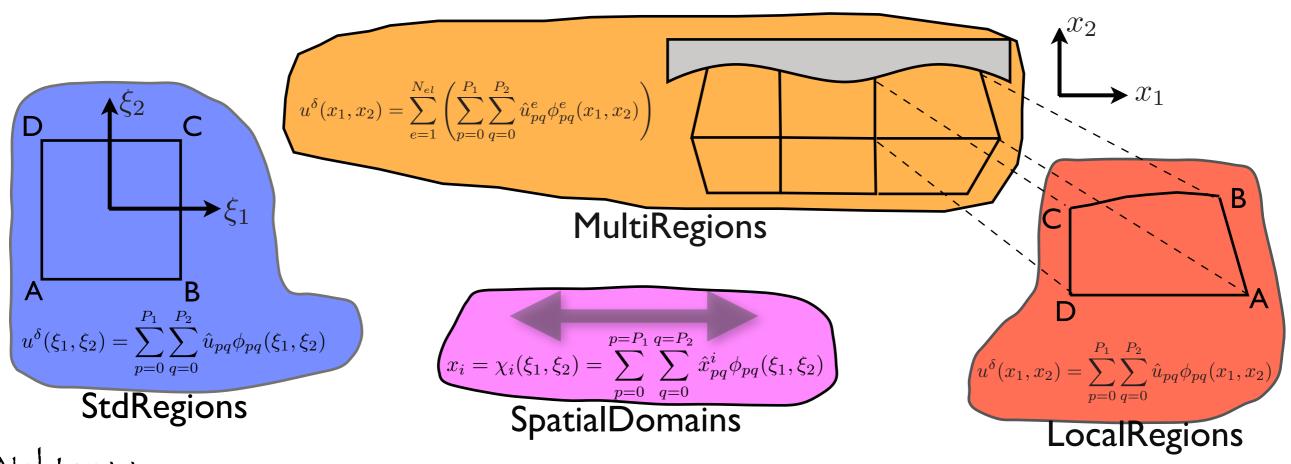
$$\sum_{i} \hat{v}_{j} \left\{ \sum_{j} \int_{0}^{l} \left[\frac{\partial \Phi_{i}^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_{j}^{\mathcal{H}}}{\partial x} + \lambda \Phi_{i}^{\mathcal{H}} \Phi_{j}^{\mathcal{H}} \right] \hat{u}_{j} dx = \int \Phi_{i}^{\mathcal{H}} f^{*} dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(x) f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$



$$\sum_{i} \hat{v}_{j} \left\{ \sum_{j} \int_{0}^{l} \left[\frac{\partial \Phi_{i}^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_{j}^{\mathcal{H}}}{\partial x} + \lambda \Phi_{i}^{\mathcal{H}} \Phi_{j}^{\mathcal{H}} \right] \hat{u}_{j} dx = \int \Phi_{i}^{\mathcal{H}} f^{*} dx \right\}$$

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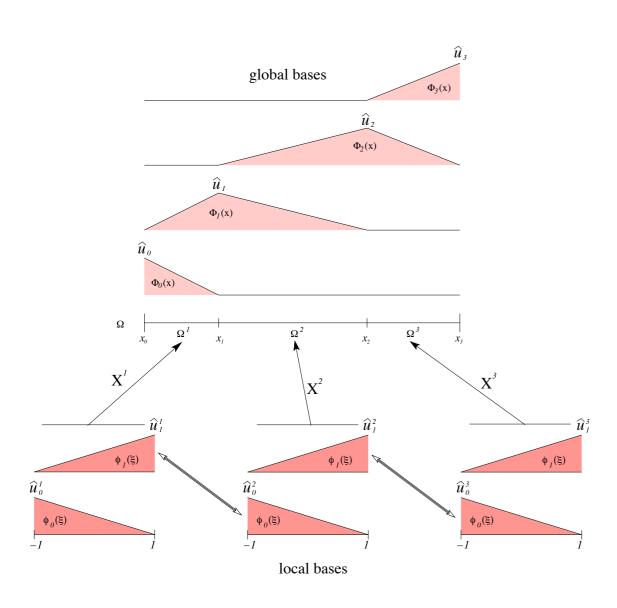


