Analysis

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2024.2.29

Preface

3.10: Chapter Continuity finished.

2024 年 3 月 30 日

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第三章 Eigenvalue and Eigenvector

3.1 Eigenvalue and Eigenvector

定义 3.1.1. Let A be a n by n matrix on the field $\mathbb{K}$. If there is some $\lambda \in \mathbb{K}$ and the **nonzero** vector $\mathbf{x} \in \mathbb{K}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

We say λ is a eigenvalue of A and $\mathbf{x}$ is the associated eignevector.

注. λ is a eigenvalue $\iff$ $(\lambda I - A)\mathbf{x} = 0$ has nonzero solution $\iff$ $\det(\lambda I - A) = 0$

命题 3.1.1.

$$\det(\lambda I - A) = \lambda^n - \operatorname{tr}(A) + \cdots + (-1)^n \det A$$

How to find eigenvalues? See in the video.

Examples

- $$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

  On $\mathbb{R}$, no eigenvalue, on $\mathbb{C}$, $\lambda_1 = e^{it}$, $\lambda_2 = e^{-it}$ and eigenvectors $\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}$. Notice that when $t = 0$, the case reduced to $\mathbb{R}$, and $\lambda_1 = \lambda_2$ which means the two eigenvectors are in one eigenspace, so the combination of them can span the space $\mathbb{R}^2$, so no contradition here.

- $V = C^\infty(\mathbb{R}), \delta$ is derivative transformation, the for each $\lambda \in \mathbb{R}$, since $\delta(e^{\lambda x}) = \lambda e^{\lambda x}$, so λ is a eigenvalue and $e^{\lambda x}$ is a eigenvector.
- $\varphi : \mathbb{K}[x] \ni f(x) \mapsto xf(x)$ has no eigenvalue.
- A is Markov matrix, then 1 must be a eigenvalue of A , and on $\mathbb{C}$, each eigenvalue λ satisfies $|\lambda| \leq 1$.

命题 3.1.2. If $f(A) = O$, then $f(\lambda) = 0$. In particular, $p_A(\lambda)$, we have $p_A(A) = O$

3.2 Diagonalization

定义 3.2.1. $A \in \mathbb{K}^n$ is diagonalizable.

$\iff$ A is similar to a diagonal matrix. In particular, there is some invertible matrix P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

If you let $P = (\mathbf{x_1}, \dots, \mathbf{x_n})$, then this may be cleaner:

$$A(\mathbf{x_1}, \dots, \mathbf{x_n}) = (\mathbf{x_1}, \dots, \mathbf{x_n}) \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$$

Obviously the columns of P are the eigenvectors.

$\iff$ $*^1$ Each complex eigenvalue λ is in $\mathbb{K}$ and, the corresponding algebraic multiplicity are equal to geometric multiplicity.

$\iff$ $*^2$ $V = \bigoplus V_{\lambda_i}$

Prove "eigenvectors corresponding to disinct eigenvalues are linearly independent" and the star ones. (I left the room for it, just say you love me.)

Upper Triangularization

定理 3.2.1. $\forall A \in M_n(\mathbb{C})$, there is invertible matrix P and a upper triangular matrix D, such that $P^{-1}AP = D$

证明. On $\mathbb{C}$ we have an amazing property: You can find a eigenvalue all the time. So just pick one of them λ_1 and corresponding eigenvector $\mathbf{x_1}$.

$$A(\mathbf{x_1}, X) = (\mathbf{x_1}, X) \begin{pmatrix} \lambda_1 & * \\ 0 & A_1 \end{pmatrix}$$

By induction, the left if easy. □

推论 3.2.2. $A \in M_n(\mathbb{K})$, $\lambda_1, \dots \lambda_n \in \mathbb{C}$ is all the complex eigenvalues, for each $f(x) \in \mathbb{K}[x]$, we have $f(\lambda_1), \dots f(\lambda_n)$ are all the eigenvalues of $f(A)$.

证明. We can upper trangularize A.

$$A = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}$$

$$f(A) = \begin{pmatrix} f(\lambda_1) & * & * \\ & \ddots & * \\ & & f(\lambda_n) \end{pmatrix}$$

Thus, left is obvious. □

定理 3.2.3. (Cayley-Hamilton) $A \in M_n(\mathbb{K})$, $p(x)$ is the characteristic polynomial, then $p(A) = O$.

