数学分析作业

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Week 14

1. (1)

$$\int_0^{\frac{\pi}{2}} \cos mx \sin nx \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin(n+m)x + \sin(n-m))\mathrm{d}x = -\frac{1}{2(n+m)}\cos(n+m)x\Big|_0^{\frac{\pi}{2}} - \frac{1}{2(n-m)}\cos(n-m)x\Big|_0^{\frac{\pi}{2}}$$
$$= -\frac{1}{2(n+m)}\cos\frac{(n+m)\pi}{2} + \frac{1}{2(n+m)} - \frac{1}{2(n-m)}\cos\frac{(n-m)\pi}{2} + -\frac{1}{2(n-m)}$$

(2) 令 $t = x^2$

$$\int x(2-x^2)^{12}\mathrm{d}x = \frac{1}{2}\int (2-t)^{12}\mathrm{d}t = -\frac{1}{26}(2-t)^{13} + C = -\frac{1}{26}(2-x^2)^{13} + C$$

$$\int_0^1 x(2-x^2)^{12}\mathrm{d}x = -\frac{1}{26}(2-x^2)^{13}\Big|_0^1 = \frac{2^{13}-1}{26}$$

2. 因为 $f(x)$ 连续且 $f(x) > 0$, $\int_0^x xf(t)\mathrm{d}t - \int_0^x tf(t)\mathrm{d}t$, 故

$$\psi'(x) = f(x)\frac{x\int_0^x f(t)\mathrm{d}t - \int_0^x tf(t)\mathrm{d}t}{(\int_0^x f(t)\mathrm{d}t)^2} > 0$$

所以 $\psi(x)$ 是 $(0,\infty)$ 上的严格单调递增函数.

3. 两边同时求导

$$f(x) = \frac{1}{2}f(x) + \frac{1}{2}xf'(x)$$

整理得

$$f(x) - xf'(x) = 0$$

故 $(\frac{f(x)}{x})' = \frac{xf'(x)-f(x)}{x^2} = 0$, $\frac{f(x)}{x}$ 为一常数.

4. 定义函数

$$F(x) = \int_x^{bx} f(t)t\mathrm{d}t, F'(x) = bf(bx) - f(x)$$

, 由于和 x 的取值无关, 故 $F'(x) = 0$, 即

$$bxf(bx) = xf(x), \forall x \in (0,\infty), \forall b > 0$$

故 $xf(x)$ 为一常数, 存在 c 使得 $f(x) = c/x$

5. 设 $x = x_0$ 时 $|f(x)|$ 取到极大值, 根据第一积分中值定理, 存在 $\xi \in [a,b]$

$$\frac{1}{b-a}\left|\int_a^{bf} f(x)\mathrm{d}x\right| = f(\xi)$$

故

$$|f(x_0)|-|f(\xi)|\le|f(x_0)-f(\xi)|=\left|\int_\xi^{x_0}f'(x)\mathrm{d}x\right|\le\int_{\min\{\xi,x_0\}}^{\max\{\xi,x_0\}}|f'(x)|\,\mathrm{d}x\le\int_a^b|f'(x)|\,\mathrm{d}x$$

6. 两边同时求导得

$$2xf(x^2)-f(x)=f(x)\iff x^2f(x^2)=xf(x),\forall x\in>0$$

令 $g(x)=xf(x)$, 由上可知 $g(x)=g(x^{2n}),n\in\mathbb{Z}$. 可知 $g(1)=f(1)$ 假设存在 $x_0,g(x_0)\neq f(1)$, 不妨记 $g(x_0)+c=f(1),c\neq0$, 由题设知 $g(x)$ 是连续函数, 故 $\lim_{n\to\infty}g(x_0^{-2n})+c=g(1)+c=f(1)$, 矛盾, 故 $g(x)\equiv f(1)$, 满足题意的所有 $f(x)$ 为 $f(x)=\frac{a}{x}$,a 为任意实数.

