Notes of “Ordinary differential equations (Arnold)”

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September 27, 2024

Preface

These notes were written for the course “Differential equations” (48 lectures one semester, each lecture is 45 minutes) I taught in Tsinghua 2016 fall and “Ordinary differential equations” in 2024 fall (64 lectures one semester, each lecture is 45 minutes), and it is based on Arnold’ book [Arn06].

Many students pointed out the gaps, typos in the notes, which are all my fault not the missing of Arnold’s book. Especially I would like to thank Weiying Li for proofreading of the notes and many suggestions on the former draft.

Chapter 1

The first order ODE

Definition 1.1 *For a function $v : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^2$ is an open domain, we define the following equation as the first order differential equation:*

$$\frac{dy}{dt} = v(t, y), \qquad \text{where } y : \mathbb{R} \to \mathbb{R}. \tag{1.1}$$

*The function $\varphi : I \to \mathbb{R}$ is called **a solution to (1.1) on I**, if $\frac{d\varphi}{dt} = v(t, \varphi(t))$ for $t \in I$ and I is an open interval of $\mathbb{R}$.*

If the solution φ to (1.1) satisfies $\varphi(t_0) = y_0$, we say that φ satisfies the initial condition (t_0, y_0).

The simplest case of (1.1) is that $v(t, y)$ is $v(t)$ or $v(y)$. The $v(t)$ follows from the integral method directly as follows.

Example 1.2 *For $t_0, y_0 \in \mathbb{R}$, solve*

$$\begin{cases} \frac{dy}{dt} & = v(t) , \\ y(t_0) & = y_0 , \end{cases} \tag{1.2}$$

*we get the **Barrow's formula** for the solution $\varphi(t) = y_0 + \int_{t_0}^{t} v(s) ds$.*

Remark 1.3 *Note the above formula assumes that v is a Riemannian integrable function, hence we can use the Newton-Leibniz formula to get the solution.*

If we define $v(t)$ as follows:

$$v(t) = \begin{cases} 1 , & t \in \mathbb{Q}, \\ 0 , & t \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Note v is not Riemannian integrable. And in fact there is no solution to (1.2) for such v, because the derivative function of any differentiable function admits intermediate value property.

1.1 The first order autonomous ODE

The $v(y)$ case is the following definition.

Definition 1.4 *For a function $v : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}$ is an open interval, we say that*

$$\frac{dy}{dt} = v(y) \tag{1.3}$$

is 1***st order autonomous ODE**.*

For (1.3), if $v(y_0) = 0$, then we say that y_0 is **an equilibrium of (1.3)**.

We firstly deals with the autonomous ODE with initial condition as equilibrium.

Theorem 1.5 *Let $v : U \to \mathbb{R}$ be a C^1 function with $v(y_0) = 0$, where $y_0 \in U$ is some fixed point and $U \subseteq \mathbb{R}$ is an open interval, then for each $t_0 \in \mathbb{R}$ and any open interval $I \subseteq \mathbb{R}$ containing t_0, the ODE (1.9) has a unique solution $\varphi : I \to U$ as follows:*

$$\varphi(t) \equiv y_0, \qquad \forall t \in I. \tag{1.4}$$

Remark 1.6 *Note that C^1 of v is necessary for the uniqueness of the solution. Because $v(y) = y^{\frac{2}{3}}$ is not C^1 in $(-1, 1) \subseteq \mathbb{R}$, and $\varphi_1 \equiv 0$ and $\varphi_2(t) = (\frac{t}{3})^3$ are both solutions to (1.9) with $t_0 = 0, y_0 = 0$.*

Proof: Note $\varphi(t) \equiv y_0$ is obviously a solution, we only need to show the uniqueness part.

By contradiction. If $v(y_0) = 0$ and the solution is not unique. Then there is ψ defined on I, which solves (1.9) and satisfies

$$\psi(t_1) = y_1, \qquad v(y_1) \neq 0.$$

Without loss of generality, we can assume $t_1 > t_0$, the case $t_1 < t_0$ can be dealt similarly as follows.

Let $t_2 = \max\{t \in [t_0, t_1] : v(\psi(t)) = 0\}$, then

$$\begin{aligned} t_0 \leq t_2 < t_1, &\qquad v(\psi(t_2)) = 0, \\ v(\psi(t)) \neq 0, &\qquad \forall t \in (t_2, t_1]. \end{aligned} \tag{1.5}$$

Without loss of generality, assume that $v(\psi(t)) > 0$ in $(t_2, t_1]$. From $\frac{d}{dt}\psi(t) = v(\psi(t))$, we get that ψ is strictly increasing on $(t_2, t_1]$.

Hence taking the integral, we have

$$t_1 - t_3 = \int_{y_3}^{y_1} \frac{1}{v(s)} ds, \qquad \psi(t_3) = y_3, \qquad \forall t_3 \in (t_2, t_1). \tag{1.6}$$

Let $y_2 = \psi(t_2)$.Note $[y_2, y_1] \subseteq U$ and $v \in C^1(U)$, then assume $k = \max\limits_{s \in [y_2, y_1]} |v'(s)|$, note $k \in (0, \infty)$.

From the mean value theorem, we get

$$|v(s)| = |v(s) - v(y_2)| \leq k \cdot |s - y_2|. \tag{1.7}$$

By (1.6) and (1.7), we obtain

$$|t_1 - t_3| \geq |\int_{y_1}^{y_3} \frac{1}{k(s - y_2)} ds| = \frac{1}{k} |\ln|\frac{y_3 - y_2}{y_2 - y_1}||. \tag{1.8}$$

Let $t_3 \to t_2$ in (1.8), note $y_2 = \psi(t_2) = \lim\limits_{t_3 \to t_2} \psi(t_3) = \lim\limits_{t_3 \to t_2} y_3$; hence we get

$$|t_1 - t_2| \geq \infty.$$

It is the contradiction. ∎

Remark 1.7 *For autonomous ODE with initial condition $(t_0, y_0) \in \mathbb{R} \times U$,*

$$\begin{cases} \frac{dy}{dt} &= v(y)\,, \\ y(t_0) &= y_0\,, \end{cases} \tag{1.9}$$

*we say that the solution $\varphi : I \to \mathbb{R}$ to (1.9) is a **most extended solution** to (1.9), if we have $\tilde{I} \subseteq I$ for any solution $\psi : \tilde{I} \to \mathbb{R}$ to (1.9). Note we did not prove the existence of the most extended solution to (1.9) so far.*

Example 1.8 *We consider the following example:*

$$\begin{cases} \frac{dy}{dt} &= y^2 + 1 \ , \\ y(t_0) &= y_0 \ , \end{cases} \tag{1.10}$$

We get a most extended solution $\varphi(t)$ to (1.10) defined as follows:

$$\varphi : (t_0 - \tan^{-1}(y_0) - \frac{\pi}{2}, t_0 - \tan^{-1}(y_0) + \frac{\pi}{2}) \to \mathbb{R}, \qquad \varphi(t) = \tan(t - (t_0 - \tan^{-1}(y_0))).$$

The next result discusses the autonomous ODE with initial condition, which is not equilibrium.

