ME780 Assignment 1

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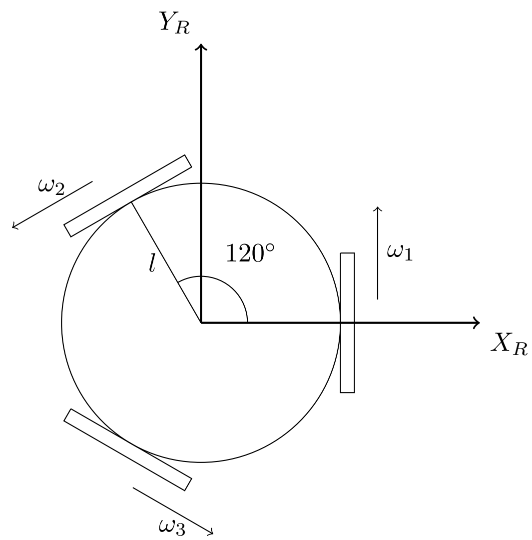
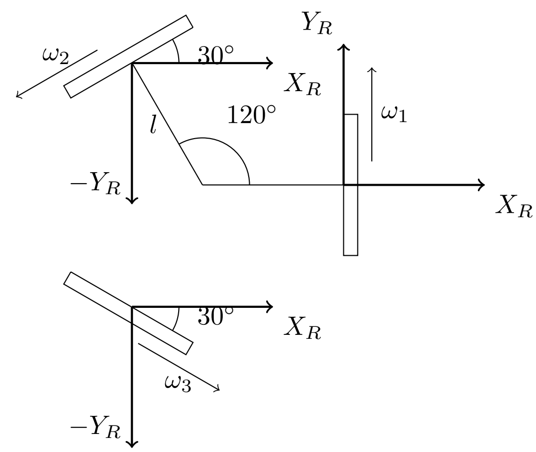
1 Derivation of the Omni-directional Robot



(a) Omni-Directional Robot in Body Frame



(b) Body Frame w.r.t Each Wheel

Figure 1: Omni-Directional Robot

Here we derive the kinematics of the Omni-directional robot. We followed the generalized approach in [1] in resolving how each wheel contributes to the robot chassis. The generalized equation is:

$$\xi_I = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{1}$$

Where:

- ξ_I: State in the body frame
- $R(\theta)$: Rotation matrix
- J_1: Wheel constraints
- J_2: Diagonal matrix denoting the radii r of each wheel
- ω: Angular velocity ω of each wheel

Deriving the robot kinematics for each wheel we get in the form of 1 we obtain:

$$J_1 = \begin{bmatrix} 0 & 1 & l \\ -\cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \\ \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \end{bmatrix} \begin{matrix} \} Wheel1 \\ \} Wheel2 \\ \} Wheel3 \end{matrix} \tag{2}$$

$$J_2 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \tag{3}$$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{4}$$

Putting it all together we have:

$$\xi_I = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{5}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & l \\ -\cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \\ \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \end{bmatrix}^{-1} \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3l} & \frac{1}{3l} & \frac{1}{3l} \end{bmatrix} \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\sin(\theta) & -\frac{1}{\sqrt{3}}\cos(\theta) - \frac{1}{3}\sin(\theta) & \frac{1}{\sqrt{3}}\cos(\theta) - \frac{1}{3}\sin(\theta) \\ \frac{2}{3}\cos(\theta) & \frac{1}{\sqrt{3}}\sin(\theta) - \frac{1}{3}\cos(\theta) & -\frac{1}{\sqrt{3}}\sin(\theta) - \frac{1}{3}\cos(\theta) \\ \frac{1}{3l} & \frac{1}{3l} & \frac{1}{3l} \end{bmatrix} \begin{bmatrix} r\omega_1 \\ r\omega_2 \\ r\omega_3 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\sin(\theta)r\omega_1 - \frac{1}{\sqrt{3}}\cos(\theta)r\omega_2 - \frac{1}{3}\sin(\theta)r\omega_2 + \frac{1}{\sqrt{3}}\cos(\theta)r\omega_3 - \frac{1}{3}\sin(\theta)r\omega_3 \\ \frac{2}{3}\cos(\theta)r\omega_1 + \frac{1}{\sqrt{3}}\sin(\theta)r\omega_2 - \frac{1}{3}\cos(\theta)r\omega_2 - \frac{1}{\sqrt{3}}\sin(\theta)r\omega_3 - \frac{1}{3}\cos(\theta)r\omega_3 \\ \frac{1}{3l}r\omega_1 + \frac{1}{3l}r\omega_2 + \frac{1}{3l}r\omega_3 \end{bmatrix} \tag{9}$$

$$\tag{10}$$

$$\dot{g}(\theta, \omega) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \tag{11}$$

Where:

- $\dot{g}$ is our velocity motion model of the omni-directional robot
- θ is the angle (or bearing) of the robot in the inertia frame.
- ω is the wheel inputs of the robot $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

2 Applying the Derived Robot Model

2.1 Predefined Rotation Inputs

Applying the following rotation inputs to the wheel for 15 seconds:

- $\omega_1 = -15.5$ rads^{-1}
- $\omega_2 = 10.5$ rads^{-1}
- $\omega_3 = 1.5$ rads^{-1}

We obtain the following motion.

