Analysis

Chengwuming

2024.2.29

Preface

3.10: Chapter Continuity finished.

March 27, 2024

Contents

Chapter 3

Foundations

3.1 Basic Logic and Set Theory

2024.2.5, the very first day of my review on Analysis, as well as my first practice of English.

Some definitions

$$(A \implies B) := (\neg A) \vee B$$

This explains that when the premise is not satisfied, the conclusion is automatically right. What's more, we deduce the empty set possesses every property.

$$\emptyset := \{x \in X : x \neq x\}$$

Proposition 3.1.1. $\emptyset$ is unique.

Proof. suppose we have two empty set $\emptyset_1$ and $\emptyset_2$, then the statement

$$(x \in \emptyset_1) \implies (x \in \emptyset_2)$$

is true, which means $\emptyset_1 \subseteq \emptyset_2$ and vice versa, thus we have $\emptyset_1 = \emptyset_2$ □

Families of Sets

Notice that the index set is always **nonempty**.

Proposition 3.1.2 (de Morgan's laws)**.**

$$(\bigcup_{\alpha\in A} A_\alpha)^c = \bigcap_{\alpha\in A} A_\alpha^c$$

$$(\bigcap_{\alpha\in A} A_\alpha)^c = \bigcup_{\alpha\in A} A_\alpha^c$$

Proof. From the definition, we know

$$\bigcup_{\alpha\in A} A_\alpha := \{x \in X; \exists\alpha \in A : x \in A_\alpha\}$$

whose complement is

$$\{x \in X; \neg(\exists\alpha \in A : x \in A_\alpha)\}$$

So we have

$$(\bigcap_{\alpha\in A} A_\alpha)^c = \{x \in X; \forall\alpha \in A : x \in A_\alpha^c\} = \bigcap_{\alpha\in A} A_\alpha^c$$

□

Remark. The concept set is not strictly defined yet.

3.2 Function, Relation and Operation

Definition 3.2.1. A function is an ordered triple (X, G, Y), where $G \subseteq X \times Y$. For each $x \in X$, there is exactly one $y \in Y$.

What if $X = \emptyset$? By the definition, $G = \emptyset$, we call the unique function $\emptyset \colon \emptyset \to Y$ the **empty function**.If $Y = \emptyset$ and $X \neq \emptyset$, there are no functions can be defined on it.

Some definitions

identity function, inclusion, restriction, extension, characteristic function, **fiber**, preimage

...

We denote the set of all functions from X to Y by $\mathrm{Funct}(X, Y)$, or Y^X. (Think about $\mathbb{R}^n$)

Proposition 3.2.1. Let $f \colon X \to Y$ be a function,

1. f is injective $\iff$ $\exists h \colon Y \to X$ such that $h \circ f = id_X$
2. f is surjective $\iff$ $\exists h \colon Y \to X$ such that $f \circ h = id_Y$

Proof. Trivial. (I'm too lazy right now and I will prove it someday) □

Example 3.2.1. For each nonempty set X, the function

$$\mathcal{P}(X) \to \{0, 1\}^X \ , \quad A \mapsto \chi_A$$

is bijective.

Proof. □

Relation

So many definitions and notations:

A relation R on X, diagonal $\Delta_X := \{(x, x); x \in X\}$, reflexive, transitive, symmetric, anti-symmetric, **equivalence** $\sim$, equivalence class $[x] := \{y \in X; y \sim x\}$, representative, partition, the set $X/\sim\ := \{[x]; x \in X\}$(called "X modulo $\sim$"), (canonical) quotient function, partial order, total order...

Easy to forget, easy to review.

Monotone, bounded on bounded sets. (You never heard these concept before, delete this when you are familier with them)

Operation... Not today.

3.3 Natural Numbers

2024.2.6

The Peano Axioms

The natural number consist of a set $\mathbb{N}$, a distinguished element $0 \in \mathbb{N}$, and a function $\nu : \mathbb{N} \to \mathbb{N}^{\times}$ with:

1. ν is injective.
2. If a subset N of $\mathbb{N}$ contains 0 and if $\nu(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$, then $N = \mathbb{N}$.

We can prove that $\nu|_{N^{\times}}$ is bijective.

Axioms 2 is one form of **principle of induction**.

The existence of the system $(N, 0, \nu)$ should be proved. One way to prove that, according to Dedekind, is the check the exsitence of the infinite system, because he proved "Any infinite system contains a model $(N, 0, \nu)$ for the natural numbers" using the comprehension axiom: the exsitence of the set

$$M := \{x; x \text{ is a set which satisfies some certain property}\}$$

By giving the example $M := \{x\ ; (x \text{ is a set}) \wedge (x \notin x)\}$ we have

$$M \in M \iff M \notin M$$

We call it antinomies. Thus, we need the *Infinity Axiom: An inductive set exists.* Here an **inductive set** is the set N which contains $\emptyset$ and such that for all $z \in N$, $z \cup \{z\}$ is also in N. Then we have the set

$$\mathbb{N} := \bigcap \{m; m \text{ is an inductive set}\}$$

satisfies the axioms we gave first this section (consider $\nu(z) := z \cup \{z\}$).

The natural numbers are unique up to **isomorphism**, which means there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}'$, such that $\varphi(0) = 0'$ and $\nu = \varphi^{-1} \circ \nu' \circ \varphi$.

Remark. The talk about classes and set, I don't understand, I'm so stupid. (Page 30, Analysis, Amann)

The Arithmetic of Natural Numbers and Induction Principle

Do this part in the future.

3.4 Countability

2024.2.9 Today is ChuXi!!! Happy new year nyaa!!

Permutations

A **permutation** is a bijection from a finite set to itself. We denote the set of all permutations of X by $\mathcal{S}_X$.

Proposition 3.4.1. If X is an n element set, then $\mathrm{Num}(\mathcal{S}_X) = n!$.

Proof. We prove this by induction. For a $n+1$ element, we can exchange it with $a_1, a_2, \dots a_n$, so we have n choices and a nonexchange choice. So there are $(n+1) \times n!$ permutations.