Theorem 1.9 *Let $v : U \to \mathbb{R}$ be a C^1 function, where $U \subseteq \mathbb{R}$ is an open interval, then for each $(t_0, y_0) \in \mathbb{R} \times U$ with $v(y_0) \neq 0$, the equation (1.9) has a most extended solution $\varphi : I_{t_0,y_0} \to \mathbb{R}$, where I_{t_0,y_0} is an open interval containing t_0. Furthermore*

(a) . The function φ satisfies the following:

$$t - t_0 = \int_{y_0}^{\varphi(t)} \frac{1}{v(s)} ds, \qquad \forall t \in I_{t_0,y_0}, \tag{1.11}$$

(b) . For any open interval $I \subseteq I_{t_0,y_0}$ with $t_0 \in I$, any solution to (1.9) on I is equal to $\varphi\big|_I$.

Proof: **Step (1)**. Without loss of generality, we assume $v(y_0) > 0$ in the rest argument. Define $F : U_{y_0} \to \mathbb{R}$ as follows:

$$\begin{aligned} a(y_0) &:= \inf\{y \in U : v(s) \cdot v(y_0) > 0, \forall\ s \in (y, y_0)\}, \\ b(y_0) &:= \sup\{y \in U : v(s) \cdot v(y_0) > 0, \forall\ s \in (y_0, y)\}, \\ U_{y_0} &= (a(y_0), b(y_0)), \qquad F(y) = \int_{y_0}^{y} \frac{1}{v(s)} ds. \end{aligned}$$

Then $F \in C^2(U_{y_0})$ and $F' > 0$, we get

$$\begin{aligned} &F(U_{y_0}) = (a, b) \subseteq \mathbb{R}, \\ &a = \lim_{y \to a(y_0)+} F(y) \in \mathbb{R}^- \cup \{-\infty\}, \\ &b = \lim_{y \to b(y_0)-} F(y) \in \mathbb{R}^+ \cup \{+\infty\}. \end{aligned} \tag{1.12}$$

Define $\varphi_0 : I_{t_0,y_0} \to \mathbb{R}$ as follows:

$$I_{t_0,y_0} = (t_0 + a, t_0 + b), \qquad \varphi_0(t) = F^{-1}(t - t_0),$$

where F^{-1} in the inverse function of the strictly increasing function F. And we have $t_0 \in I_{t_0,y_0}$.
Definition of φ_0 yields

$$F(\varphi_0(t)) = t - t_0.$$

Taking derivative with respect to t in the above equation on both sides of the equation, we get

$$\varphi_0'(t) = v(\varphi_0(t)), \qquad \forall t \in I_{t_0,y_0}.$$

Hence φ_0 is a solution to (1.9) on I_{t_0,y_0}.

Step (2). On the other hand, assume φ is a solution to (1.9) on I_{t_0,y_0}. If $v(\varphi(t_1)) = 0$ for some $t_1 \in I_{t_0,y_0}$, then from Theorem 1.5, we get $\varphi(t) \equiv \varphi(t_1)$ on I_{t_0,y_0}. Now we get

$$0 < v(y_0) = v(\varphi(t_0)) = v(\varphi(t_1)) = 0,$$

which is the contradiction. Hence $0 \notin v(\varphi(I_{t_0,y_0}))$.

Now from (1.9) we get

$$\frac{\varphi'(s)}{v(\varphi(s))} = 1, \qquad \forall s \in I_{t_0,y_0}.$$

Taking the integration of the above equation with respect to s, we get

$$\int_{t_0}^{t} \frac{\varphi'(s)}{v(\varphi(s))} ds = t - t_0.$$

Changing the variable in the above integral and note $\varphi(t_0) = y_0$, we obtain

$$\int_{y_0}^{\varphi(t)} \frac{1}{v(y)} dy = t - t_0,$$

which implies

$$\varphi(t) = F^{-1}(t - t_0) = \varphi_0(t).$$

Therefore φ_0 is the unique solution to (1.9) on I_{t_0,y_0}.

Similar argument yields that $\varphi_0\big|_I$ is the unique solution to (1.9) on any open interval $I \subseteq I_{t_0,y_0}$ with $t_0 \in I$.

Step (3). Next we show that φ_0 is the most extended solution to (1.9). By contradiction, if $\varphi : I \to \mathbb{R}$ is a solution to (1.9) with $I \backslash I_{t_0,y_0} \neq \emptyset$.

Without loss of generality, we assume $t_1 > t_0 + b$ and $t_1 \in I$.

Note $\varphi\big|_{[t_0,t_0+b)} = \varphi_0(t) = F^{-1}(t - t_0)$ by the uniqueness part in Step (2). Hence using (1.12) and F is strictly increasing, we have

$$\infty > \varphi(t_0 + b) = \lim_{t \to (t_0+b)-} \varphi_0(t) = \lim_{t \to (t_0+b)-} F^{-1}(t - t_0) = b(y_0). \tag{1.13}$$

Because φ is a solution to (1.9) on I and $t_0 + b \in I$, we get $\varphi(t_0 + b) \in U$.

Now $b(y_0) \in U$ by (1.13), which implies $v(b(y_0)) = 0$ by the definition of $b(y_0)$ and the continuity of v. Therefore we get

$$v(\varphi(t_0 + b)) = 0. \tag{1.14}$$

By (1.14) and Theorem 1.5, we get $\varphi\big|_I \equiv \varphi(t_0 + b) = b(y_0)$.

However $0 < v(y_0) = v(\varphi(t_0)) = v(\varphi(t_0 + b)) = 0$, which is the contradiction! ■

Exercise 1 *Problem* 1, 2 *in Section* 1.8 *of* [Arn06]. *Problem* 1 − − − 5 *in Section* 2.3 *of* [Arn06].

1.2 The integral curve and stability

Definition 1.10 *For a solution $\varphi(t)$ to*

$$\begin{cases} \frac{dy}{dt} &= v(t, y)\,, \\ y(t_0) &= y_0\,, \end{cases} \tag{1.15}$$

defined on I, we say that φ ***can be extended on*** *I_1, where $I \subseteq I_1 \subseteq \mathbb{R}$; if there is a solution $\tilde{\varphi}$ to (1.15) defined on I_1 and $\tilde{\varphi}\big|_I = \varphi$.*

We say φ ***can be extended forward (respectively backward) indefinitely*** *if*

$$I \subseteq I_1 \quad \textit{and} \quad \sup\{t : t \in I_1\} = \infty \ (\textit{respectively} \ \inf\{t : t \in I_1\} = -\infty);$$

if there is a solution $\tilde{\varphi}$ to (1.15) defined on I_1 and $\tilde{\varphi}\big|_I = \varphi$.

For an equilibrium y_0 of (1.3),

(a) . If for any $\epsilon > 0$, there is $\delta > 0$, such that for any $y_1 \in (y_0 - \delta, y_0 + \delta)$, any solution $\varphi(t)$ to

$$\begin{cases} \frac{dy}{dt} & = v(y)\,, \\ y(t_0) & = y_1\,, \end{cases} \tag{1.16}$$

on $I \subseteq \mathbb{R}$, can be extended forward indefinitely, denoted the extended solution as $\tilde{\varphi}$, such that

$$\sup_{t>t_0} |\tilde{\varphi}(t) - y_0| < \epsilon,$$

then we say that y_0 is a **stable (or Lyapunov stable) equilibrium** of (1.3).

(b) . If y_0 is not a stable equilibrium of (1.3), then we say that y_0 is a **unstable equilibrium** of (1.3).