证明. We show the complex case here. We know there is invertible complex matrix Q and upper trangular complex matrix U such that

$$Q^{-1}AQ = U$$

Hence, $p_A(x) = p_U(x) = p(x)$, and $p(U) = O$ by lemma(special case for upper trangular matrix, see in ppt).

Another way to proof: Write out

$$B(\lambda)(\lambda I - A) = |\lambda I - A|\, I = p(\lambda)I$$

And $B(\lambda) = \lambda^{n-1}B_{n-1} + \cdots B_0, B_i \in M_n(\mathbb{K})$ □

Take the example as practice:

Let $A \in M_m(\mathbb{C}), B \in M_n(\mathbb{C})$,$AX = XB$ has nonzero solution if and only if A, B have the some same eigenvalue.

3.3 Minimal Polynomial

The minimal polynomial is the one with smallest degree annihilating 1-leading coeffcient, or to say, the minimal element under the order $|$.

引理 3.3.1. For each annihilating polynomial f of A , $m_A(x)|f(x)$.

证明.

$$f(x) = m_A(x)q(x) + r(x), \quad \deg r < \deg m_A$$

$r(x)$ must be 0 since $r(A) = O$ which contradicts with $\deg r < \deg m_A$. □

推论 3.3.2. $m_A(x)$ and $p_A(x)$ have the same root in $\mathbb{C}$.

命题 3.3.1.

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}$$

where A_i are square matrix. Thus,

$$m_A(x) = [m_{A_1}(x), m_{A_2}(x), \dots m_{A_s}(x)]$$

证明.

$$\gamma(x) := [m_{A_1}(x), m_{A_2}(x), \dots m_{A_s}(x)]$$

it suffices to show $\gamma|m_A$ and $m_A|\gamma$. □

引理 3.3.3. For $f, g \in \mathbb{K}[x]$, $\varphi \in \mathrm{Hom}(V, V)$, if $(f, g) = 1$, then we have

$$\ker(f(\varphi)g(\varphi)) = \ker f(\varphi) \oplus \ker g(\varphi)$$

定理 3.3.4. (Primary decomposition) For φ of V on $\mathbb{C}$,

$$m_\varphi(x) = (x-\lambda_1)^{d_1}(x-\lambda_2)^{d_2}\cdots(x-\lambda_s)^{d_s}$$

We have

$$V = \bigoplus_{i=1}^{s} \ker(\varphi - \lambda_i I_V)^{d_i}$$

证明. From 1.3.3 . □

推论 3.3.5. φ is diagnoalizable if and only if its minimal polynomial is in this form:

$$m_A(x) = \prod_{i=1}^{t}(x-\lambda_i)$$

where λ_i are different eigenvalues.

Example

- For $A^2 = A$, then $f(x) = x(x-1)$, and $m_A(x)$ can only be $f(x)$ which is the product of linear divisors. Thus, A is diagnoalizable.
- It is easy to show that A and B are not similar but have the same minimal polynomial or characteristic polynomial.

3.4 λ - Matrix

You should get familiar with $M_n(\mathbb{K}[\lambda])$ first.

On $\mathbb{K}[x]$, the invertible elements are the elements in $\mathbb{K}$. We can expand the definition of invertible $\lambda-$matrix, whose **determinant** are the invertible elements in $\mathbb{K}[x]$.

The **rank** is defined by the minimum order of a non-zero minor, notice full rank doesn't imply invertibility.

引理 3.4.1. $M(\lambda) \in M_n(\mathbb{K}[\lambda])$ is a non-zero λ-matrix, let $A \in M_n(\mathbb{K})$, then there exist $Q(\lambda), S(\lambda)$ and $R, T \in M_n(\mathbb{K})$ such that

$$M(\lambda) = (\lambda I_n - A)Q(\lambda) + R$$

$$M(\lambda) = S(\lambda)(\lambda I_n - A) + T$$

证明. We denote $M(\lambda) = M_m\lambda^m + \cdots + M_0$. Consider $Q(\lambda) = M_m\lambda^{m-1}$. By induction the proof left is easy. □

定理 3.4.2. $A \sim B$ if and only if $\lambda I_n - A, \lambda I_n - B$ are equivalent.

证明. ' $\Longrightarrow$ ' Let $P^{-1}AP = B$, obviously $P^{-1}(\lambda I_n - A)P = (\lambda I_n - B)$, thus they are equivalent.

