QFT 学习笔记

啊吧啊吧

作者：Chenhao Peng
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ElegantLaTeX Program

目录

第一章 基础概念

这一个章节主要讲一下理论力学中的一些重要概念，他们在 QFT 中会经常被用到。

1.1 哈密顿与拉格朗日力学

1.1.1 拉格朗日方程的导出

如果认为最小作用量原理是一个公理，我们假如已经知道拉格朗日量可以写为:

$$L = L(q_i, \dot{q}_i) \tag{1.1}$$

当然，在物理的解中,q_i 实际上是时间的函数，也就是说 $q_i = q_i(t)$。

定理 1.1 最小作用量原理 定义作用量为:

$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)dt \tag{1.2}$$

能够使得作用量取极小的映射 $q_i(t)$ 被认为是合理的广义坐标的运动轨道。这个轨道应该满足 Euler-Lagrange 方程。

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) = 0 \tag{1.3}$$

下面是证明

证明 当作用量取极小值时，作用量的变分是 0，下面通过作用量取极值的条件来推导出广义坐标应该遵循的随着时间的演化规律。

假设满足使得作用量最小的广义坐标随着时间的变化关系可以记为:$q_i(t)$, 当广义坐标随着时间的变化有了改变时，即变为了 $q_i(t) + \delta q_i(t)$ 时，作用量有了一个改变 δS.

$$\begin{aligned}
\delta \dot{q}_i(t) &= \delta \frac{dq_i(t)}{dt} \\
&= \frac{d(q_i(t) + \delta q_i(t))}{dt} - \frac{dq_i(t)}{dt} \\
&= \frac{d}{dt}\delta q_i(t) \\
\delta S &= \int_{t_2}^{t_1} (\frac{\partial L}{\partial q_i}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i(t))dt \\
&= \int_{t_2}^{t_1} (\frac{\partial L}{\partial q_i}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i(t))dt \\
&= \int_{t_2}^{t_1} \delta q_i(t)(\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}))dt + (\delta q_i \frac{\partial L}{\partial q_i})|_{t_1}^{t_2}
\end{aligned} \tag{1.4}$$

由于要求作用量的变分为 0, 而且对于广义坐标的微扰有要求: $\delta q_i(t_1) = \delta q_i(t_2) = 0$。于是就得到了 Euler-Lagrange 方程。

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) = 0 \tag{1.5}$$

□

1.1.2 Legendre 变换, 正则方程, 柏松括号

legendre 变换：如果有函数 $f = f(x_1, x_2, x_3)$, 则他的全微分可以写为 $df = u_1 dx_1 + u_2 dx_2 \ldots$ 其中,$u_i = \frac{\partial f}{\partial x_i}$, 如果定义函数 $g = u_i x_i - f$, 则 g 可以很轻松地写为 $x_1 \ldots x_{i-1}, u_i . x_{i+1} \ldots$ 的函数. 这是因为:

$$dg = x_i du - u_1 dx_1 \ldots - u_{i-1} dx_{i-1} - u_{i+1} dx_{i+1} \ldots \tag{1.6}$$

如果定义广义动量为:$p_i = \frac{\partial L(q_i,\dot{q}_i,t)}{\partial \dot{q}_i}$, 此时可以改写拉格朗日方程的变量为 (q_i, p_i)。为了能够方便地找到可以写为新变量的全微分，我们定义哈密顿量:$H(q_i,p_i) = p_i\dot{q}_i - L$, 这样，H 的全微分可以写为:

$$dH = \dot{q}_i dp_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt \tag{1.7}$$

如果考虑到 $dp_i = \frac{dp_i}{dt}dt, dq_i = \frac{dq_i}{dt}dt$，并且将 Lagrange 方程 $\frac{\partial L}{\partial q_i} = \frac{dp_i}{dt}$, 带入:

$$\delta H = \dot{q}_i\delta p_i - \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial t}\delta t$$

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t}dt dH = \dot{q}_i\dot{p}_i dt - \frac{dp_i}{dt}\dot{q}_i dt - \frac{\partial L}{\partial t}dt \quad \text{这里的 } dH \text{ 是由时间变化 } dt \text{ 引起的} \tag{1.8}$$

$$= -\frac{\partial L}{\partial t}dt$$

正则运动方程 (正则运动方程结合了) 由上面的式子产生:

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i\end{aligned} \tag{1.9}$$

柏松括号: 柏松括号定义为

$$\{f,g\} = \Sigma_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \tag{1.10}$$

有几个比较特殊的柏松括号:(最后一步均是考虑到了正则运动方程)

$$\begin{aligned}\{q_i,H\} &= \Sigma_j \frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \\ &= \frac{\partial H}{\partial p_i} \\ &= \dot{q}_i \\ \{p_i,H\} &= \Sigma_j \frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q_j} \\ &= -\frac{\partial H}{\partial q_i} \\ &= \dot{p}_i\end{aligned} \tag{1.11}$$

1.2 场论基础

这一节讲一些场论的基础知识

1.2.1 Basic and Hamilton Formalism

泛函求导引入 F 是场 ϕ 的泛函, 场是从空间坐标到数域的映射 (这里用一维空间来计算)。

$$F = F[\phi] \tag{1.12}$$

泛函导数的定义:

$$\delta F[\phi] = \int dx \frac{\delta F[\phi]}{\delta\phi(x)}\delta\phi(x) \tag{1.13}$$

如果场的微小变换是:

$$\delta\phi(x) = \epsilon\delta(x-y) \tag{1.14}$$

泛函 F 变换:

$$\begin{aligned}\delta F[\phi] = F[\phi+\epsilon\delta(x-y)] - F[\phi] = \int dx\, \frac{\delta F[\phi]}{\delta\phi(x)}\epsilon\delta(x-y) \\ = \epsilon\frac{\delta F[\phi]}{\delta\phi(y)}\end{aligned} \tag{1.15}$$

上面的式子被当作泛函导数的定义 (用 · 代替了 x 是因为 x 是 silent argument):

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{F[\phi+\epsilon\delta(\cdot-y)] - F[\phi]}{\epsilon} \tag{1.16}$$

∘ 1.

如果

$$F[\phi] = G[\phi]H[\phi] \tag{1.17}$$

那么

$$\begin{aligned}\frac{\delta F[\phi]}{\delta\phi(x)} &= \lim_{\epsilon\to 0}\frac{G[\phi+\epsilon\delta(\cdot-x)]H[\phi+\epsilon\delta(\cdot-x)] - G[\phi]H[\phi]}{\epsilon} \\ &= \frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] + G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)}\end{aligned} \tag{1.18}$$

∘ 2.

如果

$$F = F[g(\phi)] \tag{1.19}$$

那么:

$$\frac{\delta F[g(\phi)]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{F\left[g\left(\phi+\epsilon\delta(\cdot-y)\right)\right] - F[g(\phi)]}{\epsilon} \tag{1.20}$$

观察函数:

$$g\left(\phi(x)+\epsilon\delta(x-y)\right) \tag{1.21}$$

在 $x = y+\alpha$,α 是小量时:

$$g\left(\phi(x)+\epsilon\delta(x-y)\right) = g(\phi(x)) + \frac{dg}{d\phi}|_{\phi=\phi(y)}\epsilon\delta(x-y) \tag{1.22}$$

于是:

$$\frac{\delta F[g(\phi)]}{\delta\phi(y)} = \frac{dg}{d\phi}|_{\phi=\phi(y)}\frac{F[g+\frac{dg}{d\phi}|_{\phi=\phi(y)}\epsilon\delta(\cdot-y)] - F[g(\phi)]}{\frac{dg}{d\phi}|_{\phi=\phi(y)}\epsilon} \tag{1.23}$$

也就是:

$$\frac{\delta F[g(\phi)]}{\delta\phi(y)} = \frac{dg}{d\phi}|_{\phi=\phi(y)}\frac{\delta F[g]}{\delta g(\phi(y))} \tag{1.24}$$

∘3.

当

$$F[\phi] = \int dx(\phi(x))^n \tag{1.25}$$

泛函微分:

$$\begin{aligned}\frac{\delta F[\phi]}{\delta\phi(y)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int dx\left((\phi(x)+\epsilon\delta(x-y))^n - \phi(x)^n\right) \\ &= \int dx\; n(\phi(x))^{n-1}\delta(x-y) \quad (\epsilon\text{ 变小的速度是最快的}) \\ &= n(\phi(y))^{n-1}\end{aligned} \tag{1.26}$$

∘4.

$$\frac{\delta}{\delta\phi(y)}\int dx g(\phi(x)) = g'(\phi(y)) \tag{1.27}$$

∘5.

当

$$F[\phi] = \int dx (\frac{d\phi(x)}{dx})^n \tag{1.28}$$

此时:

$$\begin{aligned}\frac{\delta F[\phi]}{\delta\phi(y)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx \left(\frac{d}{dx}\left[\phi(x)+\epsilon\delta(x-y)\right]\right)^n - \int dx \left(\frac{d}{dx}\phi(x)\right)\right)\\ &= n\int dx \ (\frac{d\phi}{dx})^{n-1}\frac{d}{dx}\delta(x-y)\\ &= -n\frac{d}{dx}\left[\left(\frac{d\phi}{dx}\right)^{n-1}\right]|_y\end{aligned} \tag{1.29}$$

∘6.

泛函:

$$\begin{aligned}\frac{\delta}{\delta\phi(y)}\int dx \ h(\frac{d\phi}{dx}) &= \frac{1}{\epsilon}\left[\int dx \ h\left(\frac{d\phi}{dx}+\epsilon\frac{d}{dx}\delta(x-y)\right) - \int dx \ h(\frac{d\phi}{dx})\right]\\ &= \frac{1}{\epsilon}\left[\int dx \ h\left(\frac{d\phi}{dx}\right) + \int dx\frac{dh}{d\left(\frac{d\phi}{dx}\right)}\epsilon\frac{d}{dx}\delta(x-y) - \int dx \ h(\frac{d\phi}{dx})\right]\\ &= \int dx\frac{dh}{d\left(\frac{d\phi}{dx}\right)}\frac{d}{dx}\delta(x-y)\\ &= -\frac{d}{dx}\left(\frac{dh}{d(\frac{d\phi}{dx})}\right)|_y\end{aligned} \tag{1.30}$$

微元的角度的泛函求导　Lagrangian 是场和场的时间导数的泛函:

$$L = L[\phi, \dot{\phi}] \tag{1.31}$$

将空间划分成很多个小区域，每一个区域用 i 来标记, 定义:

$$\phi_i(t) = \frac{1}{\Delta V_i}\int_{\Delta V_i} d^3x\phi(x,t) \tag{1.32}$$

于是泛函变成了多变量的函数:

$$L = L(\phi_i, \dot{\phi}_i) \tag{1.33}$$

它的微分:

$$\delta L(\phi_i, \dot{\phi}_i) = \sum_i\left(\frac{1}{\Delta V_i}\frac{\delta L}{\delta\phi_i}\delta\phi_i + \frac{1}{\Delta V_i}\frac{\delta L}{\delta\dot{\phi}_i}\delta\dot{\phi}_i\right)\Delta V_i \tag{1.34}$$

由泛函微分的定义:

$$\delta L = \int dx \ \left(\frac{\delta L}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\dot{\phi}}\delta\dot{\phi}\right) \tag{1.35}$$

于是:

$$\frac{\delta L}{\delta\phi} = \lim_{\Delta V_i\to 0}\frac{1}{\Delta V_i}\frac{\delta L}{\delta\phi_i} \tag{1.36}$$

$$\frac{\delta L}{\delta\dot{\phi}} = \lim_{\Delta V_i\to 0}\frac{1}{\Delta V_i}\frac{\delta L}{\delta\dot{\phi}_i} \tag{1.37}$$

Euler-Lagrange 方程　作用量定义为 Lagrangian 对时间的积分:

$$W = \int dt \ L[\phi, \dot{\phi}] \tag{1.38}$$

最小作用量原理:

$$\begin{aligned}\delta W &= \int dt d^3x \left(\frac{\delta L}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\dot\phi}\delta\dot\phi\right)\\ &= \int dt d^3x \left(\frac{\delta L}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\dot\phi}\frac{d}{dt}(\delta\phi)\right)\\ &= \int dt d^3x \left(\frac{\delta L}{\delta\phi}\delta\phi - \frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi}\right)\delta\phi + \frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi}\delta\phi\right)\right)\\ &= \int dt d^3x \left(\frac{\delta L}{\delta\phi} - \frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi}\right)\right)\delta\phi\\ &= 0\end{aligned} \tag{1.39}$$

于是, Euler-Lagrange 方程 (场的演化方程):

$$\frac{\delta L}{\delta\phi} - \frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi}\right) = 0 \tag{1.40}$$

Lagrangian 的泛函导数和 Lagrangian 密度的关系–E-L 方程另一种形式 如果 Lagrangian 可以写成:

$$L[\phi,\dot\phi] = \int d^3x \mathcal{L}\left(\phi, \nabla\phi, \dot\phi\right) \tag{1.41}$$

其中,ϕ 是场，是空间的函数，并且场会随着时间演化。当场函数有变化时:

$$\delta L = \int d^3x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\delta\nabla\phi + \frac{\partial\mathcal{L}}{\partial\dot\phi}\delta\dot\phi\right) \tag{1.42}$$

考虑到:

$$\delta\nabla\phi = \nabla\delta\phi \tag{1.43}$$

与分部积分 (由高斯定理，散度部分变成面积分后成 0 了):

$$\begin{aligned}\delta L &= \int d^3x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\delta\phi\right) - \nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial\dot\phi}\delta\dot\phi\right)\\ &= \int d^3x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial\dot\phi}\delta\dot\phi\right)\end{aligned} \tag{1.44}$$

于是观察得到泛函导数是:

$$\frac{\delta L}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right) \tag{1.45}$$

$$\frac{\delta L}{\delta\dot\phi} = \frac{\partial\mathcal{L}}{\partial\dot\phi} \tag{1.46}$$

Euler-Langrange 方程变为:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right) - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right) = 0 \tag{1.47}$$

更加紧凑的形式是:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) = 0 \tag{1.48}$$

Hamilton Formalism 定义经典共轭场 (canonical conjugate field)

$$\pi(x,t) = \frac{\delta L}{\delta\dot\phi} = \lim_{\Delta V_i\to 0}\frac{1}{\Delta V_i}\frac{\delta L}{\delta\dot\phi_i} = \frac{\partial\mathcal{L}}{\partial\dot\phi} \tag{1.49}$$

由 Euler-Lagrange 方程:

$$\dot\pi(x,t) = \frac{\delta L}{\delta\phi} \tag{1.50}$$

定义 Hamiltonian:

$$H[\pi,\phi] = \int d^3x \left(\pi(x,t)\dot\phi(x,t)\right) - L(t) \tag{1.51}$$

则:

$$
\begin{aligned}
\delta H &= \int d^3x \left(\dot{\phi}\delta\pi + \pi\delta\dot{\phi} - \frac{\delta L}{\delta\phi}\delta\phi - \frac{\delta L}{\delta\dot{\phi}}\delta\dot{\phi}\right) \\
&= \int d^3x \left(\dot{\phi}\delta\pi + \pi\delta\dot{\phi} - \frac{\delta L}{\delta\phi}\delta\phi - \pi\delta\dot{\phi}\right) \\
&= \int d^3x \left(\dot{\phi}\delta\pi - \frac{\delta L}{\delta\phi}\delta\phi\right)
\end{aligned} \tag{1.52}
$$

这样做相当于做了变量代换:

$$
L[\phi, \dot{\phi}] \Rightarrow H[\pi, \phi] \tag{1.53}
$$

将 Euler-Lagrange 方程带入 $\frac{\delta L}{\delta\phi} - \frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\phi}}\right) = 0$，$\dot{\pi}(x,t) = \frac{\delta L}{\delta\phi}$ 于是 Hamilton 方程:

$$
\frac{\delta H}{\delta\pi} = \dot{\phi} \quad \frac{\delta H}{\delta\phi} = -\dot{\pi} \tag{1.54}
$$

满足 E-L 方程的 Hamiltonian 的变化是:

$$
\delta H = \int d^3x \left(\dot{\phi}\delta\pi - \dot{\pi}\delta\phi\right) \tag{1.55}
$$

π, ϕ 是随时间变换的场。于是 $H = H(t)$。现在考虑它随时间的变化。

$$
\delta H = \int d^3x \left(\dot{\phi}\dot{\pi} - \dot{\pi}\dot{\phi}\right) dt = 0 \tag{1.56}
$$

Hamiltonian 密度:

$$
\mathcal{H}(x,t) = \pi(x,t)\dot{\phi}(x,t) - \mathcal{L}(\phi, \nabla\phi, \dot{\phi}) \tag{1.57}
$$

它是这些的函数:

$$
\mathcal{H} = \mathcal{H}(\phi, \pi, \nabla\phi, \nabla\pi) \tag{1.58}
$$

用分部积分:

$$
\frac{\delta H}{\delta\phi} = \frac{\partial\mathcal{H}}{\partial\phi} - \nabla\left(\frac{\partial\mathcal{H}}{\partial(\nabla\phi)}\right) \tag{1.59}
$$

$$
\frac{\delta H}{\delta\pi} = \frac{\partial\mathcal{H}}{\partial\pi} - \nabla\left(\frac{\partial\mathcal{H}}{\partial(\nabla\pi)}\right) \tag{1.60}
$$

定义 Poission 括号: 如果 F 和 G 都是 ϕ 和 π 的泛函

$$
\{F, G\}_{P,B} = \int d^3x \left(\frac{\delta F}{\delta\phi}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\pi}\frac{\delta G}{\delta\phi}\right) \tag{1.61}
$$

可以发现:

$$
\dot{F} = \{F, H\}_{P,B} \tag{1.62}
$$

与:

$$
\{\phi(x), \pi(x')\}_{P,B} = \delta^3(x - x') \tag{1.63}
$$

1.2.2 Noether 定理

假设场有一个 lagrangian 密度可以表示为：$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ 我们可以先把他理解为一个 L 函数, 里面有五个变量，分别叫做 $\phi, \partial_\mu\phi$, 这五个量通过一个函数和时空变量 x 联系在了一起。也就是说 ϕ 和 $\partial_\mu\phi$ 可以看作是五个映射。而且这五个映射其实在确定了 ϕ 后剩下四个都确定了。

现在考虑场发生了一个变化，这个变化有两个方面。1. 坐标 $x \to x'$ 2. 函数 $\phi(x) \to \phi'(x')$, 变化就像这样：

$$
\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial_\mu\phi(x)) \tag{1.64}
$$

$$
\mathcal{L}'(x') = \mathcal{L}(\phi'(x'), \partial'_\mu\phi'(x')) \tag{1.65}
$$

命题 1.1 在这种变化下，在原本时空区域 R 中的作用量为 $\int_R d^4x\mathcal{L}(x)$, 在发生变化后, 对作用量的时空积分变为

了 R', 同时, 拉格朗日量密度也变化为 $\mathcal{L}'(x')$, 此时, 这一部分的作用量的差值可以写为:

$$\Delta S = \int_R d^4x \left\{ \partial_\mu \left(\mathcal{L}(\delta x)^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}\phi \right) + \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}) \right) \bar{\delta}\phi \right\} \tag{1.66}$$

证明

首先考察场坐标的变化, 可以简单的写为 $x' = x + \delta x$ 接下来考察场函数和场函数的偏导的变化, 首先看场函数的变化定义, 映射 ϕ 的变化写为:$\phi'(x) - \phi(x) = \bar{\delta}\phi(x)$.

$$\begin{aligned} \delta\phi = \phi'(x') - \phi(x) &= \phi'(x') - \phi'(x) + \phi'(x) - \phi(x) \\ &= \left(\partial_\mu \phi'(x)\right) \delta x^\mu + \bar{\delta}\phi(x) \\ &= \left(\partial_\mu \phi(x)\right) \delta x^\mu + \bar{\delta}\phi(x) \text{忽略二阶小量} \end{aligned} \tag{1.67}$$

这样, 相当于拉格朗日量密度中的第一个变量的改变就确定了, 接下来考虑 lagrangian density's change Because of the change of the $\partial_\mu \phi$. Thus, we need to calculate $\partial'_\mu \phi'(x') - \partial_\mu \phi(x)$.

$$\begin{aligned} \delta\partial_\mu \phi &= \partial'_\mu \phi' - \partial_\mu \phi \\ &= \partial'_\mu (\phi(x') + \bar{\delta}\phi(x')) - \partial_\mu \phi(x) \\ &= \partial_\mu \phi|_x^{x'} + \partial_\mu \bar{\delta}\phi(x) \\ &= \partial_\mu \partial_\rho \phi(x)(\delta x)^\rho + \partial_\mu \bar{\delta}\phi(x) \end{aligned} \tag{1.68}$$

Then, we consider the Jacobi determinant.$\mathcal{J} = det(\frac{\partial x'^\mu}{\partial x^\nu})$。(雅可比行列式的行和列指标分别由 μ,ν 决定) If we look close at Jacobi matrix, we can find that, it can be written in the form $I + A$, which is the first order of $exp(A)$, it is easy to find that $A = \frac{\partial \delta x^\mu}{\partial x^\nu}$.