Definition 1.11 *(a) . The set $U = \{y : v(t, y)$ is defined for some $t\}$ is **the phase space of (1.1)**. The set $\Omega = \{(t, y) : v(t, y)$ is defined$\}$ is called **the extended phase space of (1.1)**.*

(b) . For (1.1), the map

$$\vec{v}(t, y) = (1, v(t, y)) : \mathbb{R}^2 \to \mathbb{R}^2,$$

*is called the **direction field of** v **(or the equation (1.1))**.*

*(c) . The graph $\{(t, \varphi(t)) : t \in I\}$ is called **an integral curve of (1.1)**.*

Example 1.12 *If $v(y) = k \cdot y$, where $k > 0$, then the solution to (1.9) is $y = y_0 \cdot e^{k(t-t_0)}$. The solution with respect to initial condition (t_0, y_0) is unique by Theorem 1.5 and Theorem 1.9, hence all integral curves are disjoint.*

And $y = 0$ is the unique equilibrium of this ODE, which is not stable.

Example 1.13 *If $v(y) = y(1 - y) - c$, where $c > 0$, then the integral curves and stability of equilibrium to (1.3) are discussed in the following three cases:*

(a) . If $c = \frac{1}{4}$, we get

$$y(t) = \frac{1}{2} + \frac{1}{\frac{1}{y_0 - \frac{1}{2}} + t - t_0},$$

and $\frac{1}{2}$ is an unstable equilibrium.

(b) . If $c > \frac{1}{4}$, we get that $v(y) = \frac{1}{4} - c - (y - \frac{1}{2})^2 < 0$, which implies the non-existence of equilibrium.

(c) . If $c \in (0, \frac{1}{4})$, we get two equilibrium points $A < B$ and

$$|\frac{y - \frac{1}{2} - \sqrt{\frac{1}{4} - c}}{y - \frac{1}{2} + \sqrt{\frac{1}{4} - c}}| = |\frac{y_0 - \frac{1}{2} - \sqrt{\frac{1}{4} - c}}{y_0 - \frac{1}{2} + \sqrt{\frac{1}{4} - c}}| \cdot \exp^{-2\sqrt{\frac{1}{t} - c}(t-t_0)}.$$

Furthermore A is unstable and B is stable.

Definition 1.14 *The equilibrium y_0 of (1.3) is **asymptotically stable** if it is (Lyapunov) stable and there is a constant $\tau > 0$, such that for any solution φ to (1.3) with $|y_1 - y_0| < \tau$, we have*

$$\lim_{t\to\infty} \varphi(t) = y_0.$$

Exercise 2 *Problem* 1 *in Section* 1.7 *of* [Arn06]. *Problem* 1, 2 *in Section* 2.6 *of* [Arn06].

1.3 The 1st order homogeneous linear ODE

Definition 1.15 *If $v(t,y) = f(t)\cdot y$ for some continuous function $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an open interval, then we say that (1.1) is* ***1st order homogeneous linear ODE****.*

Theorem 1.16 *The 1st order homogeneous linear ODE*

$$\begin{cases} \frac{dy}{dt} & = f(t)\cdot y\,, \\ y(t_0) & = y_0\,, \end{cases} \tag{1.17}$$

has the unique solution

$$y(t) = y_0 \cdot \exp^{\int_{t_0}^{t} f(\zeta)d\zeta}. \tag{1.18}$$

Proof: It is straightforward to verify (1.18) is a solution to (1.17).

Because (1.17) is linear in y, to prove the uniqueness, we are reduced to show there is only zero solution to

$$\begin{cases} \frac{dy}{dt} & = f(t)\cdot y\,, \\ y(t_0) & = 0\,, \end{cases} \tag{1.19}$$

By contradiction. Assume $\psi : I \to \mathbb{R}$ is a solution to (1.19) and $\psi(t_1) \neq 0$ for some $t_1 \in I$.

Without loss of generality, assume $t_1 > t_0$ and define

$$t_2 = \max\{t \in [t_0, t_1] : \psi(t) = 0\}.$$

Then we have

$$\psi(t_2) = 0, \qquad \psi(t) \neq 0, \qquad \forall t \in (t_2, t_1].$$

Now we have

$$\frac{\psi'}{\psi}(t) = f(t), \qquad \forall t \in (t_2, t_1].$$

Taking the integral of the above equation with respect to t from t_3 to t_1, where $t_3 \in (t_2, t_1]$ is freely chosen; we obtain

$$\psi(t_1) = \psi(t_3)\cdot \exp^{\int_{t_3}^{t_1} f(t)dt}.$$

Let $t_3 \to t_2$ in the above, we get $\psi(t_1) = 0$, it is the contradiction. ∎

Remark 1.17 *Note another important 1-dim curve is $\mathbb{S}^1(\frac{T}{2\pi})$, which is the round circle with radius $\frac{T}{2\pi}$. Any function $F : \mathbb{S}^1(\frac{T}{2\pi}) \to \mathbb{R}$ satisfies*

$$F(\gamma(s)) = F(\gamma(s+T)), \qquad \gamma(s) = \frac{T}{2\pi}\cdot\Big(\cos(\frac{s}{T}\cdot 2\pi), \sin(\frac{s}{T}\cdot 2\pi)\Big) : \mathbb{R} \to \mathbb{S}^1(\frac{T}{2\pi}).$$

Note $F\circ\gamma$ satisfies periodic condition on $\mathbb{R}$.

Definition 1.18 *For $f \in C(\mathbb{R})$ and a fixed constant $T > 0$, the equation*

$$\begin{cases} \frac{dy(t)}{dt} & = f(t)\cdot y(t), \qquad \forall t \in \mathbb{R}, \\ f(t+T) & = f(t), \qquad \forall t \in \mathbb{R}, \end{cases} \tag{1.20}$$

is called ***1st order linear equation with T-periodic coefficient****. Any solution $\varphi : \mathbb{R} \to \mathbb{R}$ to (1.20) satisfying $\varphi(t) = \varphi(t+T)$ for any $t \in \mathbb{R}$, is called* ***a periodic solution to (1.20)****.*

Notation 1.19 *For $t_0, y_0 \in \mathbb{R}$, we denote $\varphi(t, t_0, y_0)$ as a solution to*

$$\begin{cases} \frac{dy}{dt} & = v(t, y) , \\ y(t_0) & = y_0 , \end{cases}$$

on some $I \subseteq \mathbb{R}$ with $t_0 \in I$.

Definition 1.20 *For ODE*

$$\frac{dy}{dt} = v(t, y) \tag{1.21}$$

*A solution $\varphi(t, t_0, y_0)$ to (1.21) is called **a stable solution to (1.21)** if its domain includes $[t_0, \infty)$, and for any $\epsilon > 0$, there is $\delta > 0$, such that for any $y_1 \in (y_0 - \delta, y_0 + \delta)$, any solution $\varphi(t, t_0, y_1)$ to (1.21), can be extended forward indefinitely, (denoted the extended solution as $\tilde{\varphi}(t, t_0, y_1)$), such that*

$$\sup_{t > t_0} |\tilde{\varphi}(t, t_0, y_1) - \varphi(t, t_0, y_0)| < \epsilon.$$

*A stable solution $\varphi(t, t_0, y_0)$ to (1.21) is called **asymptotically stable** if there is $\delta > 0$, such that for any $y_1 \in (y_0 - \delta, y_0 + \delta)$, any solution $\varphi(t, t_0, y_1)$ to (1.21), can be extended forward indefinitely, (denoted the extended solution as $\tilde{\varphi}(t, t_0, y_1)$), such that*

$$\lim_{t \to \infty} |\tilde{\varphi}(t, t_0, y_1) - \varphi(t, t_0, y_0)| = 0.$$

Lemma 1.21 *For (1.20), we define $\lambda := \exp^{\int_0^T f(\zeta) d\zeta}$, then*

(1) if $\lambda = 1$, then for any $y_0 \in \mathbb{R}$, there is a unique periodic solution φ to (1.20) with $\varphi(0) = y_0$; and 0 is a stable solution but not asymptotic stable.