'$\Longleftarrow$' Let $M(\lambda)A_\lambda N(\lambda) = B_\lambda$ with $M(\lambda), N(\lambda)$ invertible. We can write

$$M(\lambda) = B_\lambda Q(\lambda) + R$$

by lemma 1.4.1, thus

$$B_\lambda Q(\lambda)A_\lambda + RA_\lambda = B_\lambda N(\lambda)^{-1}$$

and

$$RA_\lambda = B_\lambda(N(\lambda)^{-1} - Q(\lambda)A_\lambda)$$

The order of λ of A_λ, B_λ are 1, so $N(\lambda)^{-1} - Q(\lambda)A_\lambda$ must be some matrix in $M_n(\mathbb{K})$, and,

$$R(\lambda I_n - A) = (\lambda I_n - B)P$$

Thus, $R = P$, it suffices to show P is invertible. Let $N(\lambda) = S(\lambda)(\lambda I_n - B) + T$, □

Smith Normal Form

Know what smith normal form is. That is, a diagonal matrix

$$\mathrm{diag}\{d_1(\lambda), d_2(\lambda), \ldots, d_n(\lambda)\}$$

with $d_i(\lambda)|d_{i+1}(\lambda)$ for all proper i.

Technics: How to transform a matrix to its smith normal form?

1. **Elementary transformation**. Details are in your video.
2. Read the **invariant divisors** first, and write the smith normal form from that.

命题 3.4.1.

$$A(\lambda) = \begin{pmatrix} A_1(\lambda) & 0 \\ 0 & A_2(\lambda) \end{pmatrix}$$

Then the elementary divisors of $A(\lambda)$ are equal to the union of elementary divisors of $A_1(\lambda)$ and $A_2(\lambda)$.

Now you can find the elementary divisors only by transform the matrix to a diagnoal matrix, but not smith normal form.

证明. See in ppt 8. □

Divisors

定义 3.4.1. Tell me what are

- determinantal divisors
- invariant divisors
- elementary divisors

These are invariant under elementary transformation, hence the concepts of equivalent classes are well defined. And we have the uniqueness of smith normal form as a corollary immediately.

证明. Almost all are trivial. See ppt 7.5 □

3.5 Invariants Under Field Extensions

Similarity

The calculation of the greatest common divisor does not depend on the choice of the field, hence $\lambda I - A$ and $\lambda I - B$ have the same determinantal divisors of all orders, and therefore they are similar.

第四章 Group

4.1 Group

定义 4.1.1. We call an ordered pair $(G, *)$ a group. If, associativity, identity, inverse.

命题 4.1.1. Generalized assoctiativity:

Examples

- **Dihedral Groups** D_{2n}

$$D_{2n} = \left\langle r, s \,\middle|\, r^n = s^2 = 1, rs = sr^{-1} \right\rangle$$

- **Symmetric Groups** S_n

  The cycle decomposition. $(a_1, a_2, \ldots a_m)$

- **Matrix Groups** $GL_n(F)$

- **The Quaternion Group** Q_8

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

Homomorphisms and Isomorphisms

定理 4.1.1. The uniquely determined homomorphism from the representation

第五章 Differentiation

5.1 Differential and Derivative

We will now discuss all content within the framework of finite-dimensional normed real linear spaces. We know that norms are equivalent in $\mathbb{R}^n$ (the induced topological spaces are the same). Without further explanation, we will consider $X = (\mathbb{R}^n, \| \|\_X)$ and $Y = (\mathbb{R}^m, \| \|\_Y)$, where the norms are Euclidean. $D \subseteq X$ is a region (or, weaker, an open set—what propositions would not hold?). $f : D \to Y$.

定义 5.1.1. If $\mathbf{x} \in D$, there exists a linear mapping $A : X \to Y$ such that

$$\lim_{\|\mathbf{h}\| \to 0} \frac{\|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

We say that f is differentiable at $\mathbf{x}$, and the linear mapping A is the derivative of f at $\mathbf{x}$, denoted by $f'(\mathbf{x})$. If f is differentiable at every point of D, we say that f is differentiable on D.

注. Equivalent formulation: $f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + f'(\mathbf{x})(\mathbf{h}) + o(\|\mathbf{h}\|),\ \mathbf{h} \to 0$

注. The derivative is a linear mapping. If $f : \mathbb{R}^n \to \mathbb{R}^m$, then $f'(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, and the corresponding Jacobian matrix (the matrix of $f'(\mathbf{x})$ in the standard basis) is of size $m \times n$. Furthermore, we can view f' as a mapping from D to $\mathcal{L}(X, Y)$.