There is a math theory says that det(exp(A))=exp(tr(A)). in this case, we can say that $\mathcal{J} = det(exp(A)) = exp(tr(A)) = 1 + tr(A) = 1 + \partial_\mu \delta x^\mu$.

Then, we come to the difference of the action:

$$\begin{aligned} \Delta S &= \int_R d^4x \left[\left(1 + \partial_\mu \delta x^\mu\right) \left\{ \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \left(\bar{\delta}\phi + (\partial_\mu \phi)\delta x^\mu\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left((\partial_\mu \partial_\rho \phi)\delta x^\rho + \partial_\mu \bar{\delta}\phi\right) \right\} - \mathcal{L} \right] \\ &= \int_R d^4x \left\{ (\partial_\mu \delta x^\mu)\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}(\partial_\mu \phi)\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\mu \partial_\rho \phi)\delta x^\rho + \frac{\partial \mathcal{L}}{\partial \phi}\bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \bar{\delta}\phi \right\} \text{ 忽略了二阶小量} \\ &= \int_R d^4x \left\{ (\partial_\mu \delta x^\mu)\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}(\partial_\mu \phi)\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\mu \partial \rho \phi)\delta x^\rho + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}) \right] \bar{\delta}\phi + \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}\phi) \right\} \\ &= \int_R d^4x \left\{ \partial_\mu (\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}\phi) + (\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}))\bar{\delta}\phi \right\} \end{aligned} \tag{1.69}$$

we noticed that in most of the fields, the second term will be zero. If we think that $\Delta S = 0$, then, the first term will be zero (because the choose of R is no limit).

在这里, 坐标都是 x^μ 的形式, 当然也可以用度归算符进行升降指标。同时, 一般把第一项里面的 $\bar{\delta}\phi$ 写成:

$$\bar{\delta}\phi = \delta\phi - \left(\partial_\lambda \phi(x)\right) \delta x^\lambda \tag{1.70}$$

于是, Current density (流密度) j^μ 可以写为:

$$j^\mu = \mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\delta\phi_r - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \left(\partial_\lambda \phi_r(x)\right) \delta x^\lambda. \tag{1.71}$$

因为积分对任何坐标变动和场的变动都为 0, 于是被积函数处处为 0。因为上面积分的第二项的 (integrand) 处处为零, 所以第一项的 integrand 也要处处为零。连续方程 (Equation of Continuity) 可以写为:

$$\partial_\mu j^\mu = 0. \tag{1.72}$$

□

Noether 定律的表述原文: Each continuous symmetry transformation leads to a conservation law. The conserved quantity G can be obtained from the Lagrange density。

可见，它的关键在于连续对称变化 (Continuous Symmetry Transformation)。

有一个问题，为什么作用量不变意味着"物理规律不变 or 存在对称性"

连续方程 (Equation of continuity) 等价于守恒定律 (Conservation law): 对连续方程 (Equation of Continuity) 进行空间部分的积分:

$$\int d^3x \partial_0 j^0 + \int d^3x \nabla \cdot \mathbf{j} = 0 \tag{1.73}$$

利用 Gauss 定理:

$$\int d^3x \partial_0 j^0 + \oiint d\mathbf{s} \cdot \mathbf{j} = 0 \tag{1.74}$$

因为在边界上被积函数 (integrand) 是 0, 所以面积分自然是 0 于是连续方程说明:

$$G \equiv \int d^3x j^0 \tag{1.75}$$

是不随时间变化的。这个量不随时间变化，是守恒量 (Conserved Quantity)。

时空平移对称性对应能动量守恒 时空平移对称性的坐标变换:

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} \tag{1.76}$$

场的形状 (shape of the fields) 是不改变的:

$$\phi'_r(x') = \phi_r(x) \Rightarrow \delta\phi_r = 0 \tag{1.77}$$

Current density 是:

$$\begin{aligned} j^{\mu} &= \mathcal{L}\delta x^{\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\lambda \phi(x))\,\delta x^{\lambda} \\ &= \mathcal{L}\epsilon^{\mu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\lambda \phi(x))\,\epsilon^{\lambda} \end{aligned} \tag{1.78}$$

如果只有 $\epsilon^{\nu} \neq 0$, 则:(第二项对 ν 并没有求和约定, 第一项对 ν 指标有求和约定)

$$j^{\mu} = \mathcal{L}\delta^{\mu}_{\nu}\epsilon^{\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi(x))\,\epsilon^{\nu} \tag{1.79}$$

可以定义 (有时候会用到 $\delta^{\mu}_{\nu} = g^{\mu}{}_{\nu}$):

$$\Theta^{\mu}_{\nu} = -\mathcal{L}g^{\mu}{}_{\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi(x)) \tag{1.80}$$

把 ν 变到上指标:

$$\Theta^{\mu\nu} = -\mathcal{L}g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial^{\nu} \phi(x)) \tag{1.81}$$

满足:

$$\partial_\mu \Theta^{\mu\nu} = 0 \tag{1.82}$$

有四个守恒量, 写为:

$$P^{\nu} = \left(\frac{E}{c}, \mathbf{P}\right) = \frac{1}{c}\int_{v} d^3x\, \Theta^{0\nu} \tag{1.83}$$

Θ 的下指标形式:

$$\Theta_{\mu\nu} = -\mathcal{L}g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi)}(\partial_\nu \phi) \tag{1.84}$$

Lorentz 不变性对应角动量和自旋守恒 Lorentz 变换下的坐标变化:

$$x'^{\mu} = x^{\mu} + \delta w^{\mu\nu} x_{\nu} \tag{1.85}$$

为了满足四维空间中的距离不变,δw 是反对称的 ($\delta w^{\mu\nu} = -\delta w^{\nu\mu}$), 这是因为:

$$\begin{aligned} x'^{\mu}x'_{\mu} &= (x^{\mu} + \delta w^{\mu\sigma}x_{\sigma})(x_{\mu} + \delta w_{\mu}^{\ \tau}x_{\tau}) \\ &= x^{\mu}x_{\mu} + \delta w^{\mu\nu}x_{\nu}x_{\mu} + \delta w^{\mu\nu}x_{\mu}x_{\nu} \\ &= x^{\mu}x_{\mu} + (\delta w^{\mu\nu} + \delta w^{\nu\mu})\, x_{\mu}x_{\nu} \\ &= x^{\mu}x_{\mu} \end{aligned} \tag{1.86}$$

在 Lorentz 变换下, 场的变换是:

$$\phi'_r(x') = \phi_r(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})_{rs}\phi_s(x) \tag{1.87}$$

这里的 $I^{\mu\nu}$ 是 Lorentz 群表示的生成元,$r\ s$ 是生成元矩阵的行列指标。生成元 $I^{\mu\nu}$ 对于 Lorentz 指标是反对成的, 也就是 $I^{\mu\nu} = -I^{\nu\mu}$。

于是, 对于 Current density 是:

$$\begin{aligned} j^{\mu} &= \mathcal{L}\delta x^{\mu} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\left(\partial_{\lambda}\phi(x)\right)\delta x^{\lambda} \\ &= \mathcal{L}\delta w^{\mu\nu}x_{\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\frac{1}{2}\delta w_{\sigma\nu}(I^{\sigma\nu})_{rs}\phi_s - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\left(\partial_{\lambda}\phi_r(x)\right)\delta w^{\lambda\nu}x_{\nu} \end{aligned} \tag{1.88}$$

他的下指标形式:

$$\begin{aligned} j_{\mu} &= \mathcal{L}\ g_{\mu\nu}\delta w^{\nu\lambda}x_{\lambda} + \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{1}{2}\delta w_{\sigma\nu}(I^{\sigma\nu})_{rs}\phi_s - \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi_r)}\left(\partial_{\nu}\phi_r\right)\delta w^{\nu\lambda}x_{\lambda} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{1}{2}\delta w_{\sigma\nu}(I^{\sigma\nu})_{rs}\phi_s - \left(-\mathcal{L}\ g_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi_r)}\left(\partial_{\nu}\phi_r\right)\right)\delta w^{\nu\lambda}x_{\lambda} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{1}{2}\delta w_{\sigma\nu}(I^{\sigma\nu})_{rs}\phi_s - \Theta_{\mu\nu}\delta w^{\nu\lambda}x_{\lambda} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{1}{2}\delta w_{\sigma\nu}(I^{\sigma\nu})_{rs}\phi_s - \Theta_{\mu\lambda}\delta w^{\lambda\nu}x_{\nu} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{1}{2}\delta w_{\sigma\nu}(I^{\sigma\nu})_{rs}\phi_s - \frac{1}{2}\delta w^{\nu\lambda}\left(\Theta_{\mu\nu}x_{\lambda} - \Theta_{\mu\lambda}x_{\nu}\right) \text{因为 w 矩阵是反对称的} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{1}{2}\delta w^{\nu\lambda}(I_{\nu\lambda})_{rs}\phi_s - \frac{1}{2}\delta w^{\nu\lambda}\left(\Theta_{\mu\nu}x_{\lambda} - \Theta_{\mu\lambda}x_{\nu}\right) \\ &= \frac{1}{2}\delta w^{\nu\lambda}M_{\mu\nu\lambda} \end{aligned} \tag{1.89}$$

其中:

$$M_{\mu\nu\lambda} = \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}(I_{\nu\lambda})_{rs}\phi_s - \left(\Theta_{\mu\nu}x_{\lambda} - \Theta_{\mu\lambda}x_{\nu}\right) \tag{1.90}$$

定义反对称 (Anti-Symmetric) 张量 (逆变指标下的守恒荷):

$$M_{\nu\lambda} = \int d^3x\left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}(I_{\nu\lambda})_{rs}\phi_s - (\Theta_{0\nu}x_{\lambda} - \Theta_{0\lambda}x_{\nu})\right) \tag{1.91}$$

对于 $\nu\lambda$ 的空间部分, 注意到:

$$\Theta_{\mu\nu} = -\mathcal{L}g_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\left(\partial_{\nu}\phi_r\right) \tag{1.92}$$

那么:

$$\Theta_{0\nu}x_{\lambda} = \left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi_r)}\left(\partial_{\nu}\phi_r\right) - \mathcal{L}g_{0\nu}\right)x_{\lambda} \tag{1.93}$$

$$\Theta_{0\lambda}x_{\nu} = \left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi_r)}\left(\partial_{\lambda}\phi_r\right) - \mathcal{L}g_{0\lambda}\right)x_{\nu} \tag{1.94}$$

这个时候:

$$\begin{aligned} M_{nl} &= \int d^3x\left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}(I_{nl})_{rs}\phi_s - (\Theta_{0n}x_l - \Theta_{0l}x_n)\right) \\ &= \int d^3x\left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}(I_{nl})_{rs}\phi_s - \left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi_r)}\left(\partial_n\phi_r\right) - \mathcal{L}g_{0n}\right)x_l + \left(\frac{\partial\mathcal{L}}{\partial(\partial^0\phi_r)}\left(\partial_l\phi_r\right) - \mathcal{L}g_{0l}\right)x_n\right) \end{aligned} \tag{1.95}$$

一般写为:

$$M_{nl} = L_{nl} + S_{nl} \tag{1.96}$$

其中:

$$\begin{aligned} L_{nl} &= \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} (\partial_l \phi_r) x_n - \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} (\partial_n \phi_r) x_l \right) \\ &= \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} \left(x_n \frac{\partial}{\partial x^l} - x_l \frac{\partial}{\partial x^n} \right) \phi_r \end{aligned} \tag{1.97}$$

$$S_{nl} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi)} (I_{nl})_{rs} \phi_s \tag{1.98}$$

Internal Symmetries 这个时候没有坐标变换, 场的变换是:

$$\phi'_r(x) = \phi_r(x) + i\epsilon \sum_s \lambda_{rs} \phi_s(x) \tag{1.99}$$

$$\delta \phi_r = i\epsilon \sum_s \lambda_{rs} \phi_s(x) \tag{1.100}$$

守恒流是:

$$\begin{aligned} j_\mu &= \mathcal{L}\, \delta x_\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta \phi - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} (\partial_\lambda \phi(x))\, \delta x^\lambda \\ &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta \phi \\ &= i \sum_{rs} \epsilon \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)} \lambda_{rs} \phi_s \end{aligned} \tag{1.101}$$

守恒荷是:

$$Q = j^0 = i \sum_{rs} \epsilon \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_r)} \lambda_{rs} \phi_s = i \sum_{rs} \epsilon \pi_r \lambda_{rs} \phi_s \tag{1.102}$$

一个特殊的情况是对于相位改变:

$$\phi' = \phi + i\epsilon\phi \tag{1.103}$$

$$\phi'^* = \phi^* - i\epsilon\phi^* \tag{1.104}$$

将 (ϕ_1, ϕ_2) 分别叫作 (ϕ_1, ϕ_2), 于是 $\lambda_{11} = 1, \lambda_{22} = -1$。于是:

$$j_\mu = (-i) \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \phi - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} \phi^* \right) \tag{1.105}$$

Symmetrized Energy-Momentum Tensor: 考虑修正过后的能动量张量 (Modified Tensor)

$$T_{\mu\nu} = \Theta_{\mu\nu} + \partial^\sigma \chi_{\sigma\mu\nu} \tag{1.106}$$

为了保证修正后的能动量张量满足守恒流方程, 要求 $\chi_{\sigma\mu\nu} = -\chi_{\mu\sigma\nu}$, 这是因为:

$$\begin{aligned} \partial^\mu T_{\mu\nu} &= \partial^\mu \Theta_{\mu\nu} + \partial^\mu \partial^\sigma \chi_{\sigma\mu\nu} \\ &= \partial^\mu \Theta_{\mu\nu} + \frac{1}{2} \partial^\mu \partial^\sigma \left(\chi_{\sigma\mu\nu} + \chi_{\mu\sigma\nu} \right) \\ &= \partial^\mu \Theta_{\mu\nu} = 0 \end{aligned} \tag{1.107}$$

1.3 量子力学基础

1.3.1 不同的绘景 Picture

1.3.1.1 Schrodinger Picture

总结 Schrodinger 绘景的一些性质:

1) 态矢满足 Schrodinger 方程 $\hat{H}(\hat{q},\hat{p})|\Psi(t)>^S = i\hbar\frac{\partial}{\partial t}|\Psi(t)>^S$
2) 定义坐标本征态 $\hat{q}|q>=q|q>$
3) 定义动量本征态 $\hat{p}|p>=p|p>$
4) 坐标本征态正交 $<q'|q>=\delta(q'-q)$
5) 动量本征态正交 $<p'|p>=2\pi\hbar\delta(p'-p)$
6) $\int dp\frac{1}{2\pi\hbar}|p\rangle\langle p|=1$
7) $\int dq|q\rangle\langle q|=1$
8) $<q|p>=e^{i\frac{pq}{\hbar}}$ (详细见 sakurai 笔记，前面没有 normalize factor 是因为动量的正交关系里面有了 factor)
9) $<q|\Psi(t)>^S=\psi(q,t)$
10) Schrodinger 方程的解可以写成: $|\Psi(t)>^S = e^{-i\hat{H}t/\hbar}|\Psi(0)>^S$

证明 可以用上面的 (5) 推到 (6):(证明归一化可以用正交性推出来)

首先考虑积分:

$$(\int dp|p\rangle\langle p|)^2=\int dpdp'|p'\rangle\langle p'|p\rangle\langle p|=\int dp'dp\ \ |p'\rangle 2\pi\hbar\delta(p'-p)\langle p|=\int dp\ 2\pi\hbar\ |p\rangle\langle p| \tag{1.108}$$

上面的式子仅仅在第二个等号用到了 5)。可以推导出 6)

$$\begin{aligned}2\pi\hbar&=\int dp|p\rangle\langle p|\\ \int dp\frac{1}{2\pi\hbar}|p\rangle\langle p|&=1\end{aligned} \tag{1.109}$$

同样的, 这个方法也可以从 (4) 推到 (7)。

1.3.1.2 Heisenberg Picture

在 Schrodinger 绘景中，态矢量演化:$|\alpha,t\rangle=U(t,t_0)|\alpha,t_0\rangle$, 算符是 $A^S(t)$. 演化算符满足 Schrodinger 方程:

$$i\hbar\partial_t U(t,t_0)=H^S U(t,t_0). \tag{1.110}$$

其中,Hamiltonian 是厄米算符, U 是幺正算符. 也可以将态矢定住不变，而改变算符，也就是:$|\alpha,t\rangle^H=|\alpha,t_0\rangle^H=|\alpha,t_0\rangle^S$. 同时算符会发生额外的演化:$A^H(t)=U^\dagger(t,t_0)A^S(t)U(t,t_0)$.

如果 A^S 不含时间, 那么 Heisenberg 绘景中的力学量的演化满足:

$$\begin{aligned}i\hbar\partial_t A^H(t)&=U^\dagger(t,t_0)A^S U(t,t_0)\\&=-U^\dagger(t,t_0)H_S A^S U(t,t_0)+U^\dagger(t,t_0)A^S H_S U(t,t_0)\\&=-U^\dagger H_S UU^\dagger A^S U+U^\dagger A^S UU^\dagger H_S U\\&=-H^H A^H+A^H H^H\\&=[A^H,H^H]\end{aligned} \tag{1.111}$$

1.3.1.3 Interaction Picture 相互作用绘景

当系统的 Hamiltonian 可以写为:

$$H^S=H_0^S+H_1^S \tag{1.112}$$

其中,H_0^S **不包含时间项** (所以在薛定谔绘景以及相互作用绘景中是同样的算符), H_1^S 包含时间项。在 Schrodinger 绘景中，态矢量演化满足

$$|\alpha,t\rangle^S = W(t,t_0)|\alpha,t_0\rangle. \tag{1.113}$$

. 其中,$W(t,t_0)$ 满足运动方程.

$$i\hbar\frac{\partial}{\partial t}W(t,t_0) = H^S W(t,t_0) \tag{1.114}$$

相互作用绘景中的算符及其演化:

定义相互作用绘景中的态矢量:

$$\begin{aligned}|\alpha,t\rangle^I &= U_0(t,t_0)^\dagger|\alpha,t\rangle^S \\ &= U_0(t,t_0)^\dagger W(t,t_0)|\alpha,t_0\rangle,\end{aligned} \tag{1.115}$$

其中，$U(t,t_0)$ 是没有相互作用时 Schrodinger Pic 中的时间演化算符。

U 满足演化方程:

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H_0^S U(t,t_0) \tag{1.116}$$

为了保证期望值不变, 在这个绘景中, 力学量算符应该满足要求:

$$\begin{aligned}\langle\alpha,t|^I A^I(t)|\alpha,t\rangle^I &= \langle\alpha,t|^S A^S(t)|\alpha,t\rangle^S \\ &= \langle\alpha,t|^I U^\dagger(t,t_0)A^S(t)U(t,t_0)|\alpha,t\rangle^I\end{aligned} \tag{1.117}$$

于是:

$$A^I(t) = U^\dagger(t,t_0)A^S(t)U(t,t_0) \tag{1.118}$$

如果 A^S 和时间无关，$A^I(t)$ 满足 **Heisenberg 运动方程**:

$$i\hbar\partial_t A^I(t) = [A^I, H_0^I] = [A^I, H_0^S] \tag{1.119}$$

相互作用绘景中的态矢量及其演化:

定义相互作用绘景中的态矢量:

$$\begin{aligned}|\alpha,t\rangle^I &= e^{-\frac{H_0^S(t-t_0)}{i\hbar}}|\alpha,t\rangle^S \\ &= e^{-\frac{H_0^S(t-t_0)}{i\hbar}}W(t,t_0)|\alpha,t_0\rangle.\end{aligned} \tag{1.120}$$

随时间演化:

$$\begin{aligned}i\hbar\partial_t|\alpha,t\rangle^I &= -H_0^S|\alpha,t\rangle^I + e^{-\frac{H_0^S(t-t_0)}{i\hbar}}H^S W(t,t_0)|\alpha,t_0\rangle \\ &= (-e^{-\frac{H_0^S(t-t_0)}{i\hbar}}H_0^S e^{\frac{H_0^S(t-t_0)}{i\hbar}} + e^{-\frac{H_0^S(t-t_0)}{i\hbar}}H^S e^{\frac{H_0^S(t-t_0)}{i\hbar}})|\alpha,t\rangle^I \\ &= (-H_0 + H^I)|\alpha,t\rangle^I \\ &= H_1^I|\alpha,t\rangle^I\end{aligned} \tag{1.121}$$