7. 由于 $\lim_{x\to\infty}f(x)=a$, 故存在 $m>0$, $x>m$ 时 $-\frac{a}{2}<f(x)-a<\frac{a}{2}$ 故任意的 $M>0$, $x>\frac{2M}{a}$ 时

$$\int_0^x f(t)\mathrm{d}x>\int_0^x\frac{a}{2}\mathrm{d}x>M$$

故 $\lim_{x\to\infty}\int_0^x f(t)\mathrm{d}x=\infty$, 可用 L'Hospital 法则

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)\mathrm{d}t=\lim_{x\to\infty}f(x)=a$$

8. 令

$$F(x)=\int_0^x f^3(t)\,\mathrm{d}t,G(x)=\int_0^x f(t)\,\mathrm{d}t,\ H(x)=F(x)-G^2(x)$$

即证 $H(1)\le0$

$$H'(x)=f^3(x)-2G(x)f(x)=f(x)(f^2(x)-2G(x)),\ H'(0)=0$$

令 $u(x)=f^2(x)-2G(x)$, 由于 $f'(x)\ge0,\forall x\in[0,1]$, $f(0)=0$, 故 $f(x)\ge0,\forall x\in[0,1]$, 只需证 $u(x)\le0$, 这样 $H'(x)\le0$, 递减, 结合 $H(0)=0$ 得到 $H(1)\le0$.

$$u'(x)=2f(x)f'(x)-2f(x)=f(x)(f'(x)-1)\le0$$

故 $u(x)$ 递减, $u(x)\le u(0)=0$, 证毕.

9.

$$\int_0^1|f(x)-f'(x)|\,\mathrm{d}x=\int_0^1\left|\frac{f(x)-f'(x)}{e^x}\right|e^x\mathrm{d}x\ge\int_0^1\left|(\frac{f(x)}{e^x})'\right|1\mathrm{d}x\ge\left|\int_0^1(\frac{f(x)}{e^x})'\mathrm{d}x\right|=\left|\frac{f(x)}{e^x}\Big|_0^1\right|=\frac{1}{e}$$

10.

$$f'(x)=\frac{1}{x^2+f^2(x)}>0,f(1)=1\implies f(x)\ge1,x\ge1$$

$$f(t)=f(1)+\int_1^t f'(x)\mathrm{d}x\le1+\int_1^t\frac{1}{x^2+1}\mathrm{d}x=1+\arctan t-\frac{\pi}{4}$$

故 $f(x)$ 有界, $\lim_{x\to\infty}f(x)$ 存在.

$$\lim_{x\to\infty}f(x)\le1-\frac{\pi}{4}+\lim_{x\to\infty}\arctan x=1+\frac{\pi}{4}$$

7.4

1. (1)

$$\int_{-\pi}^{\pi}x^2\cos x\mathrm{d}x=x^2\sin x\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}2x\sin x\mathrm{d}x=2x\cos x\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}2\cos x\mathrm{d}x=-4\pi$$

(2)

$$\int_0^{\sqrt{3}} x\arctan x\mathrm{d}x = \frac{1}{2}x^2\arctan x\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{1}{2}\frac{x^2}{x^2+1}\mathrm{d}x = \frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{1+x^2}\mathrm{d}x = \frac{5\pi}{6} - \frac{\sqrt{3}}{2}$$

(3)

$$\int_{1/e}^{e} |\log x|\,\mathrm{d}x = \int_{1/e}^{1} -\log x\mathrm{d}x + \int_1^{e} \log x\mathrm{d}x = -x(\log x - 1)\Big|_{1/e}^{1} + x(\log x - 1)\Big|_1^{\mathrm{e}} = 2 - \frac{2}{\mathrm{e}}$$

(4) 假定 $n > 0$(否则在 0 处无定义, 不是黎曼积分)

$$\int_0^1 x^n\log x\mathrm{d}x = \frac{1}{n+1}x^{n+1}\log x\Big|_0^1 - \frac{1}{n+1}\int_0^1 x^n = -\frac{1}{(n+1)^2}$$

(5)

$$\int_0^a x^2\sqrt{a^2-x^2}\mathrm{d}x = \int_0^{\frac{\pi}{2}} a^4\sin^2 t\cos^2 t\mathrm{d}t = \int_0^{\frac{\pi}{2}} \frac{a^4}{4}\sin^2 2t\mathrm{d}t = \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4t}{2}\mathrm{d}t = \frac{\pi a^4}{16}$$

