数学分析作业

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Week 13

1.(1)

$$F(x)=\frac{1}{(x+1)(x+2)^2(x+3)^3}=\frac{a}{x+1}+\frac{b_1}{x+2}+\frac{b_2}{(x+2)^2}+\frac{c_1}{(x+3)}+\frac{c_2}{(x+3)^2}+\frac{c_2}{(x+3)^3}$$

$$a=\lim_{x\to-1}(x+1)F(x)=\frac{1}{8},\ b_2=\lim_{x\to-2}(x+2)^2F(x)=-1,\ b_1=lim_{x\to-2}(x+2)(F(x)-\frac{-1}{(x+2)^2})=2,$$

$$c_3=lim_{x\to-3}(x+3)^3F(x)=-\frac{1}{2},\ c_2=\lim_{x\to-3}(x+3)^2(F(x)-\frac{-\frac{1}{2}}{(x+3)^3})=-\frac{5}{4}$$

$$c_1=\lim_{x\to-3}(x+3)(F(x)-\frac{-\frac{1}{2}}{(x+3)^3}-\frac{-\frac{5}{4}}{(x+3)^2})=-\frac{17}{8}$$

$$\int\frac{\mathrm{d}x}{(x+1)(x+2)^2(x+3)^3}=\frac{1}{8}\log|x+1|+2\log|x+2|+\frac{1}{x+2}-\frac{17}{8}\log|x+3|+\frac{5}{4(x+3)}+\frac{1}{4(x+3)^2}+C$$

(2)

$$\begin{aligned}\int\frac{\mathrm{d}x}{(x^2+9)^3}&=\frac{1}{36}\frac{x}{(x^2+9)^2}+\frac{1}{12}\int\frac{\mathrm{d}x}{(x^2+9)^2}\\&=\frac{x}{36(x^2+9)^2}+\frac{1}{12}(\frac{1}{18}\frac{x}{(x^2+9)}+\frac{1}{18}\int\frac{\mathrm{d}x}{(x^2+9)})\\&=\frac{x}{36(x^2+9)^2}+\frac{1}{216}\frac{x}{(x^2+9)}+\frac{1}{648}\arctan\Big(\frac{x}{3}\Big)+C\end{aligned}$$

(3)

$$\begin{aligned}\int\frac{1+x^2}{1+x^4}\mathrm{d}x&=\frac{1}{2}\int(\frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{x^2-\sqrt{2}x+1})\mathrm{d}x\\&=\frac{1}{2}\int(\frac{1}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}+\frac{1}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}})\mathrm{d}x\\&=\frac{\sqrt{2}}{2}(\arctan\Big(\sqrt{2}x+1\Big)+\arctan\Big(\sqrt{2}x-1\Big))+C\end{aligned}$$

(4)

$$\int\frac{\mathrm{d}x}{x^4(1+x^2)}=\int(\frac{1}{x^4}-\frac{1}{x^2}+\frac{1}{x^2+1})=-\frac{1}{x^3}+\frac{1}{x}+\arctan x+C$$

6.4 1. (1) 令 $t=\tan x$

$$\begin{aligned}\int\frac{\mathrm{d}x}{\cos^4x+\sin^4x}=\int\frac{1+t^2}{1+t^4}\mathrm{d}t&=\frac{\sqrt{2}}{2}(\arctan\Big(\sqrt{2}t+1\Big)+\arctan\Big(\sqrt{2}t-1\Big))+C\\&=\frac{\sqrt{2}}{2}(\arctan\Big(\sqrt{2}\tan x+1\Big)+\arctan\Big(\sqrt{2}\tan x-1\Big))+C\end{aligned}$$

(2)

$$\begin{aligned}\int\frac{\mathrm{d}x}{\cos^4 x\sin^4 x}&=\int\frac{(\cos^2 x+\sin^2 x)^3}{\cos^4 x\sin^4 x}\mathrm{d}x=\int\frac{(1+t^2)^3}{t^4}\mathrm{d}t\\&=\int(t^2+3+\frac{3}{t^2}+\frac{1}{t^4})\mathrm{d}t=\frac{1}{3}t^3+3t-\frac{3}{t}-\frac{1}{3t^3}+C\\&=\frac{1}{3}\tan^3 x+3\tan x-\frac{3}{\tan x}-\frac{1}{3\tan^3 x}+C\end{aligned}$$