定义时间演化算符:

$$|\alpha,t\rangle^I = U(t,t_0)|\alpha,t_0\rangle \tag{1.122}$$

演化算符满足演化方程:

$$i\hbar\partial_t U(t,t_0) = H_1^I U(t,t_0) \tag{1.123}$$

Heisenberg 绘景和 Interacting 绘景之间的联系

通过 Interacting 绘景的演化算符得到**态矢量**之间的关系:

$$|\alpha,t\rangle^I = U(t,t_0)|\alpha,t_0\rangle = U(t,t_0)|\alpha,t\rangle^H \tag{1.124}$$

对于算符，为了保证期望值在两个绘景中相同:

$$\begin{aligned}\langle\alpha,t|^I A^I|\alpha,t\rangle^I &= \langle\alpha,t|^H A^H|\alpha,t\rangle^H \\ &= \langle\alpha,t|^I U(t,t_0)A^H U^\dagger(t,t_0)|\alpha,t\rangle^I\end{aligned} \tag{1.125}$$

于是**算符之间的关系**:

$$A^I = U(t,t_0)A^H U^\dagger(t,t_0) \tag{1.126}$$

Dyson 级数求解相互作用绘景中的演化算符

相互作用绘景中的演化算符满足演化方程:

$$i\frac{\partial}{\partial t}U(t,t_0) = H_1^I(t)U(t,t_0) \tag{1.127}$$

他的解一定满足积分式:

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} H_1^I(t)U(t,t_0)dt \tag{1.128}$$

可以进行猜解

$$U(t,t_0) = 1 + \Sigma_{n=1}(-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n H_1^I(t_1)...H_1^I(t_n) \tag{1.129}$$

证明 上面的级数有个名字叫做 Dyson 级数，可以直接带入证明:

$$\begin{aligned}&1 + (-i)\int_{t_0}^{t_f} H_1^I(t)U(t,t_0)dt \\ &= 1 + (-i)\int_{t_0}^{t_f} dt H_1^I(t)\left[1 + \Sigma_{n=1}(-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n H_1^I(t_1)...H_1^I(t_n)\right] \\ &= 1 + (-i)\int_{t_0}^{t_f} dt H_1^I(t) \\ &+ \Sigma_{n=1}(-i)^{n+1}\int_{t_0}^{t_f} dt\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n H_1^I(t)H_1^I(t_1)...H_1^I(t_n) \\ &= U(t_f,t_0)\end{aligned} \tag{1.130}$$

□

如果引入时序算符，这个算符是让被作用的函数按照时间顺序排列。可以把时间演化算符写成:

$$\begin{aligned}U(t,t_0) &= \Sigma_{n=0}^{+\infty}\frac{(-i)^n}{n!}\int_t^{t_0} dt_1 ... \int_t^{t_0} dt_n T\left(H_1^I(t_1)...H_1^I(t_n)\right) \\ &= Texp[-i\int_t^{t_0} dt' H_1^I(t')] \\ &= Texp[-i\int_t^{t_0} dt'\int d^3x \mathcal{H}_1^I(x)]\end{aligned} \tag{1.131}$$

1.3.1.3.1 标量场 如果规定在 $t_0, O^H(t_0) = O^I(t_0) = O^S$，在 Heisenber Pic 和 Interaction Pic 之间的算符的变换是。$V(t,t_0) = e^{iH_0(t-t_0)}W(t,t_0)$ 这样的 $V(t,t_0)$ 也是 $U(t,t_0)$ 的一个定义 (因为他们都可以让 Interact Pic 中的态矢演化)。emmm.

这样，Heisenberg Pic 算符和 Interact Pic 算符之间的关系就是

$$U(t,t_0)^\dagger O^I(t)U(t,t_0) = O^H(t) \tag{1.132}$$

现在说一个事情就是我们习以为常的量子标量场的解 (就是写成产生算符和湮灭算符的形式)，他是在没有势能 (相互作用) 的时候的解。我们现在说这个解是 $\phi_0(x)$，是 Interacting Pic 下的算符，那么可以转换为 Heisenberg Pic 下的算符通过上面的式子。这个在 Schwartz 的 P87

$$\phi(x) = U^\dagger(t,t_0)\phi_0(x)U(t,t_0) \tag{1.133}$$

$$|-\rangle_H = U^\dagger(t,t_0)|-\rangle_I \tag{1.134}$$

我们现在说真空态, 这个态矢在 Interacting Pic 下面叫做 $|0\rangle$, 在 Heisenberg Pic 下叫做 $|\Omega\rangle$。
这两个态矢的联系按照上面的式子应该叫做:

$$|\Omega\rangle = U^\dagger(t,t_0)|0\rangle = U(t_0,t)|0\rangle \tag{1.135}$$

$$\langle\Omega| = \langle 0|U(t,t_0) \tag{1.136}$$

现在考虑 n-point Function:

$$\begin{aligned}
&\langle\Omega|\phi(x_1)\phi(x_2)...\phi(x_n)|\Omega\rangle \\
&= \langle 0|U(t,t_0)U(t_0,t_1)\phi_0(x_1)U(t_1,t_0)U(t_0,t_2)...U(t_0,t_n)\phi_0(x_n)U(t_n,t_0)U(t_0,t)|0\rangle \\
&= \langle 0|U_{t,1}\phi_0(x_1)U_{1,2}...U_{n-1,n}\phi_0(x_n)U_{n,t}|0\rangle
\end{aligned} \tag{1.137}$$

现在 Schwartz P88 有一个操作, 就是说左边的 t 应该是 $+\infty$, 右边的 t 叫做 $-\infty$, 这样左右的真空不是同一时间的?...

$$\begin{aligned}
&\langle 0|U_{+\infty,1}\phi_0(x_1)U(t_1,t_0)U(t_0,t_2)...U(t_0,t_n)\phi_0(x_n)U(t_n,t_0)U(t_0,t)|0\rangle \\
&= \langle 0|U_{+\infty,1}\phi_0(x_1)U_{1,2}...U_{n-1,n}\phi_0(x_n)U_{n,-\infty}|0\rangle
\end{aligned} \tag{1.138}$$

然后, 不知道怎么地，反正就是归纳出了一个结论是这样的：主要是不知道为什么 U 可以被提出来

$$\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle = \langle 0|T\{\phi_0(x_1)...\phi_0(x_n)U_{+\infty,-\infty}\}|0\rangle \tag{1.139}$$

然后为了归一化，一般是这样来写 N-POINT function

$$\begin{aligned}
\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle &= \frac{\langle 0|T\{\phi_0(x_1)...\phi_0(x_n)U_{+\infty,-\infty}\}|0\rangle}{\langle 0|U_{+\infty,-\infty}|0\rangle} \\
&= \frac{\langle 0|T\{\phi_0(x_1)...\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{int}(\phi_0)}\}|0\rangle}{\langle 0|e^{i\int d^4x\mathcal{L}_{int}(\phi_0)}|0\rangle}
\end{aligned} \tag{1.140}$$

第二章 Spin-0 Field

自然单位制

由于 $[\hbar][T]^{-1}=[E]=[m][L]^2[T]^{-2}$

$$\begin{cases}[\hbar]=[m][L]^2[T]^{-1}\\ [c]=[L][T]^{-1}\end{cases}$$

任何单位可写为：

$$[m]^{\alpha}[L]^{\beta}[T]^{\gamma}=[m]^{x}[\hbar]^{y}[c]^{z}.$$

取 $[\hbar]=[c]=1$

则 $[m]=[E]=[L]^{-1}$

可用 $[m]^{x}$ 来表示所有物理量的单位.

$g=(1,-1,-1,-1)$

Non-charged Klein-Gordon Field quantization

- Klein-Gordon Field 的 Hamiltonian 导出

$$\mathcal{L}(\phi,\dot\phi,\nabla\phi)=\frac{1}{2}\hbar^2\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x^\mu}-\frac{1}{2}m^2c^2\phi^2\ ;\ L=L[\phi,\dot\phi] \qquad -(1)$$

Euler-Lagrange 方程

$$\frac{\partial\mathcal{L}}{\partial\phi}=\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right) \qquad -(2)$$

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right|\ (\Box+m^2)\phi(x)=0 \qquad \Box=\partial^\mu\partial_\mu$$

正则共轭场:(自然单位制 $\hbar=c=1$)

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial\dot\phi}=\dot\phi$$

Hamiltonian 密度:

$$\mathcal{H}=\pi\dot\phi-\mathcal{L}=\frac{1}{2}\left(\pi^2+(\nabla\phi)^2+m^2\phi^2\right)$$

Hamiltonian

$$H=\int d^3x\cdot\frac{1}{2}\left(\pi^2(x)+(\nabla\phi(x))^2+m^2\phi^2(x)\right)=H[\pi,\phi]$$

- 量子化

$$\hat H=\int d^3x\ \frac{1}{2}\left(\hat\pi(\vec x,t)^2+(\nabla\hat\phi(\vec x,t))^2+m^2\hat\phi(\vec x,t)^2\right)$$

Equal Time Commutation Relation $[\hat\phi(\vec x,t),\hat\pi(\vec x',t)]=i\delta^{(3)}(\vec x-\vec x')$

$$[\hat\phi(\vec x,t),\hat\phi(\vec x',t)]=[\hat\pi(\vec x,t),\hat\pi(\vec x',t)]=0$$

Hamilton 运动方程:

$$\frac{\partial}{\partial t}\hat\phi(\vec x,t)=-i[\hat\phi(\vec x,t),\hat H]=\hat\pi(\vec x,t)$$

$$\frac{\partial}{\partial t}\hat\pi(\vec x,t)=-i[\hat\pi(\vec x,t),\hat H]=(\nabla^2-m^2)\hat\phi(\vec x,t)$$

$$\frac{\partial^2}{\partial t^2}\hat\phi(\vec x,t)=(\nabla^2-m^2)\hat\phi(\vec x,t)$$

场按基底展开 $\omega_p=\sqrt{\vec p^2+m^2}$; $-\omega_p^2\,u_p(\vec x)=(\nabla^2-m^2)u_p(\vec x)$ (不含时解) $u_p(\vec x)=N_p e^{i\vec p\cdot\vec x}$ $N_p=\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}$

$$\hat\phi(\vec x,t)=\int d^3p\cdot\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}\cdot e^{i\vec p\cdot\vec x}\ \hat a_p(t)=\int d^3p\cdot u_p(\vec x)\cdot\hat a_p(t)=\int d^3p\cdot N_p\cdot e^{i\vec p\cdot\vec x}\ \hat a_p(t) \qquad -(1)$$

完备条件:

$$\int d^3x\cdot\frac{1}{\sqrt{2(\vec p'^2+m^2)^{1/2}(2\pi)^3}}\ e^{-i\vec p'\cdot\vec x}\ \hat\phi(\vec x,t)=\int d^3x\frac{1}{\sqrt{2(\vec p'^2+m^2)^{1/2}(2\pi)^3}}\cdot e^{-i\vec p'\cdot\vec x}\cdot\int d^3p\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}\cdot e^{i\vec p\cdot\vec x}\cdot\hat a_p(t)$$

$$=\int d^3x\,d^3p\ e^{i(\vec p-\vec p')\cdot\vec x}\ \frac{1}{2\ (2\pi)^3}\cdot\frac{1}{(\vec p'^2+m^2)^{1/4}(\vec p^2+m^2)^{1/4}}\ \hat a_p(t)$$

$$=\int d^3p\ \frac{1}{2}\ \frac{1}{(\vec p'^2+m^2)^{1/4}(\vec p^2+m^2)^{1/4}}\ \delta^{(3)}(\vec p-\vec p')\ \hat a_p(t)$$

$$=\frac{1}{2}\ \frac{1}{(\vec p'^2+m^2)^{1/2}}\cdot\hat a_{p'}(t)$$

$$\hat a_{p'}(t)=\sqrt{2\cdot(\vec p'^2+m^2)^{1/2}}\ \frac{1}{(2\pi)^{3/2}}\ \int d^3x\cdot e^{-i\vec p'\cdot\vec x}\ \hat\phi(\vec x,t) \qquad -(2)$$

可用(1)的形式来表示 $\hat\phi(\vec x,t)=\int d^3p\ \frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}\ e^{i\vec p\cdot\vec x}\cdot\left(\sqrt{2(\vec p^2+m^2)^{1/2}}\ \frac{1}{(2\pi)^{3/2}}\int d^3x'\cdot e^{-i\vec p\cdot\vec x'}\ \hat\phi(\vec x',t)\right)$

$$=\int d^3p\cdot d^3x'\ \frac{1}{(2\pi)^3}\cdot e^{i\vec p\cdot(\vec x-\vec x')}\ \hat\phi(\vec x',t)$$

$$=\int d^3x'\ \ \delta^{(3)}(x-x')\,\hat\phi(\vec x',t)$$

$$=\hat\phi(\vec x,t) \qquad -(3)$$

说明了(1)的展开形式合理!完备性: $\int d^3p\ e^{i\vec p\cdot\vec x}\ e^{-i\vec p\cdot\vec x'}=(2\pi)^3\ \delta^{(3)}(x-x')$

○ $\hat{a}_p(t)$ 的运动方程

$$\ddot{\hat{a}}_p(t) = -(\vec{p}^2+m^2)\hat{a}_p(t)$$

解

$$\hat{a}_p(t) = \hat{a}_p^{(1)}e^{-iW_pt} + \hat{a}_p^{(2)}e^{iW_pt} \qquad W_p=\sqrt{\vec{p}^2+m^2}$$

由于是实数场 $\phi^*=\phi$，ϕ 是 Hermitian $\phi^\dagger=\phi$

$$\hat{\phi}(\vec{x},t) = \int d^3p \frac{1}{\sqrt{2\cdot W_p(2\pi)^3}}\cdot\left(\hat{a}_p^{(1)}e^{i(\vec{p}\cdot\vec{x}-W_pt)} + \hat{a}_p^{(2)}e^{i(\vec{p}\cdot\vec{x}+W_pt)}\right)$$

$$= \int d^3p \frac{1}{\sqrt{2\cdot W_p(2\pi)^3}}\cdot\left(\hat{a}_p^{(1)}e^{i(\vec{p}\cdot\vec{x}-W_pt)} + \hat{a}_{-p}^{(2)}e^{i(-\vec{p}\cdot\vec{x}+W_pt)}\right)$$

$$(\hat{a}_p^{(1)})^\dagger = \hat{a}_{-p}^{(2)}$$

$$\hat{\phi}(x,t) = \int d^3p \frac{1}{\sqrt{2W_p(2\pi)^3}}\left(\hat{a}_p e^{i(\vec{p}\cdot\vec{x}-W_pt)} + \hat{a}_p^\dagger e^{-i(\vec{p}\cdot\vec{x}-W_pt)}\right)$$

$$\hat{\pi}(x,t) = \frac{\partial}{\partial t}\hat{\phi}(x,t) = \int d^3p \frac{1}{\sqrt{2W_p(2\pi)^3}}\cdot(-iW_pt)\left(\hat{a}_p e^{i(\vec{p}\cdot\vec{x}-W_pt)} - \hat{a}_p^\dagger e^{-i(\vec{p}\cdot\vec{x}-W_pt)}\right)$$

○ $\hat{a}_p$ 的对易关系：

$$[\hat{a}_p,\hat{a}_{p'}^\dagger] = \delta^{(3)}(\vec{p}-\vec{p}') \qquad [\hat{a}_p,\hat{a}_{p'}] = [\hat{a}_p^\dagger,\hat{a}_{p'}^\dagger]=0$$

○ 定义 $U_p(x,t) = N_p e^{-ip\cdot x} = \frac{1}{\sqrt{2W_p(2\pi)^3}}e^{-i(W_pt-p\cdot x)}$

$$(\partial_0^2-\nabla^2+m^2)U_p(x,t)=0$$

• 此时：

正能量 指 $-iWt$

$$\hat{\phi}(x,t) = \int d^3p\left(\hat{a}_p U_p(x,t) + \hat{a}_p^\dagger U_p^*(x,t)\right) = \hat{\phi}^{(+)}(x,t) + \hat{\phi}^{(-)}(x,t)$$

$$\hat{\pi}(x,t) = \frac{\partial}{\partial t}\hat{\phi}(x,t) = \int d^3p\cdot(-iW_pt)\left(\hat{a}_p U_p(x,t) - \hat{a}_p^\dagger U_p^*(x,t)\right) = \hat{\pi}^{(+)}(x,t) - \hat{\pi}^{(-)}(x,t)$$

○ 定义 scalar product of two Klein-Gordon Wave functions ϕ and χ.

$$(\phi,\chi) = i\int d^3x\ \phi^*(x,t)\overleftrightarrow{\partial_0}\chi(x,t)$$

$$= i\int d^3x\cdot\left(\phi^*(x,t)\frac{\partial\chi(x,t)}{\partial t} - \frac{\partial\phi^*(x,t)}{\partial t}\chi(x,t)\right)$$

○ 表达 $\hat{a}_p$ 的简单形式：

$$\hat{a}_p = i\int d^3x\ U_p^*(x,t)\overleftrightarrow{\partial_0}\hat{\phi}(x,t)$$

$$\hat{a}_p^\dagger = -i\int d^3x\ U_p(x,t)\overleftrightarrow{\partial_0}\hat{\phi}(x,t)$$

○ Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3p\ W_p\left(\hat{a}_p^\dagger\hat{a}_p + \hat{a}_p\hat{a}_p^\dagger\right) = \frac{1}{2}\int d^3p\ W_p\left(2\hat{a}_p^\dagger\hat{a}_p + \delta^{(3)}(0)\right)$$

• Normal ordering 引入：(生成算符似乎总在左)

$$:\hat{\phi}\hat{\chi}: = \hat{\phi}^{(-)}\hat{\chi}^{(-)} + \hat{\phi}^{(-)}\hat{\chi}^{(+)} + \hat{\chi}^{(-)}\hat{\phi}^{(+)} + \hat{\phi}^{(+)}\hat{\chi}^{(+)}$$

为什么叫 $\hat{a}_p^\dagger$ 生成算符，

设有态 $|E\rangle$ 是 $\hat{H}$ 本征态，

则：$\hat{H}|E\rangle = E|E\rangle$

$$\hat{H}\hat{a}_{p_1}^\dagger|E\rangle = \hat{a}_{p_1}^\dagger\hat{H}|E\rangle + W_{p_1}\cdot\hat{a}_{p_1}^\dagger|E\rangle$$

$$= \underline{(E+W_{p_1})}\cdot\hat{a}_{p_1}^\dagger|E\rangle$$

⇓ 生成了一个 p_1 动量的粒子.

$\hat{a}_p$ 叫湮灭算符同理！

真空态：$|0\rangle$：$\hat{a}_p|0\rangle = 0$；

○ 动量算符

经典场论，时空平移 → 能动量张量.

$$P_\mu = \int d^3x\, \Theta_{0\mu} = \int d^3x \left(\pi \frac{\partial \phi}{\partial x^\mu} - g_{0\mu} \mathcal{L} \right)$$

$$\vec{P} = -\int d^3x\, \pi\, \nabla\phi \qquad (相当于(P_x, P_y, P_z))$$

量子化：

$$\hat{P} = -\frac{1}{2}\int d^3x \left(\hat{\pi}(x,t)\nabla\hat{\phi}(x,t) + \nabla\hat{\phi}(x,t)\hat{\pi}(x,t) \right)$$

$$= \frac{1}{2}\int d^3p\, \vec{p}\, (\hat{a}_p^\dagger \hat{a}_p + \hat{a}_p \hat{a}_p^\dagger)$$

○ Noether 定理量子化出的角动量算符

$$\hat{L} = -\frac{1}{2}\int d^3x : \left(\hat{\pi}\, x\times\nabla\hat{\phi} + (x\times\nabla\hat{\phi})\hat{\pi} \right) :$$

$$= i\int d^3p\ \hat{a}_p^\dagger (p\times\nabla_p)\hat{a}_p$$

Charged KG 场.