Another way to consider. For each $j \in \{1, 2, \dots n+1\}$, there are exactly $n!$ permutations which send a_j to a_1.

The case $X = \emptyset$ is trivial. □

Proposition 3.4.2. There is no surjection from set X to $\mathcal{P}(X)$.

Proof. Let $\varphi : X \to \mathcal{P}(X)$ be a function, consider the subset of X

$$A := \{x \in X \ ; x \notin \varphi(x)\}$$

Then we have that A is not in $\mathrm{Im}(\varphi)$. Indeed we suppose that there exists y with $\varphi(y) = A$, follow the definition we have

$$y \in A \iff y \notin A$$

□

Hence we have the simple inequipotent example.

Please get familier with the notation X^Y

3.5 Group, Ring, Field, Poly

3.6 The Rational Numbers

3.7 The Real Numbers

3.8 Vector Space, Affine Space,

3.9 Convergence

Metric Spaces

Definition 3.9.1. Let X be a set, a function $d : X \times X \to R^+$ is called a metric on X if the following hold:

In metric space (X, d) we define the open(closed) ball.

$$\mathbb{B}(a, r) := \mathbb{B}_X(a, r) := \{x \in X; f(a, x) < r\}$$

$x \in X$ is important! Since you may make mistakes when the space is induced from a big space. (Try to give an example.)

Proposition 3.9.1.

$$d(x, y) \geq |d(x, z) - d(z, y)|, \forall x, y, z \in X$$

Discrete metric, SNCF-metric...

Two metric are called **equivalent** if, for each $x \in X$ and $\varepsilon > 0$, we have r_1, r_2 such that

$$\mathbb{B}_1(x, r_1) \subseteq \mathbb{B}_2(x, \varepsilon) \quad , \quad \mathbb{B}_2(x, r_2) \subseteq \mathbb{B}_1(x, \varepsilon)$$

Sequences

The set $\mathbb{K}^{\mathbb{N}}$ is an algebra denoted by s, and the convergence sequences consist a subalgebra denoted by c, and the null sequences consist a subalgebra of c denoted by c_0. We have

$$\lim : c \to \mathbb{K}, \qquad (x_n) \mapsto \lim x_n$$

is an algebra homomorphism.

3.10 Normed Vector Space

Norm

Definition 3.10.1. Tell me what is a norm.

We can induce a metric from a norm. Hence any normed vector space is also a metric space.

Chapter 4

Continuity

4.1 Continuity

Definition 4.1.1. Let X, Y be metric spaces, $f : X \to Y$ is continuous.

$\iff$ For each neighborhood V of $f(x_0) \in Y$, there exists a neighborhood U of $x_0 \in X$ such that

$$f(U) \subseteq V$$

$\iff$ There is some $\delta := \delta(x_0, \varepsilon)$ such that for all $x \in X$ with $d(x_0, x) < \delta$, we have $f(f(x_0), f(x)) < \varepsilon$.

$\iff$ (Sequentially continuous) For every sequence (x_k) in X such that $\lim x_k = x$, we have $\lim f(x_k) = f(x)$.

The concept **continuity** is based on metric now, since we can use neighborhood and sequences to characterize it. But we actually can define continuity on general topological space, using the concept of filter. We shall discuss this at the end of the chapter.

Make yourself less confusing, it is a good habit to check the metric space whenever we say something is continuous.

We can define some kind of stronger continuous, which are useful.

Definition 4.1.2. (Uniformly contiuous) $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } d(x, y) < \delta(\varepsilon) \implies d(f(x), f(y)) < \varepsilon,$$

We shall prove easily that if X is compact, then contiuous map f is automatically uniformly contiuous.

Definition 4.1.3. (Lipschitz continuous) $f : X \to Y$ is Lipschitz continuous if there is a constant $\alpha > 0$(Lipschitz constant) that

$$d(f(x), f(y)) \leq \alpha d(x, y), \quad x, y \in X$$

The addition, multiplication(on $\mathbb{K}$), conposition of two continuous funcition is continuous.

Examples

Canonical projections are Lipschitz contiuous. In particular, the projections $pr_k := \mathbb{R}^n \to \mathbb{R}$ is Lipschitz contiuous.

The functions $z \mapsto \mathrm{Re}(z), z \mapsto \mathrm{Im}(z), z \mapsto \bar{z}$ is Lipschitz contiuous.

Norms are Lipschitz continuous (in the norm-induced metric space).

Proof.

$$|\|x\| - \|y\|| \leq \|x - y\|, x, y \in E$$

□

The scalar product is continuous.

Rational functions are continuous.

Polynomials in n variables are continuous.

The function f defined by some power series with positive radius of convergence ρ, is continuous on $\rho\mathbb{B}$.

$\det : M_n(\mathbb{K}) \to \mathbb{K}$ is continuous.

The content of one-sided continuity is trivial and not that important. Please refer to page 228 (Analysis 1, Amann).

Theorem 4.1.1. (Cauchy criterion) $\lim_{x\to x_0} f(x)$ exists if and only if for each $\varepsilon > 0$, there is $\delta > 0$ such that $x_1, x_2 \in D \cap \mathring{\mathbb{B}}(x_0, \delta)$, we have

$$|f(x_1) - f(x_2)| < \varepsilon$$

It actually means limit exists if and only if when (x_n) is a cauchy sequence in D, we have $(f(x_n))$ is cauchy sequence in Y.

4.2 Topology

It is for a deeper understanding of continuous. So far, we have three ways to characterize continous in a metric space.

Definition 4.2.1. Function f is continue if and only if

(1)

(2)

(3)

Fill in the blank as a practice, mwahahahaha...

After the review, let start our journey in **topological space**. I suppose you have already known what is a general topological space, and, we are going to talk about the topological space induced from a metric for simplicity's sake.