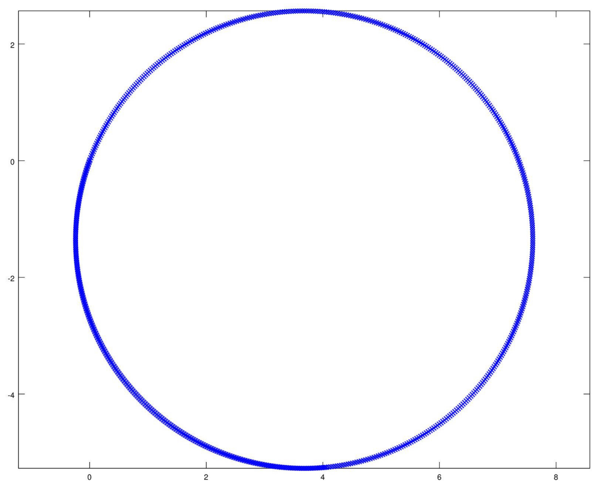


Figure 2: Traverse with inputs: $\omega_1 = -15.5$ rads^{-1} $\omega_2 = 10.5$ rads^{-1} $\omega_3 = 1.5$ rads^{-1}

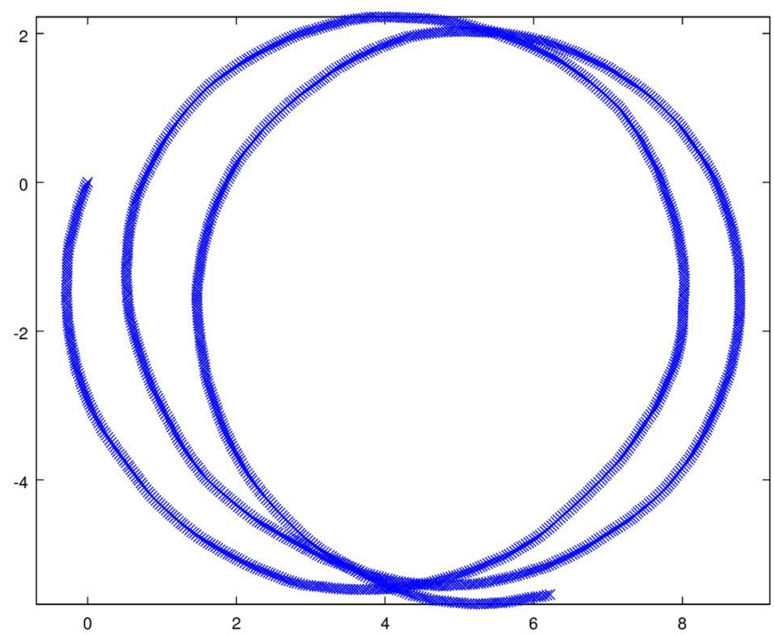
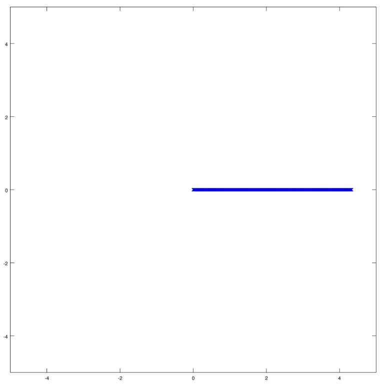
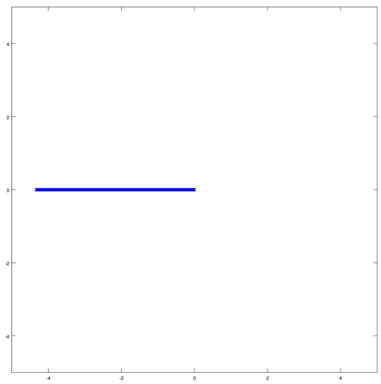


Figure 3: Same as Fig 2 but with Additive Gaussian Noise

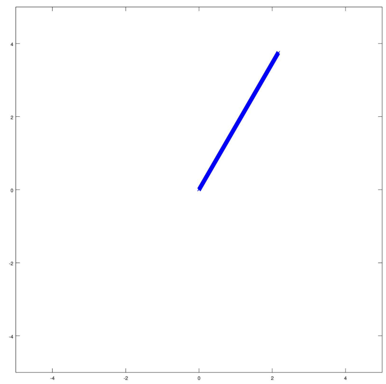
2.2 Traversing East, West, North East, South East and Inplace



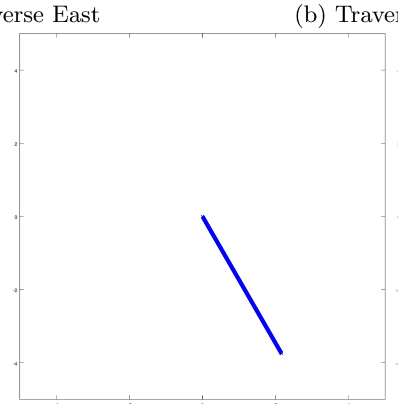
(a) Traverse East



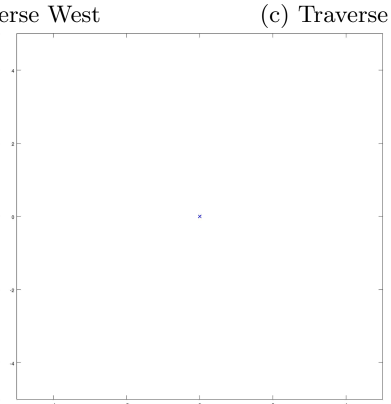
(b) Traverse West



(c) Traverse North East

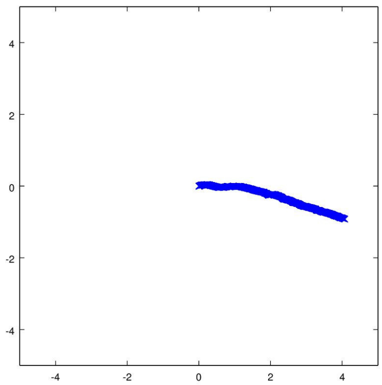


(d) Traverse South East

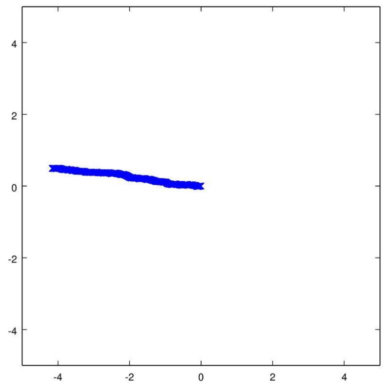


(e) Traverse Inplace

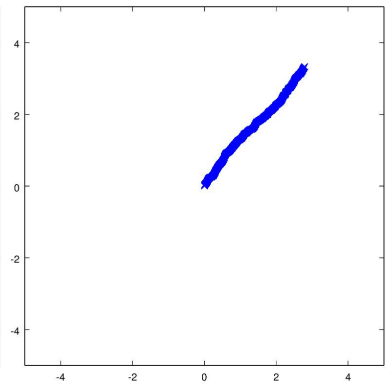
Figure 4: Traversing East, West, North East, South East and Inplace