(2) if $\lambda > 1$, then there is no periodic solution φ to (1.20) except 0, and 0 is not a stable solution.

(3) if $\lambda < 1$, then there is no periodic solution φ to (1.20) except 0, and 0 is an asymptotic stable solution.

Proof: **Step (1)**. When $\lambda = 1$, For any initial value $\varphi(0) = y_0$, we have the solution $\varphi(t) = y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta}$. The uniqueness comes from Theorem 1.16.

Now we check that $\varphi(t) = y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta}$ is a periodic solution. Since

$$\varphi(t + T) - \varphi(t) = y_0 \cdot \exp^{\int_0^{t+T} f(\zeta) d\zeta} - y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta} = y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta} (\exp^{\int_t^{t+T} f(\zeta) d\zeta} - 1),$$

by $f(t) = f(t + T)$, we have $\exp^{\int_t^{t+T} f(\zeta) d\zeta} = \exp^{\int_0^T f(\zeta) d\zeta} = 1$. So there is $\varphi(t + T) - \varphi(t) = 0$.

For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2} \cdot \inf_{t \in [0,T]} (\exp^{-\int_0^T f(\zeta) d\zeta})$.

When $|y_0| < \delta$, for the most extended solution φ_1 of (1.20) with the initial value $\varphi_1(0) = y_0$, we have $|\varphi_1(t)| = |y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta}| < \frac{\epsilon}{2}$. So 0 is a stable solution.

On the other hand, for any $y_0 \neq 0$, there is some constant $c_1 > 0$ such that

$$|\varphi_1(t)| = |y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta}| = |y_0 \cdot \exp^{\int_{t + [\frac{t}{T}]T}^t f(\zeta) d\zeta}| \geq |y_0| \inf_{t \in [0,T]} |\exp^{\int_0^t f(\zeta) d\zeta}| \geq c_1 > 0.$$

So 0 is not asymptotic stable.

Step (2). When $\lambda > 1$, for any initial value $\varphi(0) = y_0 \neq 0$, we have

$$\varphi(t + T) - \varphi(t) = y_0 \cdot \exp^{\int_0^t f(\zeta) d\zeta} (\exp^{\int_t^{t+T} f(\zeta) d\zeta} - 1) \neq 0. \tag{1.22}$$

So the unique periodic solution is 0.

Since for any initial value $\varphi(0) = y_0 \neq 0$, there is

$$\lim_{t\to\infty} |\varphi(t)| = \lim_{t\to\infty} |y_0 \cdot \exp^{\int_0^t f(\zeta)d\zeta}| \geq \lim_{t\to\infty} |y_0|\lambda^{[\frac{t}{T}]} \inf_{t\in[0,T]} |\exp^{\int_0^T f(\zeta)d\zeta}| = +\infty.$$

So 0 is not a stable solution.

Step (3). When $\lambda < 1$, similar to (1.22), we have that the unique periodic solution is 0.

And for any initial value $\varphi(0) = y_0 \neq 0$, we have

$$\lim_{t\to\infty} |\varphi(t)| = \lim_{t\to\infty} |y_0 \cdot \exp^{\int_0^t f(\zeta)d\zeta}| \leq \lim_{t\to\infty} |y_0|\lambda^{[\frac{t}{T}]} = 0.$$

So 0 is an asymptotic stable solution. ■

1.4 The 1st order inhomogeneous linear ODE

Definition 1.22 *If $v(t, y) = f(t) \cdot y + g(t)$ for some continuous functions $f, g \in C(\mathbb{R})$, then we say that (1.1) is* ***1st order inhomogeneous linear ODE****.*

Theorem 1.23 *The 1st order inhomogeneous linear ODE*

$$\begin{cases} \frac{dy}{dt} & = f(t) \cdot y + g(t)\ , \\ y(t_0) & = 0\ , \end{cases} \tag{1.23}$$

has the unique solution

$$y(t) = \int_{t_0}^{t} g(s) \cdot \exp^{\int_s^t f(\zeta)d\zeta}\, ds. \tag{1.24}$$

Remark 1.24 *There are three methods to get the formula (1.24).*

(a) . The easiest one is to make the exact differential. Note $e^{-\int_{t_0}^t f(\zeta)d\zeta}(y' - f \cdot y) = (e^{-\int_{t_0}^t f(\zeta)d\zeta} \cdot y)'$, then solve (1.23) by multiplying both sides of the equation with $e^{-\int_{t_0}^t f(\zeta)d\zeta}$ and taking integral.

(b) . The second method is called "variation of parameters", which is more flexible although it is a little bit complicated. We can assume the solution has the form $\varphi(t) = C(t) \cdot \varphi_0(t)$, where $\varphi_0(t)$ is the solution to (1.23) with $g \equiv 0$. Then direct computation yields the formula for $C(t)$, and $\varphi(t)$.

(c) . The third method is by Green's function for ODE, which is delicate but more conceptual.

Proof: It is easy to verify that (1.24) is a solution to (1.23).

If there is another solution $\tilde{y}$ to (1.23), then $\psi := y - \tilde{y}$ is a solution to

$$\begin{cases} \psi'(t) & = f(t) \cdot \psi(t)\ , \\ \psi(t_0) & = 0\ , \end{cases} \tag{1.25}$$

From Theorem 1.16, we get $\psi \equiv 0$, the uniqueness part follows. ■

Definition 1.25 *We define the* ***mollifier*** *$\eta : \mathbb{R} \to \mathbb{R}$ as the following:*

$$\eta(x) = \begin{cases} c \cdot e^{\frac{1}{|x|^2-1}}\ , & if\ |x| < 1 \\ 0\ , & if\ |x| \geq 1 \end{cases}$$

where c is chosen such that $\int_{\mathbb{R}} \eta(x)dx = 1$. For any $\epsilon > 0$, we use the notation:

$$\eta_\epsilon(x) = \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$$

Lemma 1.26 *For any $g \in C(R)$, we have*

$$g(t) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \eta_\epsilon(t-\xi) \cdot g(\xi) d\xi, \qquad \forall t \in \mathbb{R}.$$

Proof: It is from direct computation and the definition of η_ϵ. ■

Lemma 1.27 *If $\xi > 0$, the most extended solution $\varphi_\epsilon(\cdot, \xi)$ to*

$$\begin{cases} y'(t) & = f(t) \cdot y(t) + \eta_\epsilon(t-\xi) , \\ y(0) & = 0 , \end{cases} \tag{1.26}$$

satisfies

$$\lim_{\epsilon \to 0} \varphi_\epsilon(t,\xi) = G(t,\xi) = \begin{cases} 0 , & \text{if } t < \xi \\ e^{\int_\xi^t f(\zeta) d\zeta} , & \text{if } t > \xi. \end{cases} \tag{1.27}$$

And for any $g \in C(\mathbb{R})$, we have

$$\lim_{\epsilon \to 0} \int_0^{t+\epsilon} \varphi_\epsilon(t,\xi) \cdot g(\xi) d\xi = \int_0^t G(t,\xi) \cdot g(\xi) d\xi. \tag{1.28}$$