- Lagrangian 到 Hamiltonian.

$$\mathcal{L} = \frac{\partial\phi^*}{\partial x_\mu}\frac{\partial\phi}{\partial x^\mu} - m^2\phi^*\phi \qquad L = L[\phi, \phi^*, \dot{\phi}, \dot{\phi}^*]$$

正则共轭场

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*$$

$$\pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi}$$

Hamiltonian:

$$H = \int d^3x\,(\pi\partial_0\phi + \pi^*\partial_0\phi^* - \mathcal{L})$$

$$= \int d^3x\,(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi\phi^*)$$

$$\hat{H} = \int d^3x\,(\pi^\dagger\pi + \nabla\phi^\dagger\nabla\phi + m^2\phi\phi^\dagger)$$

量子化:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\delta^{(3)}(x-x') = [\hat{\phi}^\dagger(x,t), \hat{\pi}^\dagger(x',t)]$$

展开:

$$\hat{\phi}(x,t) = \int d^3p\cdot(\hat{a}_p u_p(\vec{x},t) + \hat{b}_p^\dagger u_p^*(\vec{x},t))$$

$$\hat{\phi}^\dagger(x,t) = \int d^3p\,(\hat{a}_p^\dagger u_p^*(\vec{x},t) + \hat{b}_p u_p(\vec{x},t))$$

生成湮灭算符对易:

$$[\hat{a}_p, \hat{a}_{p'}^\dagger] = [\hat{b}_p, \hat{b}_{p'}^\dagger] = \delta^{(3)}(p-p') \quad \text{others} = 0$$

Hamiltonian:

$$\hat{H} = :\int d^3p\,\omega_p(\hat{a}_p\hat{a}_p^\dagger + \hat{b}_p^\dagger\hat{b}_p):$$

$$= \int d^3p\,\omega_p(\hat{a}_p^\dagger\hat{a}_p + \hat{b}_p^\dagger\hat{b}_p)$$

动量:

$$\hat{P} = \int d^3p\,\vec{p}\,(\hat{a}_p^\dagger\hat{a}_p + \hat{b}_p^\dagger\hat{b}_p) = \int d^3p\,\vec{p}\,(\hat{n}_p^{(a)} + \hat{n}_p^{(b)})$$

角动量

$$\hat{L} = -i\int d^3p\cdot[\hat{a}_p^\dagger(p\times\nabla_p)\hat{a}_p + \hat{b}_p^\dagger(p\times\nabla_p)\hat{b}_p]$$

- 诺特定理. 经典场的变化:

$$\phi' = \phi e^{i\alpha} \quad \phi^{*\prime} = \phi^* e^{-i\alpha}$$

$$Q = \int d^3x\, j^0(x) = -i\int d^3x\cdot\left(\frac{\partial\mathcal{L}}{\partial\pi^*}\phi - \frac{\partial\mathcal{L}}{\partial\pi}\phi^*\right)$$

$$= -i\int d^3x\,(\pi\phi - \pi^*\phi^*) \qquad \pi = \dot{\phi}^* \quad \pi^* = \dot{\phi}$$

$$= i\int d^3x\,\phi^*\overleftrightarrow{\partial_0}\phi = (\phi, \phi)$$

量子:

$$\hat{Q} = -i\int d^3x :(\hat{\pi}\hat{\phi} - \hat{\pi}^\dagger\hat{\phi}^\dagger):$$

$$= \int d^3p\,(\hat{a}_p^\dagger\hat{a}_p - \hat{b}_p^\dagger\hat{b}_p)$$

不变对易条件 Invariant Commutation Relation.

° Charged Klein-Gordon 场 场算符 $\hat\phi$, $\hat\phi^\dagger$ 之间的对易关系.

$$i\Delta(x-y) := [\hat\phi(x), \hat\phi^\dagger(y)] \qquad (1)$$

(4.58):

$$\hat\phi(x) = \int d^3p\,(\hat a_p u_p(x) + \hat b_p^\dagger u_p^*(x)) = \hat\phi^{(+)}(x) + \hat\phi^{(-)}(x) \qquad (2)$$

代入(1):

$$i\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \qquad p^0 = \omega_p = \sqrt{\vec p^2 + m^2}$$

$$= i\Delta^{(+)}(x-y) + i\Delta^{(-)}(x-y) \qquad (3)$$

$$\Delta(x-y) = -\int \frac{d^3p}{(2\pi)^3}\frac{\sin p\cdot(x-y)}{\omega_p} \quad \} \text{视为定义式!} \qquad (4)$$

1° $\Delta(x-y)$ 是 Lorentz 不变的. 取 $z = x-y$

(3)

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-i(\omega_p z_0 - \vec p\cdot\vec z)} - e^{i(\omega_p z_0 - \vec p\cdot\vec z)}\right)$$

$$= \int \frac{d^4p}{(2\pi)^3}\frac{1}{2\omega_p}\left(\delta(p_0-\omega_p) - \delta(p_0+\omega_p)\right)\cdot e^{-i(p_0 z_0 - \vec p\cdot\vec z)}$$

$$= \int \frac{d^4p}{(2\pi)^3}\frac{\mathrm{sgn}(p_0)}{2\omega_p}\left(\delta(p_0-\omega_p)+\delta(p_0+\omega_p)\right)e^{-ip\cdot z}$$

$$= \int \frac{d^4p}{(2\pi)^3}\,\mathrm{sgn}(p_0)\,\delta(p^2-m^2)\,e^{-ip\cdot z}$$

每一项都 Lorentz invariant.

$$\frac{1}{2\omega_p}\left(\delta(p_0-\omega_p)+\delta(p_0+\omega_p)\right) = \delta\left((p_0-\omega_p)(p_0+\omega_p)\right) = \delta(p_0^2-\omega_p^2)$$

$$= \delta(p_0^2 - \vec p^2 - m^2)$$

$$= \delta(p^2 - m^2)$$

2° $\Delta(x-y)$ 是奇 function.

由(4)

3° 边界条件. boundary condition at vanishing time difference:

a. $\Delta(0,\vec x) = 0$ 由(3) (将体积分转换为球积分)

b. $\frac{\partial}{\partial x^0}\Delta(x^0,\vec x)|_{x^0=0} = -\delta^{(3)}(\vec x)$

由(3)

$$\frac{\partial}{\partial x^0}\Delta(x^0,\vec x)|_{x^0=0} = \frac{1}{i}\cdot\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\cdot\left(-i\omega_p e^{-ip\cdot x} - (i\omega_p)e^{ip\cdot x}\right)\Big|_{x_0=0}$$

$$= -\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(e^{i\vec p\cdot\vec x} + e^{-i\vec p\cdot\vec x}\right) = -\delta^{(3)}(\vec x)$$

c. $\nabla\Delta(x^0,x)|_{x^0=0} = \int\frac{d^3p}{(2\pi)^3}\frac{\vec p\cos(\vec p\cdot\vec x)}{\omega_p} = 0$ (体积分化为球积分可证!)

d. $x^0=0$ 时 $\left\{\begin{array}{l}\text{i 所有空间求导}\\ \text{ii 所有时间2阶导}\end{array}\right|$ 都会消失.

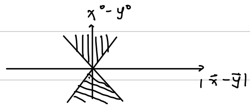
4° $\Delta(x)$ 满足各向同性 KG 方程. homogeneous K-G-Equation

$$(\Box + m^2)\Delta(x) = 0$$

因: $\Delta(x) = \frac{1}{i}[\hat\phi(x), \hat\phi^\dagger(0)]$, $\hat\phi(x)$ 满足 KGE

5. $\Delta(x-y)=0$ for $(x-y)^2<0$ (类空间隔)

由 1°, 3°(a).



◦ 对于双线性量 observable Grenier (4.118) / microcausality.

$\hat{O}(x) = \hat{\phi}^\dagger(x)\, O(x)\, \hat{\phi}(x)$

$[\hat{O}(x), \hat{O}(y)] = O(x)\, O(y) \left(\hat{\phi}^\dagger(x)\hat{\phi}(y) + \hat{\phi}^\dagger(y)\hat{\phi}(x) \right) i\Delta(x-y)$. $\begin{cases} =0 & (x-y)^2<0 \\ \neq 0 & (x-y)^2>0 \end{cases}$

标量场 Feynman Propagator

○ Charged K-G 场：/ $\Delta_F(x-y)$ 的具体表达

$$i\Delta_F(x-y) = \langle 0|T(\hat\phi(x)\,\hat\phi^\dagger(y))|0\rangle$$

$$T(\hat A(x),\hat B(y)) = \hat A(x)\hat B(y)\Theta(x_0-y_0) \pm \hat B(y)\hat A(x)\,\Theta(y_0-x_0)$$

+: bosonic　　—: fermionic.

$x^0>y^0$ 时：$\langle 0|T(\hat\phi(x)\hat\phi^\dagger(y))|0\rangle = \langle 0|\hat\phi(x)\hat\phi^\dagger(y)|0\rangle = \langle 0|\hat\phi^{(+)}(x)\,\hat\phi^{\dagger(-)}(y)|0\rangle$　　$(x_0>y_0)$

$$\hat\phi^{(+)}(x) = \int d^3p\,\hat a_p\,u_p(x) \qquad \hat\phi^{\dagger(+)}(x) = \int d^3p\,\hat a_p^\dagger\,u_p^*(x)$$

$$\hat\phi^{(-)}(x) = \int d^3p\,\hat b_p^\dagger\,u_p^*(x) \qquad \hat\phi^{\dagger(-)}(x) = \int d^3p\,b_p\,u_p(x)$$

$$i\Delta_F(x-y) = \int d^3p\cdot u_p(x)\,u_p^*(y) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip\cdot(x-y)} = i\Delta^{(+)}(x-y) \quad (x_0>y_0)$$

$$= \langle 0|\hat\phi^\dagger(y)\hat\phi(x)|0\rangle = \langle 0|\hat\phi^\dagger(y)\,\hat\phi^{(-)}(x)|0\rangle = \int d^3p\cdot u_p(y)\,u_p^*(x)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\cdot e^{ip\cdot(x-y)}$$

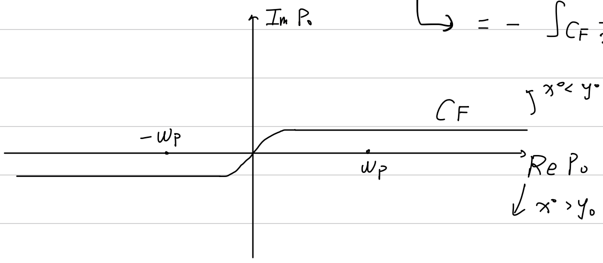
$$= -i\Delta^{(-)}(x-y) \qquad (x_0<y_0)$$

$$i\Delta_F(x-y) = \Theta(x_0-y_0)\,i\Delta^{(+)}(x-y) - \Theta(y_0-x_0)\,i\Delta^{(-)}(x-y)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(\Theta(x_0-y_0)e^{-ip\cdot(x-y)} + \Theta(y_0-x_0)e^{ip\cdot(x-y)}\right) \longleftarrow p=(\omega_p,\vec p)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\underbrace{\left(\Theta(x_0-y_0)e^{-i\omega_p(x^0-y^0)}\cdot e^{i\vec p\cdot(\vec x-\vec y)} + \Theta(y_0-x_0)e^{i\omega_p(x^0-y^0)}e^{i\vec p\cdot(\vec x-\vec y)}\right)}$$

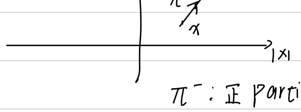
$$\hookrightarrow\ = -\int_{C_F}\frac{dp^0}{2\pi i}\frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p)(p^0+\omega_p)}\cdot e^{i\vec p\cdot\vec x}$$



$$\Delta_F(x-y) = \int_{C_F}\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2} = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{1}{p^2-m^2+i\epsilon}$$

$\langle 0|T(\phi(y)\,\phi^\dagger(x))|0\rangle$

○ 物理解释：$\phi^\dagger(x)$ 在 charged KG 场中代表生成了一个粒子，在 x，则 $\langle 0|\phi(y)\phi^\dagger(x)|0\rangle$ 代表 x 生成的粒子和 y 生成粒子内积。若 $x^0<y^0$，代表正粒子从 x 传到 y 的几率！
$x^0<y^0$，代表反粒子从 y 传到 x 的几率　$\langle 0|\phi^\dagger(x)\,\phi(y)|0\rangle$



π^-：正 particle

π^+：反粒子.

第三章 Spin-1/2Field Dirac 方程

3.1 Dirac 方程的导出

Dirac 方程导出 类似于 Schrodinger 方程，考虑相对论波函数满足 Dirac 方程

$$\hat{E}\psi = (\boldsymbol{\alpha}\cdot\hat{\boldsymbol{P}} + \beta m)\psi \tag{3.1}$$

认为能量和动量算符分别为 $\hat{E} = i\hbar\frac{\partial}{\partial t}$ $\hat{\boldsymbol{P}} = -i\hbar\nabla$ Dirac 方程就是 (没有用四维指标，如果用了的话应该是 $\hat{P}^\mu = i\hbar\partial^\mu$):

$$i\frac{\partial}{\partial t}\psi = (-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi \tag{3.2}$$

平方得到:

$$\begin{aligned}
-\frac{\partial^2}{\partial t^2}\psi =&(-i\boldsymbol{\alpha}\cdot\nabla + \beta m)(-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi \\
=&-\alpha_x^2\partial_x^2 - \alpha_y^2\partial_y^2 - \alpha_z^2\partial_z^2 + \beta^2 m^2\psi \\
&-(\alpha_x\alpha_y + \alpha_y\alpha_x)\partial_x\partial_y\psi - (\alpha_x\alpha_z + \alpha_z\alpha_x)\partial_x\partial_z\psi - (\alpha_y\alpha_z + \alpha_z\alpha_y)\partial_y\partial_z\psi \\
&-im(\alpha_x\beta + \beta\alpha_x)\partial_x\psi - im(\alpha_y\beta + \beta\alpha_y)\partial_y\psi - im(\alpha_z\beta + \beta\alpha_z)\partial_z\psi
\end{aligned} \tag{3.3}$$

为了满足狭义相对论的 Einstein 能量-动量关系–其实也就是满足 Klein-Gordan 方程

$$\left\{\hat{E}^2\psi = (\hat{\boldsymbol{P}}^2 + m^2)\psi\right. \tag{3.4}$$

这也就要求了 α 和 β 的代数关系:

$$\begin{aligned}
\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1 \quad \beta^2 = 1 \\
\left\{\alpha_i, \alpha_j\right\} = 0 \quad (i \neq j) \\
\left\{\alpha_j, \beta\right\} = 0 \\
\beta^\dagger = \beta \quad \alpha_i^\dagger = \alpha_i \quad \text{由于 } \hat{E} \text{ 的幺正性}
\end{aligned} \tag{3.5}$$

这些都是矩阵。ψ 被称为 Dirac spinor

Dirac 方程的协变形式,γ 矩阵 接下来，引出 γ 矩阵。给 Dirac 方程同时乘以 β 得到:

$$i\beta\alpha_x\frac{\partial}{\partial x} + i\beta\alpha_y\frac{\partial}{\partial y} + i\beta\alpha_z\frac{\partial}{\partial z} + i\beta\frac{\partial}{\partial t}\psi - \beta^2 m\psi = 0 \tag{3.6}$$

定义 Γ 矩阵:

$$\gamma^0 \equiv \beta \quad \gamma^1 \equiv \beta\alpha_x \quad \gamma^2 \equiv \beta\alpha_y \quad \gamma^3 \equiv \beta\alpha_z \tag{3.7}$$

Dirac 方程可以写为更为简单的形式:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3.8}$$

对 α 和 β 矩阵的要求可以转变为对 γ 矩阵的要求。

$$\begin{aligned}
\{\beta, \beta\alpha_i\} = \beta\beta\alpha_i + \beta\alpha_i\beta = \alpha_i - \alpha_i = 0 \\
\{\beta, \beta\} = \beta^2 + \beta^2 = 2 \\
\{\beta\alpha_i, \beta\alpha_i\} = 2\beta\alpha_i\beta\alpha_i = -2\beta^2\alpha_i^2 = -2 \\
\left\{\beta\alpha_i, \beta\alpha_j\right\} = \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i = -\beta^2\left(\alpha_i\alpha_j + \alpha_j\alpha_i\right) = 0 \\
\beta^\dagger = \beta \quad (\beta\alpha_i)^\dagger = \alpha_i\beta = -\beta\alpha_i
\end{aligned} \tag{3.9}$$

也就是:

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\}&=2g^{\mu\nu}\\ \gamma^{0\dagger}=\gamma^0\quad \gamma^{k\dagger}&=-\gamma^k\end{aligned}\tag{3.10}$$

命题 3.1 (Dirac 方程)

$$(i\gamma^\mu\partial_\mu-m)\psi=0\tag{3.11}$$

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\}&=2g^{\mu\nu}\\ \gamma^{0\dagger}=\gamma^0\quad \gamma^{k\dagger}&=-\gamma^k\end{aligned}\tag{3.12}$$

3.2 * Pauli's Fundamental Theorem

不同 γ 表示之间的相似性 想要验证。对于不同的 γ 矩阵表示 γ_μ 和 γ'_μ。它们之间满足相似性:$\gamma'_\mu=S\gamma_\mu S^{-1}$

(0) 构建 16 个 4×4 矩阵

$$\begin{aligned}\Gamma_A=&\mathbb{I},\ \gamma_0,\ i\gamma_1,\ i\gamma_2,\ i\gamma_3,\\ &i\gamma_2\gamma_3,\ i\gamma_3\gamma_1,\ i\gamma_1\gamma_2,\ \gamma_1\gamma_0,\ \gamma_2\gamma_0,\ \gamma_3\gamma_0,\\ &\gamma_1\gamma_2\gamma_3,\ i\gamma_1\gamma_2\gamma_0,\ i\gamma_3\gamma_1\gamma_0,\ i\gamma_2\gamma_3\gamma_0\\ &i\gamma_1\gamma_2\gamma_3\gamma_0\quad (A=1\cdots16)\end{aligned}\tag{3.13}$$

另外，定义矩阵 γ_5:

$$\gamma_5=i\gamma_1\gamma_2\gamma_3\gamma_0\tag{3.14}$$

满足几个性质:

$$\Gamma_A^2=1\quad \{\gamma_5,\gamma_\mu\}=0\quad \Gamma_A\Gamma_B=Const\,\Gamma_C\ (A\neq B\Rightarrow C\neq 1\text{；固定 A 的同时 B 遍历 16}\Rightarrow\text{C 遍历 16})\tag{3.15}$$

(1) 对于所有的 Γ_A $A\neq1$, 一定可以找到一个 Γ_B s.t.:

$$\Gamma_B\Gamma_A\Gamma_B=-\Gamma_A\tag{3.16}$$

A	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B	9	4	3	3	4	5	3	2	2	2	2	6	6	7	2

(2) 对于所有的 Γ_A $(A=2\cdots16)$, $tr(\Gamma_A)=0$

$$tr(-\Gamma_A)=tr(\Gamma_B\Gamma_A\Gamma_B)=tr(\Gamma_B^2\Gamma_A)=tr(\Gamma_A)=0\tag{3.17}$$

(3) Γ_A 是相互线性无关的。也就是要证明:

$$\sum_{A=1}^{16}a_A\Gamma_A=0\Rightarrow a_A=0\ (A=1\cdots16)\tag{3.18}$$

证明 认为

$$\sum_{A=1}^{16}a_A\Gamma_A=0\tag{3.19}$$

于是:

$$\begin{aligned}
\Gamma_B(\sum_{A=1}^{16} a_A\Gamma_A) = a_B\Gamma_B^2 + \sum_{A\neq B} a_A\Gamma_B\Gamma_A = 0 \\
a_B + \sum_{C\neq 1} a_A\ Const\ \Gamma_C = 0 \\
4a_B + \sum_{C\neq 1} a_A\ Const\ tr(\Gamma_C) = 0 \\
a_B = 0\ (B = 1\cdots 16)
\end{aligned} \tag{3.20}$$

□

(4) 所有的 4×4 矩阵可以用 Γ_A 展开。因为上面说已经找到了 16 个线性无关的矩阵基底。

$$\chi = \sum_A \chi_A\Gamma_A, \quad \chi_B = \frac{1}{4}tr(\Gamma_B\chi) \tag{3.21}$$

(5) 如果有一个矩阵和所有的 Γ_A 都对易，那么它正比于 $\mathbb{I}$。认为:

$$\begin{aligned}
\chi = \chi_B\Gamma_B + \sum_{A\neq B}\chi_A\Gamma_A|_{B\neq 1} \\
\Gamma_C\Gamma_B\Gamma_C = -\Gamma_B \\
\Gamma_C\chi = \chi\Gamma_C \Rightarrow \chi = \Gamma_C\chi\Gamma_C
\end{aligned} \tag{3.22}$$

于是:

$$\begin{aligned}
\chi_B\Gamma_B + \sum_{A\neq B}\chi_A\Gamma_A|_{B\neq 1} = \chi_B\Gamma_C\Gamma_B\Gamma_C + \sum_{A\neq B}\chi_A\Gamma_C\Gamma_A\Gamma_C|_{B\neq 1} \\
\chi_B tr(\Gamma_B^2) + \sum_{A\neq B}\chi_A tr(\Gamma_B\Gamma_A)|_{B\neq 1} = \chi_B tr(\Gamma_B\Gamma_C\Gamma_B\Gamma_C) + \sum_{A\neq B}\chi_A tr(\Gamma_B\Gamma_C\Gamma_A\Gamma_C)|_{B\neq 1} \\
4\chi_B + \sum_{A\neq B}\chi_A tr(\Gamma_B\Gamma_A)|_{B\neq 1} = -4\chi_B + \sum_{A\neq B}\chi_A tr(-\Gamma_B\Gamma_A)|_{B\neq 1}\ \ (\Gamma_A\Gamma_B \sim \Gamma_D|_{D\neq 1}) \\
\chi_B = 0
\end{aligned} \tag{3.23}$$