Open Sets

We can define **open set** and **interior point** dualistically using the concept **neighbourhood**, which is actually based on metric. Do you remember that

$$\|\cdot\|_1 \sim \|\cdot\|_2 \sim \|\cdot\|_\infty$$

Therefore, it's not that important which metric we chose.

$$\text{metric} \longrightarrow \text{ball} \longrightarrow \text{neighbourhood} \longrightarrow \text{open set \& interior point}$$

But which metric space we are talking in is quiet important! For example, an open interval J is open in $\mathbb{R}$, but it is NOT open in $\mathbb{R}^2$(Raise more examples later).

Definition 4.2.2. Set A is **open** in X.

$\iff$ $A = \mathring{A}$

$\iff$ each $x \in A$ is an interior point of A.

$\iff$ A is a neighbourhood of each of its point.

$\iff$ More preciously, $\forall x \in A, \ \exists \mathbb{B}(x, \delta) \subseteq A$.

Here we defined interior point first.

The open set, as defined by form, align with the criteria for a topology.

Theorem 4.2.1. $\mathcal{T} := \{O \subseteq X; O \text{ is open}\}$ is a topology.

Proof. 1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$: trivial.

2. $\bigcup_\alpha O_\alpha$ is open: because $\forall x \in O_{\alpha_0} \subseteq \bigcup_\alpha O_\alpha$ where O_{α_0} itself is a neighbourhood of x.

3. $\bigcap_{k=0}^{n} O_k$ is open: choose the smallest ball.

□

Closed Sets

Definition 4.2.3. A is closed in X.

$\iff$ A^c is open in X.

$\iff$ $A = \overline{A}$.

$\iff$ A contains all its limits point.

$\iff$ Every sequence in A which converges in X, has its limit in A.

Grasping the connection between accumulation point of A and interior point of A^c.

Accumulation point is different from limit point.

The Closure, Interior and the Boundary of a set

We can say the closure of A is the smallest closed set which contains A, and the interior of A the biggest open set which is contained in A.

We can also define the closure of A by $\overline{A} := \{x \in X; x \text{ is an accumulation point of } A\}$ and the interior of A by $\mathring{A} := \{x \in X; x \text{ is an interior point of } A\}$. Both are equivalent.

Proof. $\overline{A} \subseteq \text{cl}(A)$ is trivial.

$\text{cl}(A) \subseteq \overline{A}$: let $x \notin \overline{A}$ (The case $\overline{A} = X$ is trivial), then there is an open set $U \ni x$ such that $U \cap A = \emptyset$ by the definition of $\overline{A}$. Thus, $x \in U \subseteq \text{cl}(A)^c$ (Use De Morgan's Law)

$\mathring{A}$ is similar. □

There are some trivial properties to remember, I will delete this part the day you handle them well.

Proposition 4.2.1. Let A and B are subsets of X

1. $A \subseteq B \implies \overline{A} \subseteq \overline{B}$
2. $\overline{(\overline{A})} = \overline{A}$
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Notice that the function $f : A \mapsto \overline{A}$ is increasing and idempotent.

$\mathring{A}$ is similar.

Definition 4.2.4. ... $x \in \partial A$

$\iff$ x is neither an interior point of A nor an exterior point of A.

$\iff$ $x \in \overline{A} \setminus \mathring{A}$.

$\iff$ $x \in \overline{A} \cap (\mathring{A})^c$

Hausdorff Condition

There are topological spaces which the condition fails, but we can prove that the metric-induced topological space satisfies the Hausdorff condition.

Proof. The key is $d(x, y) > 0$ when $x \neq y$, then you can make two balls with radius $r = d(x, y)/2$. □

Amann: one consequence of the Hausdor condition is

$$\bigcap\{U \ ; \ U \in \mathcal{U}_X(x)\} = \{x\} \ , \ x \in X$$

So far, I have no idea what this is useful for.

Any one element subset of a metric space is closed, we can prove this easily applying the Hausdorff condition.

Continuity

Now we back to continuity.

Definition 4.2.5. $f : X \to Y$ is continuous.

$\iff$ $f^{-1}(O)$ is open in X for each open set $O \in Y$.

$\iff$ $f^{-1}(A)$ is closed in X for each closed set $A \in Y$.

Proof. $1 \implies 2$: $\forall x \in f^{-1}(O)$, there is a neighborhood $U_x \in \mathcal{U}_X(x)$ such that $f(U_x) \subseteq O$ since f is continuous. Consider $\bigcup U_x$.

$2 \iff 3$: $f^{-1}(A^c) = f^{-1}(A)^c$.

$2 \implies 1$: $U := f^{-1}(V)$ is open, $f(U) \subseteq V$. □

Or to say, the set valued function $f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$ (contained in $\mathcal{T}_X$).

Some trivial examples, like fiber is closed, subspace of $\mathbb{K}^n$ is closed... You can find more.

Characterization of Limits

Let X and Y be metric space,$D \subseteq X$, $f : D \to Y$. When we say $\lim_{x\to x_0} f(x) = y$, we actually mean that any sequence (x_n) in X which converges to x_0, we have $(f(x_n))$ converges to y.

Another way to characterize it is, for each neighborhood V of y, there is a NOCENTER neighborhood $\mathring{U}$ of x_0, we have $f(\mathring{U} \cap D) \subseteq V$.

Consider the translation of $f(U \cap D) \not\subseteq V$:

$$f(\mathbb{B}_X(x_0, \frac{1}{n}) \cap D) \cap V^c \neq \emptyset$$

to prove easily.

Now we natually have contiuous extensions.

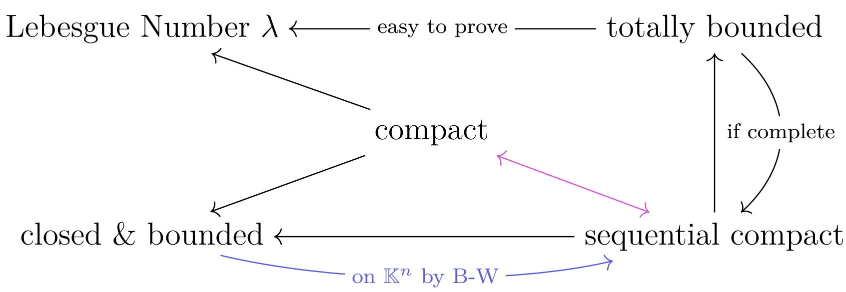
4.3 Compactness

To start with, ensure you have already handled following concepts well.