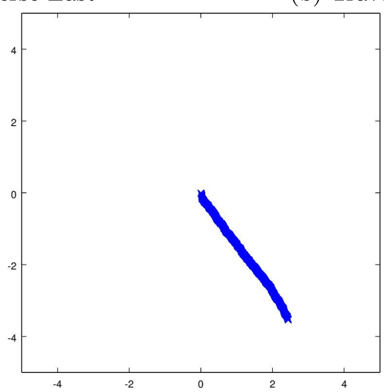
(a) Traverse East



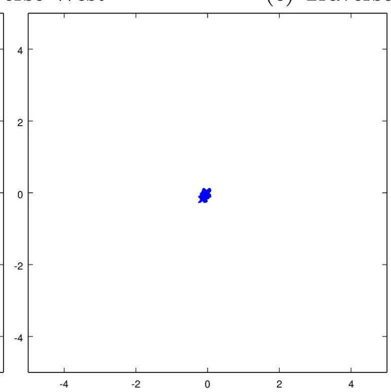
(b) Traverse West



(c) Traverse North East



(d) Traverse South East



(e) Traverse Inplace

Figure 5: Same as Fig 4 but with Additive Gaussian Noise

The inputs that drove the robot to traverse in the directions shown in Fig 4 and Fig 5 for 15 seconds are as follows:

- East: $\omega_1 = 0$, $\omega_2 = -1$, $\omega_3 = 1$
- West: $\omega_1 = 0$, $\omega_2 = 1$, $\omega_3 = -1$
- North East: $\omega_1 = 1$, $\omega_2 = -1$, $\omega_3 = 0$
- South East: $\omega_1 = -1$, $\omega_2 = 0$, $\omega_3 = 1$
- Inplace: $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 1$

2.3 Moving in a Circle

To traverse in a circle of diameter of 2m, we had to be a little creative with the rotation inputs of the wheels. To do this we applied the following intuition:

1. Constrained the robot's y-axis movement to 0, and so $\dot{y}$ is $\dot{y} = 0$. Use wheels 2 and 3 as fixed wheels pushing the robot forward, and use wheel 1 as a steering wheel. (See fig 1)
2. For the robot to traverse a circle, the robot must traverse the circumference of a circle, i.e. $2\pi R$, where R is the radius of the circle we want the robot to make $d = 2\pi R$. Converting that to velocity, the equation becomes $\dot{x} = \frac{2\pi R}{t}$
3. For the robot to make a full circle, it must change angle $\theta = 2\pi$. Converting this to the angular rate the equation becomes $\theta = \frac{2\pi}{t}$

Putting the above all together we have the following, originally we had:

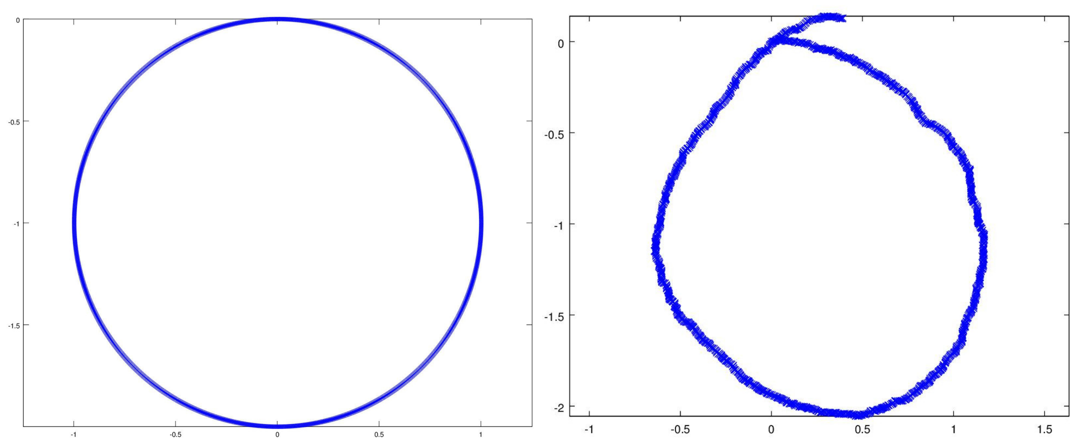
$$\xi_I = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{12}$$

Where we are trying to figure out what the ω inputs are to drive the robot to make a full circle in the inertial frame, we already know what $R(\theta)$, J_1, J_2 are, the only unknown left is ξ_I, but using the above intuition we can relate what it should be to make a circle.

$$\omega = \xi_I R(\theta) J_1 J_2^{-1} \tag{13}$$

$$\xi_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{2\pi R}{t} \\ 0 \\ \frac{2\pi}{t} \end{bmatrix} \tag{14}$$

Using the above equation we found the ω inputs required to traverse a 2m diameter circle in 15 seconds is $\omega_1 = 0.50265$, $\omega_2 = -0.94838$ $\omega_3 = 1.95369$ starting from the origin $(0, 0)$.