(3)

$$\begin{aligned}\int\frac{\sin x}{\cos^3 x+\sin^3 x}&=\int\frac{t}{1+t^3}\mathrm{d}t=-\int\frac{1}{3}\frac{\mathrm{d}t}{(1+t)}+\frac{1}{3}\int\frac{t+1}{t^2-t+1}\mathrm{d}t\\&=-\frac{1}{3}\log|t+1|+\frac{1}{3}\int\frac{t-\frac{1}{2}+\frac{3}{2}}{(t-\frac{1}{2})^2+\frac{3}{4}}\mathrm{d}t\\&=-\frac{1}{3}\log|t+1|+\frac{1}{6}\log\Big(\frac{3}{4}+(t-\frac{1}{2})^2\Big)+\frac{\sqrt{3}}{3}\arctan\left(\frac{2\sqrt{3}}{3}(t-\frac{1}{2})\right)+C\\&=-\frac{1}{3}\log|\tan x+1|+\frac{1}{6}\log\Big(\frac{3}{4}+(\tan x-\frac{1}{2})^2\Big)+\frac{\sqrt{3}}{3}\arctan\left(\frac{2\sqrt{3}}{3}(\tan x-\frac{1}{2})\right)+C\end{aligned}$$

(4) 令 $t=\sqrt{\tan x},x=\arctan t^2$

$$\begin{aligned}&\int\sqrt{\tan x}\mathrm{d}x=\int\frac{2t^2}{1+t^4}\mathrm{d}t=\int(\frac{-\frac{\sqrt{2}}{2}(t+\frac{\sqrt{2}}{2})+\frac{1}{2}}{(t+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}+\frac{\frac{\sqrt{2}}{2}(t-\frac{\sqrt{2}}{2})+\frac{1}{2}}{(t-\frac{\sqrt{2}}{2})^2+\frac{1}{2}})\mathrm{d}t\\&=\frac{\sqrt{2}}{2}\arctan\big(\sqrt{2}t+1\big)-\frac{\sqrt{2}}{4}\log\left((t+\frac{\sqrt{2}}{2})^2+\frac{1}{2}\right)+\frac{\sqrt{2}}{2}\arctan\big(\sqrt{2}t-1\big)+\frac{\sqrt{2}}{4}\log\left((t-\frac{\sqrt{2}}{2})^2+\frac{1}{2}\right)\\&=\frac{\sqrt{2}}{2}\arctan\big(\sqrt{2}\sqrt{\tan x}+1\big)+\frac{\sqrt{2}}{2}\arctan\big(\sqrt{2}\sqrt{\tan x}-1\big)+\frac{\sqrt{2}}{4}\log\left(\frac{(\sqrt{\tan x}-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}{(\sqrt{\tan x}+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)\end{aligned}$$

(5)

$$\begin{aligned}\int\frac{\mathrm{d}x}{1+\epsilon\cos x}=\int\frac{\frac{2}{1+t^2}}{1+\epsilon\frac{1-t^2}{1+t^2}}\mathrm{d}t=\frac{2}{1-\epsilon}\int\frac{\mathrm{d}t}{\frac{1+\epsilon}{1-\epsilon}+t^2}&=\frac{2}{\sqrt{1-\epsilon^2}}\arctan\sqrt{\frac{1-\epsilon}{1+\epsilon}}t+C\\&=\frac{2}{\sqrt{1-\epsilon^2}}\arctan\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{x}{2}\right)+C\end{aligned}$$

2. (1) 令 $t^3=1+x$

$$\begin{aligned}\int\frac{\mathrm{d}x}{1+\sqrt[3]{1+x}}=\int\frac{3t^2}{1+t}\mathrm{d}t=\int(3t-3+\frac{3}{1+t})\mathrm{d}t&=\frac{3}{2}t^2-3t+\log|1+t|+C\\&=\frac{3}{2}(1+x)^{\frac{2}{3}}-3(1+x)^{\frac{1}{3}}+\log\Big|1+(1+x)^{\frac{1}{3}}\Big|+C\end{aligned}$$