Remark 1.28 *From the definition of $\varphi_\epsilon(t,\xi)$, consider the distinct points $\xi_i \in (0, t+\epsilon)$ with $|\xi_i - \xi_{i+1}| = \delta$, then $\sum_i \varphi_\epsilon(t,\xi_i) g(\xi_i)\delta$ is a solution to*

$$y' = f \cdot y + \sum_i \eta_\epsilon(t-\xi_i) g(\xi_i)\delta.$$

Taking $\delta \to 0$ in the above, we get that

$$\int_0^{t+\epsilon} \varphi_\epsilon(t,\xi) g(\xi) d\xi \quad \text{is a solution to} \quad y' = f \cdot y + \int_0^{t+\epsilon} \eta_\epsilon(t-\xi) g(\xi) d\xi.$$

Let $\epsilon \to 0$ in the above, we expect that

$$\int_0^t G(t,\xi) g(\xi) d\xi \quad \text{is a solution to} \quad y'(t) = f(t) \cdot y(t) + g(t).$$

Note in the proof of Lemma 1.27, we assume the existence of $\varphi_\epsilon(\cdot,\xi)$, which can be is guranteed by Theorem 1.23. But this will lead the above explanation into a closed-loop argument.

In fact, in the more general case, we will firstly show the existence of solution (this will be done for general ODEs in later chapters), then using the method in the proof of Lemma 1.27 to get the property of the solution to inhomogeneous differential equations.

Proof: **Step (1)**. From the definition of η_ϵ, we know that for $\xi > \epsilon$, then $\varphi_\epsilon(\cdot,\xi)\big|_{(-\infty,\xi-\epsilon)}$ is a solution to

$$\begin{cases} y'(t) & = f(t) \cdot y(t) , \\ y(0) & = 0 , \end{cases}$$

on $(-\infty, \xi - \epsilon)$. From Theorem 1.16, we get that

$$\varphi_\epsilon(t,\xi) = 0, \qquad \text{if } t < \xi - \epsilon. \tag{1.29}$$

Step (2). Again from the defintion of η_ϵ, we know that $\varphi_\epsilon(\cdot,\xi)\big|_{(\xi+\epsilon,\infty)}$ is a solution to

$$\begin{cases} y'(t) & = f(t) \cdot y(t) , \\ y(\xi+\epsilon) & = \varphi_\epsilon(\xi+\epsilon,\xi) , \end{cases}$$

on $(\xi+\epsilon,\infty)$. By Theorem 1.16, we get

$$\varphi_\epsilon(t,\xi)=\varphi_\epsilon(\xi+\epsilon,\xi)\cdot\exp^{\int_{\xi+\epsilon}^{t}f(\zeta)d\zeta},\qquad\forall t>\xi+\epsilon. \tag{1.30}$$

Step (3). If $t<\xi$, from (1.29) we get

$$\lim_{\epsilon\to0}\varphi_\epsilon(t,\xi)=0. \tag{1.31}$$

Note

$$\varphi_\epsilon(\xi+\epsilon,\xi)=\int_{\xi-\epsilon}^{\xi+\epsilon}\varphi_\epsilon'(t)dt=\int_{\xi-\epsilon}^{\xi+\epsilon}[f\cdot\varphi_\epsilon+\eta_\epsilon(t-\xi)]dt=1+\int_{\xi-\epsilon}^{\xi+\epsilon}[f\cdot\varphi_\epsilon](s)ds. \tag{1.32}$$

By (1.30) and (1.32), for $t>\xi$, we obtain

$$\begin{aligned}\lim_{\epsilon\to0}\varphi_\epsilon(t,\xi)&=\lim_{\epsilon\to0}\Big(1+\int_{\xi-\epsilon}^{\xi+\epsilon}[f\cdot\varphi_\epsilon](s)ds\Big)\cdot\exp^{\int_{\xi+\epsilon}^{t}f(\zeta)d\zeta}\\&=\exp^{\int_{\xi}^{t}f(\zeta)d\zeta}+\exp^{\int_{\xi}^{t}f(\zeta)d\zeta}\cdot\lim_{\epsilon\to0}\int_{\xi-\epsilon}^{\xi+\epsilon}[f\cdot\varphi_\epsilon](s)ds.\end{aligned} \tag{1.33}$$

Let $c_1:=\sup_{s\in[\xi-1,\xi+1]}|f(s)|$, $c_2=\sup_{t\in[\xi-\epsilon,\xi+\epsilon]}|\varphi_\epsilon(t,\xi)|$. Note $\varphi_\epsilon(\cdot,\xi)$ is continuous, then there is $t_M\in[\xi-\epsilon,\xi+\epsilon]$ such that $c_2=\varphi_\epsilon(t_M,\xi)$.

Note

$$c_2=|\varphi_\epsilon(t_M,\xi)|=|\int_{\xi-\epsilon}^{t_M}\varphi_\epsilon'dt|\le1+|\int_{\xi-\epsilon}^{t_M}f\cdot\varphi_\epsilon(t,\xi)dt|\le1+c_1\cdot c_2\cdot(2\epsilon).$$

Hence if $\epsilon<\frac{1}{4c_1}$, we get

$$c_2\le\frac{1}{1-2c_1\epsilon}\le2. \tag{1.34}$$

Now we obtain

$$\Big|\lim_{\epsilon\to0}\int_{\xi-\epsilon}^{\xi+\epsilon}[f\cdot\varphi_\epsilon](s)ds\Big|\le\lim_{\epsilon\to0}2c_1\epsilon=0. \tag{1.35}$$

Plugging (1.35) into (1.33), we get

$$\lim_{\epsilon\to0}\varphi_\epsilon(t,\xi)=\exp^{\int_{\xi}^{t}f(\zeta)d\zeta},\qquad\text{if } t>\xi. \tag{1.36}$$

Finally the equation (1.28) follows from (1.31), (1.36) and (1.34). ■

Definition 1.29 *If $f,g:\mathbb{R}\to\mathbb{R}$ and $T>0$ is a fixed constant, the equation*

$$\begin{cases}\frac{dy(t)}{dt}&=f(t)\cdot y(t)+g(t),\qquad\forall t\in\mathbb{R},\\ f(t+T)&=f(t),\qquad g(t+T)=g(t),\qquad\forall t\in\mathbb{R},\end{cases} \tag{1.37}$$

is called 1***st order inhomogenous linear equation with T-periodic coefficient****. Any solution $\varphi:\mathbb{R}\to\mathbb{R}$ is called* ***periodic solution to (1.37)****.*

Theorem 1.30 *For (1.37), we define $\lambda:=\exp^{\int_0^T f(\zeta)d\zeta}$, and we have*

(a) . If $\int_0^T g(\xi)\cdot\exp^{\int_\xi^T f(\zeta)d\zeta}d\xi=0$, then

(a.1) there exists a periodic solution φ if and only if $\lambda=1$.

(a.2) if $\lambda = 1$, then for any $y_0 \in \mathbb{R}$, there is a unique periodic solution φ with $\varphi(0) = y_0$.

(b) . If $\int_0^T g(\xi) \cdot \exp^{\int_\xi^T f(\zeta)d\zeta}\, d\xi \neq 0$,

(b.1) there exists a periodic solution φ if and only if $\lambda \neq 1$.