那么:

$$\chi \sim \mathbb{I} \tag{3.24}$$

(6) 能找到两组 Dirac 基底 γ_μ 和 γ'_μ, 它们构成 16 维空间中的基底 Γ_A 和 Γ'_A 那么可以定义 $S = \sum_B \Gamma'_B F\Gamma_B$ (F是任意 4 维矩阵) 使得:

$$\Gamma'_A S = S\Gamma_A \tag{3.25}$$

证明

$$\Gamma'_A S\Gamma_A = \sum_B \Gamma'_A\Gamma'_B F\Gamma_B\Gamma_A \tag{3.26}$$

考虑到:

$$\left\{\Gamma_B\Gamma_A = \epsilon_C\Gamma_C \quad \Gamma_A\Gamma_B = \frac{1}{\epsilon_C}\Gamma_C\right. \tag{3.27}$$

于是:

$$上式 = \sum_C \Gamma'_C F\Gamma_C = S \tag{3.28}$$

也就是:

$$\Gamma'_A S = S\Gamma_A \tag{3.29}$$

□

(7) 上一点提到的 S 矩阵可以不是 0 矩阵。为了证明这一点，先取一个特殊的 F 矩阵:

$$F_{\nu\sigma} = 1 \quad others = 0 \tag{3.30}$$

$$S_{\mu\rho} = \sum_B (\Gamma'_B F \Gamma_B)_{\mu\rho} = \sum_B (\Gamma'_B)_{\mu\nu} (\Gamma_B)_{\sigma\rho} \tag{3.31}$$

如果不论怎么取 F，S 都是 0。就意味着:

$$\sum_B (\Gamma'_B)_{\mu\nu} \Gamma_B = 0 \tag{3.32}$$

但是因为前面提到 Γ_B 是线性无关的。所以这个等式不成立。

(8) 可以合理的取 F, 让 S 的行列式不等于 0（可逆）。下面是证明，首先构造

$$S = \sum_B \Gamma'_B F \Gamma_B \quad T = \sum_B \Gamma_B G \Gamma'_B \tag{3.33}$$

满足:

$$\Gamma_A T = T \Gamma'_A \quad \Gamma'_A S = S \Gamma_A \tag{3.34}$$

那么:

$$\Gamma_A T S = T \Gamma'_A S = T S \Gamma_A \tag{3.35}$$

于是 TS 对易于所有的 Γ_A。于是:

$$TS = k\mathbb{I} \tag{3.36}$$

如果 $det(S) = 0$ 那么 $k = 0$。

$$TS = \sum_{B=1} T\ \Gamma'_B F \Gamma_B = 0 \tag{3.37}$$

取:

$$F_{\nu\sigma} = 1 \quad others = 0 \tag{3.38}$$

那么:

$$\sum_B (T\Gamma'_B)_{\mu\nu} (\Gamma_B)_{\sigma\rho} = 0 \quad (\forall \mu, \nu, \sigma, \rho) \tag{3.39}$$

于是:

$$\begin{gathered} \sum_B (T\Gamma'_B)_{\mu\nu} \Gamma_B = 0 \quad (\forall \mu, \nu) \\ \Downarrow \\ (T\Gamma'_B)_{\mu\nu} = 0 \quad (\forall \mu, \nu) \\ \Downarrow \\ T = 0 \end{gathered} \tag{3.40}$$

也就是说，只要 $T \neq 0$, 那么一定可以找到 S，使得 $det(T)det(S) \neq 0$。

$$\gamma'_\mu = S \gamma_\mu S^{-1} \tag{3.41}$$

相似变换可以用幺正算符表示 当 $\gamma_\mu^\dagger = g_{\mu\mu}\gamma_\mu, \gamma_\mu'^\dagger = g_{\mu\mu}\gamma'_\mu$ 时。通过前面的证明已经找到了 S 使得:

$$\gamma'_\mu = S \gamma_\mu S^{-1} \tag{3.42}$$

定义:

$$V = (detS)^{-1/4} S \quad \gamma'_\mu = V \gamma_\mu V^{-1} \quad det(V) = 1 \tag{3.43}$$

先证明: 如果 $\gamma'_\mu = V_1 \gamma_\mu V_1^{-1}, \gamma'_\mu = V_2 \gamma_\mu V_2^{-1}$。那么: $V_1 = kV_2$。

证明

$$\begin{aligned} V_1\gamma_\mu V_1^{-1} &= V_2\gamma_\mu V_2^{-1} \\ V_2^{-1}V_1\gamma_\mu &= \gamma_\mu V_2^{-1}V_1 \\ V_2^{-1}V_1\Gamma_A &= \Gamma_A V_2^{-1}V_1 \end{aligned} \tag{3.44}$$

由前面提到的 Shur's lemma:

$$V_1 = kV_2 \tag{3.45}$$

由于 $det(V_1) = det(V_2) = 1$, 于是: $V_2 = V_1 \exp(im\pi/2) \quad (m = 0, 1, 2, 3)$

□

由于

$$\left\{\gamma'_\mu = V\gamma_\mu V^{-1}\right. \tag{3.46}$$

那么

$$\gamma'^\dagger_\mu = (V^\dagger)^{-1}\gamma^\dagger_\mu V^\dagger \tag{3.47}$$

考虑:

$$\left\{\gamma^\dagger_\mu = g_{\mu\mu}\gamma_\mu\right. \tag{3.48}$$

$$\gamma'_\mu = (V^\dagger)^{-1}\gamma_\mu V^\dagger \tag{3.49}$$

于是:

$$\begin{aligned} &(V^\dagger)^{-1} = V\exp(im\pi/2) \quad (m = 0, 1, 2, 3) \\ &V^\dagger V = \exp(im\pi/2) \\ &(V^\dagger V)_{ii} = \sum_j V_{ji}V^*_{ji} > 0 \Rightarrow m = 0 \quad (V^\dagger)^{-1} = V, V^\dagger = V^{-1} \end{aligned} \tag{3.50}$$

也就找到了一个幺正变换。

命题 3.2 (Dirac 矩阵表示之间的幺正变化)

$$\gamma'_\mu = V\gamma_\mu V^\dagger \tag{3.51}$$

其中:

$$V = (detS)^{-1/4}S \quad (S = \sum_B \Gamma'_B F\Gamma_B) \tag{3.52}$$

□

3.3 Dirac 方程的 Lorentz 协变性

$\psi'(x')$ 与 $\psi(x)$ 之间的转变

$$\psi'(x') = S(\Lambda)\psi(x) \Rightarrow \psi(x) = S^{-1}(\Lambda)\psi'(x') \tag{3.53}$$

满足:

$$S^{-1}(\Lambda) = S(\Lambda^{-1}) \tag{3.54}$$

S 满足的基础关系 考虑 Dirac 方程:

$$\begin{aligned} (i\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} - m_0c)\psi(x) &= 0 \\ (i\hbar\gamma^\mu S^{-1}(\Lambda)\frac{\partial}{\partial x^\mu} - m_0cS^{-1}(\Lambda))\psi'(x') &= 0 \\ \left(i\hbar S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\frac{\partial}{\partial x^\mu} - m_0c\right)\psi'(x') &= 0 \end{aligned} \tag{3.55}$$

考虑到:

$$\left\{x'^{\nu} = \Lambda^{\nu}_{\ \mu} x^{\mu}\right. \tag{3.56}$$

于是:

$$\left\{\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\nu}} = \Lambda^{\nu}_{\ \mu} \frac{\partial}{\partial x'^{\nu}}\right. \tag{3.57}$$

$$\left(i\hbar S(\Lambda)\gamma^{\mu} S^{-1}(\Lambda)\Lambda^{\nu}_{\ \mu} \frac{\partial}{\partial x'^{\nu}} - m_0 c\right)\psi'(x') = 0 \tag{3.58}$$

要保证在所有的参考系中的方程形式一样:

$$S(\Lambda)\gamma^{\mu} S^{-1}(\Lambda)\Lambda^{\nu}_{\ \mu} = \gamma^{\nu} \tag{3.59}$$

利用 Λ 矩阵的正交性:

$$\left\{\Lambda^{\nu}_{\ \mu}\Lambda^{\rho}_{\nu} = \delta^{\rho}_{\mu}\right. \tag{3.60}$$

$$S(\Lambda)\gamma^{\rho} S^{-1}(\Lambda) = \Lambda^{\rho}_{\nu}\gamma^{\nu} \tag{3.61}$$

S 生成算符的对易关系 首先构建在微小 Lorentz 变换下的 S 算符:

$$S(\Delta w^{\mu\nu}) = \mathbb{I} - \frac{i}{4}\sigma_{\mu\nu}\Delta w^{\mu\nu} \quad (\sigma_{\mu\nu} = -\sigma_{\nu\mu}) \tag{3.62}$$

$$S^{-1}(\Delta w^{\mu\nu}) = \mathbb{I} + \frac{i}{4}\sigma_{\mu\nu}\Delta w^{\mu\nu} \tag{3.63}$$

由前文对 S 算符的要求:

$$\left\{S(\Lambda)\gamma^{\rho} S^{-1}(\Lambda) = \Lambda^{\rho}_{\nu}\gamma^{\nu}\right. \tag{3.64}$$

$$\begin{aligned}
\left(\mathbb{I} - \frac{i}{4}\sigma_{\alpha\beta}\Delta w^{\alpha\beta}\right)\gamma^{\nu}\left(\mathbb{I} + \frac{i}{4}\sigma_{\alpha\beta}\Delta w^{\alpha\beta}\right) &= \Lambda_{\mu}^{\ \nu}\gamma^{\mu} \\
\left(\mathbb{I} - \frac{i}{4}\sigma_{\alpha\beta}\Delta w^{\alpha\beta}\right)\gamma^{\nu}\left(\mathbb{I} + \frac{i}{4}\sigma_{\alpha\beta}\Delta w^{\alpha\beta}\right) &= \left(\delta^{\nu}_{\mu} + \Delta w_{\mu}^{\ \nu}\right)\gamma^{\mu} \\
-\frac{i}{4}\Delta w^{\alpha\beta}\left(\sigma_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\sigma_{\alpha\beta}\right) &= \Delta w_{\mu}^{\ \nu}\gamma^{\mu} \\
-\frac{i}{4}\Delta w^{\alpha\beta}\left(\sigma_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\sigma_{\alpha\beta}\right) &= \Delta w^{\beta\nu}\gamma_{\beta} \\
-\frac{i}{4}\Delta w^{\alpha\beta}\left(\sigma_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\sigma_{\alpha\beta}\right) &= \Delta w^{\beta\alpha} g^{\nu}_{\ \alpha}\gamma_{\beta} \\
-\frac{i}{4}\Delta w^{\alpha\beta}\left(\sigma_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\sigma_{\alpha\beta}\right) &= \frac{1}{2}\Delta w^{\beta\alpha}\left(g^{\nu}_{\ \alpha}\gamma_{\beta} - g^{\nu}_{\ \beta}\gamma_{\alpha}\right)
\end{aligned} \tag{3.65}$$

于是得到 S 生成算符的对易关系:

$$2i\left(g^{\nu}_{\ \beta}\gamma_{\alpha} - g^{\nu}_{\ \alpha}\gamma_{\beta}\right) = [\sigma_{\alpha\beta}, \gamma^{\nu}] \tag{3.66}$$

找到满足条件的 S 的生成算符 直接假定:

$$\sigma_{\alpha\beta} = \frac{i}{2}\left[\gamma_{\alpha}, \gamma_{\beta}\right] \tag{3.67}$$

$$\begin{aligned}
[\sigma_{\alpha\beta}, \gamma^{\nu}] &= \left[\frac{i}{2}\left[\gamma_{\alpha}, \gamma_{\beta}\right], \gamma^{\nu}\right] \\
&= \frac{i}{2}\left[\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}, \gamma^{\nu}\right] \\
&= \frac{i}{2}\left(\left[\gamma_{\alpha}\gamma_{\beta}, \gamma^{\nu}\right] - \left[\gamma_{\beta}\gamma_{\alpha}, \gamma^{\nu}\right]\right)
\end{aligned} \tag{3.68}$$

考虑到 γ 矩阵满足的基本性质:

$$\left\{\gamma_{\alpha}\gamma_{\beta} + \gamma_{\beta}\gamma_{\alpha} = 2g_{\alpha\beta} \Rightarrow \gamma_{\beta}\gamma_{\alpha} = 2g_{\alpha\beta} - \gamma_{\alpha}\gamma_{\beta}\right. \tag{3.69}$$

$$\begin{aligned} \text{上式} &= \frac{i}{2}\left(2\left[\gamma_\alpha\gamma_\beta, \gamma^\nu\right] - 2\left[g_{\alpha\beta}\mathbb{I}, \gamma^\nu\right]\right) \\ &= i\left(\gamma_\alpha\gamma_\beta\gamma^\nu - \gamma^\nu\gamma_\alpha\gamma_\beta\right) \end{aligned} \tag{3.70}$$

考虑到:

$$\begin{cases} \gamma_\beta\gamma^\nu + \gamma^\nu\gamma_\beta &= 2g_\beta^{\ \nu} \\ \gamma_\alpha\gamma^\nu + \gamma^\nu\gamma_\alpha &= 2g_\alpha^{\ \nu} \end{cases} \tag{3.71}$$

$$\begin{aligned} \text{上式} &= i\left(-\gamma_\alpha\gamma^\nu\gamma_\beta + 2g_\beta^{\ \nu}\gamma_\alpha - \gamma^\nu\gamma_\alpha\gamma_\beta\right) \\ &= i\left(\gamma^\nu\gamma_\alpha\gamma_\beta - 2g_\alpha^{\ \nu}\gamma_\beta + 2g_\beta^{\ \nu}\gamma_\alpha - \gamma^\nu\gamma_\alpha\gamma_\beta\right) \\ &= 2i\left(g^\nu_{\ \beta}\gamma_\alpha - g^\nu_{\ \alpha}\gamma_\beta\right) \end{aligned} \tag{3.72}$$

也就是，满足生成元算符的对易条件:

$$2i\left(g^\nu_{\ \beta}\gamma_\alpha - g^\nu_{\ \alpha}\gamma_\beta\right) = \left[\sigma_{\alpha\beta}, \gamma^\nu\right] \tag{3.73}$$

3.4 Dirac 自由粒子的 spinor

简单起见:

$$\gamma^0 \equiv \beta \quad \gamma^1 \equiv \beta\alpha_x \quad \gamma^2 \equiv \beta\alpha_y \quad \gamma^3 \equiv \beta\alpha_z \tag{3.74}$$

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma^2 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.75}$$

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{3.76}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.77}$$

总角动量与自旋的引入

定义自旋，求其与哈密顿量之间的对易关系　定义自旋算符为 (同时也是 Σ 算符的定义):

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{3.78}$$

观察自旋算符和 Dirac 哈密顿量之间的对易关系:

$$\left[H_D, S_x\right] = \left[\left(\boldsymbol{\alpha}\cdot\mathbf{P} + m_0\beta\right), \frac{1}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\right] \tag{3.79}$$

考虑到:

$$\begin{cases} \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \end{cases} \tag{3.80}$$

于是:

$$上式 = \left[\alpha_x P_x + \alpha_y P_y + \alpha_z P_z, \frac{1}{2}\Sigma_x\right] \tag{3.81}$$

考虑到 Pauli 矩阵之间的对易性质:

$$\left\{\left[\sigma_i, \sigma_j\right] = 2i\epsilon_{ijk}\epsilon_k\right. \tag{3.82}$$

得到:

$$\left\{[\alpha_i, \Sigma_x] = \begin{pmatrix} 0 & \sigma_i\sigma_x - \sigma_x\sigma_i \\ \sigma_i\sigma_x - \sigma_x\sigma_i & 0 \end{pmatrix} = 2i\epsilon_{ixk}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = 2i\epsilon_{ixk}\alpha_k\right. \tag{3.83}$$

于是:

$$上式 = -iP_y\alpha_z + iP_z\alpha_y = i\left(\boldsymbol{\alpha} \times \boldsymbol{P}\right)_x \tag{3.84}$$

总结起来:

$$[H_D, \boldsymbol{S}] = i(\boldsymbol{\alpha} \times \boldsymbol{P}) \tag{3.85}$$

轨道角动量与 H_D 之间的对易关系 轨道角动量算符定义为

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P} \tag{3.86}$$

和哈密顿量的对易:

$$\begin{aligned}
[H_D, \boldsymbol{L}] &= \left[\alpha_x P_x + \alpha_y P_y + \alpha_z P_z, \boldsymbol{L}\right] \\
[H_D, L_x] &= \left[\alpha_x P_x + \alpha_y P_y + \alpha_z P_z, yP_z - zP_y\right] \\
&= -i\alpha_y P_z + i\alpha_z P_y \\
&= -i\left(\boldsymbol{\alpha} \times P\right)_x
\end{aligned} \tag{3.87}$$

合理推断:

$$[H_D, \boldsymbol{L}] = -i\left(\boldsymbol{\alpha} \times P\right) \tag{3.88}$$

角动量算符和 H_D 之间的对易关系

$$[H_D, \boldsymbol{J}] = 0 \tag{3.89}$$

S 与 L 之间的对易关系 首先观察 $\boldsymbol{S}$ 之间的对易关系:

$$\left[S_i, S_j\right] = \frac{1}{4}\left[\Sigma_i, \Sigma_j\right] = i\frac{1}{2}\epsilon_{ijk}\Sigma_k = i\epsilon_{ijk}S_k \tag{3.90}$$

$\boldsymbol{S}$ 和 $\boldsymbol{L}$ 之间是对易的:

$$[S_i, L_k] = 0 \tag{3.91}$$

最后观察 $\boldsymbol{L}$ 之间的对易关系:

$$\left[L_i, L_j\right] = i\epsilon_{ijl}L_k \tag{3.92}$$

自旋 $\frac{1}{2}$ 发现:

$$|\boldsymbol{S}|^2 = \frac{1}{4}|\boldsymbol{\Sigma}|^2 = \frac{1}{4}\left(1+1+1\right)\mathbb{I} = \frac{3}{4}\mathbb{I} \tag{3.93}$$

于是可以改写 Feynman

$$s = \frac{1}{2} \quad s(s+1) = \frac{3}{4} \tag{3.94}$$

共同本征态

能量和动量共同本征态 考虑 Dirac 方程:

$$i\hbar\frac{\partial\psi}{\partial t} = H_D\psi = (\boldsymbol{\alpha}\cdot\mathbf{P} + m_0\beta)\,\psi \tag{3.95}$$

可以发现 Dirac 能量算符 H_D 和动量算符 $\boldsymbol{P}$ 是对易的。所以可以找到他们的共同本征态。认为:

$$\psi(x) = \begin{bmatrix}\varphi_0\\ \chi_0\end{bmatrix}\exp(i\boldsymbol{p}\cdot\boldsymbol{x})\exp(-i\epsilon t) \tag{3.96}$$

$$H_D\psi(x) = \epsilon\psi(x) \quad \boldsymbol{P}\psi(x) = \boldsymbol{p}\psi(x) \tag{3.97}$$

现在带入 Dirac 方程得到:

$$\begin{aligned}\epsilon\begin{bmatrix}\varphi_0\\ \chi_0\end{bmatrix} &= \left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + m_0\beta\right)\begin{bmatrix}\varphi_0\\ \chi_0\end{bmatrix}\\ &= \left(\begin{bmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{bmatrix}\cdot\boldsymbol{p} + m_0\begin{bmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{bmatrix}\right)\begin{bmatrix}\varphi_0\\ \chi_0\end{bmatrix}\end{aligned} \tag{3.98}$$

得到线性方程组:

$$\begin{aligned}(\epsilon - m_0)\ \varphi_0 - \boldsymbol{\sigma}\cdot\boldsymbol{p}\ \chi_0 = 0\\ -\boldsymbol{\sigma}\cdot\boldsymbol{p}\ \varphi_0 + (\epsilon + m_0)\ \chi_0 = 0\end{aligned} \tag{3.99}$$

要得到不为 0 的解，行列式是 0:

$$\det\begin{bmatrix}(\epsilon - m_0) & -\boldsymbol{\sigma}\cdot\boldsymbol{p}\\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & (\epsilon + m_0)\end{bmatrix} = 0 \tag{3.100}$$