(6)

$$\int_0^a \log\Big(x+\sqrt{a^2+x^2}\Big)\mathrm{d}x = x\log\Big(x+\sqrt{a^2+x^2}\Big)\Big|_0^a - \int_0^a \frac{x}{\sqrt{a^2+x^2}}$$

$$= a\log\Big(a+\sqrt{2}a\Big) - \sqrt{a^2+x^2}\Big|_0^a = a\log\Big(a+\sqrt{2}a\Big) - \sqrt{2}a + a$$

(7) 令 $\sin x = t, u = t^2$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x\sin x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x = \int_0^1 \frac{t}{a^2+(b^2-a^2)t^2}\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{1}{a^2+(b^2-a^2)u}\mathrm{d}u$$

$$= \frac{1}{2(b^2-a^2)}\log\left|a^2+(b^2-a^2)u\right|\Big|_0^1 = \frac{1}{b^2-a^2}\log\left|\frac{b}{a}\right|$$

2. 令 $x = -t$

$$\int_{-a}^0 f(x)\mathrm{d}x = \int_a^0 f(-t)\mathrm{d}(-t) = \int_0^a f(t)\mathrm{d}t$$

故

$$\int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$$

3.

$$\int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \int_0^{\frac{\pi}{2}-\delta} \sin^n x\mathrm{d}x + \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \sin^n x\mathrm{d}x \le r^n(\frac{\pi}{2}-\delta) + \delta$$

其中 $0 < \delta < \frac{\pi}{2}, r = \sin(\frac{\pi}{2}-\delta) < 1$, 故对于任意的 $\varepsilon > 0$, 可以取 $\delta < \varepsilon$, 再取 n 充分大使得 $r^n < \varepsilon$, 此时

$$\int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x \le 2\varepsilon$$

故

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = 0$$

4. 由于

$$\frac{(2n-1)!!}{(2n)!!} < \frac{(2n)!!}{(2n+1)!!}$$

故

$$0 < \frac{(2n-1)!!}{(2n)!!} < \sqrt{\frac{(2n-1)!!}{(2n)!!}\frac{(2n)!!}{(2n+1)!!}} = \sqrt{\frac{1}{2n+1}}$$

注意到左右两边当 $n \to \infty$ 时趋于 0, 故

$$\lim_{n\to\infty} \frac{(2n)!!}{(2n+1)!!} = 0$$

5. 因为 $f \in C([-1,1])$, 令 $x(t) = \pi - t$, $x(t)$ 可微且 $x'(t)$ Reimann 可积

$$\int_0^\pi x f(\sin x)\mathrm{d}x = \int_\pi^0 (\pi - t) f(\sin(\pi - t))\mathrm{d}(\pi - t) = -\int_0^\pi t f(\sin t)\mathrm{d}t + \pi \int_0^\pi f(\sin t)\mathrm{d}t$$

故

$$\int_0^\pi f(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\mathrm{d}x$$

6. 由于 $f \in C([-1,1])$, 根据 Lagrange 中值定理

$$\begin{aligned}\int_0^2 f(x)\mathrm{d}x &= \int_0^1 f(x)\mathrm{d}x + \int_1^2 f(x)\mathrm{d}x = \int_0^1 f'(\xi_1)(x-0)\mathrm{d}x + \int_0^1 f(0)\mathrm{d}x + \int_1^2 f'(\xi_2)(x-2)\mathrm{d}x + \int_1^2 f(2)\mathrm{d}x \\ &= 2 + \frac{f'(\xi_1) + f'(\xi_2)}{2} \in [2 + \inf f'(x), 2 + \sup f'(x)]\end{aligned}$$

由题意, $2 + \inf f'(x) = 1, 2 + \sup f'(x) = 3$, 得证

7. 令 $F(x) = \int_0^x \frac{\sin t}{t}\mathrm{d}t$, 即证明

$$\int_0^{2\pi} (F(2\pi) - F(x))\mathrm{d}x = 0 \iff \int_0^{2\pi} F(x)\mathrm{d}x = 2\pi F(2\pi)$$

又

$$\int_0^{2\pi} F(x)\mathrm{d}x = xF(x)\Big|_0^{2\pi} - \int_0^{2\pi} xF'(x)\mathrm{d}x = 2\pi F(2\pi) - \int_0^{2\pi} \sin x\mathrm{d}x = 2\pi F(2\pi)$$