(b.2) if $\lambda \neq 1$, then there is a unique periodic solution φ; furthermore

$$\varphi(0) = \frac{1}{1-\lambda} \cdot \int_0^T g(\xi) \cdot \exp^{\int_\xi^T f(\zeta)d\zeta}\, d\xi.$$

Proof: It follows from Theorem 1.16 and Theorem 1.23 directly. ■

Exercise 3 *Problem* 1 *in Section* 3.4*, Problem* 1, 2 *in Section* 3.5 *of* [Arn06]*. Problem* 1, 2 *in Chapter* 1 *of* [CL55].

Chapter 2

Methods of integrating ODEs

Besides the autonomous ODE, the first order linear (inhomogeneous) ODE; the following ode

$$\frac{dy}{dt} = f(t) \cdot g(y),$$

which is called **ODE with separable variables**, can be solved by integration method as follows. Simplifying the above ODE, we get

$$\frac{\frac{dy}{dt}}{g(y)} = f(t), \tag{2.1}$$

taking the integral of both sides of (2.1) with respect to t, we get the solution implicitly.

This chapter focuses on the ODE (or ODEs), which can be solved by reducing to ODE (or ODEs) with separable variables.

2.1 Change of variables in ODEs

Definition 2.1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **homogeneous of degree** k, where $k \in \mathbb{R}$; if it satisfies*

$$f(e^s \cdot x) = e^{ks} f(x), \qquad \forall x \in \mathbb{R}^n, s \in \mathbb{R}.$$

Example 2.2 *If $\frac{dy}{dt} = f(t, y)$, where f is homogeneous of degree 0, then define $v = \frac{y}{t}$, we get*

$$\frac{dy}{dt} = v + t \cdot \frac{dv}{dt}.$$

On the other hand, from homogeneous of degree 0 assumption on f, we have

$$f(t, y) = f(1, \frac{y}{t}) = f(1, v).$$

By the above, we obtain

$$\frac{dv}{dt} = \frac{f(1, v) - v}{t},$$

which implies

$$\frac{\frac{dv}{dt}}{f(1, v) - v} = \frac{1}{t}.$$

The above ODE can be solved by taking integration with respect to t on both sides.

Definition 2.3 *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called* ***quasi-homogeneous of degree*** k ***with weight*** α,β*, where $\alpha, \beta, k \in \mathbb{R}$; if it satisfies*

$$f(e^{\alpha s} \cdot x_1, e^{\beta s} \cdot x_2) = e^{ks} f(x_1, x_2), \qquad \forall (x_1, x_2) \in \mathbb{R}^2, s \in \mathbb{R}.$$

Example 2.4 *Assume $\frac{dy}{dt} = f(t, y)$, where f is quasi-homogeneous of degree k with weight α,β, and $k = \beta - \alpha$.*

Case (1). *If $\alpha \neq 0$, we define $v = \frac{y^\alpha}{t^\beta}$, by direct computation as in Example 2.2, we obtain*

By the above, we obtain

$$\frac{d(v^{\frac{1}{\alpha}})}{dt} = \frac{f(1, v^{\frac{1}{\alpha}}) - \frac{\beta}{\alpha} v^{\frac{1}{\alpha}}}{t},$$

which implies

$$\frac{\frac{d(v^{\frac{1}{\alpha}})}{dt}}{f(1, v^{\frac{1}{\alpha}}) - \frac{\beta}{\alpha} v^{\frac{1}{\alpha}}} = \frac{1}{t}.$$

The above ODE can be solved by taking integration with respect to t on both sides.

Case (2). *If $\alpha = 0$, if $y > 0$, we get*

$$\frac{dy}{dt} = f(t, y) = y \cdot f(t, 1),$$

which implies

$$\frac{\frac{dy}{dt}}{y} = f(t, 1).$$

The above ODE can be solved by taking integration with respect to t on both sides. If $y < 0$, similar argument yields the solution by integration method.

Question 2.5 *In the above examples, the key is the change of variables from (t, y) to (t, v), then applying integration method (combining the separable variables). For general ODE*

$$\frac{dy}{dt} = v(t, y),$$

when we can find such changes of variables to getting separable variables? That is the reason we study the action on direction field $(1, v(t, y))$ (also general vector fields) as follows.

Definition 2.6 *A* ***diffeomorphism*** *between smooth manifolds M, N (think it as $\mathbb{R}^n$) is a bijective map $f : M \to N$, which is smooth and f^{-1} is also smooth.*

For each map $\vec{v} : \mathbb{R}^m \to \mathbb{R}^n$ where $m, n \in \mathbb{Z}^+$, we say that $\vec{v}$ is **a vector field on** $\mathbb{R}^m$.

For each map $\vec{v} : \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^m$ is an open domain, we say that $\vec{v}$ is **a vector field on** Ω.

Example 2.7 *For $\frac{dy}{dt} = \frac{y}{t}$, the corresponding direction field is $\vec{v}(t, y) = (1, \frac{y}{t})$, which is a vector field on $\Omega = \{(t, y) : t \neq 0\}$.*

Let $f : M \to N$ be a smooth map, for any $x \in M$, we say that $(f_*)_x : T_xM \to T_{f(x)}N$ is the derivative of f at x, which is defined as follows:

$$(f_*)_x(\vec{v}_0) = \frac{d}{dt} f \circ \varphi(t)\Big|_{t=0}, \qquad \forall \vec{v}_0 \in T_xM, \vec{v}_0 = \frac{d}{dt}\varphi(t)\Big|_{t=0},$$

where T_xM is the tangent space of M at x and $T_{f(x)}N$ is the tagent space of N at $f(x)$, the map $\varphi : (-1, 1) \to M$ is smooth and $\varphi(0) = x$.

Notation 2.8 *Assume $\vec{v}:\Omega\to\mathbb{R}^n$ is a vector field, where $\Omega\subseteq\mathbb{R}^n$, and $f:\mathbb{R}^n\to\mathbb{R}^n$ is a diffeomorphism. Then for any $x\in\Omega$, we use the notation in the rest argument:*

$$f_*\vec{v}(x)=(f_*\vec{v})(x)=(f_*)_x(\vec{v}(x)).$$

Example 2.9 *Assume $f:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth map and $f=(f_1,\cdots,f_n)$, let*

$$\vec{v}_0=\sum_{i=1}^n v_i\frac{\partial}{\partial x_i},\qquad \varphi(t)=(x_1+tv_1,\cdots,x_n+tv_n).$$

Direct computation yields $(f_*)_x:\mathbb{R}^n\to\mathbb{R}^n$ is a matrix

$$(f_*)_x=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix},$$

Remark 2.10 *The basic philosophy of chaning variables is as follows:* ***Assume $f:\mathbb{R}^n\to\mathbb{R}^n$ is a diffeomorphism, $\vec{v}:\Omega\to\mathbb{R}^n$ is a vector field, define $\vec{w}(f(p))=f_*\vec{v}(p)$ for any $p\in\mathbb{R}^n$. Then $\vec{w}:f(\Omega)\to\mathbb{R}^n$ is a vector field. If $\varphi:I\to\Omega$ is a solution of***

$$\frac{d}{dt}\vec{x}(t)=\vec{v}(\vec{x}(t)),\qquad \textbf{\textit{where }}\vec{x}(t)\in\Omega.$$

Then $f\circ\varphi:I\to f(\Omega)$ is a solution of

$$\frac{d}{dt}\vec{y}(t)=\vec{w}(\vec{y}(t)),\qquad \textbf{\textit{where }}\vec{y}(t)\in f(\Omega).$$

The proof is direct by definition of f_*.