(2) 令 $\sqrt[6]{x+1}=t$

$$\begin{aligned}\int\frac{x}{\sqrt{x+1}+\sqrt[3]{x+1}}\mathrm{d}x=\int\frac{t^6-1}{t^2+t^3}6t^5\mathrm{d}t&=6\int(t^8-t^7+t^6-t^5+t^4-t^3)\mathrm{d}t\\&=\frac{2}{3}t^9-\frac{3}{4}t^8+\frac{6}{7}t^7-t^6+\frac{6}{5}t^5-\frac{3}{2}t^4+C\\=\frac{2}{3}(x+1)^{\frac{3}{2}}-\frac{3}{4}(x+1)^{\frac{4}{3}}+\frac{6}{7}(x+1)^{\frac{7}{6}}&-(x+1)+\frac{6}{5}(x+1)^{\frac{5}{6}}-\frac{3}{2}(x+1)^{\frac{2}{3}}+C\end{aligned}$$

(3) 令 $2x^2=t,1+t=u^2$

$$\begin{aligned}\int\frac{x^3}{\sqrt{1+2x^2}}\mathrm{d}x&=\frac{1}{8}\int\frac{t}{\sqrt{1+t}}\mathrm{d}t=\frac{1}{4}\int(u^2-1)\mathrm{d}u=\frac{1}{12}u^3-\frac{1}{4}u+C\\&=\frac{1}{12}(t+1)^{\frac{3}{2}}-\frac{1}{4}(t+1)^{\frac{1}{2}}+C=\frac{1}{12}(2x^2+1)^{\frac{3}{2}}-\frac{1}{4}(2x^2+1)^{\frac{1}{2}}+C\end{aligned}$$

(4) 存在部分分式分解

$$\frac{\mathrm{d}x}{(x+a)^m(x+b)^n}=(\sum_{k=1}^{m}\frac{A_k}{(x+a)^k}+\sum_{i=1}^{n}\frac{B_i}{(x+b)^i})$$

$$\int\frac{\mathrm{d}x}{(x+a)^m(x+b)^n}=\int(\sum_{k=1}^{m}\frac{A_k}{(x+a)^k}+\sum_{i=1}^{n}\frac{B_i}{(x+b)^i})$$

$$=A_1\log|x+a|+B_1\log|x+b|+(\sum_{k=2}^{m}\frac{1}{1-k}\frac{A_k}{(x+a)^{k-1}}+\sum_{i=2}^{n}\frac{1}{1-i}\frac{B_i}{(x+b)^{i-1}})+C$$

(5)*

$$\int\frac{\mathrm{d}x}{(x+1)^{\frac{n+1}{n}}}=\int\frac{\mathrm{d}x}{x^{n+1}(\frac{1}{x^n}+1)^{\frac{n+1}{n}}}=-\frac{1}{n}\int\frac{\mathrm{d}(\frac{1}{x^n}+1)}{(1+\frac{1}{x^n})^{\frac{n+1}{n}}}=\frac{1}{(1+\frac{1}{x^n})^{\frac{1}{n}}}+C$$

7.1

1. 注意到 $\sqrt{(x-a)(b-x)}$ 的图像是以 $\frac{b-a}{2}$ 为半径, $(\frac{a+b}{2},0)$ 为圆心的半圆, 故所求积分即为 $\frac{1}{2}\pi(\frac{b-a}{2})^2=\frac{\pi(b-a)^2}{8}$.

2.

$$S=\sum_{k=1}^{n}\frac{1}{(a+\frac{k(b-a)}{n})^2}\frac{b-a}{n}=n(b-a)\sum_{k=1}^{n}\frac{1}{((n-k)a+kb)^2}$$

$$S<n(b-a)\sum_{k=1}^{n}\frac{1}{(n-k)a+kb}\frac{1}{(n-k+1)a+(k-1)b}$$

$$=n\sum_{k=1}^{n}(\frac{1}{(n-k+1)a+(k-1)b}-\frac{1}{(n-k)a+kb})=n(\frac{1}{na}-\frac{1}{nb})=\frac{b-a}{ab}$$

$$S>n(b-a)\sum_{k=1}^{n}\frac{1}{(n-k)a+kb}\frac{1}{(n-k-1)a+(k+1)b}$$

$$=n\sum_{k=1}^{n}(\frac{1}{(n-k)a+kb}-\frac{1}{(n-k-1)a+(k+1)b})=n(\frac{1}{(n-1)a+b}-\frac{1}{-a+(n+1)b})$$