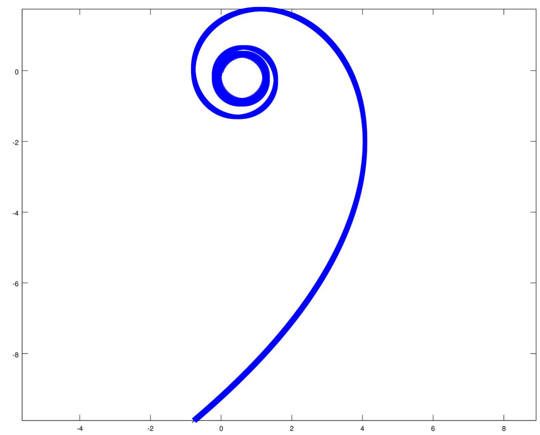


(a) Without Noise (b) With Noise

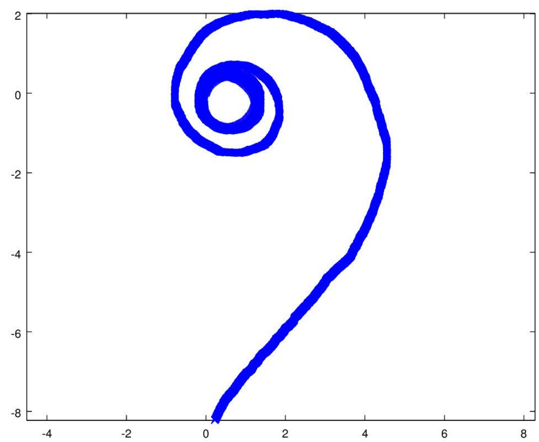
Figure 6: Omni-Robot Traversing a Circle

2.4 Traversing in a Spiral

For traversing in a spiral we used a similar approach as traversing in a circle. We constrained the robot so that it was only traversing in the x direction in the robot's body frame, and set the speed of wheel 1 to have intial angular velocity $\omega_1 = 10$, with its angular velocity set to decay at a rate of 0.999 per $dt = 0.1$. Below is a figure of the robot traversing in a spiral pattern for 50 seconds.



(a) Without Noise



(b) With Noise

Figure 7: Omni-Robot Traversing a Spiral

3 Measurement Model

Lets suppose the robot has a GPS and magnometer sensor. It is given that "The GPS output is in true North, East (and Down, not used here) and the magnetometer points to magnetic north (your declination is 9.7 degrees West in Waterloo, ON). Define additive noise distributions, with standard deviations in North and East of 0.50m, and in magnetic north of 10 degrees."

Given the above information, our measurement noise covariance matrix Q is:

$$Q = \begin{bmatrix} 0.5^2 & 0 & 0 \\ 0 & 0.5^2 & 0 \\ 0 & 0 & 0.1745^2 \end{bmatrix} \tag{15}$$

Note that 10 degrees is 0.1745 radians. With a mean μ of:

$$\mu = \begin{bmatrix} 0 \\ 0 \\ -0.1692 \end{bmatrix} \tag{16}$$

Note that 9.7 degrees is 0.1692 radians. Now that we have μ and Q, the Gaussian Additive Noise then becomes:

$$\delta_t \sim \mathcal{N}(\mu, Q) \tag{17}$$

To complete the measurement model it becomes:

$$y(t) = C(t)x(t) + \delta_t \tag{18}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \delta_t \tag{19}$$

4 Defining the Extended Kalman Filter for the Omni-directional Robot

The Extended Kalman filter (EKF) has two stages, the prediction and measurement updates. To complete the EKF we need to linearize the motion and measurement model, we detail that in the following sections.

4.1 Linearizing the Motion Model

Normally if our motion model and/or measurement model is linear we can simply use the Kalman Filter, however in the non-linear case we have to use the Extended Kalman Filter (EKF). In our case we have to linearize the non-linear motion model $g(x_{t-1}, u_t)$ using 1st order Taylor Series Expansion:

$$g(x_{t-1}, u_t) \approx g(\mu_{t-1}, u_t) + \left. \frac{\partial g(x_{t-1}, u_t)}{x_{t-1}} \right|_{x_{t-1}=\mu_{t-1}} (x_{t-1} - \mu_{t-1}) \tag{20}$$

$$= g(\mu_{t-1}, u_t) + G_t \cdot (x_{t-1} - \mu_{t-1}) \tag{21}$$

Where G_t is the first order differential of the non-linear motion model, a.k.a the Jacobian of $g(x_{t-1}, u_t)$.

$$G_t = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{bmatrix} \tag{22}$$

Since we only know $\dot{g}(\theta, \omega)$ we need to convert it to $g(x_{t-1}, u_t)$, it is simply:

$$g(x_{t-1}, u_t) = x_{t-1} + \dot{g}(\theta, \omega) dt \tag{23}$$

$$= \begin{bmatrix} x_{1,t-1} + \dot{g_1}(\theta, \omega) dt \\ x_{2,t-1} + \dot{g_2}(\theta, \omega) dt \\ x_{3,t-1} + \dot{g_3}(\theta, \omega) dt \end{bmatrix} \tag{24}$$

Now that we have $g(x_{t-1}, u_t)$ we can work out the Jacobian. One important note, the partial derivate of x_3 in our case is the bearing θ of the robot, the only term affected in $\dot{g}(\theta, \omega)$ is the rotation matrix $R(\theta)$, since $\dot{g}(\theta, \omega)$ is:

$$\dot{g}(\theta, \omega) = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{25}$$

Where $R(\theta)$, J_1, J_2 and ω are (derived in Section 1):

$$R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

$$J_1^{-1} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3l} & \frac{1}{3l} & \frac{1}{3l} \end{bmatrix} \tag{27}$$

$$J_2 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \tag{28}$$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{29}$$

Therefore to differentiate rotation matrix $R(\theta)$ with respect to x_3 (or θ), we can treat J_1 J_2 and ω as constant vector A. The partial derivative of $R(\theta)$ then becomes:

$$\frac{\partial R(\theta)^{-1}}{\partial x_3} = \begin{bmatrix} -\sin(x_3) & \cos(x_3) & 0 \\ -\cos(x_3) & -\sin(x_3) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{30}$$

$$A = J_1^{-1} J_2 \omega \tag{31}$$

Or in terms of g and x.