Characterization

Definition 4.3.1.
1. compact
2. sequential compact
3. totally bounded

We introduce the image



The purple one is the core. Let try to prove that.

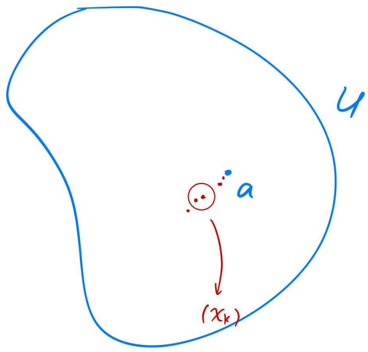
Proposition 4.3.1. Subset D of metric space X is compact if and only if D is sequential compact.

Proof. Two ways to prove. You can use "totally bounded + Lebesgue λ to prove. See in your homework (maybe someday I can copy that right here). We show another way here.

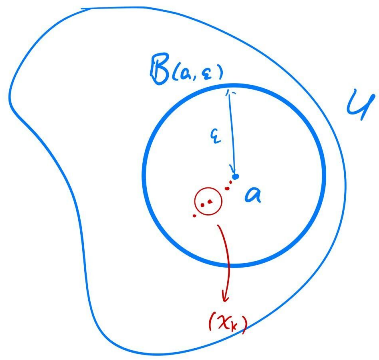
' $\Longrightarrow$ ' is not that hard. Suppose D is not sequential compact, we construct a open cover $\mathcal{U}$ with each set U contains at most finitely many terms of the sequence , which immediately contradict to $\mathcal{U}$ have a finite subcover.

'$\Longleftarrow$', We first prove that a sequential compact set must be totally bounded, if not , by choosing point out of the union of "balls who have point in them already", we have a sequence with each x_i, x_j , $d(x_i, x_j) <$ the radius of the balls, that, impossible to converge, leading to a contradition.

Then we prove it. Let $\mathcal{U}$ a cover and , for each $k \in \mathbb{N}^{\times}$,there is a finite cover of radius $\frac{1}{k}$. Suppose there is no finite $\bigcup U$ covers one of balls B_k(if not, then finite subcover exists), Choose the sequence of center of B_k ,(x_n) converges to $a \in D$, $a \in U_a \in \mathcal{U}$



there is some ε,$\mathbb{B}(a, \varepsilon) \subseteq U_a$, you can choose big enough k to meet:



It is easy to show the contradition. □

Two ways to find Lebesgue λ

- Assuming for a sequentially compact metric space (X, d), there does not exist such a Lebesgue number, then for any $n \in \mathbb{N}$, there exists a set $A_n \subseteq X$ such that $d(A_n) < \frac{1}{n}$, satisfying $\forall i \in \mathcal{I}$, $A_n \not\subset U_i$. Then there exists a sequence (x_n) where $x_n \in A_n$; take a convergent subsequence, x_{k_n}, which converges to $a \in X$. There exists i_0 such that $a \in U_{i_0}$. According to the definition of an open set, there exists δ such that $\mathbb{B}(a, \delta) \subseteq U_{i_0}$. We can take n sufficiently large such that $d(a, x_{k_n}) < \frac{\delta}{2}$ and $\frac{1}{n} < \frac{\delta}{2}$, then for any $y \in A_n$, $d(a, y) < d(a, x_{k_n}) + d(x_{k_n}, y) \leq \delta$. Thus, $A_n \subseteq \mathbb{B}(a, \delta)$. This is a contradiction.
- $\delta := \sup(\frac{1}{n} \sum_{i=1}^{n} d(x, C_i))$ where $C_i := X \setminus U_i$.

Applications

Theorem 4.3.1. Continuous images of compact sets are compact.

Proof. trivial. □

This is really useful. As a colollary, the **extreme value theorm** is important, too. This implies, the image of norm funcion of S^{n-1} has maximum and infimum, thus we can define norms on linear maps using the first one , and notice the equivalence of all norms on $\mathbb{K}^n$.

Corollary 4.3.2. All norms on $\mathbb{K}^n$ are equivalence.

Proof. See in your homework.(copy someday) □

And more...

Let $f : D \to \mathbb{R}^m, D \subseteq \mathbb{R}^n$ be continuous.

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m; x \in D, y = f(x)\}$$

Then if D is closed, we have Γ_f is closed. If D is compact, Γ_f is compact, too.

Suppose we don't know about f, but, Γ_f is closed , then f must be continuous.

4.4 Connectivity

Definition 4.4.1. A metric space X is called **connected**

$\iff$ $\nexists O_1, O_2 \subseteq X$, open, nonempty, with $O_1 \cap O_2 = \emptyset$ and $O_1 \cup O_2 = X$.

$\iff$ X is the only nonempty subset of X which is both open and closed.

$\iff$ For any nonempty subset A, B, if $A \cup B = X$, then we have $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

The tool is useful:

$$A \cup B = X, A \cap B = \emptyset \implies A^c = B$$

Using the tool, the proof is not hard.

As a corollary , we have a trick: to prove that each element x of a connected set X has property E, one way is to prove the set

$$O := \{x \in X; E(x)\}$$

is open, closed , and nonempty.

(ii) $\iff$ (iii), we put it in homework.

Connectivity on $\mathbb{R}$

A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof. Left as a practice. (I've even left room for it, just say you love me.)

□

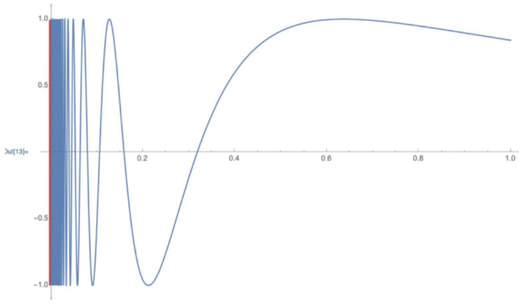
Theorem 4.4.1. The continuous images of connected sets are connected.