Outline

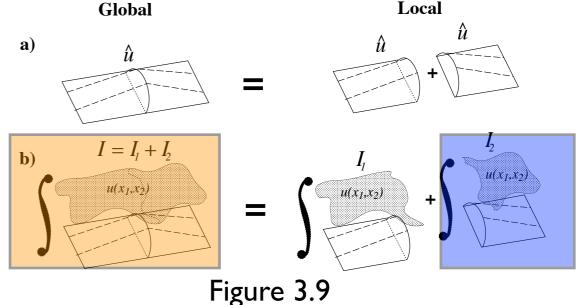
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Global Assembly



$$\max[1][i] = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \qquad \max[2][i] = \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} \qquad \max[3][i] = \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\}$$



$$egin{aligned} I^e[i] &= \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi \ & f[i] &= \int \Phi_i f^* dx \end{aligned}$$

Do
$$e = 1, N_{el}$$

Do $i = 0. N_{m}^{e} - 1$

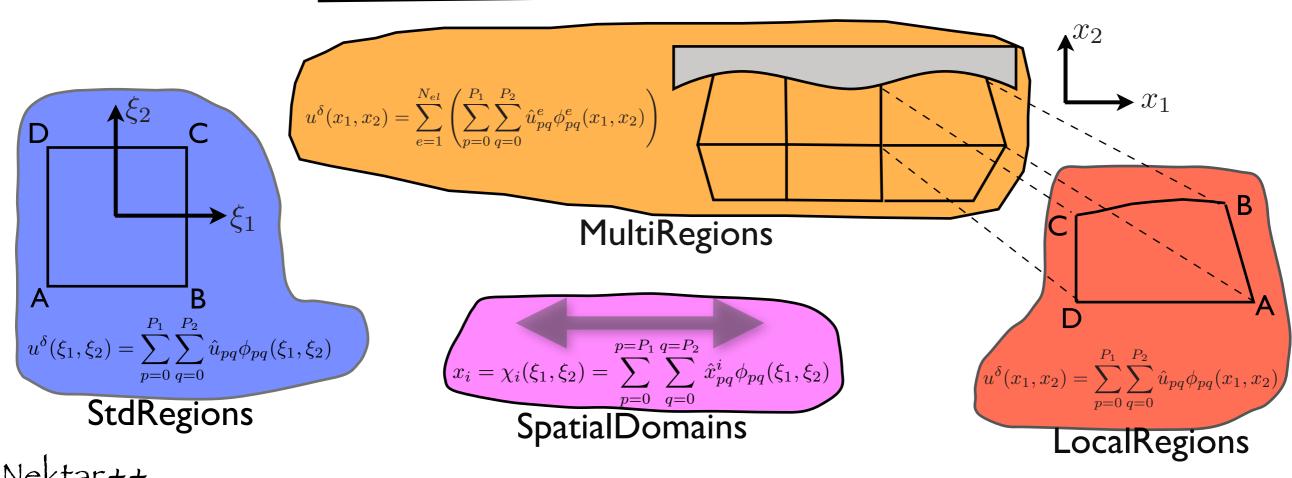
$$f[\text{map}[e][i]] = f[\text{map}[e][i]] + I^{e}[i]$$
continue
continue

Course Notes: Sections 1.3.1.4 & 3.2.1

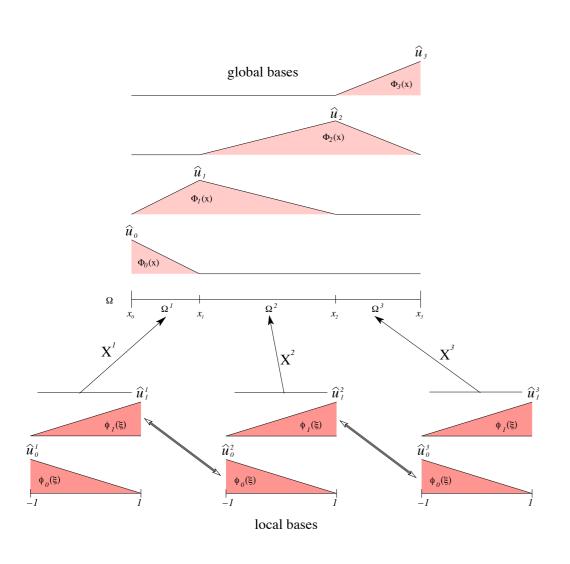
Matrix Construction

$$\sum_{i} \hat{v}_{j} \left\{ \sum_{j} \int_{0}^{l} \left[\frac{\partial \Phi_{i}^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_{j}^{\mathcal{H}}}{\partial x} + \lambda \Phi_{i}^{\mathcal{H}} \Phi_{j}^{\mathcal{H}} \right] \hat{u}_{j} dx = \int \Phi_{i}^{\mathcal{H}} f^{*} dx \right\}$$