考虑到 Pauli 矩阵的性质:

$$\left\{(\boldsymbol{\sigma}\cdot\boldsymbol{A})\,(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}\mathbb{I} + i\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B})\right. \tag{3.101}$$

得到:

$$\epsilon^2 = m_0^2 + \boldsymbol{p}^2 \Rightarrow \epsilon = \lambda E_p \quad \left(E_p = \sqrt{m_0^2 + \boldsymbol{p}^2};\ \ \lambda = \pm 1\right) \tag{3.102}$$

于是:

$$\chi_0 = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0 + \epsilon}\,\varphi_0 \quad \text{可以假定}\varphi_0 = U \text{ 满足:}U^\dagger U = 1 \tag{3.103}$$

最终得到能量和动量共同本征态的 Spinor 是:

$$\Psi_{\boldsymbol{p},\lambda}(x) = N_{\boldsymbol{p},\lambda}\begin{pmatrix}U\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p}\ U\end{pmatrix}\frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \tag{3.104}$$

可以验证，为了满足归一化和正交条件:(通过直接计算，正交已经满足。不过需要通过调整 N 的值来归一化)

$$\left\{\int d^3x\ \Psi^\dagger_{\boldsymbol{p},\lambda}(x)\Psi_{\boldsymbol{p}',\lambda'}(x) = \delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')\right. \tag{3.105}$$

得到:

$$N = \sqrt{\frac{m_0 + \lambda E_p}{2\lambda E_p}} \tag{3.106}$$

能量/动量/螺旋度的共同本征态 定义螺旋度 Helicity:

$$\Lambda_S = \boldsymbol{S}\cdot\frac{\boldsymbol{P}}{|\boldsymbol{p}|} \tag{3.107}$$

考虑到:

$$\left\{H_D = \boldsymbol{\alpha}\cdot\boldsymbol{P} + m_0\beta \quad \boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\Sigma}\right. \tag{3.108}$$

对易关系:

$$[H_D, \boldsymbol{\Sigma}\cdot\boldsymbol{P}] = [\boldsymbol{\alpha}\cdot\boldsymbol{P} + m_0\beta, \boldsymbol{\Sigma}\cdot\boldsymbol{P}] \tag{3.109}$$

考虑到:

$$\begin{cases}\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}\end{cases} \tag{3.110}$$

$$\begin{aligned}
\text{上式} &= [\boldsymbol{\alpha}\cdot\boldsymbol{P}, \boldsymbol{\Sigma}\cdot\boldsymbol{P}] \\
&= \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{P} \\ \boldsymbol{\sigma}\cdot\boldsymbol{P} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{P} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{P} \end{pmatrix} - \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{P} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{P} \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{P} \\ \boldsymbol{\sigma}\cdot\boldsymbol{P} & 0 \end{pmatrix} \\
&= 0
\end{aligned} \tag{3.111}$$

考虑螺旋度的平方:

$$\Lambda_S^2 = (\boldsymbol{S}\cdot\boldsymbol{P})^2 \frac{1}{|\boldsymbol{p}|^2} = \frac{1}{4} \qquad \{(\boldsymbol{\Sigma}\cdot\boldsymbol{A})(\boldsymbol{\Sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}\mathbb{I} + i\boldsymbol{\Sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B})\} \tag{3.112}$$

所以如果 spinor 是螺旋度的本征态,

$$\Lambda_S\Psi(x) = \lambda_S\Psi(x) \tag{3.113}$$

那么:

$$\lambda_S = \pm\frac{1}{2} \tag{3.114}$$

所以 $\Lambda_S, H_D, \boldsymbol{P}$ 可以拥有共同的本征态。接下来要寻找他们的共同本征态。前面已经得到了能量和动量的共同本征态:

$$\Psi_{\boldsymbol{p},\lambda}(x) = N\begin{pmatrix} U \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p} U \end{pmatrix} \frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \qquad N = \sqrt{\frac{m_0+\lambda E_p}{2\lambda E_p}} \tag{3.115}$$

要通过合理的选取 U, 使得它也是螺旋度的本征态:

$$\Lambda_S\Psi_{\boldsymbol{p},\lambda}(x) = \pm\frac{1}{2}\Psi_{\boldsymbol{p},\lambda}(x) \tag{3.116}$$

$$\begin{aligned}
\frac{1}{2|\boldsymbol{p}|}\boldsymbol{\Sigma}\cdot\boldsymbol{p}\begin{pmatrix} U \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p} U \end{pmatrix} &= \pm\frac{1}{2}\begin{pmatrix} U \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p} U \end{pmatrix} \\
\frac{1}{2|\boldsymbol{p}|}\begin{pmatrix} \boldsymbol{\sigma}\cdot p & 0 \\ 0 & \boldsymbol{\sigma}\cdot p \end{pmatrix}\begin{pmatrix} U \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p} U \end{pmatrix} &= \pm\frac{1}{2}\begin{pmatrix} U \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p} U \end{pmatrix}
\end{aligned} \tag{3.117}$$

研究列向量中的前两个:

$$\frac{1}{2|\boldsymbol{p}|}\boldsymbol{\sigma}\cdot\boldsymbol{p}U = \pm\frac{1}{2}U \tag{3.118}$$

考虑到:

$$\begin{cases}\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\end{cases} \tag{3.119}$$

带入得到:

$$\frac{1}{2|\boldsymbol{p}|}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & p_z \end{pmatrix} U = \pm\frac{1}{2}U \tag{3.120}$$

认为:

$$\begin{cases}p_z = |\boldsymbol{p}|\cos\theta \quad p_x = |\boldsymbol{p}|\sin\theta\cos\phi \quad p_y = |\boldsymbol{p}|\sin\theta\sin\phi\end{cases} \tag{3.121}$$

带入得到:

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} U = \pm U \tag{3.122}$$

并且要让 U 满足归一关系:

$$\left\{U^{\dagger}U = 1\right. \tag{3.123}$$

得到:

$$(\pm 1 - \cos\theta)U_1 = \sin\theta e^{-i\phi}U_2 \tag{3.124}$$

向上螺旋态 $\lambda_S = 1/2$ 此时:

$$(1 - \cos\theta)U_1 = \sin\theta e^{-i\phi}U_2 \tag{3.125}$$

考虑到 U 向量的归一化条件:

$$\begin{aligned}
U_1U_1^*\left(1 + \frac{(1-\cos\theta)^2}{\sin^2\theta}\right) = 1 \\
U_1U_1^* = \cos^2\frac{\theta}{2} \\
U_1 = \cos\frac{\theta}{2} \\
U = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}
\end{aligned} \tag{3.126}$$

$$\Psi_{\boldsymbol{p},\lambda,\uparrow}(x) = N\begin{pmatrix} U \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m_0+\lambda E_p}\ U \end{pmatrix}\frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \quad N = \sqrt{\frac{m_0 + \lambda E_p}{2\lambda E_p}} \tag{3.127}$$

带入 U, 化简之后得到:

$$\Psi_{\boldsymbol{p},\lambda,\uparrow}(x) = N\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\cos\frac{\theta}{2} \\ \frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}\frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \quad N = \sqrt{\frac{m_0 + \lambda E_p}{2\lambda E_p}} \tag{3.128}$$

向下螺旋态 $\lambda_s = -1/2$ 此时:

$$(-1 - \cos\theta)U_1 = \sin\theta e^{-i\phi}U_2 \tag{3.129}$$

考虑到 U 向量的归一化条件:

$$U_1U_1^*\left(1 + \frac{(1+\cos\theta)^2}{\sin^2\theta}\right) = 1 \tag{3.130}$$

得到:

$$U_1U_1^* = \sin^2\frac{\theta}{2} \tag{3.131}$$

于是:

$$U = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \tag{3.132}$$

于是，向上螺旋态的 spinor 是:

$$\Psi_{\boldsymbol{p},\lambda,\downarrow}(x) = N\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\sin\frac{\theta}{2} \\ -\frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}\frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \quad N = \sqrt{\frac{m_0 + \lambda E_p}{2\lambda E_p}} \tag{3.133}$$

至此，确定了同时是 $H_D, \boldsymbol{P}, \Lambda_S$ 的本征态 Spinor:

$$\Psi_{\boldsymbol{p},\lambda,\uparrow}(x) = N \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\cos\frac{\theta}{2} \\ \frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \quad N = \sqrt{\frac{m_0 + \lambda E_p}{2\lambda E_p}} \tag{3.134}$$

$$\Psi_{\boldsymbol{p},\lambda,\downarrow}(x) = N \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\sin\frac{\theta}{2} \\ -\frac{|\boldsymbol{p}|}{m_0+\lambda E_p}\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \frac{\exp\left(i(\boldsymbol{p}\cdot\boldsymbol{x} - \lambda E_p t)\right)}{(2\pi\hbar)^{3/2}} \quad N = \sqrt{\frac{m_0 + \lambda E_p}{2\lambda E_p}} \tag{3.135}$$

并且满足归一化条件:

$$\int d^3x\, \Psi^\dagger_{\boldsymbol{p},\lambda,\lambda_S}(x)\Psi_{\boldsymbol{p}',\lambda',\lambda'_S}(x) = \delta_{\lambda_S,\lambda'_S}\delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') \tag{3.136}$$

这里不看 在 Mark Thomson 的粒子物理里面，直接定义了 normalized particle Spinor 和 normalized anti-particle Spinor。首先是 particle spinor:

$$u_\uparrow = \sqrt{E_p + m_0}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0+E_p}\cos\frac{\theta}{2} \\ \frac{|\boldsymbol{p}|}{m_0+E_p}\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E_p + m_0}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0+E_p}\sin\frac{\theta}{2} \\ -\frac{|\boldsymbol{p}|}{m_0+E_p}\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \tag{3.137}$$

至于 anti-particle Spinor 注意到:

$$\sqrt{\frac{E_p - m_0}{2E_p}}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0-E_p}\cos\frac{\theta}{2} \\ \frac{|\boldsymbol{p}|}{m_0-E_p}\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = \sqrt{\frac{E_p + m_0}{2E_p}}\begin{pmatrix} \frac{|\boldsymbol{p}|}{m_0+E_p}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \quad ; \quad \sqrt{\frac{E_p - m_0}{2E_p}}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \\ \frac{|\boldsymbol{p}|}{m_0-E_p}\sin\frac{\theta}{2} \\ -\frac{|\boldsymbol{p}|}{m_0-E_p}\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} = \sqrt{\frac{E_p + m_0}{2E_p}}\begin{pmatrix} -\frac{|\boldsymbol{p}|}{m_0+E_p}\sin\frac{\theta}{2} \\ \frac{|\boldsymbol{p}|}{m_0+E_p}\cos\frac{\theta}{2}e^{i\phi} \\ -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \tag{3.138}$$

定义 anti-particle helicity spinor:

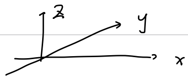
$$v_\uparrow = \sqrt{E + m_0}\begin{pmatrix} \frac{|\boldsymbol{p}|}{m_0+E_p}\sin\frac{\theta}{2} \\ -\frac{|\boldsymbol{p}|}{m_0+E_p}\cos\frac{\theta}{2}e^{i\phi} \\ \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E_p + m_0}\begin{pmatrix} \frac{|\boldsymbol{p}|}{m_0+E_p}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \tag{3.139}$$

到这里，和 Mark Thomson 对不上

Solution of Dirac Equation from Lorentz Transformation!

Plane Wave in arbitrary direction.

∘ Rest Frame solution(静粒子解)



静系 rest frame 中静止自由粒子 Dirac Equation:

$$\begin{cases} (i\gamma^\mu \partial_\mu - m_0)\psi = 0 \\ (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m)\psi = E_D\psi = i\frac{\partial}{\partial t}\psi \end{cases}$$

rest condition: β具体表示:

$$\Big\}\ \hat{\vec{p}}\,\psi^r = 0 \qquad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$\beta m\ \psi^r = i\frac{\partial}{\partial t}\psi^r$$

$$m\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\psi^r = i\frac{\partial}{\partial t}\psi^r$$

$$\Downarrow$$

$$\psi^r = w^r(0)\cdot \exp(-i\varepsilon_r\cdot m_0 t)$$

$$w^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad w^3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad w^4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\varepsilon_r = \begin{cases} +1 & r = 1 \text{ or } 2 \text{ 正能解} \\ -1 & r = 3 \text{ or } 4 \text{ 负能解.} \end{cases}$$

∘ plane Wave for moving particle. 由静止粒子解到运动粒子解.

— S 和 Λ 间关系.

Minimal Lorentz Transformation leads to S op:

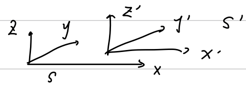
$$S = \mathbb{I} - \frac{i}{4}\cdot \sigma_{\mu\nu}\cdot \Delta w^{\mu\nu}$$

Minimal Lorentz Transformation represented by Generator.

$$\Lambda(\Delta w)^{\mu\nu} = \mathbb{I} + \Delta w^{\mu\nu} = \mathbb{I} + w_1 I_1 \cdots + w_6 I_6$$

不考虑转动: → Lorentz Trans "$\frac{w}{N}$"

$$\begin{cases} \Lambda(\Delta w) = \mathbb{I} + \Delta w = \mathbb{I} + w_1 I_1 + w_2 I_2 + w_3 I_3 \\ \qquad\qquad = \mathbb{I} + \frac{w}{N}(\cos\alpha I_1 + \cos\beta I_2 + \cos\gamma\cdot I_3) \\ S(\Delta w) = \mathbb{I} - \frac{i}{4}\sigma_{\mu\nu}\left(\frac{w}{N}(\cos\alpha I_1^{\mu\nu} + \cos\beta I_2^{\mu\nu} + \cos\gamma I_3^{\mu\nu})\right) \end{cases}$$



代表 S' 相对 S velocity $\frac{w}{N}(\cos\alpha, \cos\beta, \cos\gamma)$ (N 很大)

将上面微变动做 N 次

$$\begin{cases} \Lambda(w) = \exp(w(\cos\alpha I_1 + \cos\beta I_2 + \cos\gamma I_3)) \\ S(w) = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\, w\cdot(\cos\alpha I_1^{\mu\nu} + \cos\beta I_2^{\mu\nu} + \cos\gamma I_3^{\mu\nu})\right) \end{cases}$$

Lorentz Trans 中 w 和 β 的关系:

$\tanh(w) = \beta$ (若仅有 V_x, 易看出.) ← Latex Lorentz Trans 知识!

Lorentz Trans 生成元 operator

$$I_1^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad I_2^{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad I_3^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

— σ 表达　和理的 $\sigma_{\mu\nu}$ 表达式: (Latex QFT 知识)

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \qquad \gamma^0 = \beta \quad \gamma^1 = \beta\alpha_1 \quad \gamma^2 = \beta\alpha_2 \quad \gamma^3 = \beta\alpha_3$$

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

— S(w) 的表达.

$$S(w) = \exp\left[-\frac{i}{4} w\, \sigma_{\mu\nu} \cdot \begin{pmatrix} 0 & \cos\alpha & \cos\beta & \cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \exp\left(-\frac{i}{4} w \cdot 2 \cdot (\sigma_{01}\cos\alpha + \sigma_{02}\cos\beta + \sigma_{03}\cos\gamma)\right)$$

由 $\sigma_{\mu\nu}$ 的定义:

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\sigma_{0i} = \frac{i}{2}[\gamma_0, \gamma_i] = -\frac{i}{2}[\gamma^0, \gamma^i] = -\frac{i}{2}[\beta, \beta\alpha_i] = -\frac{i}{2}(\beta\beta\alpha_i - \beta\alpha_i\beta)$$

$$\{\alpha_i, \beta\} = 0$$

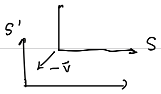
$$\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1, \quad \{\alpha_i, \alpha_j\} = 0$$

$$= -\frac{i}{2}(\alpha_i + \alpha_i) = -i\alpha_i$$

$$S(w) = \exp\left(-\frac{i}{4} w \cdot 2 \cdot (-i)(\alpha_x \cos\alpha + \alpha_y \cos\beta + \alpha_z \cos\gamma)\right)$$

$$= \exp\left(+\frac{1}{2} w \cdot \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)$$

其中, v 是在 S' 系中 Dirac 粒子的运动 velocity! $\left((\cos\alpha, \cos\beta, \cos\gamma) = -\frac{\vec{v}}{|\vec{v}|} \quad \tanh w = \beta > 0\right)$



— S(w) 的具体表达

$$S(w) = \exp\left(+\frac{1}{2} w\, \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right) = 1 + \frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|} + \frac{1}{2!}\left(\frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^2 \cdots$$

$$= 1 + \frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|} + \frac{1}{2} \frac{w^2}{4|\vec{v}|^2} (\vec{\alpha} \cdot \vec{v})^2 + \cdots$$

$$(\vec{\alpha} \cdot \vec{v})^2 = (\alpha_i V_i)(\alpha_j V_j) = \alpha_i \alpha_j V_i V_j = \frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) V_i V_j$$

$$= \frac{1}{2}\{\alpha_i, \alpha_j\} V_i V_j$$

$$= \frac{1}{2} \cdot 2 \cdot \delta_{ij} \cdot V_i V_j$$

$$= V_i V_i$$

$$= |\vec{V}|^2$$

$$\left(\vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^2 = 1$$

$$= 1 + \frac{1}{1!} \cdot \left(\frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right) + \frac{1}{3!}\left(\frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^3 + \frac{1}{5!} \cdot \left(\frac{1}{2} \cdot w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^5 \cdots$$

$$+ \frac{1}{2!}\left(\frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^2 + \frac{1}{4!} \cdot \left(\frac{1}{2} \cdot w \cdot \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^4 + \frac{1}{6!}\left(\frac{1}{2} w \vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)^6 \cdots$$

$$= 1 + \left(\vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right)\left(\frac{1}{2} w + \frac{1}{3!}\left(\frac{1}{2} w\right)^3 + \frac{1}{5!}\left(\frac{1}{2} w\right)^5 \cdots\right)$$

$$+ \left(\frac{1}{2!} \cdot \left(\frac{1}{2} w\right)^2 + \frac{1}{4!}\left(\frac{1}{2} w\right)^4 + \frac{1}{6!}\left(\frac{1}{2} w\right)^6 \cdots\right)$$

$$= \left(\vec{\alpha} \cdot \frac{\vec{v}}{|\vec{v}|}\right) \cdot \sinh\left(\frac{1}{2} w\right) + \cosh\left(\frac{1}{2} w\right)$$

$$\tanh(w) = \beta = \frac{\sinh(w)}{\cosh(w)} \;;\; 2\cosh\left(\frac{w}{2}\right)\sinh\left(\frac{w}{2}\right) = \sinh(w) \;;\; \cosh(w) = 2\cosh^2\left(\frac{w}{2}\right) - 1 = 2\sinh^2\left(\frac{w}{2}\right) + 1$$

$$\cosh^2\left(\frac{w}{2}\right) = 1 + \sinh^2\left(\frac{w}{2}\right)$$

$$\beta = \frac{2\cosh\left(\frac{w}{2}\right)\sinh\left(\frac{w}{2}\right)}{2\cosh^2\left(\frac{w}{2}\right) - 1}$$

$$= \frac{2\cosh\left(\frac{w}{2}\right)\sqrt{\cosh^2\left(\frac{w}{2}\right) - 1}}{2\cosh^2\left(\frac{w}{2}\right) - 1}$$

$$\beta^2\left(2\cosh^2\left(\frac{w}{2}\right)-1\right)^2 = 4\cosh^2\left(\frac{w}{2}\right)\left(\cosh^2\left(\frac{w}{2}\right)-1\right)$$

$$\beta^2(2x-1)^2 = 4x\cdot(x-1) \qquad \cosh^2\left(\frac{w}{2}\right)=x$$

$$\beta^2(4x^2+1-4x) = 4x^2-4x$$

$$4(1-\beta^2)x^2+4(\beta^2-1)x-\beta^2=0$$

$$x = \frac{-4(\beta^2-1)\pm\sqrt{16(\beta^2-1)^2+16\beta^2(1-\beta^2)}}{8(1-\beta^2)}$$

$$= \frac{-4(\beta^2-1)\pm\sqrt{16\beta^4+16-32\beta^2+16\beta^2-16\beta^4}}{8(1-\beta^2)}$$

$$= \frac{-4(\beta^2-1)\pm\sqrt{16(1-\beta^2)}}{8(1-\beta^2)}$$

$$= \frac{4\pm4\sqrt{1-\beta^2}\,\frac{1}{1-\beta^2}}{8}$$

$$= \frac{1}{2}\left(1\pm\frac{1}{\sqrt{1-\beta^2}}\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-\beta^2}}\right)$$

$$\cosh^2\left(\frac{w}{2}\right) = \frac{1}{2}\left(1+\frac{E}{m_0}\right) = \frac{E+m_0}{2m_0}$$