Take it as a proof of generalized intermediate value theorem.

Path Connectivity

A continuous function $w : [\alpha, \beta] \to X$ is a **continuous path** connecting the **end point** $w(\alpha), w(\beta)$.

Path connected set is connected, but connected set may not be path connected.



In particular, $X = E$ is a normed vector space, a 'special straight path' $[\![a, b]\!]$ is defined by the image of function v

$$v : [0, 1] \to E, \ t \mapsto (1 - t)a + tb$$

A subset Y of E is **convex** if $[\![a, b]\!] \subseteq X, \ \forall a, b \in Y$. Obviously every convex subset of E is path connected.

As a practice of the trick mentioned before, try to prove the next theorem, from which we have in a normed vector space, connected and path connected on open set are equivalent.

Let X be a nonempty, open and connected subset, , then any pair of points of X can be connected by a polygonal path in X.

Proof. $\square$

4.5 Functions on $\mathbb{R}^n$

$\mathbb{R}$ case

Continuous images of intervals are intervials.

Proof. The connected subset in $\mathbb{R}$ is intervals. □

Monotone function $f : I \to \mathbb{R}$ is contiuous except perhaps at countably many jump discontinuities.

Proof. The Set

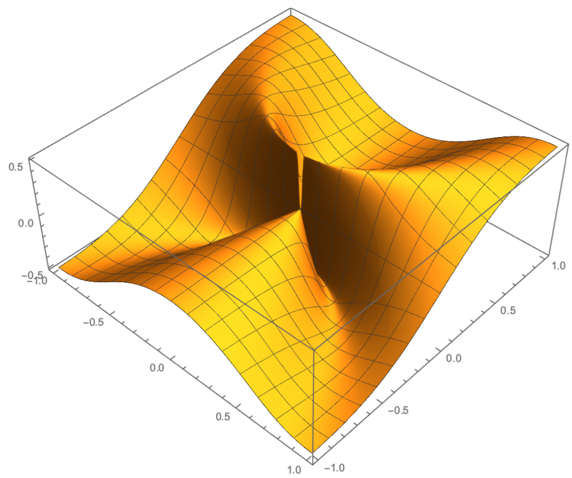
$$M := \{t_0 \in \mathring{I}; f(t_0-) \neq f(t_0+)\}$$

is countable, cause

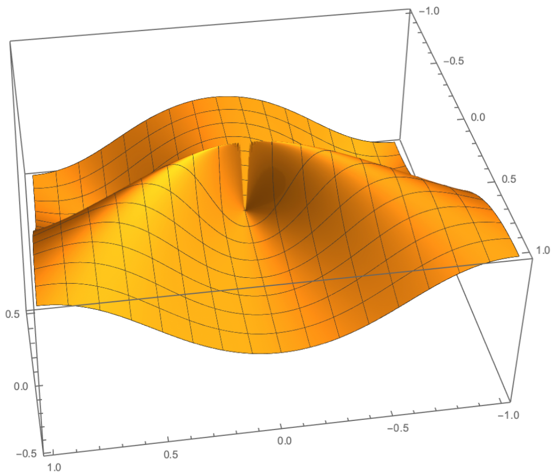
$$r : M \to \mathbb{Q}, t \mapsto r(t)$$

is injective when we define $r(t) \in \mathbb{Q} \cap (f(t-), f(+))$ □

$\mathbb{R}^n$ case



,



In the special case,

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} \ , \ x^2 + y^2 \neq 0 \\ 0 \ , \ x = y = 0 \end{cases}$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exists, even if

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) := \lim_{x\to 0}(\lim_{y\to 0} f(x,y)) = \lim_{y\to 0}\lim_{x\to 0} f(x,y) =: \lim_{y\to 0}(\lim_{x\to 0} f(x,y))$$

Proposition 4.5.1. If

$$\lim_{(x,y)\to(0,0)} f(x,y) = A < +\infty$$

And when $|y|$ is small enough, the limit

$$\lim_{x\to 0} f(x,y)$$

exists, we have

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = A$$

Proof. Consider their meaning within the framework of the $\varepsilon - \delta$ definition. $\square$

4.6 The Exponential and Related Functions

Euler's Formula

First, $\exp, \sin, \cos$ are define by power series, with the infinite radii of convergence.

Theorem 4.6.1. $e^{w+z} = e^w e^z$

Proof. That is because the absolutely convergent of $\sum x^j/j!$, then,

$$e^{w+z} = (\sum_{j=0}^{\infty} \frac{x^j}{j!})(\sum_{k=0}^{\infty} \frac{y^k}{k!}) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!})$$

Binomial formula:

$$= \frac{1}{n!} \sum_{k=0}^{n} \frac{n! x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} C_n^k = \frac{1}{n!}(x+y)^m$$

$\square$

And are all continuous. Because the property of power series with positive radius.

Theorem 4.6.2. (Euler)

$$e^{i\ z} = \cos z + i \sin z$$

We have the "high school properties" immediately, just from

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Remark. 1. sin is odd, cos is even

2. $e^z \neq 0, e^{-z} = 1\ e^z, \overline{e^z} = e^{\overline{z}}$

3.
$$\sin z - \sin w = 2 \cos \frac{z+w}{2} \sin \frac{z-w}{2}$$
$$\cos z - \cos w = -2 \sin \frac{z+w}{2} \sin \frac{z-w}{2}$$

4. $\sin^2 z + \cos^2 z = 1$

We are familier with these , right?

The Exponential Function on $\mathbb{R}$

Refer to p280

The Exponential Function on $i\mathbb{R}$

Define the function $\mathrm{cis} : \mathbb{R} \to \mathbb{C}, t \mapsto e^{it}$, obviously the image of cis is contain in S^1.

(i) We show the image of the cosine function is

$$\cos(\mathbb{R}) = \mathrm{pr}_1[\mathrm{cis}(\mathbb{R})] = [-1, 1]$$

The left = derived from what we have talked, and , $\mathrm{pr}_1[\mathrm{cis}(\mathbb{R})] \subseteq [-1, 1]$ is obvious.