From the definition of differentiability, we can write the following symmetric form:

$$f(\mathbf{x}) - f(\mathbf{x_0}) + o(\|\mathbf{x} - \mathbf{x_0}\|) = f'(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0} + o(\|\mathbf{x} - \mathbf{x_0}\|))$$

定义 5.1.2. Let $\mathcal{V} = C(D, Y)$, for $\mathbf{x_0} \in D$, define the linear space $\mathcal{N}_{\mathbf{x_0}}$ and its subspace:

$$\mathcal{N}_{\mathbf{x_0}} := \{f \in \mathcal{V} | \lim_{\mathbf{x}\to\mathbf{x_0}} f(\mathbf{x}) = \mathbf{0}\},\ \mathcal{M}_{x_0} := \{f \in \mathcal{V} | \lim_{\mathbf{x}\to\mathbf{x_0}} \frac{f(\mathbf{x})}{\|\mathbf{x} - \mathbf{x_0}\|}\}$$

Define the quotient space $\Omega_{\mathbf{x_0}} := \mathcal{N}_{x_0}/\mathcal{M}_{x_0}$, and the differential mapping

$$\mathrm{d} : \mathcal{V} \to \Omega_{\mathbf{x_0}}, \quad f \mapsto \mathrm{d}f(\mathbf{x_0}) := f(\mathbf{x}) - f(\mathbf{x_0}) + \mathcal{M}_{\mathbf{x_0}}$$

If Y_1, Y_2 are two finite-dimensional normed linear spaces, and $A : Y_1 \to Y_2$ is a linear mapping, we can verify that the linear mapping on the quotient space $\Omega_{\mathbf{x_0}\mathbf{x_0}}$ is well-defined:

$$A : \Omega_{\mathbf{x_0}}(D, Y_1) \to \Omega_{\mathbf{x_0}}(D, Y_2), \quad f + \mathcal{M}_{\mathbf{x_0}}(D, Y_1) \mapsto f + \mathcal{M}_{\mathbf{x_0}}(D, Y_2)$$

(By verifying that $A(\Omega_{\mathbf{x_0}}(D, Y_1)) \subseteq \Omega_{\mathbf{x_0}}(D, Y_2)$, and the mapping is independent of the choice of representatives) At this point, if we denote $\mathbf{x}$ as the identity mapping id_X restricted to D, then f is differentiable at $\mathbf{x_0}$ if and only if there exists a linear mapping $f(\mathbf{x_0}) : X \to Y$ such that in $\Omega_{\mathbf{x_0}}(D, Y)$ we have

$$\mathrm{d}f(x_0) = f'(\mathbf{x_0})(\mathrm{d}\mathbf{x}(x_0))$$

定理 5.1.1. If g, f are differentiable at $\mathbf{x_0}$, and $g(\mathbf{x_0})$ is differentiable, then $h = f \circ g$ is differentiable at $\mathbf{x_0}$, and $h'(\mathbf{x_0}) = f'(g(\mathbf{x_0})) \circ g'(\mathbf{x_0})$

引理 5.1.2. d is a linear mapping: $(af + bg)' = af' + bg'$

例 5.1.1. Operator Norm

第六章　Integrals

6.1　Partial Fraction Decomposition, Rational Function Integration, Transformation of Rational Functions

December 6th, sunny weather

Partial Fraction Decomposition

For a proper rational function $\frac{f(x)}{g(x)}$, according to the Fundamental Theorem of Algebra,

$$g(x) = \prod_{i=1}^{n} h_i(x) \prod_{j=1}^{m} l_j(x)$$

where $h_i(x)$ is a linear polynomial over the real numbers, $l_i(x)$ is an irreducible quadratic polynomial over the real numbers (which may be equal), and there exist constants A_i, B_i, C_i that satisfy

$$\frac{f(x)}{g(x)} = \sum_{i=1}^{n} \sum_{k_1=1}^{r} \frac{A_i}{h_i^{k_1}(x)} + \sum_{j=1}^{m} \sum_{k_2=1}^{s} \frac{B_j x + C_j}{l_j^s(x)}$$

where r, s are the corresponding factor repetition times. Simply put, this is about writing a proper rational function as a sum of fractions, where the numerators are constants or linear functions, and the denominators are irreducible polynomials or their powers. The proof of this proposition is somewhat complicated and involves the use of mathematical induction to reduce the degree of the denominator, along with some techniques.