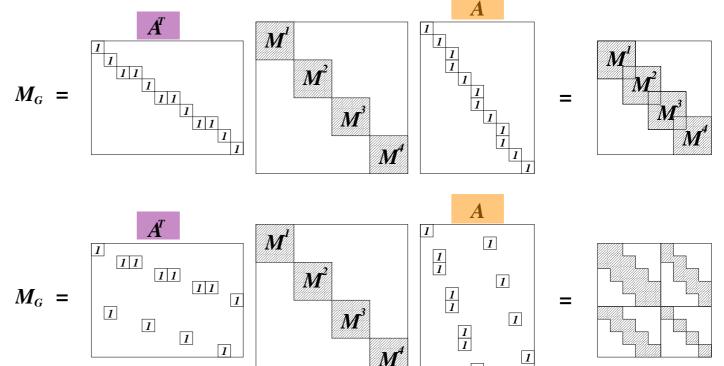
$$\mathbf{M}[i][j] = \int_{\Omega} \Phi_i \Phi_j dx = \sum_{e}^{nel} \sum_{p} \sum_{q} \int_{\Omega^e} \phi_p(x) \phi_q(x) dx$$
$$= \sum_{e}^{nel} \sum_{p} \sum_{q} \int_{-1}^{1} \phi_p(\chi^e(\xi)) \phi_q(\chi^e(\xi)) j^e d\xi$$



Matrix construction



$$\mathrm{map}[1][i] = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \qquad \mathrm{map}[2][i] = \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} \qquad \mathrm{map}[3][i] = \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\}$$



$$\begin{array}{l}
\text{Do } e = 1, N_{el} \\
\text{Do } i = 0, N_m^e - 1 \\
\hat{\boldsymbol{u}}^e[i] = \hat{\boldsymbol{u}}_g[\text{map}[e][i]] \\
\text{continue}
\end{array} \right\} \Leftrightarrow \hat{\boldsymbol{u}}_l = \mathcal{A}\hat{\boldsymbol{u}}_g,$$

$$\begin{cases} \text{Do } e = 1, N_{el} \\ \text{Do } i = 0, N_m^e - 1 \\ \hat{\boldsymbol{u}}_g[\text{map}[e][i]] = \hat{\boldsymbol{u}}_g[\text{map}[e][i]] + \hat{\boldsymbol{u}}^e[i] \\ \text{continue} \end{cases} \Leftrightarrow \hat{\boldsymbol{u}}_g = \boldsymbol{\mathcal{A}}^T \hat{\boldsymbol{u}}_l$$

Course Notes: Sections 1.3.1.4 & 3.2.1

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 - Elemental formulation
 - Global assembly
- 2D Formulation of Helmholtz problem
 - Global assembly ———

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2/3D Helmholtz problem

$$\nabla^2 u - \lambda u = -f$$

Integral formulation:
$$\int_{\Omega} v \nabla^2 u \, d\mathbf{x} - \int_{\Omega} \lambda u \, d\mathbf{x} = - \int_{\Omega} v f \, d\mathbf{x}$$

Divergence theorem:

$$\int_{\Omega} v \nabla^2 u \, d\mathbf{x} = \oint v \frac{\partial u}{\partial n} \, ds - \int_{\Omega} \nabla v \nabla u \, d\mathbf{x}$$

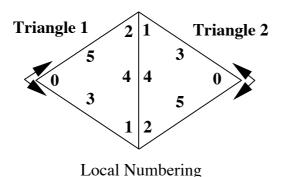
Weak form:
$$\int_{\Omega} \nabla v \nabla u + \lambda u \, d\mathbf{x} = \int_{\Omega} v f \, d\mathbf{x} + \oint_{\partial \Omega} v \frac{\partial u}{\partial n} \, ds$$

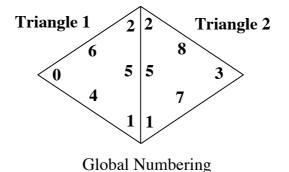
Dirichlet BC: $u = u^{\mathcal{D}} + u^{\mathcal{H}}$