Now we have $\cos 0 = 1$, so, suppose $I = \mathrm{pr}_1[\mathrm{cis}(\mathbb{R})] = [a, 1]$ or $(a, 1]$. a must be -1, if not: $a_0 = (a + 1)/2$, consider $\mathrm{pr}_1(z_0^2) = 2a_0^2 - 1$ and we have a contradition.

Next we show $I = [a, 1]$. Cause we have already know that there is some t_0, $\cos t_0 = 0 \implies \cos 2t_0 = -1$

(ii) Choose $z \in S^1$, there exists $t \in \mathbb{R}$

$$\mathrm{Re}\, z = \mathrm{Re}\, e^{it}$$

Thus, together with the truth $|z| = |e^{it}| = 1$, we have $z = e^{it}$or $\overline{z} = e^{it} \implies z = e^{-it}$, so $S^1 \subseteq \mathrm{cis}(\mathbb{R})$, hence,

$$\mathrm{cis}(\mathbb{R}) = S^1$$

The Definition of π and its Consequences

Definition 4.6.1.

$$\pi := \frac{1}{2} \min\{t > 0; e^{it} = 1\}$$

π is well defined since the set

$$M := \{t > 0; e^{it} = 1\}$$

is nonempty, closed and bounded below. We immediately have some properties.

Proposition 4.6.1. (i) $e^z = 1 \iff z \in 2\pi i\mathbb{Z}$

(ii) $e^z = -1 \iff z \in \pi i + 2\pi i\mathbb{Z}$

So e^z is periodic with period $2\pi i$.

Proposition 4.6.2.

$$\operatorname{cis}|[a, a+2\pi) : [a, a+2\pi) \to S^1$$

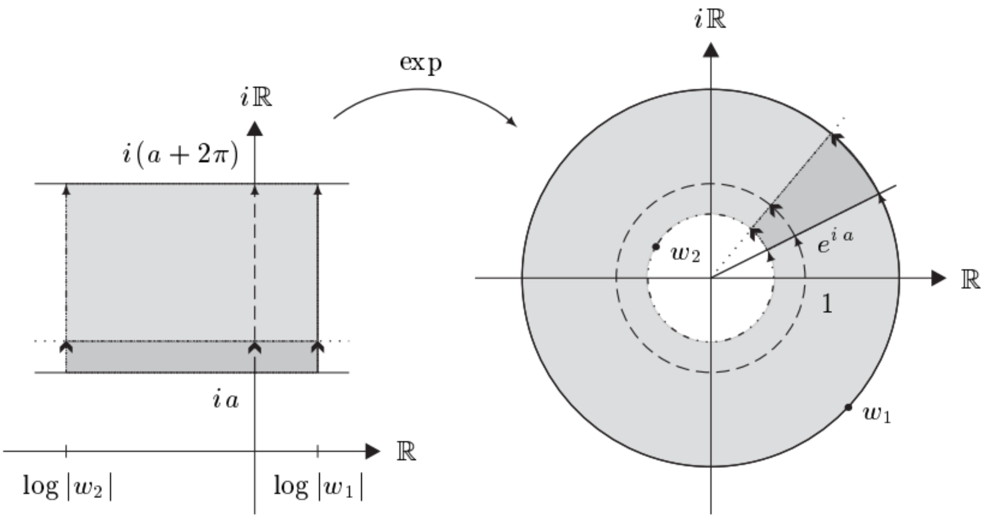
is bijection for each $a \in \mathbb{R}$.

Proof. Try yourself.

□

Then, we have sine and cosine are the familier function we have ever seen before.

The Complex Exponential Function



For $a \in \mathbb{R}$, let I_α be an inverval of $[a, a+2\pi)$,

$$\exp(\mathbb{R} + iI_\alpha) : \mathbb{R} + iI_\alpha \to \mathbb{C}^\times, \ z \mapsto e^z$$

is continuous and bijective. Hence, we have each $z \in \mathbb{C}$ can be represented uniquely by

$$z = |z|\, e^{i\alpha}$$

with $\alpha \in [0, 2\pi)$. We denote α as $\arg_N(z)$, $|z|$ as $e^{\log|z|}$. Finally , we can define two set valued functions.

$$\mathrm{Arg}(w) := \arg_N(w) + 2\pi\mathbb{Z}$$

$$\mathrm{Log}(w) := \log|w| + i\,\mathrm{Arg}(w)$$

$$z^w := e^{w\,\mathrm{Log}\,z}$$

Group $(S^1, \cdot)$ and $(\mathbb{R}, +)/(2\pi\mathbb{Z})$ are isomorphic.

Chapter 5

Differentiability

5.1 Definition

If we want to talk about the linear approximation of functions, what structures do we need to add?

In the following, E, F are Banach spaces on field $\mathbb{K}$. (In fact, is E is not complete, we need to ensure that x_0 is a limit point of X.)

Definition 5.1.1. $f : X \to F$ is differentiable at $x_0 \in X$

$\iff$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \partial f(x_0)(x - x_0)}{\|x - x_0\|} = 0$$

$\iff$ there is a function $r(x) : X \to F$ that is continuous at x_0 satisfies $r(x_0) = 0$ such that

$$f(x) = f(x_0) + \partial f(x_0)(x - x_0) + r(x) \|x - x_0\|$$

If $f : X \to F$ is differentiable at every point of X, we say f is differentiable and call the map

$$\partial f : X \to \mathcal{L}(E, F), x \mapsto \partial f(x)$$

the derivative of f, when $\mathcal{L}(E, F)$ is complete, we can meaningfully speak of the continuity of ∂f. If $\partial f \in C(X, \mathcal{L}(E, F))$, we say f is contiuously differentiable.

Definition 5.1.2. If $\mathbf{x} \in D$, there exists a linear mapping $A : X \to Y$ such that

$$\lim_{\|\mathbf{h}\| \to 0} \frac{\|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

We say that f is differentiable at $\mathbf{x}$, and the linear mapping A is the derivative of f at $\mathbf{x}$, denoted by $f'(\mathbf{x})$. If f is differentiable at every point of D, we say that f is differentiable on D.