$$\frac{\partial g}{\partial x_3} = \frac{\partial R(\theta)^{-1}}{\partial x_3} A \tag{32}$$

$$= \begin{bmatrix} -\sin(x_3) & \cos(x_3) & 0 \\ -\cos(x_3) & -\sin(x_3) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} -A_1 sin(x_3) + A_2 cos(x_3) \\ -A_1 cos(x_3) - A_2 sin(x_3) \\ 0 \end{bmatrix} \tag{34}$$

Completing our Jacobian G_t we get:

$$G_t = \begin{bmatrix} 1 & 0 & -A_1 sin(x_3) + A_2 cos(x_3) \\ 0 & 1 & -A_1 cos(x_3) - A_2 sin(x_3) \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

4.2 Linearizing the Measurement Model

As for the measurement model, it does not require linearization since in Section 3 we derived a linear measurement model. Therefore $H_t = C_t$.

$$H_t = C_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

4.3 Summary

Now that we have defined what $g(x_{t-1})$, G_t and H_t (along with intial values for the states) we have enough information to incorporate them into the EKF model.

5 Implementing the Extended Kalman Filter

In the EKF we have used the following values:

$$
\begin{aligned}
\mu_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
u_0 &= \begin{bmatrix} -15.5 \\ -10.5 \\ 1.5 \end{bmatrix} \\
H_t &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
G_t &= \begin{bmatrix} 1 & 0 & -A_1 sin(x_3) - A_2 cos(x_3) \\ 0 & 1 & A_1 cos(x_3) - A_2 sin(x_3) \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -A_1 sin(x_3) - A_2 cos(x_3) \\ 0 & 1 & A_1 cos(x_3) - A_2 sin(x_3) \\ 0 & 0 & 1 \end{bmatrix}^{-1} A \\
A_t &= J_1^{-1} J_2 \omega \\
&= \begin{bmatrix} 0 & 1 & l \\ -\cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \\ \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \end{bmatrix}^{-1} \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
\end{aligned}
$$

Running the models for 15 seconds with the above input parameters for the EKF we obtain the following plots.