$$\cosh\left(\frac{w}{2}\right) = \sqrt{\frac{E+m_0}{2m_0}}$$

$$\beta = \tanh(w) = \frac{2\cosh\left(\frac{w}{2}\right)\sinh\left(\frac{w}{2}\right)}{\cosh^2\left(\frac{w}{2}\right)+\sinh^2\left(\frac{w}{2}\right)} = \frac{2\tanh\left(\frac{w}{2}\right)}{1+\tanh^2\left(\frac{w}{2}\right)}$$

$$\Downarrow\ x=\tanh(w/2)$$

$$\beta\cdot x^2-2x+\beta=0$$

$$x = \frac{4\pm\sqrt{4-4\beta^2}}{2\beta}$$

$$= \frac{1\pm\sqrt{1-\beta^2}}{\beta} \qquad \left(\because \tanh\frac{w}{2}<1\right)$$

$$= \frac{1-\sqrt{1-\beta^2}}{\beta} = \frac{1-\frac{m_0}{E}}{|\vec{p}|\cdot\frac{1}{E}} = \frac{E-m_0}{|\vec{p}|}$$

$$\tanh\left(\frac{w}{2}\right) = \frac{E-m_0}{|\vec{p}|} = \frac{(E-m_0)|\vec{p}|}{|\vec{p}|^2} = \frac{|\vec{p}|}{E+m_0}$$

$$\cosh\left(\frac{w}{2}\right) = \sqrt{\frac{E+m}{2m}} \qquad \sinh\left(\frac{w}{2}\right) = \cosh\left(\frac{w}{2}\right)\tanh\left(\frac{w}{2}\right) = \sqrt{\frac{E+m}{2m}}\cdot\frac{|\vec{p}|}{E+m_0}$$

$$= \left(\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\cdot\sinh\left(\frac{w}{2}\right)+\cosh\left(\frac{w}{2}\right) \qquad \alpha_i=\begin{pmatrix}0&\sigma_i\\\sigma_i&0\end{pmatrix} \quad \alpha_x=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \quad \alpha_y=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix} \quad \alpha_z=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\cdot\mathbb{I}+\sqrt{\frac{E+m}{2m}}\cdot\frac{|\vec{p}|}{E+m}\cdot\begin{pmatrix}0&0&\frac{p_z}{|p|}&\frac{p_x-ip_y}{|p|}\\0&0&\frac{p_x+ip_y}{|p|}&-\frac{p_z}{|p|}\\\frac{p_z}{|p|}&\frac{p_x-ip_y}{|p|}&0&0\\\frac{p_x-ip_y}{|p|}&-\frac{p_z}{|p|}&0&0\end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\begin{pmatrix}1&0&\frac{p_z}{E+m}&\frac{p_-}{E+m}\\0&1&\frac{p_+}{E+m}&-\frac{p_z}{E+m}\\\frac{p_z}{E+m}&\frac{p_-}{E+m}&1&0\\\frac{p_+}{E+m}&-\frac{p_z}{E+m}&0&-1\end{pmatrix} \qquad \left\}\begin{aligned}p_+&=p_x+ip_y\\p_-&=p_x-ip_y\end{aligned}\right.$$

对于 S' 系中的 spinor (表示 $\vec{p}$ 粒子的 spinor):

$$\psi^{r\prime}(x') = S\cdot\psi^r(x)$$

$$= S\ w^r(0)\cdot\exp(-i\,\varepsilon_r\cdot m_0 t)$$

$$= S\ w^r(0)\cdot\exp(-i\,\varepsilon_r\cdot p_\mu x^\mu) = S\,w^r(0)\cdot\exp(-i\,\varepsilon_r\,p'_\mu x'^\mu)$$

这里 p_μ 和 p'_μ 是同一个.

$$= \sqrt{\frac{E+m}{2m}} \cdot \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & -\frac{p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix} \cdot W^r(0) \cdot \exp(-i\varepsilon_r p_\mu x'^\mu)$$

$$= W^r(\vec{p}) \exp(-i\varepsilon_r p_\mu \cdot x'^\mu)$$

$$[W^1(\vec{p}), W^2(\vec{p}), W^3(\vec{p}), W^4(\vec{p})] = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & -\frac{p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix}$$

- $W^r(\vec{p})$ 满足： $(\not{p} - \varepsilon_r m) W^r(\vec{p}) = 0$

proof:

在 S' 系中, Dirac Equation 能够得到满足!

$$i(\gamma^\mu \partial_\mu - m)\psi = 0$$

而：

$$\psi^r(x) = W^r(\vec{p}) \cdot \exp(-i\varepsilon_r p_\mu x^\mu)$$

1° $\varepsilon_r = +1$ $(r = 1 \text{ or } 2)$:

$$(\gamma^\mu p_\mu - m) W^r(\vec{p}) = 0$$

2° $\varepsilon_r = -1$ $(r = 3 \text{ or } 4)$:

$$(-\gamma^\mu p_\mu - m) W^r(\vec{p}) = 0 \Rightarrow (\gamma^\mu p_\mu + m) W^r(\vec{p}) = 0$$

$$(\gamma^\mu p_\mu - \varepsilon_r m) W^r(\vec{p}) = 0 \qquad (\not{p} - \varepsilon_r m) W^r(\vec{p}) = 0$$

End of Proof.

- $W^r(\vec{p})$ 满足： $\overline{W}^r(\vec{p}) (\not{p} - \varepsilon_r m_0 c) = 0$

Proof： 由上一个 $W^r(\vec{p})$ 的性质：

$$(\not{p} - \varepsilon_r m) W^r(\vec{p}) = 0$$

Complex Conjugate :

$$W^{r\dagger}(\vec{p}) (\not{p} - \varepsilon_r m)^\dagger = 0$$

$$W^{r\dagger}(\vec{p}) (\gamma^{\mu\dagger} p_\mu - \varepsilon_r m) = 0$$

$$\Downarrow \longleftarrow \; \gamma^{0\dagger} = \gamma^0 \quad \gamma^{k\dagger} = -\gamma^k$$

$$W^{r\dagger}(\vec{p}) (\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3 - \varepsilon_r m) = 0$$

$$\Downarrow \leftarrow \; \gamma^0 \cdot \gamma^0 = \mathbb{I} \quad \{\gamma^0, \gamma^k\} = 0$$

$$W^{r\dagger}(\vec{p}) (\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3 - \varepsilon_r m) \cdot \gamma^0$$

$$= W^{r\dagger}(\vec{p}) \gamma^0 (\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 - \varepsilon_r m)$$

$$= \overline{W}(\vec{p}) \; (\not{p} - m_0) = 0$$

End of Proof

- Normalisation relation (Direct calculation!)

$$\overline{w}^{r}(\vec{p})\, w^{r'}(\vec{p}) = \delta_{rr'}\,\varepsilon_{r}$$
$$w^{r\dagger}(\varepsilon_r\vec{p})\, w^{r'}(\varepsilon_{r'}\vec{p}) = \frac{E}{m}\,\delta_{rr'}$$

这样的正交性保证了粒子spinor之间的正交性：

$$\psi_{\vec{p}}^{1,2}(x) = w^{1,2}(\vec{p})\exp(-iP_0x^0 + i\vec{p}\cdot\vec{x}) \rightarrow \text{能量 } P_0,\ \text{动量 } \vec{p}$$
$$\psi_{-\vec{p}}^{3,4}(x) = w^{3,4}(-\vec{p})\exp(iP_0x^0 + i\vec{p}\cdot\vec{x}) \rightarrow \text{能量} -P_0,\ \text{动量 } \vec{p}$$

$$\langle \psi_{\vec{p}'}^{r'} | \psi_{\vec{p}}^{r} \rangle = \int d^3x\ w^{r'\dagger}(\vec{p}')\, w^{r}(\vec{p})\cdot\exp(-i(\varepsilon_r p_\mu x^\mu - \varepsilon_{r'} p'_\mu x^\mu))$$
$$= w^{r'\dagger}(\vec{p}')\, w^{r}(\vec{p})\cdot(2\pi)^3\cdot\delta^{(3)}(\varepsilon_r\vec{p} - \varepsilon_{r'}\vec{p}')\cdot\exp(-i(\varepsilon_r P_0 - \varepsilon_{r'}P_0')x^0)$$
$$= \frac{E}{m}(2\pi)^3\cdot\delta^{(3)}(\vec{p}' - \vec{p})\,\delta_{rr'}$$

- closure relation：(2.4)

$$\sum_{r=1}^{4} \varepsilon_r\, w_\alpha^{r}(\vec{p})\, \overline{w}_\beta^{r}(\vec{p}) = \delta_{\alpha\beta}$$
$$\sum_{r=1}^{4} w_\alpha^{r}(\varepsilon_r\vec{p})\, w_\beta^{r\dagger}(\varepsilon_r\vec{p}) = \frac{E}{m}\,\delta_{\alpha\beta}$$

3.6 用 Lorentz 变换从静粒子波函数到动粒子波函数

静止系中的波函数 假设在静止参考系中的波函数写为:

$$\psi^r(x) = w^r(0)\exp(-i\epsilon_r p^\mu_{r.s}x_\mu) \quad \epsilon_r = 1 \ \text{ for } \ r = 1,2 \ \ \epsilon_r = -1 \ \text{ for } \ r = 3,4 \tag{3.140}$$

因为是在静止参考系中，此时:

$$p_{r.s} = (m, 0) \tag{3.141}$$

在静止参考系中定义四维向量 s

$$s_{r.s} = (0, \boldsymbol{s}_{r.s.}) \tag{3.142}$$

定义 (先不写具体形式，知道有这个东西，u 对应的的能量本征值是正，v 对应的的能量本征值是负):

$$\begin{gathered} w^1(0) = u(p_{r.s.}, \boldsymbol{s}_{r.s.}) \quad w^2(0) = u(p_{r.s.}, -\boldsymbol{s}_{r.s.}) \\ w^3(0) = v(p_{r.s.}, -\boldsymbol{s}_{r.s.}) \quad w^4(0) = v(p_{r.s.}, \boldsymbol{s}_{r.s.}) \end{gathered} \tag{3.143}$$

当然，也需要满足 Dirac-Equation:

$$i\hbar\frac{\partial\psi}{\partial t} = H_D\psi = (\boldsymbol{\alpha}\cdot\mathbf{P} + m_0\beta)\,\psi \tag{3.144}$$

其中:

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{3.145}$$

将静止系中 spinor 定义为特定方向自旋的本征值 已知自旋算符 $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma}$。要求上面定义的 spinor 满足:

$$\begin{gathered} \frac{1}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{s}_{r.s.}u(p_{r.s.}, \pm\boldsymbol{s}_{r.s.}) = \pm\frac{1}{2}u(p_{r.s.}, \pm\boldsymbol{s}_{r.s.}) \\ \frac{1}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{s}_{r.s.}v(p_{r.s.}, \pm\boldsymbol{s}_{r.s.}) = \mp\frac{1}{2}u(p_{r.s.}, \pm\boldsymbol{s}_{r.s.}) \end{gathered} \tag{3.146}$$

在静止系中，定义三维 s 向量是 (p 是在运动参考系中，粒子的动量)

$$\boldsymbol{s}_{r.s.} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \tag{3.147}$$

考虑到:

$$\begin{cases} \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ p_x = |\boldsymbol{p}|\sin\theta\cos\phi \quad p_y = |\boldsymbol{p}|\sin\theta\sin\phi \quad p_z = |\boldsymbol{p}|\cos\theta \end{cases} \tag{3.148}$$

于是:

$$\frac{1}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{s}_{r.s.} = \frac{1}{2}\frac{1}{|\boldsymbol{p}|}\begin{pmatrix} p_z & p_x - ip_y & 0 & 0 \\ p_x + ip_y & -p_z & 0 & 0 \\ 0 & 0 & p_z & p_x - ip_y \\ 0 & 0 & p_x + ip_y & -p_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} & 0 & 0 \\ \sin\theta e^{i\phi} & -\cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta e^{-i\phi} \\ 0 & 0 & \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{3.149}$$

考虑一个矩阵的本征态:

$$\begin{gathered} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \pm\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \\ U_1^*U_1 + U_2^*U_2 = 1 \end{gathered} \tag{3.150}$$

上面的子部分已经有了解:

$$U_+ = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{i\phi} \end{pmatrix} \quad U_- = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2e^{i\phi} \end{pmatrix} \tag{3.151}$$

意义在于:

$$w^1(0) = u(p_{r.s.}, \boldsymbol{s}_{r.s.}) = \begin{pmatrix} U_+ \\ 0 \end{pmatrix} \quad w^2(0) = u(p_{r.s.}, -\boldsymbol{s}_{r.s.}) = \begin{pmatrix} U_- \\ 0 \end{pmatrix}$$
$$w^3(0) = v(p_{r.s.}, -\boldsymbol{s}_{r.s.}) = \begin{pmatrix} 0 \\ U_+ \end{pmatrix} \quad w^4(0) = v(p_{r.s.}, \boldsymbol{s}_{r.s.}) = \begin{pmatrix} 0 \\ U_- \end{pmatrix} \tag{3.152}$$

运动参考系 Spinor 转换 具体过程见 ipad，对于相对于粒子静止的参考系，运动系以 $-\boldsymbol{v}$ 的速度相对运动（也就是粒子在运动系中的运动速度是 $\boldsymbol{v}$）。

首先，对于坐标变换以及动量变换:

$$(m, 0, 0, 0) \cdot x = p \cdot x' \tag{3.153}$$

其次, 在这个运动参考系中的 spinor 是:

$$\psi'(x') = S\psi(x) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & -\frac{p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix} \psi(x) \tag{3.154}$$

最终:

$$\psi'^r(x') = w^r(\boldsymbol{p}) \exp(-i\epsilon_r p^\mu x'_\mu) \tag{3.155}$$

其中:

$$\begin{pmatrix} w^1(\boldsymbol{p}) & w^2(\boldsymbol{p}) & w^3(\boldsymbol{p}) & w^4(\boldsymbol{p}) \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} U_+ & U_- & \frac{|\boldsymbol{p}|}{E+m} U_+ & -\frac{|\boldsymbol{p}|}{E+m} U_- \\ \frac{|\boldsymbol{p}|}{E+m} U_+ & -\frac{|\boldsymbol{p}|}{E+m} U_- & U_+ & U_- \end{pmatrix}$$
$$= \begin{pmatrix} u(p,s) & u(p,-s) & v(p,-s) & v(p,s) \end{pmatrix}$$
$$p = (\sqrt{|\boldsymbol{p}|^2 + m^2}, \boldsymbol{p})$$
$$U_+ = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 e^{i\phi} \end{pmatrix} \quad U_- = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 e^{i\phi} \end{pmatrix} \tag{3.156}$$

用 Lorentz 变化方法得到的波函数和前面直接求解本征态得到的波函数之间的关系是:

$$\psi'^1 = \sqrt{\frac{E}{m}} \Psi_{\boldsymbol{p},+,\uparrow} \quad \psi'^2 = \sqrt{\frac{E}{m}} \Psi_{\boldsymbol{p},+,\downarrow}$$
$$\psi'^3 = \sqrt{\frac{E}{m}} \Psi_{-\boldsymbol{p},-,\uparrow} \quad \psi'^4 = \sqrt{\frac{E}{m}} \Psi_{-\boldsymbol{p},-,\downarrow} \tag{3.157}$$

如何理解这个结果：利用波函数的坐标变化，也可以得到 H, P, Λ_S 的共同本征态。

Spin-1/2 Field.

Dirac Equation

- Dirac Equation　γ 是一个4维矩阵.

$$(i\gamma^{\mu}\partial_{\mu}-m)\psi=0$$

(5.2)式下面一段说 ψ 有4个变量,满足相对论旋量的变化规律.

Lagrangian (自带charge)

$$\mathcal{L}=\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi=i\psi^{\dagger}\dot{\psi}+i\psi^{\dagger}\vec{\alpha}\cdot\nabla\psi-m\psi^{\dagger}\beta\psi$$

$$\bar{\psi}=\psi^{\dagger}\gamma^{0}\qquad \beta=\gamma^{0}\quad \beta^{2}=1\qquad \vec{\alpha}=\gamma_{0}\cdot\vec{\gamma}$$

正则共轭场:

$$\pi_{\psi}=\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\psi^{\dagger}\qquad \pi_{\psi^{\dagger}}=\frac{\partial\mathcal{L}}{\partial\dot{\psi}^{\dagger}}=0$$

Hamiltonian

$$\mathcal{H}=\psi^{\dagger}(-i\vec{\alpha}\cdot\nabla+\beta m)\psi$$

$$H=\int d^{3}x\,\psi^{\dagger}(x,t)(-i\vec{\alpha}\cdot\vec{\nabla}+\beta m)\psi(x,t)$$

为保证 $\mathcal{L}^{*}=\mathcal{L}$

$$\mathcal{L}'\equiv\frac{1}{2}(\mathcal{L}+\mathcal{L}^{\dagger})=\frac{1}{2}\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi+\frac{1}{2}\bar{\psi}(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu}-m)\psi$$

$$=\frac{1}{2}\bar{\psi}i\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi-m\bar{\psi}\psi$$

$$\pi_{\psi}=\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=i\psi^{\dagger}\qquad \pi_{\psi^{\dagger}}=0$$

Quantization

- Equal - Time anti-commutation Rules.

$$\{\hat\psi_\alpha(x,t), \hat\psi^\dagger_\beta(x',t)\} = \delta_{\alpha\beta}\,\delta^3(x-x')$$

$$\{\hat\psi_\alpha(x,t), \hat\psi_\beta(x',t)\} = \{\hat\psi^\dagger_\alpha(x,t), \hat\psi^\dagger_\beta(x',t)\} = 0$$

- 平面波展开：

$$\psi^{(r)}_p(x,t) = (2\pi)^{-3/2}\cdot\sqrt{\frac{m}{\omega_p}}\cdot w_r(p)\cdot e^{-i\varepsilon_r(\omega_p t - \vec p\cdot\vec x)}$$

$$\begin{cases} r=1,2 & \varepsilon_r=+1 \\ r=3,4 & \varepsilon_r=-1 \end{cases} \qquad \omega_p = \sqrt{\vec p^2 + m^2}$$

$$(i\gamma^\mu\partial_\mu - m)\psi^{(r)}_p(x,t) = 0$$

$$(\gamma^\mu p_\mu - \varepsilon_r m)\,w_r(p) = 0$$

正交,归一化 $w_r(p)$ (Dirac Spinors)

$$w^\dagger_{r'}(\varepsilon_{r'}p)\,w_r(\varepsilon_r p) = \frac{\omega_p}{m}\delta_{rr'}$$

$$\bar w_{r'}(p)\,w_r(p) = \varepsilon_r\,\delta_{rr'}$$

$$\sum_{r=1}^{4} w_{r\alpha}(\varepsilon_r p)\,w^\dagger_{r\beta}(\varepsilon_r p) = \frac{\omega_p}{m}\delta_{\alpha\beta}$$

$$\sum_{r=1}^{4} \varepsilon_r\, w_{r\alpha}(p)\,\bar w_{r\beta}(p) = \delta_{\alpha\beta}$$

展开：

$$\hat\psi(x,t) = \sum_{r=1}^{4}\int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_p}}\cdot\hat a(p,r)\,w_r(p)\,e^{-i\varepsilon_r p\cdot x}$$

$$\hat H = \int d^3p\left(\sum_{r=1}^{2}\omega_p\,\hat a^\dagger(p,r)\hat a(p,r) - \sum_{r=3}^{4}\omega_p\,\hat a^\dagger(p,r)\hat a(p,r)\right)$$

取：

$$E_0 = -\sum_p\sum_{r=3}^{4}\omega_p$$

$$\begin{cases} \hat H' = \hat H - E_0 \\ \quad = \sum_p\left(\sum_{r=1}^{2}\omega_p\hat n_{pr} + \sum_{r=3}^{4}\omega_p\hat{\bar n}_{pr}\right) \qquad (\text{Dirac 之海}) \\ \hat{\bar n}_{pr} = 1 - \hat a^\dagger(p,r)\hat a(p,r) = \hat a(p,r)\hat a^\dagger(p,r) \end{cases}$$

Vacuum：

$$\hat n_{pr}|0\rangle = 0 \qquad r=1,2$$

重定义：

$$\hat{\bar n}_{pr}|0\rangle = 0 \qquad r=3,4$$

$$\begin{cases} u(p,+s) = w_1(p) \\ u(p,-s) = w_2(p) \\ v(p,-s) = w_3(p) \\ v(p,+s) = w_4(p) \end{cases} \qquad \begin{cases} \hat b(p,+s) = \hat a(p,1) \\ \hat b(p,-s) = \hat a(p,2) \\ \hat d^\dagger(p,-s) = \hat a(p,3) \\ \hat d^\dagger(p,+s) = \hat a(p,4) \end{cases}$$

$$\{\hat b(p,s), \hat b^\dagger(p',s')\} = \delta^{(3)}(p-p')\,\delta_{ss'} \qquad \{\hat d(p,s), \hat d^\dagger(p',s')\} = \delta^{(3)}(p-p')\,\delta_{ss'}$$

$$\hat\psi(x,t) = \sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_p}}\cdot\left(\hat b(p,s)\,u(p,s)\,e^{-ip\cdot x} + \hat d^\dagger(p,s)\cdot v(p,s)\,e^{ip\cdot x}\right)$$

Hamiltonian

$$\hat H' = \sum_s\int d^3p\;\omega_p\left(\hat b^\dagger(p,s)\hat b(p,s) + \hat d^\dagger(p,s)\hat d(p,s)\right)$$

生成反粒子.