Neumann BC: $\oint_{\partial \Omega} vg_{\mathcal{N}} ds$

Tutorial 3, exercise 1

Global assembly





$$\max[1][i] = \begin{cases} 0 \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{cases}$$

$$\max[2][i] = \begin{cases} 3\\2\\1\\8\\5\\7 \end{cases}.$$

Do
$$e = 1, N_{el}$$
Do $i = 0, N_m^e - 1$

$$\hat{\boldsymbol{u}}^e[i] = \frac{\text{sign}[e][i]}{\text{continue}} \cdot \hat{\boldsymbol{u}}_g[\text{map}[e][i]] \qquad \Rightarrow \hat{\boldsymbol{u}}_l = \mathcal{A}\hat{\boldsymbol{u}}$$
continue

Do
$$e = 1, N_{el}$$

Do $i = 0, N_m^e - 1$

$$\hat{\boldsymbol{I}}_g[\text{map}[e][i]] = \hat{\boldsymbol{I}}_g[\text{map}[e][i]]$$
+ $|\text{sign}[e][i] \cdot \hat{\boldsymbol{I}}^e[i]$
continue
continue

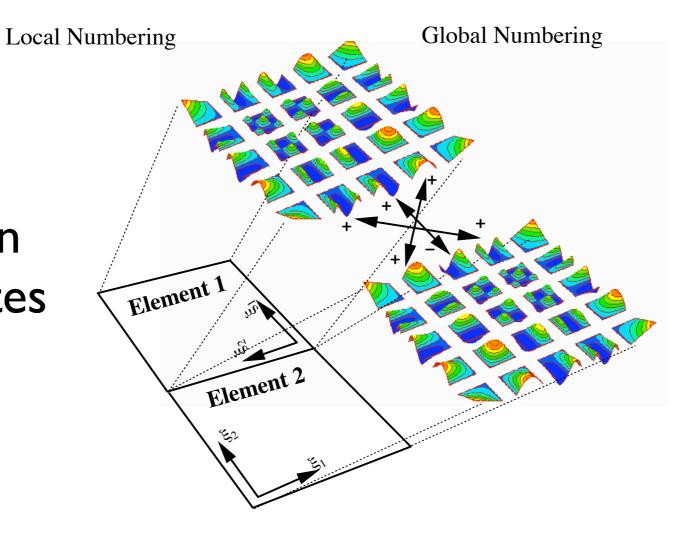
Course Notes: Section 3.2.1

2D Global Assembly: Modal

El	ement	t 1	\mathbf{E}	lement	2
1	4 5 6	0	2	7 8 9	1
789	15	14 13	10 11	12	654
2	12 11 10	3	3	15 14 13	0

Ele	ment	1	ŀ	Llemer	nt 2
1	6 7	0	0	21 22	5
9 10 11	8 17 1	16 15	15	23 16 17	20 19 18
2	14 13 12	3	3	26 25 24	4

Sign change required when local coordinates reversed



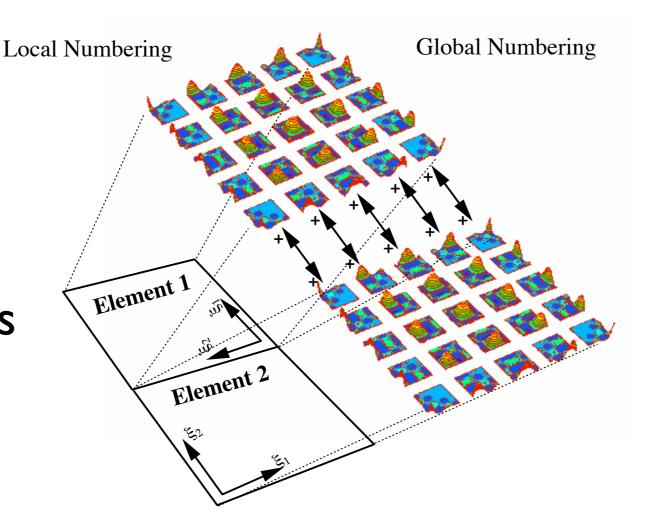
Course Notes: Section 3.2.1.1.

2D Global Assembly: Nodal

	Ele	eme	ent	1	Element 2				
1	6	5	4	0	2	9	8	7	1
7				15	10				6
8				14	11				5
9				13	12				4
2	10	11	12	2 3	3	13	14	15_	

	Ele	mei	nt 1	-	Element 2				
1	8	7	6	0	0	23	22	21	5
9				17	17				20
10				16	16				19
11				15	15				18
2	12	13	14	3	3	24	25	26	4

Numbering altered when local coordinates reversed



Course Notes: Section 3.2.1.1.

Nektar++ code