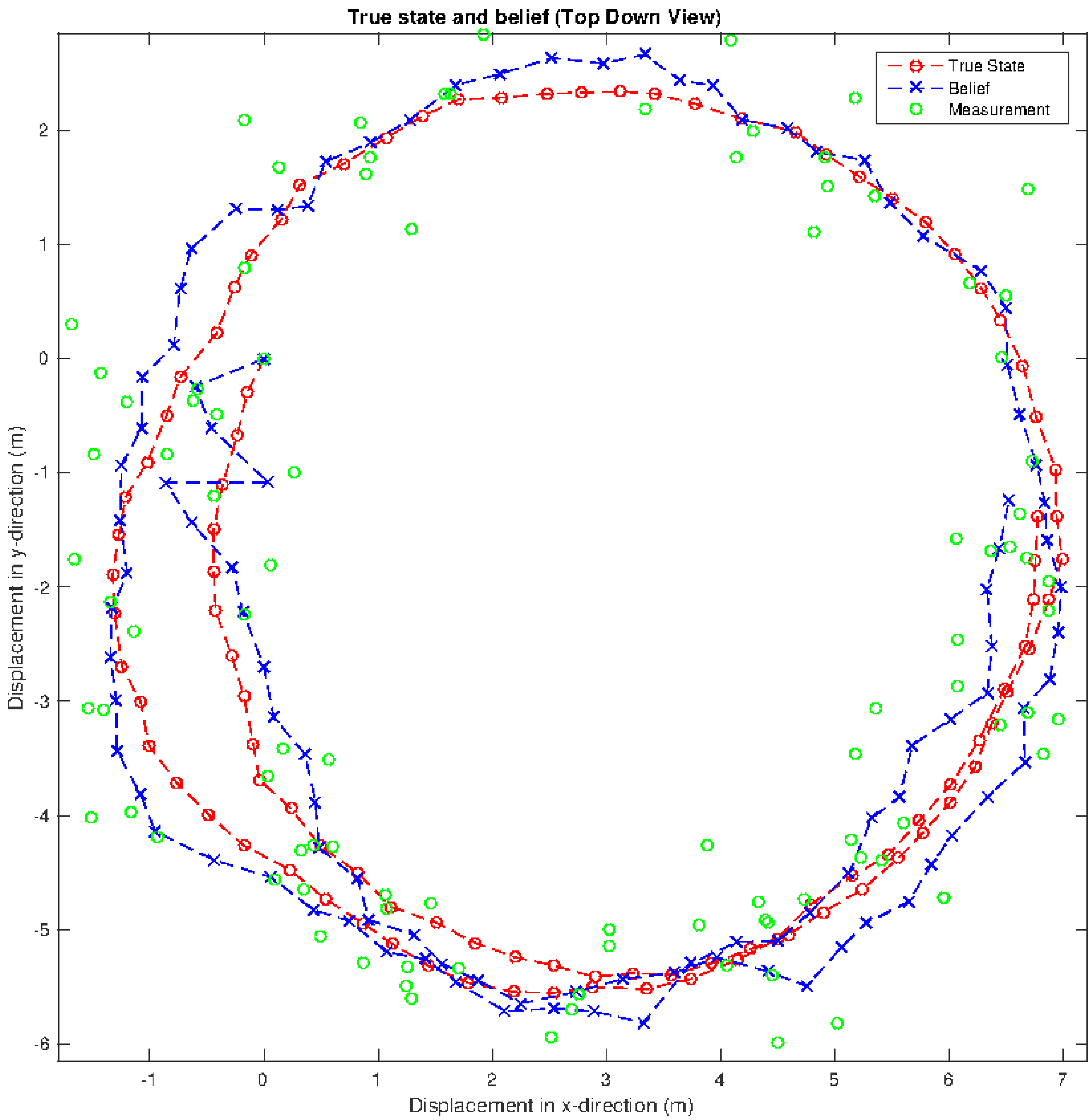


Figure 8: Top Down view of Robot Motion with EKF

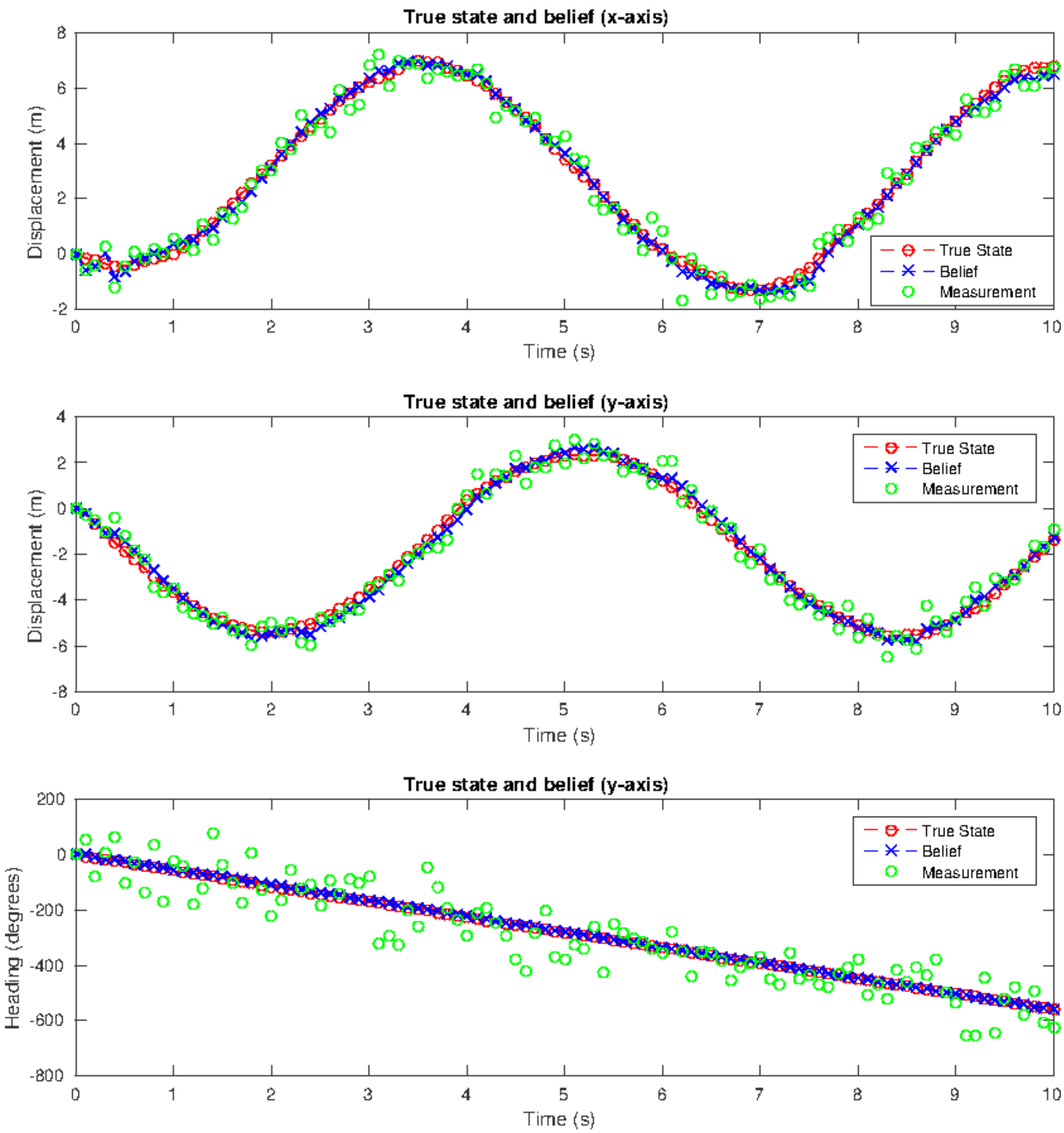


Figure 9: State of Robot Motion with EKF

6 Adding GPS corrections to the EKF

Previously it was assumed that the GPS and Magnetometer both work at a rate of 10hz, however it is given that "corrections are sent for the x, y positions of the vechicle at an update rate of 1Hz. These corections improve the GPS measurements at that instance only, with an improved standard deviation of 0.01m".

Since the corrections are coming in a lower rate (1Hz) than the normal gps and the magnometer measurements (10Hz) two Q matrices were constructed that correspond to these two sets of measurements. The H_t matrix is unchanged by this correction as it does not measure any additional states.

The covariance evolution can be seen from the two videos with this report. From Fig 12 we can observe that the 1Hz GPS corrections greatly reduces the uncertainty in the belief state.

$$Q = \begin{bmatrix} 0.5^2 & 0 & 0 \\ 0 & 0.5^2 & 0 \\ 0 & 0 & 0.1745^2 \end{bmatrix} \tag{37}$$

to (at every 1Hz):

$$Q = \begin{bmatrix} 0.1^2 & 0 & 0 \\ 0 & 0.1^2 & 0 \\ 0 & 0 & 0.1745^2 \end{bmatrix} \tag{38}$$