Remark. Equivalent formulation: $f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + f'(\mathbf{x})(\mathbf{h}) + o(\|\mathbf{h}\|),\ \mathbf{h} \to 0$

Remark. The derivative is a linear mapping. If $f : \mathbb{R}^n \to \mathbb{R}^m$, then $f'(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, and the corresponding Jacobian matrix (the matrix of $f'(\mathbf{x})$ in the standard basis) is of size $m \times n$. Furthermore, we can view f' as a mapping from D to $\mathcal{L}(X, Y)$.

From the definition of differentiability, we can write the following symmetric form:

$$f(\mathbf{x}) - f(\mathbf{x_0}) + o(\|\mathbf{x} - \mathbf{x_0}\|) = f'(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0} + o(\|\mathbf{x} - \mathbf{x_0}\|))$$

Definition 5.1.3. Let $\mathcal{V} = C(D, Y)$, for $\mathbf{x_0} \in D$, define the linear space $\mathcal{N}_{\mathbf{x_0}}$ and its subspace:

$$\mathcal{N}_{\mathbf{x_0}} := \{f \in \mathcal{V} | \lim_{\mathbf{x} \to \mathbf{x_0}} f(\mathbf{x}) = \mathbf{0}\},\ \mathcal{M}_{x_0} := \{f \in \mathcal{V} | \lim_{\mathbf{x} \to \mathbf{x_0}} \frac{f(\mathbf{x})}{\|\mathbf{x} - \mathbf{x_0}\|}\}$$

Define the quotient space $\Omega_{\mathbf{x_0}} := \mathcal{N}_{x_0} / \mathcal{M}_{x_0}$, and the differential mapping

$$\mathrm{d} : \mathcal{V} \to \Omega_{\mathbf{x_0}},\quad f \mapsto \mathrm{d}f(\mathbf{x_0}) := f(\mathbf{x}) - f(\mathbf{x_0}) + \mathcal{M}_{\mathbf{x_0}}$$

If Y_1, Y_2 are two finite-dimensional normed vector spaces, and $A : Y_1 \to Y_2$ is a linear mapping, we can verify that the linear mapping on the quotient space $\Omega_{\mathbf{x_0}\mathbf{x_0}}$ is well-defined:

$$A : \Omega_{\mathbf{x_0}}(D, Y_1) \to \Omega_{\mathbf{x_0}}(D, Y_2),\quad f + \mathcal{M}_{\mathbf{x_0}}(D, Y_1) \mapsto f + \mathcal{M}_{\mathbf{x_0}}(D, Y_2)$$

(By verifying that $A(\Omega_{\mathbf{x_0}}(D, Y_1)) \subseteq \Omega_{\mathbf{x_0}}(D, Y_2)$, and the mapping is independent of the choice of representatives) At this point, if we denote $\mathbf{x}$ as the identity mapping id_X restricted to D, then f is differentiable at $\mathbf{x_0}$ if and only if there exists a linear mapping $f(\mathbf{x_0}) : X \to Y$ such that in $\Omega_{\mathbf{x_0}}(D, Y)$ we have

$$\mathrm{d}f(x_0) = f'(\mathbf{x_0})(\mathrm{d}\mathbf{x}(x_0))$$

Theorem 5.1.1. If g, f are differentiable at $\mathbf{x_0}$, and $g(\mathbf{x_0})$ is differentiable, then $h = f \circ g$ is differentiable at $\mathbf{x_0}$, and $h'(\mathbf{x_0}) = f'(g(\mathbf{x_0})) \circ g'(\mathbf{x_0})$

Lemma 5.1.2. d is a linear mapping: $(af + bg)' = af' + bg'$

Example 5.1.1. Operator Norm

5.2 $E = \mathbb{K}$ case

Inverse function

Leibniz' rule

$$f(x, y) = \begin{cases} x^2 \sin \dfrac{1}{x}, \ x \in \mathbb{R}^\times, \\ 0 \ , \ x = 0 \end{cases}$$

$$f(x, y) = \begin{cases} e^{-1/x}, \ x > 0, \\ 0 \ , \ x \leq 0 \end{cases}$$

fermat - rolle -...hospital

convex and ineuality

The mean value theorem for vector

Theorem 5.2.1. $f \in C([a, b], E)$ is differentiable on (a, b) ,

$$\|f(b) - f(a)\| \leq \sup_{t \in (a,b)} (b - a)$$

Proof. Amann id

Youjin id

d □

5.3 $\dim E > 1$ case

Directional derivatives and Partial derivatives

$$D_v f(x_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

If $g(t) := f(x_0 + tv)$, then $g'(0) = D_v f(x_0)$. To make g well defined, $x_0 + tv \in X$ is necessary.

If differentiability is satisfied, $g'(t) = D_v f(x_0 + tv)$.

Proposition 5.3.1. Differenitable at $x_0 \implies D_v f(x_0)$ exists for every $v \in E \setminus \{0\}$

$$\partial f(x_0) v = D_v f(x_0)$$

Proof.

$$f(x_0 + tv) = f(x_0) + \partial f(x_0) tv + o(\|tv\|) = f(x_0) + t \partial f(x_0) v + o(\|tv\|)$$

□

Partial derivatives are just the directional derivatives on standard basis.