Example 2.11 *Assume $x(t)=(x_1(t),x_2(t))\in\mathbb{R}^2$, consider*

$$\frac{d}{dt}x(t)=\frac{d}{dt}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\vec{v}(x(t))=\begin{pmatrix}x_2(t)\\x_1(t)\end{pmatrix},$$

Define $f:\mathbb{R}^2\to\mathbb{R}^2$ by

$$f(x_1,x_2)=(x_1+x_2,x_1-x_2).$$

Define $X=x_1+x_2, Y=x_1-x_2$, then we get

$$\vec{w}(X,Y)=\vec{w}(f(x(t))):=(f_*)\vec{v}(x(t))=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\cdot\begin{pmatrix}x_2\\x_1\end{pmatrix}=\begin{pmatrix}x_1+x_2\\x_2-x_1\end{pmatrix}=:\begin{pmatrix}X\\-Y\end{pmatrix},$$

For the ODEs

$$\frac{d}{dt}\begin{pmatrix}X(t)\\Y(t)\end{pmatrix}=\begin{pmatrix}X(t)\\-Y(t)\end{pmatrix}=\vec{w}(X,Y),$$

we can directly get the solution $\Phi(t)=(c_1e^t,c_2e^{-t})$.

Note $(f^{-1})_*\vec{w}=\vec{v}$, applying Remark 2.10 on f^{-1} and Φ above, we get

$$f^{-1}\circ\Phi(t)=(\frac{1}{2}c_1e^t+\frac{1}{2}c_2e^{-t},\frac{1}{2}c_1e^t-\frac{1}{2}c_2e^{-t}),$$

is a solution to to $\frac{d}{dt}(x_1(t),x_2(t))=(x_2(t),x_1(t))$.

Exercise 4 *Problem* 1, 2, 5 − − − −8 *in Section* 5.1*, Problem* 2, 3 *in Section* 5.2*, Problem* 1 *in Section* 5.3*, and Problem* 1 − − − −5 *in Section* 5.4 *of* [Arn06].

2.2 One-parameter diffeomorphism group and ODE

Definition 2.12 *We say that $\{g^t\}_{t\in\mathbb{R}}$ is an **one-parameter diffeomorphism group of a smooth manifold** M, if $g^t : M \to M$ is a diffeomorphism and $g^t x$ depends smoothly on t, x, furthermore*

(a) . for any $t_1, t_2 \in \mathbb{R}$, we have $g^{t_1+t_2} = g^{t_1} \circ g^{t_2}$;

(b) . and $g^0 = Id$.

*And $\{g^t\}_{t\in\mathbb{R}}$ is also called a **phase flow with phase space** M. For $p \in M$, the orbit of the point p with respect to $G := \{g^t\}_{t\in\mathbb{R}}$ is denoted*

$$Gp = \{g^t p : g^t \in G\}.$$

*The orbits of a phase flow is called its **phase curve**.*

Example 2.13 *(a) . Define $g^s : \mathbb{R} \to \mathbb{R}$ by $g^s(x) = x + 2s$, where $s \in \mathbb{R}$, then $\{g^s\}_{s\in\mathbb{R}}$ is an one-parameter diffeomorphism group of $\mathbb{R}$.*

(b) . Define $g^s : \mathbb{R} \to \mathbb{R}$ by $g^s(x) = x + \sin s$, where $s \in \mathbb{R}$, then $\{g^s\}_{s\in\mathbb{R}}$ is not an one-parameter diffeomorphism group of $\mathbb{R}$.

(c) . Define $g^s : \mathbb{R}^n \to \mathbb{R}^n$ as $g^s(\vec{x}) = e^s \cdot \vec{x}$, where $\vec{x} \in \mathbb{R}^n$ and $s \in \mathbb{R}$. Then $\{g^s\}_{s\in\mathbb{R}}$ is an one-parameter diffeomorphism group of $\mathbb{R}^n$.

(d) . Define $g^s : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$g^s \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and $\{g^s\}_{s\in\mathbb{R}}$ is an one-parameter diffeomorphism group of $\mathbb{R}^2$.

Lemma 2.14 *Assume $\{g^t\}_{t\in\mathbb{R}}$ is an one-parameter diffeomorphism group of $\mathbb{R}^n$, and define $\varphi(t) = g^t p$ where $p \in \mathbb{R}^n$. Then $\varphi(t)$ is a solution to*

$$\begin{cases} \frac{d}{dt}\vec{x}(t) = \vec{v}(\vec{x}(t)) \,, & t \in \mathbb{R}, \vec{x}(t) \in \mathbb{R}^n, \\ \vec{x}(0) = p \,, & \end{cases}$$

where $\vec{v} = \left(\lim\limits_{\tau\to 0} \dfrac{g^\tau - g^0}{\tau}\right) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map.

Proof: Using the definition of g^s, direct computation yields

$$\frac{d}{dt}(g^t p) = \lim_{\tau\to 0} \frac{g^{t+\tau} p - g^t p}{\tau} = \lim_{\tau\to 0} \frac{g^\tau g^t p - g^t p}{\tau} = \lim_{\tau\to 0} \frac{(g^\tau - Id) \circ g^t p}{\tau} = \left(\lim_{\tau\to 0} \frac{g^\tau - g^0}{\tau}\right) \circ g^t p.$$

Recall g^t is smooth in t, we get the existence of the smooth map $\vec{v} = \left(\lim\limits_{\tau\to 0} \dfrac{g^\tau - g^0}{\tau}\right) : \mathbb{R}^n \to \mathbb{R}^n$. The concluson follows. ∎

Definition 2.15 *For a phase flow $\{g^t\}_{t\in\mathbb{R}}$ with the phase space M, the **phase velocity vector** of the flow $\{g^t\}_{t\in\mathbb{R}}$ at $x \in M$ is*

$$\vec{v}(x) = \frac{d}{dt}\Big|_{t=0}(g^t x). \tag{2.2}$$

*The map $\vec{v} : M \to M$ defined by (2.2) is called the **phase velocity field**.*

Some smooth vector field is the phase velocity field of phase flow as follows.

Example 2.16 *(a) . Let $v(y) = k \cdot y : \mathbb{R} \to \mathbb{R}$, where $k \in \mathbb{R}$ is some constant. Then v is the phase velocity field of phase flow $g^t y_0 = e^{kt} y_0$.*

(b) . Let $\vec{v}(x_1, x_2) = (x_2, -x_1) : \mathbb{R}^2 \to \mathbb{R}^2$, then the corresponding phase flow $\{g^\theta\}_{\theta \in \mathbb{R}}$ is

$$g^\theta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

Note that not every smooth vector field is the phase velocity field of some phase flow. See the following example.

Example 2.17 *Let $v(x) = x^2 : \mathbb{R} \to \mathbb{R}$, then it is a smooth vector field on $\mathbb{R}$. If v is the phase velocity field of some phase flow $\{g^t\}_{t \in \mathbb{R}}$, then from Lemma 2.14, we get that $\varphi(t) := g^t x_0 : \mathbb{R} \to \mathbb{R}$ is a solution to*

$$\dot{x} = x^2, \qquad x(0) = x_0, \tag{2.3}$$

where $x_0 \in (\mathbb{R} \backslash \{0\})$ is fixed.

On the other hand, from Theorem 1.9, we get the unique solution is $x(t) = \frac{x_0}{1 - t x_0}$. Hence $g^t x_0 = \frac{x_0}{1 - t x_0}$, note it is not an one-to-one map of $\mathbb{R}$ for $t \neq 0$. Hence $\{g^t\}$ is not a phase flow of $\mathbb{R}$, it is the contradiction.