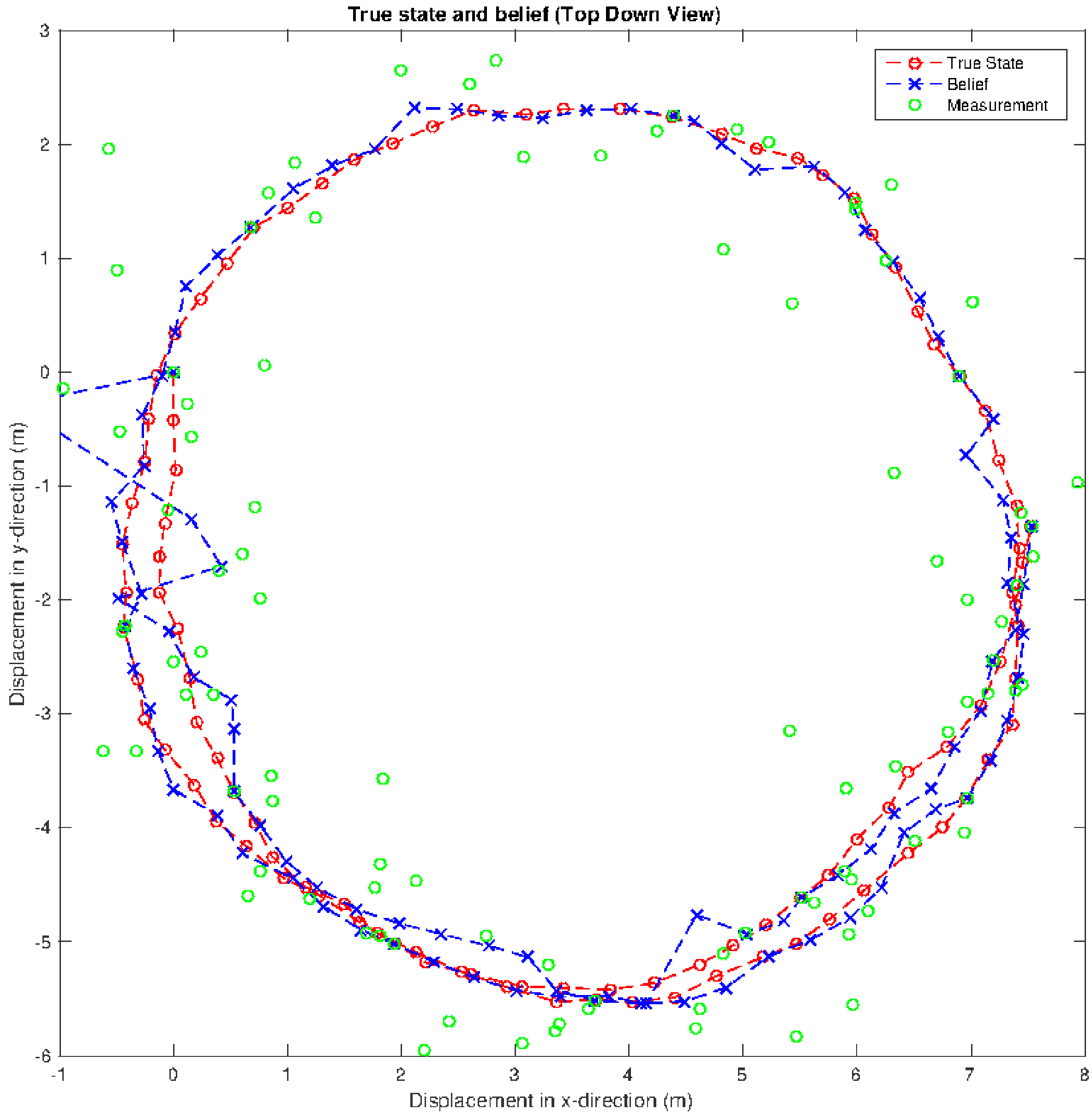


Figure 10: Top Down view of Robot Motion with EKF and GPS Corrections at 1Hz

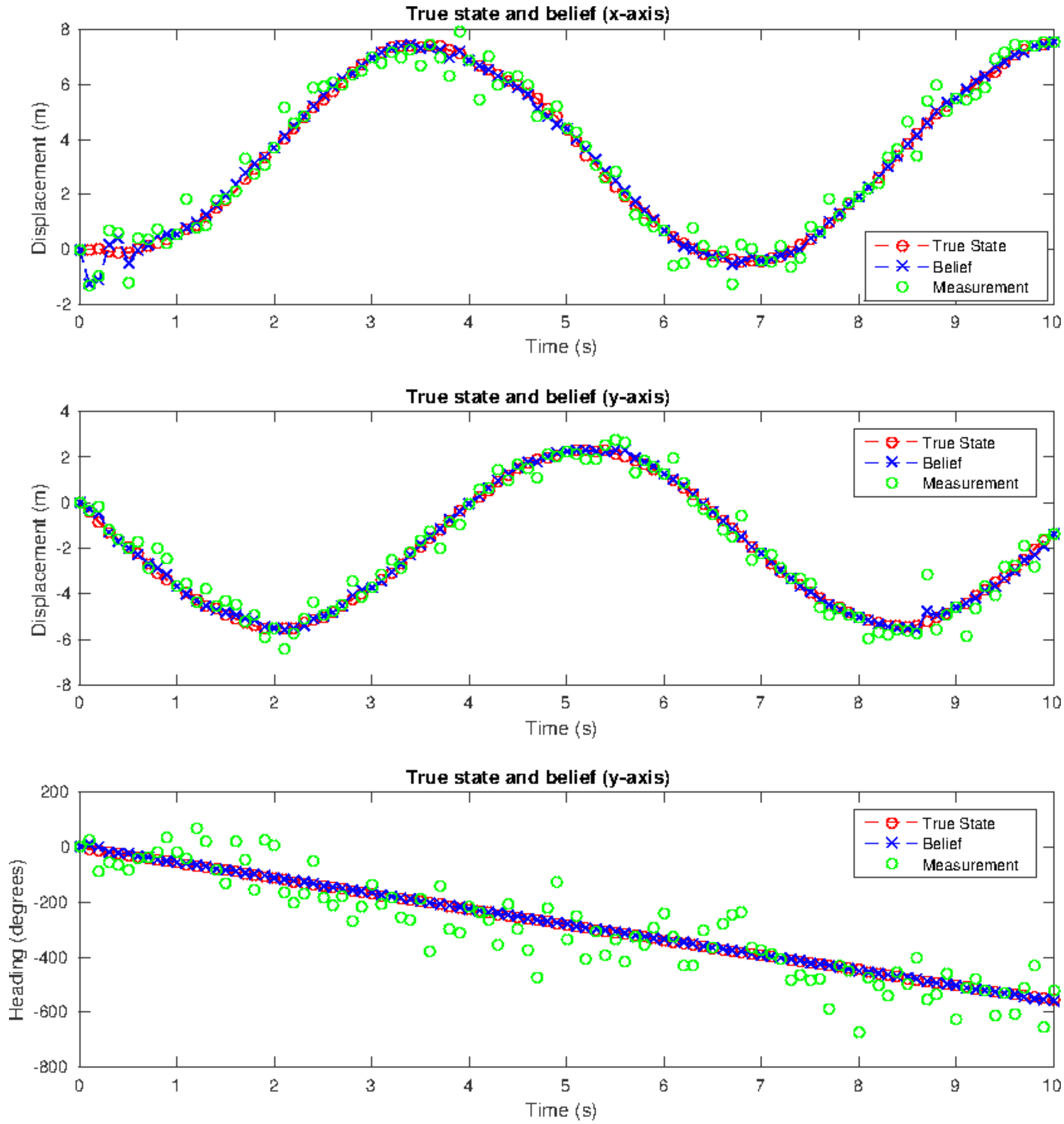
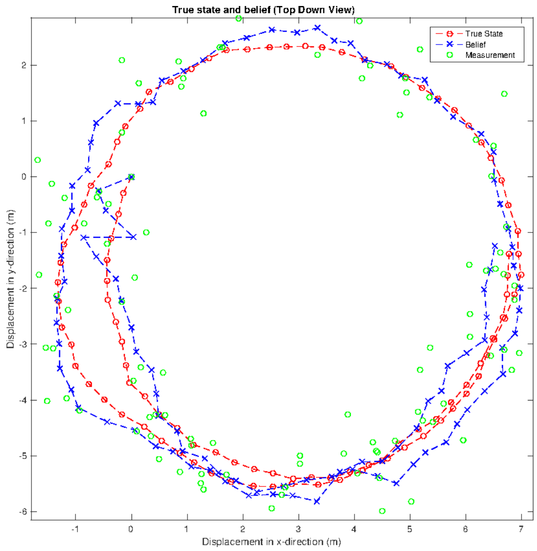
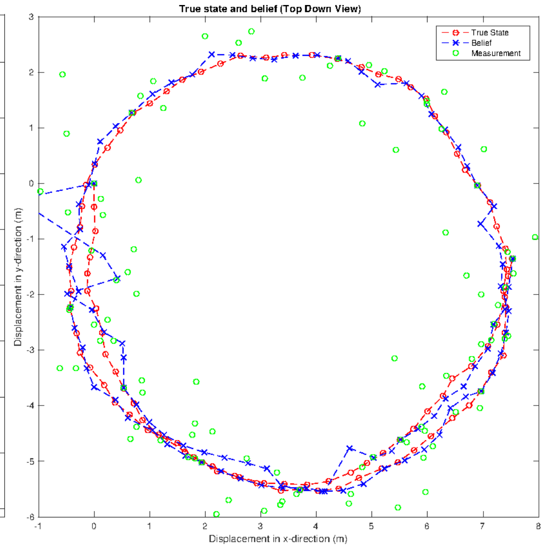


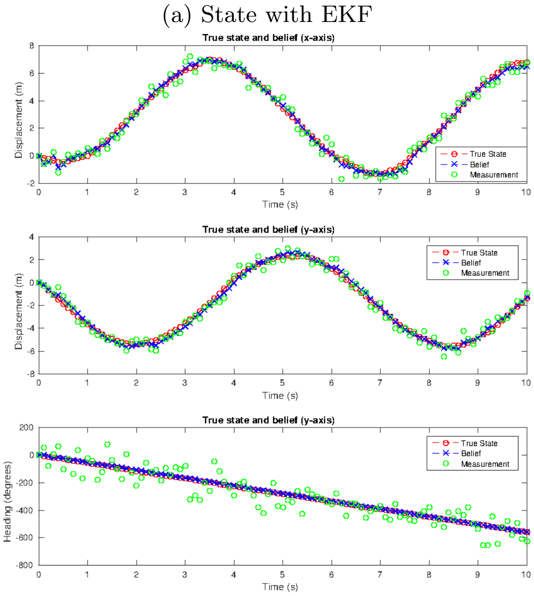
Figure 11: State of Robot Motion with EKF and GPS Corrections at 1Hz



