ME780 Assignment 1

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1 Derivation of the Omni-directional Robot

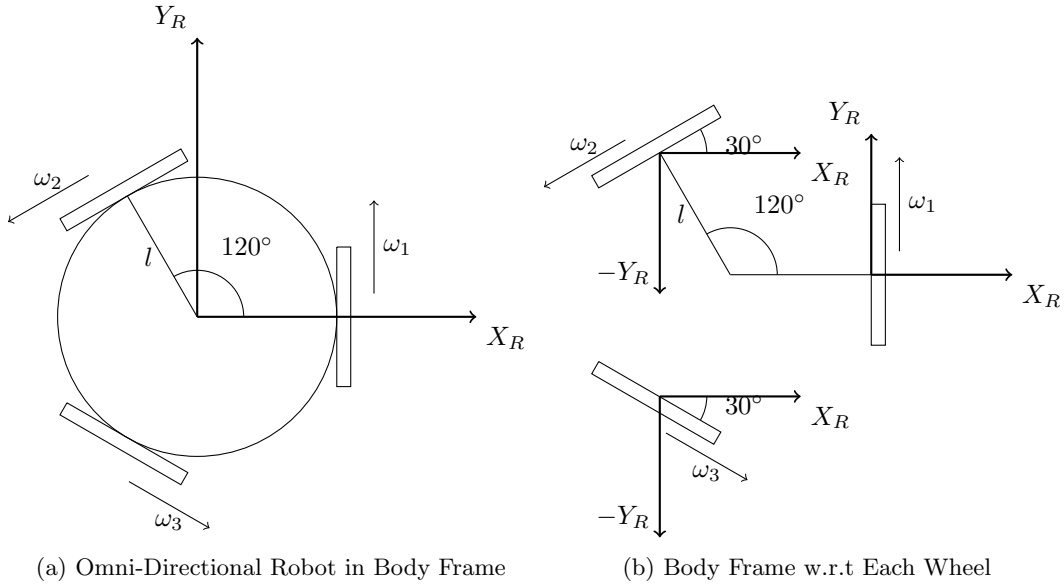(a) Omni-Directional Robot in Body Frame   (b) Body Frame w.r.t Each Wheel

Figure 1: Omni-Directional Robot

Here we derive the kinematics of the Omni-directional robot. We followed the generalized approach in [**?**] in resolving how each wheel contributes to the robot chassis. The generalized equation is:

$$\xi_I = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{1}$$

Where:

- ξ_I: State in the body frame
- $R(\theta)$: Rotation matrix
- J_1: Wheel constraints
- J_2: Diagonal matrix denoting the radii r of each wheel
- ω: Angular velocity ω of each wheel

Deriving the robot kinematics for each wheel we get in the form of 1 we obtain:

$$
J_1 = \begin{bmatrix} 0 & 1 & l \\ -\cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \\ \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \end{bmatrix} \begin{matrix} \}\,Wheel1 \\ \}\,Wheel2 \\ \}\,Wheel3 \end{matrix} \tag{2}
$$

$$
J_2 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \tag{3}
$$

$$
\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{4}
$$

Putting it all together we have:

$$
\xi_I = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{5}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & l \\ -\cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \\ \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & l \end{bmatrix}^{-1} \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{6}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3l} & \frac{1}{3l} & \frac{1}{3l} \end{bmatrix} \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{7}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\sin(\theta) & -\frac{1}{\sqrt{3}}\cos(\theta) - \frac{1}{3}\sin(\theta) & \frac{1}{\sqrt{3}}\cos(\theta) - \frac{1}{3}\sin(\theta) \\ \frac{2}{3}\cos(\theta) & \frac{1}{\sqrt{3}}\sin(\theta) - \frac{1}{3}\cos(\theta) & -\frac{1}{\sqrt{3}}\sin(\theta) - \frac{1}{3}\cos(\theta) \\ \frac{1}{3l} & \frac{1}{3l} & \frac{1}{3l} \end{bmatrix} \begin{bmatrix} r\omega_1 \\ r\omega_2 \\ r\omega_3 \end{bmatrix} \tag{8}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\sin(\theta)r\omega_1 - \frac{1}{\sqrt{3}}\cos(\theta)r\omega_2 - \frac{1}{3}\sin(\theta)r\omega_2 + \frac{1}{\sqrt{3}}\cos(\theta)r\omega_3 - \frac{1}{3}\sin(\theta)r\omega_3 \\ \frac{2}{3}\cos(\theta)r\omega_1 + \frac{1}{\sqrt{3}}\sin(\theta)r\omega_2 - \frac{1}{3}\cos(\theta)r\omega_2 - \frac{1}{\sqrt{3}}\sin(\theta)r\omega_3 - \frac{1}{3}\cos(\theta)r\omega_3 \\ \frac{1}{3l}r\omega_1 + \frac{1}{3l}r\omega_2 + \frac{1}{3l}r\omega_3 \end{bmatrix} \tag{9}
$$

$$
\tag{10}
$$

$$
\dot{g}(\theta, \omega) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \tag{11}
$$

Where:

- $\dot{g}$ is our velocity motion model of the omni-directional robot
- θ is the angle (or bearing) of the robot in the inertia frame.