Theorem 2.18 *Assume $\vec{v}(t, \vec{x})$ is C^1 in $\vec{x}$ and C^0 in t, where $(t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^n$; and $\vec{\varphi}, \vec{\psi}$ are solutions to*

$$\begin{cases} \frac{d}{dt} \vec{x}(t) = \vec{v}(t, \vec{x}(t)), & t \in \mathbb{R}, \vec{x}(t) \in \mathbb{R}^n, \\ \vec{x}(t_0) = x_0, \end{cases}$$

on the open interval $I \subseteq \mathbb{R}$, then $\vec{\varphi}(t) = \vec{\psi}(t)$ for all $t \in I$.

Proof: **Step (1)**. We only need to show that $\vec{\varphi}(t) = \vec{\psi}(t)$ for all $t \in [a, b] \subseteq I$, where $t_0 \in (a, b)$.
From the assumption on $\vec{v}$, we know that

$$L := \sup_{t \in [a,b]} \sup_{s \in [0,1]} \|D_{\vec{x}} \vec{v}(t, s\vec{\psi}(t) + (1 - s)\vec{\varphi}(t))\| < \infty,$$

where $D_{\vec{x}} \vec{v}(t, \vec{x}) = \left(\frac{\partial}{\partial x_i} v_j(t, \vec{x})\right)_{i,j}$ with $\vec{v} = (v_1, \cdots, v_n), \vec{x} = (x_1, \cdots, x_n)$ and

$$\|D_{\vec{x}} \vec{v}(t, x)\| := \sup_{\vec{v}_0 \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n} |D_{\vec{x}} \vec{v}(t, x) \vec{v}_0|_{\mathbb{R}^n}.$$

Define $f(t) = |\vec{\psi}(t) - \vec{\varphi}(t)|_{\mathbb{R}^n} : [a, b] \to \mathbb{R}$. Note

$$\begin{aligned} |f'(t)| &= |(\sqrt{\sum_{i=1}^n (\psi_i - \varphi_i)^2(t)})'| = \Big|\frac{\sum_i (\psi_i - \varphi_i) \cdot (\psi_i - \varphi_i)'}{\sqrt{\sum_{i=1}^n (\psi_i - \varphi_i)^2(t)}}\Big| \le \sqrt{\sum_i [(\psi_i - \varphi_i)']^2} = |(\vec{\psi} - \vec{\varphi})'(t)|_{\mathbb{R}^2} \\ &= |\vec{v}(t, \vec{\psi}(t)) - \vec{v}(t, \vec{\varphi}(t))|_{\mathbb{R}^n} = |\int_0^1 \frac{d}{ds} \vec{v}(t, s\vec{\psi}(t) + (1 - s)\vec{\varphi}(t)) ds| \\ &\le |\int_0^1 D_{\vec{x}} \vec{v}(t, s\vec{\psi}(t) + (1 - s)\vec{\varphi}(t)) \cdot (\vec{\psi}(t) - \vec{\varphi}(t)) ds|. \end{aligned}$$

From Hölder inequality, we have

$$\begin{aligned} &|\int_0^1 D_{\vec{x}} \vec{v}(t, s\vec{\psi}(t) + (1 - s)\vec{\varphi}(t)) \cdot (\vec{\psi}(t) - \vec{\varphi}(t)) ds| \le \Big(\int_0^1 |D_{\vec{x}} \vec{v}(t, s\vec{\psi}(t) + (1 - s)\vec{\varphi}(t)) \cdot (\vec{\psi}(t) - \vec{\varphi}(t))|^2 ds\Big)^{\frac{1}{2}} \\ &\le \Big(\int_0^1 L^2 \cdot |\vec{\psi}(t) - \vec{\varphi}(t)|^2_{\mathbb{R}^n} ds\Big)^{\frac{1}{2}} \le L \cdot f(t). \end{aligned}$$

Therefore we get

$$\begin{cases} f'(t) \le L \cdot f(t)\ , & t \in [a,b], \\ f(t_0) = 0\ , \end{cases} \tag{2.4}$$

Step (2). Now we are reduced to show $f(t) = 0$ for $t \in [a,b]$. By contradiction, there is $t_1 \in [a,b]$ such that $f(t_1) \neq 0$. Without loss of generality, we can cassume $t_1 > t_0$ in the rest argument (if $t_1 < t_0$, similar argument follows).

We define $t_2 = \sup\{t \in [a,t_1] : f(t) = 0\}$, then

$$t_0 \le t_2 < t_1, \qquad f(t_2) = 0.$$

For any $t \in (t_2, t_1)$, we have $f(t) > 0$. From (2.4), we obtain

$$\frac{f'(t)}{f(t)} \le L, \qquad \forall t \in (t_2, t_1].$$

Therefore, we get

$$f(t_1) \le f(t) e^{L(t_1 - t)}, \qquad \forall t \in (t_2, t_1).$$

Let $t \to t_2+$ in the above, it yields

$$f(t_1) \le 0.$$

This implies $f(t_1) = 0$ because of $f(t) \ge 0$, which is the contradiction. ∎

Definition 2.19 *A point $p \in \mathbb{R}^n$ is called* ***non-stationary for an one-parameter diffeomorphism group*** $\{g^s\}_{s\in\mathbb{R}}$ ***of*** $\mathbb{R}^n$*, if there is $s \in \mathbb{R}$ such that $g^s p \neq p$.*

Corollary 2.20 *If p is a non-stationary point of an one-parameter group of diffeomorphism $\{g^s\}_{s\in\mathbb{R}}$ on $\mathbb{R}^n$, then $\frac{d}{dt}\big|_{t=0}(g^t p) \neq \vec{0} \in \mathbb{R}^n$.*

Proof: By contradiction, assume $\frac{d}{dt}\big|_{t=0}(g^t p) = \vec{0} \in \mathbb{R}^n$.

By Lemma 2.14, we know that $\vec{\varphi}(t) = g^t p$ is a solution to

$$\begin{cases} \frac{d}{dt}\vec{x}(t) = \vec{v}(\vec{x}(t))\ , & t \in \mathbb{R}, \vec{x}(t) \in M, \\ \vec{x}(0) = p\ , \end{cases} \tag{2.5}$$

on $\mathbb{R}$, where $\vec{v} = \left(\lim_{\tau\to 0} \dfrac{g^\tau - g^0}{\tau}\right)$.

Note $\vec{v}(p) = \frac{d}{dt}\big|_{t=0}(g^t p) = \vec{0} \in \mathbb{R}^n$, and we get that $\vec{\psi}(t) \equiv p$ is a solution to (2.5) on $\mathbb{R}$.

By Theorem 2.18, we have that $\vec{\varphi}(t) = \vec{\psi}(t) \equiv p$, which contradicts the assumption that p is a non-stationary point of $\{g^s\}_{s\in\mathbb{R}}$. ∎

Exercise 5 *Problem* 1, 2 *in Section* 4.3*, Problem* 2, 6, 8 *in Section* 4.4 *of* [Arn06].

2.3 Integration of ODE and symmetry of direction fields

Definition 2.21 *A diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is called* ***a symmetry of a vector field*** $\vec{v} : \Omega \to \mathbb{R}^n$*, where $\Omega \subseteq \mathbb{R}^n$; if*

$$(f_* \vec{v})(t, y) = \vec{v}(f(t,y)), \qquad \forall (t,y), f(t,y) \in \Omega.$$

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