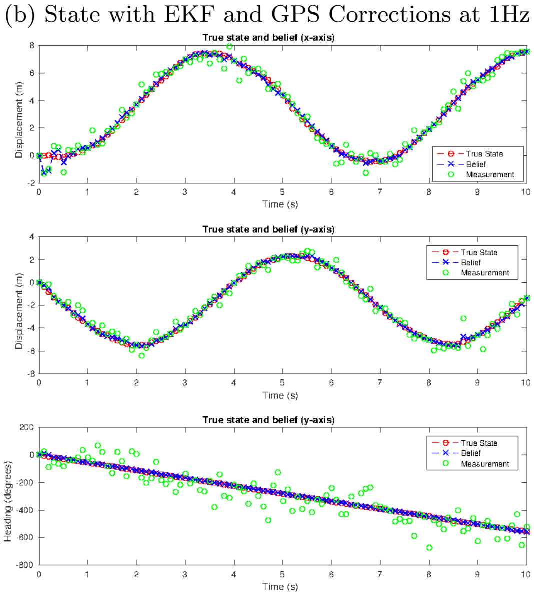
(a) State with EKF



(b) State with EKF and GPS Corrections at 1Hz



(c) State with EKF



(d) State with EKF and GPS Corrections at 1Hz

Figure 12: Comparison between EKF and EKF with GPS correction

References

[1] R. Siegwart, I. R. Nourbakhsh, and D. Scaramuzza, *Introduction to autonomous mobile robots*. MIT press, 2011.