证明. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, it suffices to prove that there exist constants

A, and a polynomial $P_1(x)$ such that

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a)^m Q_1(x)} = \frac{A}{(x-a)^m} + \frac{P_1(x)}{(x-a)^{m-1}Q_1(x)}$$
$$= \frac{AQ_1(x) + P_1(x)(x-a)}{(x-a)^m Q_1(x)}$$

This is equivalent to $P(x) - AQ_1(x)$ being divisible by $(x-a)$. Let $A = \frac{P(a)}{Q_1(a)}$. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, it suffices to prove that there exist constants B, C, and a polynomial $P_1(x)$ such that

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x^2+bx+c)^m Q_1(x)} = \frac{Bx+C}{(x^2+bx+c)^m} + \frac{P_1(x)}{(x^2+bx+c)^{m-1}Q_1(x)}$$
$$= \frac{(Bx+C)Q_1(x) + P_1(x)(x^2+bx+c)}{(x^2+bx+c)^m Q_1(x)}$$

This is equivalent to $P(x) - (Bx+C)Q_1(x)$ being divisible by (x^2+bx+c). Consider the remainders $(\alpha_1 x + \beta_1)$ and $(\alpha_2 x + \beta_2)$ when $P(x)$ and $Q_1(x)$ are divided by (x^2+bx+c), it suffices to show that

$$(\alpha_1 x + \beta_1) - (Bx+C)(\alpha_2 x + \beta_2) = B\alpha_1 x^2 + (C\alpha_1 + B\beta_1 - \alpha_2)x + C\beta_1 - \beta_2$$

is associated with (x^2+bx+c). Considering the coefficient of the quadratic term, it suffices to prove that the linear system of equations

$$\begin{cases} C\alpha_1 + B\beta_1 - \alpha_2 & = B\alpha_1 b \\ C\beta_1 - \beta_2 & = B\alpha_1 c \end{cases}$$

has a solution, that is, to prove that the determinant of the coefficients $\beta_1^2 - \alpha_1 b\beta_1 + \alpha_1^2 c = \alpha_1^2 c \neq 0$. Note that

$$\beta_1^2 - \alpha_1 b\beta_1 + \alpha_1^2 c = (\beta_1 - \frac{\alpha_1 b}{2})^2 - \frac{\alpha_1^2}{4}(b^2 - 4c) > 0$$

This is proven, and the last inequality holds because (x^2+bx+c) is irreducible, that is, $b^2 - c < 0$

□

This kind of decomposition helps us with differentiation, integration, and power series expansion.

Rational Function Integration

If you have a rational function in hand, through partial fraction decomposition, you get a series of functions that are easy to integrate. Assuming you are familiar with the integrals of these functions (go through them mentally), the problem you may have with integration is likely **not being able to factorize the denominator**. You can start by staring out a root of the denominator polynomial, and then perform polynomial division, continuing this process. Here is a systematic guide for the staring method.

注. Staring Method:

1. For a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

If there exists a rational root $\frac{r}{s}$, then s is a factor of a_n, and r is a factor of a_0.

2. If you can't stare out a rational root, ~~try the Eisenstein Criterion~~. It is likely that there are no rational roots.

3. Irrational roots might also be stearable, why not try $\sqrt{2}$?

4. If you are too focused and stare out a complex root, then its conjugate is also a root.

5. Following from 4, if you stare out a complex root, then multiply its conjugate when performing division.

For polynomial division, your advanced algebra notes have a triple-optimized paper-and-pencil algorithm, so why not go familiarize yourself with it? For finding A_m, you can do the following

$$R = \sum_i \frac{A_i}{h_i},$$

$$h_m R = \sum_{i, i \neq m} \frac{A_i}{h_i} h_m + A_m$$

Take $h_m \to 0$ to obtain, noting here that it is required that other h_i do not tend to zero, so for terms with power multiplication, multiply the higher power first. Similarly, for

$$R = \sum_i \frac{B_i x + C_i}{l_i}$$

$$l_m R = \sum_{i, i \neq m} \frac{B_i x + C_i}{l_i} l_m + B_m x + C_m$$

Take $l_m = 0$ for its two complex roots, get the linear equations system for B_m, C_m, and solve it. The only item that is not easy to integrate is

$$\int \frac{\mathrm{d}x}{(x^2 + a^2)^m}$$

Remember the recurrence relation

$$\int \frac{\mathrm{d}x}{(x^2 + a^2)^{m+1}} = \frac{1}{2ma^2} \frac{x}{(x^2 + a^2)^m} + \frac{2m-1}{2ma^2} \int \frac{\mathrm{d}x}{(x^2 + a^2)^m}$$