令 $n\to\infty$, 有 $\frac{b-a}{ab}\leq S\leq\frac{b-a}{ab}$, 故

$$\int_a^b\frac{\mathrm{d}x}{x^2}=S=\frac{b-a}{ab}$$

3. (1) n 为奇数时,

$$\frac{x^n}{1+x}=\sum_{k=0}^{n-1}(-1)^kx^k-\frac{1}{1+x}$$

$$\int_0^1\frac{x^n}{1+x}\mathrm{d}x=\sum_{k=0}^{n-1}(-1)^k\frac{1}{k+1}x^{k+1}-\log|1+x|\Big|_0^1=\sum_{k=0}^{n-1}(-1)^k\frac{1}{k+1}-\log 2$$

n 为偶数时,

$$\frac{x^n}{1+x}=\sum_{k=0}^{n-1}(-1)^{k+1}x^k-\frac{1}{1+x}$$

$$\int_0^1\frac{x^n}{1+x}\mathrm{d}x=\sum_{k=0}^{n-1}(-1)^{k+1}\frac{1}{k+1}x^{k+1}+\log|1+x|\Big|_0^1=\sum_{k=0}^{n-1}(-1)^{k+1}\frac{1}{k+1}+\log 2$$

(2)

$$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{1+\epsilon\cos x}=\frac{2}{\sqrt{1-\epsilon^2}}\arctan\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{x}{2}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{2}{\sqrt{1-\epsilon^2}}\arctan\sqrt{\frac{1-\epsilon}{1+\epsilon}}$$

4. 可取 n 充分大使得对任意的 $\varepsilon>0$

$$\int_a^b e^{-nx^2}\mathrm{d}x<\int_a^b e^{-na^2}\mathrm{d}x=e^{-na^2}(b-a)<\varepsilon$$

又 $e^{-nx^2} > 0$, 故

$$\lim_{n\to\infty}\int_a^b e^{-nx^2}\mathrm{d}x = 0$$

5. (1)

$$|a\cos x + b\sin x| \le \sqrt{a^2+b^2}$$

故

$$\int_0^{2\pi} |a\cos x + b\sin x| \le \int_0^{2\pi} \sqrt{a^2+b^2} = 2\pi\sqrt{a^2+b^2}$$

(2)

$$(x^m(1-x)^n)' = x^{m-1}(1-x)^{n-1}(m-mx-nx)$$

故 $x = \frac{m}{m+n}$ 时取极大值. 又在 $[0.1]$ 上 $x^m(1-x)^n > 0$

$$\int_0^1 x^m(1-x)^n\mathrm{d}x \le \int_0^1 \frac{m}{m+n}^m(1-\frac{m}{m+n})^n\mathrm{d}x = \frac{m^m n^n}{(m+n)^{m+n}}$$

6. (1)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n} = \int_0^{\pi}\sin x = -\cos x\Big|_0^{\pi} = 2$$

(2)

$$\text{原极限} = \int_0^1 x^p\mathrm{d}x = \frac{1}{p+1}$$

7.

$$\int_{-a}^{b}(x+a)(x-b)f(x)\,\mathrm{d}x = \int_{-a}^{b}x^2f(x)\mathrm{d}x - ab\int_{-a}^{b}f(x)\mathrm{d}x + (a-b)\int_{-a}^{b}xf(x)\mathrm{d}x$$

其中 $(x+a)(x-b)f(x) \le 0, \forall x \int[a,b]$, $(a-b)\int_{-a}^{b}xf(x)\mathrm{d}x = 0$. 故

$$\int_{-a}^{b}x^2f(x)\mathrm{d}x \le ab\int_{-a}^{b}f(x)\mathrm{d}x$$

8. (1)

$$\int_0^1\frac{(1+x)^4}{1+x^2}\mathrm{d}x = \int_0^1(x^2+4x+5-\frac{4}{x^2+1})\mathrm{d}x = \frac{1}{3}+2+5-4\arctan x\Big|_0^1 = \frac{22}{3}-\pi$$

(2)

$$\int_0^1\frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x = \int_0^1(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2})\mathrm{d}x = \frac{1}{7}-\frac{4}{6}+\frac{5}{5}-\frac{4}{3}+4-4\arctan x\Big|_0^1 = \frac{22}{7}-\pi$$

7.2

1.