- ω is the wheel inputs of the robot $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

2 Applying the Derived Robot Model

2.1 Predefined Rotation Inputs

Applying the following rotation inputs to the wheel for 15 seconds:

- $\omega_1 = -15.5$ rads^{-1}
- $\omega_2 = 10.5$ rads^{-1}
- $\omega_3 = 1.5$ rads^{-1}

We obtain the following motion.

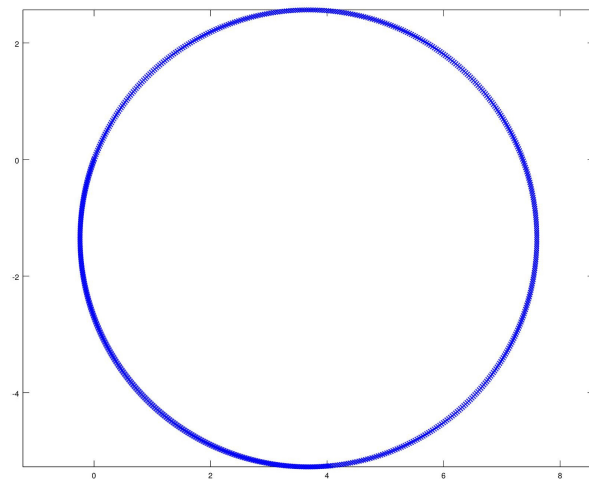Figure 2: Traverse with inputs: $\omega_1 = -15.5$ rads^{-1} $\omega_2 = 10.5$ rads^{-1} $\omega_3 = 1.5$ rads^{-1}

2.2 Traversing East, West, North East, South East and Inplace

(a) Traverse East   (b) Traverse West   (c) Traverse North East

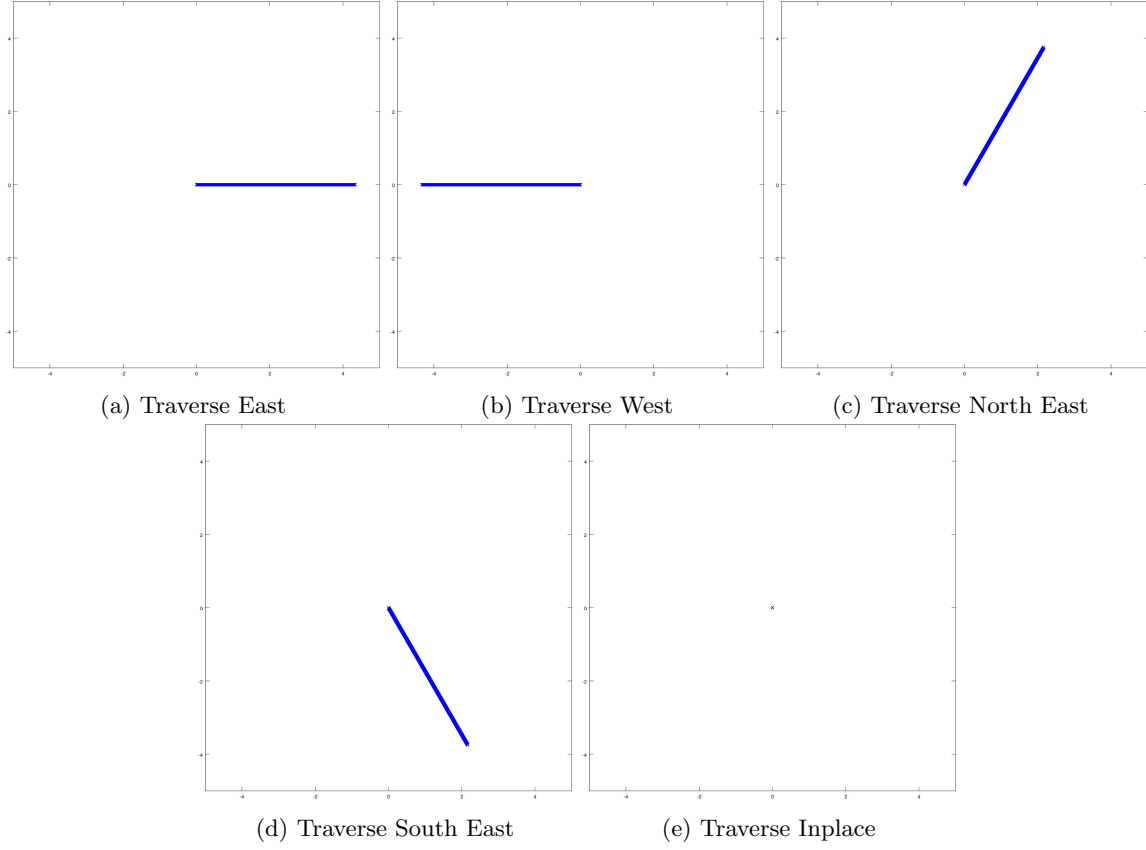(d) Traverse South East   (e) Traverse Inplace