5.4 Local Diffeomorphisms

Inverse linear maps

- $\mathcal{L}\mathrm{is}(E, F)$ is open in $\mathcal{L}(E, F)$.
- $$\mathrm{inv} : \mathcal{L}\mathrm{is}(E, F) \to \mathcal{L}(F, E), A \mapsto A^{-1}$$

  we have inv is in C^∞ class, and

  $$\partial \,\mathrm{inv}(A) H = -A^{-1} H A^{-1}, A \in \mathcal{L}\mathrm{is}(E, F), H \in \mathcal{L}(E, F)$$

The inverse function theorem

$f \in C^q(X, F)$, if at x_0, $\partial f(x_0) \in \mathcal{L}\text{is}(E, F)$, then there is neighborhood U, V,

- $f \in \text{Diff}^q(U, V)$
- $\partial f^{-1}(f(x)) = [\partial f(x)]^{-1},\ x \in U$

Proof. Have a try $\square$

Hint: (i)

(ii)

Chapter 6

Integrals

6.1 Partial Fraction Decomposition, Rational Function Integration, Transformation of Rational Functions

December 6th, sunny weather

Partial Fraction Decomposition

For a proper rational function $\frac{f(x)}{g(x)}$, according to the Fundamental Theorem of Algebra,

$$g(x) = \prod_{i=1}^{n} h_i(x) \prod_{j=1}^{m} l_j(x)$$

where $h_i(x)$ is a linear polynomial over the real numbers, $l_i(x)$ is an irreducible quadratic polynomial over the real numbers (which may be equal), and there exist constants A_i, B_i, C_i that satisfy

$$\frac{f(x)}{g(x)} = \sum_{i=1}^{n} \sum_{k_1=1}^{r} \frac{A_i}{h_i^{k_1}(x)} + \sum_{j=1}^{m} \sum_{k_2=1}^{s} \frac{B_j x + C_j}{l_j^{s}(x)}$$

where r, s are the corresponding factor repetition times. Simply put, this is about writing a proper rational function as a sum of fractions, where the numerators are constants or linear functions, and the denominators are irreducible polynomials or their powers. The proof of this proposition is somewhat complicated and involves the use of mathematical induction to reduce the degree of the denominator, along with some techniques.

Proof. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, it suffices to prove that there exist constants A, and a polynomial $P_1(x)$ such that

$$\begin{aligned}\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a)^m Q_1(x)} &= \frac{A}{(x-a)^m} + \frac{P_1(x)}{(x-a)^{m-1}Q_1(x)} \\ &= \frac{AQ_1(x) + P_1(x)(x-a)}{(x-a)^m Q_1(x)}\end{aligned}$$

This is equivalent to $P(x) - AQ_1(x)$ being divisible by $(x-a)$. Let $A = \frac{P(a)}{Q_1(a)}$. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, it suffices to prove that there exist constants B, C, and a polynomial $P_1(x)$ such that

$$\begin{aligned}\frac{P(x)}{Q(x)} = \frac{P(x)}{(x^2+bx+c)^m Q_1(x)} &= \frac{Bx+C}{(x^2+bx+c)^m} + \frac{P_1(x)}{(x^2+bx+c)^{m-1}Q_1(x)} \\ &= \frac{(Bx+C)Q_1(x) + P_1(x)(x^2+bx+c)}{(x^2+bx+c)^m Q_1(x)}\end{aligned}$$

This is equivalent to $P(x) - (Bx+C)Q_1(x)$ being divisible by (x^2+bx+c). Consider the remainders $(\alpha_1 x + \beta_1)$ and $(\alpha_2 x + \beta_2)$ when $P(x)$ and $Q_1(x)$ are divided by (x^2+bx+c), it suffices to show that

$$(\alpha_1 x + \beta_1) - (Bx+C)(\alpha_2 x + \beta_2) = B\alpha_1 x^2 + (C\alpha_1 + B\beta_1 - \alpha_2)x + C\beta_1 - \beta_2$$

is associated with (x^2+bx+c). Considering the coefficient of the quadratic term, it suffices to prove that the linear system of equations

$$\begin{cases} C\alpha_1 + B\beta_1 - \alpha_2 &= B\alpha_1 b \\ C\beta_1 - \beta_2 &= B\alpha_1 c \end{cases}$$

has a solution, that is, to prove that the determinant of the coefficients $\beta_1^2 - \alpha_1 b\beta_1 + \alpha_1^2 c = \alpha_1^2 c \neq 0$. Note that

$$\beta_1^2 - \alpha_1 b\beta_1 + \alpha_1^2 c = (\beta_1 - \frac{\alpha_1 b}{2})^2 - \frac{\alpha_1^2}{4}(b^2 - 4c) > 0$$

This is proven, and the last inequality holds because (x^2+bx+c) is irreducible, that is, $b^2 - c < 0$

□

This kind of decomposition helps us with differentiation, integration, and power series expansion.

Rational Function Integration

If you have a rational function in hand, through partial fraction decomposition, you get a series of functions that are easy to integrate. Assuming you are familiar with the integrals of these functions (go through them mentally), the problem you may have with integration is likely **not being able to factorize the denominator**. You can start by staring out a root of the denominator polynomial, and then perform polynomial division, continuing this process. Here is a systematic guide for the staring method.

Remark. Staring Method:

1. For a polynomial
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$
If there exists a rational root $\frac{r}{s}$, then s is a factor of a_n, and r is a factor of a_0.
2. If you can't stare out a rational root, ~~try the Eisenstein Criterion~~. It is likely that there are no rational roots.
3. Irrational roots might also be stearable, why not try $\sqrt{2}$?
4. If you are too focused and stare out a complex root, then its conjugate is also a root.
5. Following from 4, if you stare out a complex root, then multiply its conjugate when performing division.

For polynomial division, your advanced algebra notes have a triple-optimized paper-and-pencil algorithm, so why not go familiarize yourself with it? For finding A_m, you can do the following

$$R = \sum_i \frac{A_i}{h_i},$$

$$h_m R = \sum_{i, i \neq m} \frac{A_i}{h_i} h_m + A_m$$

Take $h_m \to 0$ to obtain, noting here that it is required that other h_i do not tend to zero, so for terms with power multiplication, multiply the higher power first. Similarly, for

$$R = \sum_i \frac{B_i x + C_i}{l_i}$$

$$l_m R = \sum_{i,i\neq m} \frac{B_i x + C_i}{l_i} l_m + B_m x + C_m$$

Take $l_m = 0$ for its two complex roots, get the linear equations system for B_m, C_m, and solve it. The only item that is not easy to integrate is

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^m}$$

Remember the recurrence relation

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^{m+1}} = \frac{1}{2ma^2}\frac{x}{(x^2+a^2)^m} + \frac{2m-1}{2ma^2}\int \frac{\mathrm{d}x}{(x^2+a^2)^m}$$