证明. 假设存在 $x_0 \in (a,b)$, $f(x) < g(x)$, 由连续函数的局部保号性, 存在 $\varepsilon > 0, \forall x \in [x_0-\varepsilon, x_0+\varepsilon] \subseteq [a,b]$, $f(x) < g(x)$,

$$0 = \int_a^b(f(x)-g(x))\mathrm{d}x = \int_a^{x_0-\varepsilon}(f(x)-g(x))\mathrm{d}x + \int_{x_0-\varepsilon}^{x_0+\varepsilon}(f(x)-g(x))\mathrm{d}x + \int_{x_0+\varepsilon}^{b}(f(x)-g(x))\mathrm{d}x > 0$$

$x_0 = a$ 或 b 时类似的, 只需将 $[x_0-\varepsilon, x_0+\varepsilon]$ 换为 $[a, a+\varepsilon]$ 或 $[b-\varepsilon, b]$ 即可. □

2. 令 $g(x) = f(x)$, 有 $f(x)g(x) = f(x)^2 \ge 0$. 利用第一题的结论得到 $f^2 = 0$, 故 $f = 0$.

3. (1) $\frac{\sin x}{x} > 0, \forall x \in (0,\pi]$, $\lim_{x\to 0}\frac{\sin x}{x} > 0$, 故 $\int_0^{\pi}\frac{\sin x}{x}\mathrm{d}x > 0$

(2) $e^x\log^3 x<0,\forall x\in[\frac{1}{2},1)$, $x=0$ 时 $e^x\log^3 x=0$ 故

$$\int_{\frac{1}{2}}^{1} e^x\log^3 x\mathrm{d}x<0$$

4. (1) $x\in[0,1]$ 时, $0<e^{-x}<e^{-x^2}$, 故 $\int_0^1 e^{-x}\mathrm{d}x<\int_0^1 e^{-x^2}\mathrm{d}x$

(2) $e^{-x^2}=e^{-(-x)^2}$, 故

$$\int_{-1}^{0} e^{-x^2}\mathrm{d}x=\int_0^1 e^{-x^2}\mathrm{d}x$$

(3) $x\in[0,1]$ 时, $0<\frac{\sin x}{1+x}<\frac{\sin x}{1+x^2}$, 故

$$\int_0^1 \frac{\sin x}{1+x}\mathrm{d}x<\int_0^1 \frac{\sin x}{1+x^2}\mathrm{d}x$$

(4) 由于 $\frac{\sin x}{x}<1,\forall x\in(0,\frac{\pi}{2})$, 故 $0<\frac{\sin^2 x}{x^2}<\frac{\sin x}{x}$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{x^2}\mathrm{d}x<\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x$$

5. 有不等式 $\frac{\sin x}{x}>\frac{2}{\pi},\forall x\in(0,\frac{\pi}{2})$, 即 $\sin x>\frac{2x}{\pi}$, $R>0$ 时,

$$\int_0^{\frac{\pi}{2}} e^{-R\sin x}\mathrm{d}x<\int_0^{\frac{\pi}{2}} e^{-\frac{2Rx}{\pi}}\mathrm{d}x=-\frac{\pi}{2R}e^{\frac{2x}{\pi}}\Big|_0^{\pi\pi/2}=\frac{\pi}{2R}(1-e^{-R})$$

$R>0$ 时,

$$\int_0^{\frac{\pi}{2}} e^{-R\sin x}\mathrm{d}x>\int_0^{\frac{\pi}{2}} e^{-\frac{2Rx}{\pi}}\mathrm{d}x=-\frac{\pi}{2R}e^{\frac{2x}{\pi}}\Big|_0^{\pi\pi/2}=\frac{\pi}{2R}(1-e^{-R})$$

$R=0$ 时, $I=\int_0^{\frac{\pi}{2}} 1\,\mathrm{d}x=\frac{\pi}{2}$