证毕

8. 令 $F(x) = \int_0^x f(t)\mathrm{d}t$, $f \in C(0,\infty)$, 故 $F(x)$ 可微, 故根据分部积分公式

$$\int_0^a F(x)\mathrm{d}x = aF(a) - 0 - \int_0^a x f(x)\mathrm{d}x = a\int_0^a f(x)\mathrm{d}x - \int_0^a x f(x)\mathrm{d}x = \int_0^a (a-x) f(x)\mathrm{d}x$$

即为所求

9. 令 $g(x) = f(x) - f(0), g(0) = 0, g(x) \in C([-1,1])$, $g(x)$ 有界. 对任意的 $0 < \varepsilon < 1$, 记 $M_\varepsilon = \sup_{x\in[-\varepsilon,\varepsilon]} |g(x)|$, $M = \sup_{x\in[-1,1]} |g(x)|$

$$\begin{aligned}\left|\int_{-1}^1 \frac{h}{h^2+x^2} g(x)\mathrm{d}x\right| &\le \left|\int_{-1}^{-\varepsilon} \frac{h}{h^2+x^2} g(x)\mathrm{d}x\right| + \left|\int_{-\varepsilon}^{\varepsilon} \frac{h}{h^2+x^2} g(x)\mathrm{d}x\right| + \left|\int_{\varepsilon}^{1} \frac{h}{h^2+x^2} g(x)\mathrm{d}x\right| \\ &\le 2\frac{hM}{h^2+\varepsilon^2} + M_\varepsilon \arctan\frac{x}{h}\Big|_{-\varepsilon}^{\varepsilon}\end{aligned}$$

令 $h \to 0$, 有 $\left|\int_{-1}^1 \frac{h}{h^2+x^2} g(x)\mathrm{d}x\right| \le M_\varepsilon \pi$, 根据 ε 的任意性, 且 $\lim_{\varepsilon\to 0} M_\varepsilon = 0$(这是因为 $g(x)$ 连续), 有

$$\int_{-1}^1 \frac{h}{h^2+x^2} g(x)\mathrm{d}x \to 0, h \to 0$$

故

$$\int_{-1}^1 \frac{h}{h^2+x^2} f(x)\mathrm{d}x = \int_{-1}^1 \frac{h}{h^2+x^2} g(x)\mathrm{d}x + \int_{-1}^1 \frac{h}{h^2+x^2} f(0)\mathrm{d}x = \int_{-1}^1 \frac{h}{h^2+x^2} g(x)\mathrm{d}x + f(0)\arctan\frac{x}{h}\Big|_{-1}^1$$

令 $h \to 0$

$$\int_{-1}^1 \frac{h}{h^2+x^2} f(x)\mathrm{d}x = 0 + f(0)\pi$$

证毕

10. 因为 $f \in C^1([a,b])$, 根据分部积分公式

$$\int_a^b f(x)\cos\lambda x \mathrm{d}x = \frac{1}{\lambda}\sin\lambda x f(x)\bigg|_a^b - \frac{1}{\lambda}\int_a^b \sin\lambda x f'(x)\mathrm{d}x$$

$f'(x)$ 在 $[a,b]$ 上有界, 故令 $\lambda \to \infty$,

$$\int_a^b f(x)\cos\lambda x \mathrm{d}x \to 0$$

证毕

11. 令 $\tan\frac{x}{2} = t$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\frac{\cos 2x+1}{2}}{\sqrt{2}\sin(x+\frac{\pi}{4})}\mathrm{d}x$$

根据对称性, 注意到

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\sqrt{2}\sin(x+\frac{\pi}{4})}\mathrm{d}x = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos 2x}{\sqrt{2}\sin(x+\frac{\pi}{4})}\mathrm{d}x$$

故

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x}\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x}\mathrm{d}x &= \frac{1}{2}\int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t \\
&= \int_0^1 \frac{1}{-(t-1)^2+2}\mathrm{d}x \\
&= -\frac{1}{2\sqrt{2}}\int_0^1 (\frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}})\mathrm{d}t \\
&= \frac{1}{2\sqrt{2}}\log\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right|\Bigg|_0^1 = -\frac{\sqrt{2}}{4}\log\left(3-2\sqrt{2}\right) \\
&= \frac{\sqrt{2}}{4}\log\left(3+2\sqrt{2}\right)
\end{aligned}$$