Figure 3: Traversing East, West, North East, South East and Inplace

The inputs that drove the robot to traverse in the directions shown in Fig 3 for 15 seconds are as follows:

- East: $\omega_1 = 0$, $\omega_2 = -1$, $\omega_3 = 1$
- West: $\omega_1 = 0$, $\omega_2 = 1$, $\omega_3 = -1$
- North East: $\omega_1 = 1$, $\omega_2 = -1$, $\omega_3 = 0$
- South East: $\omega_1 = -1$, $\omega_2 = 0$, $\omega_3 = 1$
- Inplace: $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 1$

2.3 Moving in a Circle

To traverse in a circle of diameter of 2m, we had to be a little creative with the rotation inputs of the wheels. To do this we applied the following intuition:

1. Constrained the robot's y-axis movement to 0, and so $\dot{y}$ is $\dot{y} = 0$. Use wheels 2 and 3 as fixed wheels pushing the robot forward, and use wheel 1 as a steering wheel. (See fig 1)

2. For the robot to traverse a circle, the robot must traverse the circumference of a circle, i.e. $2\pi R$, where R is the radius of the circle we want the robot to make $d = 2\pi R$. Converting that to velocity, the equation becomes $\dot{x} = \frac{2\pi R}{t}$

3. For the robot to make a full circle, it must change angle $\theta = 2\pi$. Converting this to the angular rate the equation becomes $\theta = \frac{2\pi}{t}$

Putting the above all together we have the following, originally we had:

$$\xi_I = R(\theta)^{-1} J_1^{-1} J_2 \omega \tag{12}$$

Where we are trying to figure out what the ω inputs are to drive the robot to make a full circle in the inertial frame, we already know what $R(\theta)$, J_1, J_2 are, the only unknown left is ξ_I, but using the above intuition we can relate what it should be to make a circle.

$$\omega = \xi_I R(\theta) J_1 J_2^{-1} \tag{13}$$

$$\xi_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{2\pi R}{t} \\ 0 \\ \frac{2\pi}{t} \end{bmatrix} \tag{14}$$

Using the above equation we found the ω inputs required to traverse a 2m diameter circle in 15 seconds is $\omega_1 = 0.50265$, $\omega_2 = -0.94838$ $\omega_3 = 1.95369$ starting from the origin $(0,0)$.
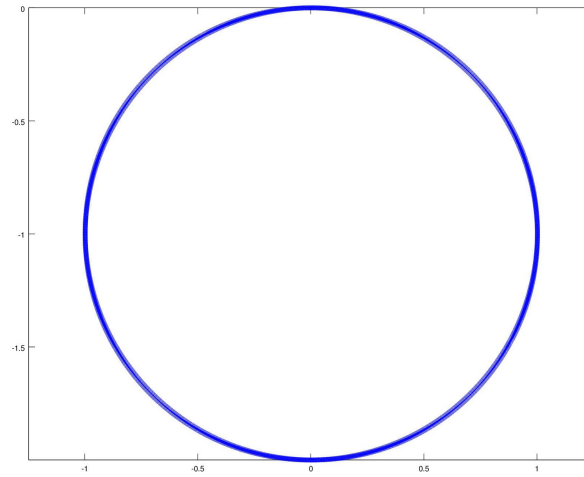

Figure 4: Omni-Robot Traversing a Circle

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2.4　Traversing in a Spiral

For traversing in a spiral we used a similar approach as traversing in a circle. We constrained the robot so that it was only traversing in the x direction in the robot's body frame, and set the speed of wheel 1 to have intial angular velocity $\omega_1 = 10$, with its angular velocity set to decay at a rate of 0.999 per $dt = 0.1$. Below is a figure of the robot traversing in a spiral pattern for 50 seconds.
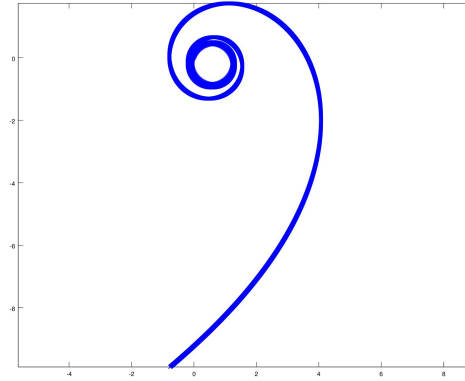

Figure 5: Omni-Robot Traversing a Spiral

3 Measurement Model

Lets suppose the robot has a GPS and magnometer sensor. It is given that "The GPS output is in true North, East (and Down, not used here) and the magnetometer points to magnetic north (your declination is 9.7 degrees West in Waterloo, ON). Define additive noise distributions, with standard deviations in North and East of 0.50m, and in magnetic north of 10 degrees."

Given the above information, our measurement noise covariance matrix Q is:

$$Q = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{15}$$

With a mean μ of:

$$\mu = \begin{bmatrix} 0 \\ 0 \\ -0.1692 \end{bmatrix} \tag{16}$$

Note that 9.7 degrees is 0.1692 radians. Now that we have μ and Q, the Gaussian Additive Noise then becomes:

$$\delta_t \sim \mathcal{N}(\mu, Q) \tag{17}$$

To complete the measurement model it becomes:

$$y(t) = C(t)x(t) + \delta_t \tag{18}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \delta_t \tag{19}$$

4 Defining the Extended Kalman Filter for the Omni-directional Robot

Normally if our motion model and/or measurement model is linear we can simply use the Kalman Filter, however in the non-linear case we have to use the Extended Kalman Filter (EKF). In our case we have to linearize the non-linear motion model $g(x_{t-1}, u_t)$, where:

$$\dot{g}(\theta, \omega) = \begin{bmatrix} \frac{2}{3}\sin(\theta)r\omega_1 - \frac{1}{\sqrt{3}}\cos(\theta)r\omega_2 - \frac{1}{3}\sin(\theta)r\omega_2 + \frac{1}{\sqrt{3}}\cos(\theta)r\omega_3 - \frac{1}{3}\sin(\theta)r\omega_3 \\ \frac{2}{3}\cos(\theta)r\omega_1 + \frac{1}{\sqrt{3}}\sin(\theta)r\omega_2 - \frac{1}{3}\cos(\theta)r\omega_2 - \frac{1}{\sqrt{3}}\sin(\theta)r\omega_3 - \frac{1}{3}\cos(\theta)r\omega_3 \\ \frac{1}{3l}r\omega_1 + \frac{1}{3l}r\omega_2 + \frac{1}{3l}r\omega_3 \end{bmatrix} \quad (20)$$

Rewriting the above in terms of state x_1, x_2, x_3, and inputs u_1, u_2, u_3, where S and C short hand for *sin* and *cos* functions.

$g(x_{t-1}, u_t) =$

$$\begin{bmatrix} x_{1,t-1} + \left(\frac{2ru_{1,t}S(x_{3,t-1})}{3} - \frac{ru_{2,t}C(x_{3,t-1})}{\sqrt{3}} - \frac{ru_{2,t}S(x_{3,t-1})}{3} + \frac{ru_{3,t}C(x_{3,t-1})}{\sqrt{3}} - \frac{ru_{3,t}S(x_{3,t-1})}{3} \right) dt \\ x_{2,t-1} + \left(\frac{2ru_{1,t}C(x_{3,t-1})}{3} + \frac{ru_{2,t}S(x_{3,t-1})}{\sqrt{3}} - \frac{ru_{2,t}C(x_{3,t-1})}{3} - \frac{ru_{3,t}S(x_{3,t-1})}{\sqrt{3}} - \frac{ru_{3,t}C(x_{3,t-1})}{3} \right) dt \\ x_{3,t-1} + \left(\frac{u_{1,t} + u_{2,t} + u_{3,t}}{3l} \right) rdt \end{bmatrix}$$

If we linearize the above w.r.t x_1, x_2, x_3 we get:

$$\frac{\delta g}{\delta x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (21)$$

$$\frac{\delta g}{\delta x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (22)$$

$$\frac{\delta g}{\delta x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (23)$$

$$(24)$$

As for the measurement model, because it is linear we do not need to linearize it.

5   Implementing the Extended Kalman Filter