Charge

$$\hat{Q} = e\int d^3x\, \hat{\psi}^\dagger(x)\hat{\psi}(x)$$

$$= e\sum_s \int d^3p\cdot\left(\hat{b}^\dagger(p,s)\hat{b}(p,s) + \hat{d}(p,s)\hat{d}^\dagger(p,s)\right)$$

Charge of Dirac sea

$$\hat{Q}_0 = e\sum_s\int d^3p\ \delta^{(3)}(0)$$

$$\hat{Q}' = \hat{Q} - Q_0 = e\sum_s\int d^3p\cdot\left(\hat{b}^\dagger(p,s)\hat{b}(p,s) - \hat{d}^\dagger(p,s)\hat{d}(p,s)\right)$$

Spin

$$\hat{\vec{S}}\cdot\frac{\vec{p}}{|p|} = \frac{1}{2}\int d^3p\left(\hat{b}^\dagger(p,s)\hat{b}(p,s) + \hat{d}^\dagger(p,s)\hat{d}(p,s) - \hat{b}^\dagger(p,-s)\hat{b}(p,-s) - \hat{d}^\dagger(p,-s)\hat{d}(p,-s)\right)$$

第四章 Maxwell 场的量子化

Spin-1 field — Maxwell and Proca Equations.

Maxwell equation

- Maxwell equation:

$$\begin{cases} \nabla \cdot E = \rho \\ \nabla \cdot B = 0 \\ \nabla \times B - \frac{\partial E}{\partial t} = \vec{j} \\ \nabla \times E + \frac{\partial B}{\partial t} = 0 \end{cases}$$

- Electro-magnetic Field strength tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

- F 的性质:

$$F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \leftrightarrow \begin{cases} B = \nabla\times A \\ E = -\frac{\partial A}{\partial t} - \nabla A_0 \end{cases} \quad A^\mu = (A_0, \vec{A})$$

$\downarrow$

F是反对称的 $F^{\mu\nu} = -F^{\mu\nu}$

- Maxwell Equation 用 F 表示:

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad j^\nu = (\rho, \vec{j})$$

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0$$

- Maxwell 方程用 A 表示:

$$\begin{cases} \partial_\mu F^{\mu\nu} = j^\nu \\ \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0 \end{cases}$$

$\Downarrow$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial_\mu \partial^\mu) A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu$$

$$\Downarrow$$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu \qquad \text{Maxwell-Eq}$$

$$\partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda) + \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) = 0$$

(恒成立)

- A 的 Gauge invariance:

$$A'^\mu = A^\mu + \partial^\mu \Lambda$$

$$F'^{\mu\nu} = \partial^\mu(A^\nu + \partial^\nu\Lambda) - \partial^\nu(A^\mu + \partial^\mu\Lambda) = F^{\mu\nu}$$

Lagrange density and conserved quantities.

○ $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$

$\{ F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2)$

$= -\frac{1}{4}(-2)(E^2 - B^2) - j_\mu A^\mu$

$= \frac{1}{2}(E^2 - B^2) - \rho A_0 + \vec{j}\cdot\vec{A}$

$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j_\mu A^\mu$

$= -\frac{1}{2}((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu)) - j_\mu A^\mu$

Action:

$$W = \int d^4x\, \mathcal{L} = \int d^4x \left[\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\nu A_\mu)(\partial^\mu A^\nu)\right) - j_\mu A^\mu\right]$$

Euler-lagrange Equation

$$\frac{\partial \mathcal{L}}{\partial(\phi)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\frac{\partial((\partial_\rho A_\sigma)(\partial^\rho A^\sigma))}{\partial(\partial_\mu A^\nu)} = \frac{\partial((\partial_\mu A^\nu)\cdot(\partial^\mu A_\nu) + \cdots)}{\partial(\partial_\mu A^\nu)} = \frac{\partial(g^{\mu\mu}g_{\nu\nu}(\partial_\mu A_\nu)^2)}{\partial(\partial_\mu A^\nu)}$$

$$= 2\,\partial^\mu A_\nu$$

$$\frac{\partial((\partial_\rho A_\sigma)(\partial^\sigma A^\rho))}{\partial(\partial_\mu A^\nu)} = \frac{\partial((\partial_\mu A^\nu)(\partial_\nu A^\mu) + (\partial_\nu A^\mu)(\partial_\mu A^\nu) + \cdots)}{\partial(\partial_\mu A^\nu)} = 2\partial_\nu A^\mu$$

Euler-lagrange Equation.

$$\frac{\partial\mathcal{L}}{\partial(A^\nu)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) = 0$$

$$(-j_\nu) - \partial_\mu\left(-\partial^\mu A_\nu + \partial_\nu A^\mu\right) = 0$$

$$(\partial_\mu\partial^\mu)A_\nu - \partial_\nu(\partial_\mu A^\mu) = j_\nu$$

$$\Box A^\nu + \partial^\nu(\partial_\mu A^\mu) = j^\nu \quad ✓$$

○ Gauge-invariance and current conservation. 之间的关系!

$W = \int d^4x\,(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu)$

$W' = \int d^4x\,[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu(A^\mu + \partial^\mu\Lambda)]$

$= \int d^4x\,[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu] - \int d^4x\, j_\mu\partial^\mu\Lambda$

$= W - \int d^4x\,[\partial^\mu(j_\mu\Lambda) - \Lambda\partial^\mu j_\mu]$

$= W + \int d^4x \cdot \Lambda(\partial^\mu j_\mu) - \underbrace{\int d^4x\cdot\partial^\mu(j_\mu\Lambda)}_{\text{surface term!}}$

$= W \qquad \partial^\mu j_\mu = 0$

。能动量张量：

$$\Theta^{\mu\nu} = -\mathcal{L}\, g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial^\nu \phi(x))$$

$$\Theta^{\mu\nu} = -\left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} - j_\rho A^\rho\right) g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)}(\partial^\nu A^\lambda)$$

$$= \left(\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} + j_\rho A^\rho\right) g^{\mu\nu} + (-\partial^\mu A_\lambda + \partial_\lambda A^\mu)(\partial^\nu A^\lambda)$$

$$= \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\cdot g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu}\cdot j_\sigma \cdot A^\sigma.$$

— Four divergence of energy-momentum tensor.

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma}\cdot(\partial^\nu A_\sigma) + g^{\mu\nu} j_\sigma A^\sigma\right)$$

$$= \frac{1}{4}\partial^\nu\cdot(F_{\alpha\beta}F^{\alpha\beta}) - \partial_\mu\left[F^{\mu\sigma}(\partial^\nu A_\sigma)\right] + \partial^\nu(j_\sigma A^\sigma)$$

Maxwell - Equation.

$$\partial_\mu F^{\mu\sigma} = j^\sigma$$

$$= \frac{1}{4}\partial^\nu(F_{\alpha\beta})F^{\alpha\beta} + \frac{1}{4}F_{\alpha\beta}\partial^\nu(F^{\alpha\beta}) - (\partial_\mu F^{\mu\sigma})(\partial^\nu A_\sigma) - F^{\mu\sigma}\cdot\partial_\mu(\partial^\nu A_\sigma) + (\partial^\nu j_\sigma)A^\sigma + j_\sigma(\partial^\nu A^\sigma)$$

$$= \frac{1}{2}\partial^\nu(F_{\alpha\beta})F^{\alpha\beta} - j^\sigma\cdot(\partial^\nu A_\sigma) - F^{\mu\sigma}\cdot\partial_\mu(\partial^\nu A_\sigma) + (\partial^\nu j_\sigma)A^\sigma + j_\sigma(\partial^\nu A^\sigma)$$

$$= \frac{1}{2}\partial^\nu(\partial_\alpha A_\beta - \partial_\beta A_\alpha)F^{\alpha\beta} - F^{\alpha\beta}\partial_\alpha(\partial^\nu A_\beta) + (\partial^\nu j_\sigma)A^\sigma$$

$$= -\frac{1}{2}\partial^\nu(\partial_\alpha A_\beta + \partial_\beta A_\alpha)F^{\alpha\beta} + (\partial^\nu j_\sigma)A^\sigma$$

反对称!

$$\partial_\mu \Theta^{\mu\nu} = (\partial^\nu j_\sigma)A^\sigma$$

— Energy - Momentum tensor is not Gauge Invariant!

$$\Theta^{\mu\nu} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\cdot g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu}\cdot j_\sigma\cdot A^\sigma.$$

Gauge transformation: $A^\mu \to A^\mu + \partial^\mu \Lambda$:

$$\Theta'^{\mu\nu} = \Theta^{\mu\nu} - F^{\mu\sigma}\cdot\partial^\nu\cdot\partial_\sigma\cdot\Lambda + g^{\mu\nu} j_\sigma\cdot\partial^\sigma\cdot\Lambda$$

$$= \Theta^{\mu\nu} - \partial_\sigma\left(F^{\mu\sigma}\cdot\partial^\nu\Lambda - g^{\mu\nu} j^\sigma \Lambda\right) + (\partial_\sigma F^{\mu\sigma})(\partial^\nu\Lambda) - g^{\mu\nu}(\partial^\sigma\cdot j_\sigma)\cdot\Lambda$$

Maxwell Equation: $\partial_\sigma F^{\mu\sigma} = -\partial_\sigma F^{\sigma\mu} = -j^\mu$

current conservation: $\partial_\sigma j^\sigma = 0$

$$= \Theta^{\mu\nu} - \partial_\sigma\left(F^{\mu\sigma}\cdot\partial^\nu\Lambda - g^{\mu\nu}\cdot j^\sigma\Lambda\right) - j^\mu(\partial^\nu\Lambda)$$

— 能动量张量可加入 Divergence of arbitrary rank-3 tensor. — Anti-symmetric in first two indices $\chi^{\sigma\mu\nu} = -\chi^{\mu\sigma\nu}$

reason:

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma \chi^{\sigma\mu\nu}$$

$$\tilde{P}^{\nu} = \int d^3x \left(\Theta^{0\nu} + \partial_\sigma \cdot \chi^{\sigma 0 \nu} \right)$$

$$= P^\nu + \int d^3x \cdot \partial_\sigma \cdot \chi^{\sigma 0 \nu}$$

$$= P^\nu + \underbrace{\int d^3x \; \partial_0 \chi^{00\nu}}_{=0\ (\chi^{00\nu} = -\chi^{00\nu})} + \underbrace{\int d^3x \cdot \partial_i \chi^{i0\nu}}_{\text{surface term.}}$$

$$= P^\nu$$

Modified energy momentum tensor:

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma \cdot \chi^{\sigma\mu\nu}$$

$$= \Theta^{\mu\nu} + \partial_\sigma \cdot (F^{\mu\sigma} A^\nu)$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma} (\partial^\nu A_\sigma) + g^{\mu\nu} \cdot j_\sigma \cdot A^\sigma + \underline{\partial_\sigma (F^{\mu\sigma} A^\nu)}$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma} (\partial^\nu A_\sigma) + F^{\mu\sigma} \cdot \partial_\sigma A^\nu + (\partial_\sigma F^{\mu\sigma}) A^\nu + g^{\mu\nu} j_\sigma \cdot A^\sigma$$

Maxwell equation:
$$(\partial_\sigma F^{\sigma\mu}) = j^\mu$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} + F^{\mu\sigma} \cdot F_\sigma{}^{\nu} - j^\mu A^\nu + g^{\mu\nu} \cdot j_\sigma \cdot A^\sigma$$

Modify 的好处: 1° $j^\mu = 0$ 时, T 是 Gauge Invariant 的!
2° $j^\mu = 0$ 时, T 是 Symmetric Tensor!

— Modified energy density:

$$w = T^{00} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \cdot g^{00} + F^{0\sigma} \cdot F_\sigma{}^{0} - j^0 A^0 + g^{00} j_\sigma A^\sigma$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$ F, definition.

$$F_{\alpha\beta} F^{\alpha\beta} = -2 (E^2 - B^2)$$ property of F.

$$= -\frac{1}{2} (E^2 - B^2) + (-E^1) \cdot (E^1) \cdot (-1) + (-E^2)(E^2)(-1) + (-E^3) E^3 \cdot (-1) - \vec{j} \cdot \vec{A}$$

$$= \frac{1}{2}(B^2+E^2) - \vec{j}\cdot\vec{A}.$$

—— Modified momentum density:

$$p^k = T^{0k} = \vec{E}\times\vec{B} - j^0\vec{A}$$

∘ Lorentz 对称性 → 角动量和自旋守恒

$$j_\mu = \frac{1}{2}\delta w^{\nu\lambda}\cdot M_{\mu\nu\lambda}$$

$$M_{\mu\nu\lambda} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}(I_{\nu\lambda})_{rs}\phi_s - (\Theta_{\mu\nu}x_\lambda - \Theta_{\mu\lambda}x_\nu)$$

$$\phi_r'(x') = \phi_r(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})_{rs}\phi_s(x)$$

$$A'^\alpha(x') = A^\alpha(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta A^\beta(x)$$

$$M_{\mu\nu\lambda} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\alpha)}(I_{\nu\lambda})^\alpha{}_\beta A^\beta - (\Theta_{\mu\nu}x_\lambda - \Theta_{\mu\lambda}x_\nu)$$

1° $$\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\alpha)} = \frac{\partial\left[-\frac{1}{2}\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu)\right) - j_\mu A^\mu\right]}{\partial(\partial^\mu A^\alpha)}$$

$$= -\partial_\mu A_\alpha + \partial_\alpha A_\mu$$

$$= F_{\alpha\mu}$$

2° $$\Theta^{\mu\nu} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\cdot g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu}\cdot j_\sigma\cdot A^\sigma.$$

3° $$A'^\alpha(x') = A^\alpha(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta A^\beta(x)$$

$$= A^\alpha(x) + \delta w^\alpha{}_\beta\cdot A^\beta(x)$$

$$\frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta = \delta w^\alpha{}_\beta$$

$$\frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta = \delta w_{\mu\nu}\cdot g^{\mu\alpha}\cdot g^\nu{}_\beta$$

$$0 = \delta w_{\mu\nu}\left(g^{\mu\alpha}\cdot g^\nu{}_\beta - \frac{1}{2}(I^{\mu\nu})^\alpha{}_\beta\right)$$

$$(I^{\mu\nu})^\alpha{}_\beta = 2g^{\mu\alpha}\cdot g^\nu{}_\beta$$

$$\Downarrow$$

反对称部分：（反对称部分在与 $\delta w_{\mu\nu}$ 缩并时才不消失！）

$$\frac{1}{2}\left[(I^{\mu\nu})^\alpha{}_\beta - (I^{\nu\mu})^\alpha{}_\beta\right] = g^{\mu\alpha}g^\nu{}_\beta - g^{\nu\alpha}g^\mu{}_\beta$$

$$(I^{\mu\nu})^\alpha{}_\beta = g^{\mu\alpha}g^\nu{}_\beta - g^{\nu\alpha}g^\mu{}_\beta$$

$$M_{\mu\nu\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\alpha)} (I_{\nu\lambda})^\alpha{}_\beta A^\beta - (\Theta_{\mu\nu}\chi_\lambda - \Theta_{\mu\lambda}\chi_\nu)$$

$$= -F_{\mu\alpha}(g_\nu{}^\alpha g_{\lambda\beta} - g_\lambda{}^\alpha g_{\nu\beta}) \cdot A^\beta - (\Theta_{\mu\nu}\cdot\chi_\lambda - \Theta_{\mu\lambda}\chi_\nu)$$

$$= \Theta_{\mu\lambda}\chi_\nu - \Theta_{\mu\nu}\chi_\lambda + F_{\mu\lambda}A_\nu - F_{\mu\nu}A_\lambda$$

$$j_\mu = \frac{1}{2}\delta W^{\nu\lambda} \cdot M_{\mu\nu\lambda}$$

守恒荷: $j_0 = \frac{1}{2}\delta W^{\nu\lambda} \cdot M_{0\nu\lambda} = \frac{1}{2}\delta W^{\nu\lambda} \cdot (\Theta_{0\lambda}\chi_\nu - \Theta_{0\nu}\chi_\lambda + F_{0\lambda}A_\nu - F_{0\nu}A_\lambda)$

spin: $S^{nl} = \int d^3x \cdot (F^{0l} \cdot A^n - F^{0n} \cdot A^l)$

three-vector spin:

$$\vec{S} = \int d^3x \; \vec{E} \times \vec{A}$$

reason:

Lorentz transformation

$$\chi^\mu \to \Lambda^\mu{}_\nu \chi^\nu = (\delta^\mu_\nu + \delta W^\mu{}_\nu) \cdot \chi^\nu$$

坐标系绕 z 轴转动:

$$\Lambda = \mathbb{I} + \Delta\varphi \cdot I_6$$

$$I_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \Delta W_{6\,\nu\lambda} \overset{\text{将第一个指标变为下指标!}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

坐标系绕 x 轴转动:

$$\Lambda = \mathbb{I} + \Delta\varphi \cdot I_4$$

$$I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \longrightarrow \Delta W_{4\,\nu\lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

坐标系绕 y 轴转动:

$$\Lambda = \mathbb{I} + \Delta\varphi I_5$$

$$I_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \Delta W_{5\,\nu\lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

仅看转动 Lorentz 变化:

$$j_0 = \frac{1}{2}\left(\overset{\Delta\varphi_4\cdot}{(\delta W_4)_{\nu\lambda}} + \overset{\Delta\varphi_5\cdot}{(\delta W_5)_{\nu\lambda}} + \overset{\Delta\varphi_6\cdot}{(\delta W_6)_{\nu\lambda}}\right) (\Theta_0{}^\lambda \chi^\nu - \Theta_0{}^\nu \chi^\lambda + F_0{}^\lambda A^\nu - F_0{}^\nu \cdot A^\lambda)$$

由于只有空间坐标转动:

$$j_0 = \frac{1}{2}\left(\Delta\varphi_4 (\delta W_4)_{nl} + \Delta\varphi_5 (\Delta W_5)_{nl} + \Delta\varphi_6 (\delta W_6)_{nl}\right) \boxed{(\Theta_0{}^l \chi^n - \Theta_0{}^n \chi^l + F_0{}^l A^n - F_0{}^n A^l)}$$

$\Downarrow$ 关于 n, l 反对称!

$$= -(\Delta\varphi_4 \cdot \delta^2_n \delta^3_l + \Delta\varphi_5 \cdot \delta^3_n \delta^1_l + \Delta\varphi_6 \cdot \delta^1_n \delta^2_l) \cdot (\Theta^{0l}\chi^n - \Theta^{0n}\chi^l + \underbrace{F^{0l}A^n - F^{0n}A^l}_{})$$

$$\Downarrow$$

$$S^{nl} = F^{0l}A^n - F^{0n}A^l.$$

plane-wave-expansion —— Maxwell field.

∘ fixed momentum field mode. $k=(k^0, \vec{k})$ $k^0 = |\vec{k}|$ (photons) $k^0 \neq |\vec{k}|$ (virtual photons)

$$A^\mu(k, x, \lambda) = N_k\, \varepsilon^\mu(k,\lambda)\cdot \exp(-i k_\mu x^\mu)$$

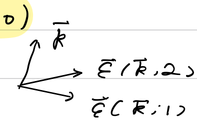
Maxwell Equation:

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu$$

∘ Field Mode (给定 k - four vector, 不一定是 $k^2=0$)

— $\lambda=1$ $\quad \varepsilon(k,1) = (0, \vec{\varepsilon}(\vec{k},1))$

— $\lambda=2$ $\quad \varepsilon(k,2) = (0, \vec{\varepsilon}(\vec{k},2))$



— $\lambda=3$ In a special frame, define:

$n=(1,0,0,0)$ ⟶ In other fram (Lorentz Transformation)

$$\varepsilon(k,3) = \frac{k - n(k\cdot n)}{((k\cdot n)^2 - k^2)^{1/2}} \xrightarrow{\text{In special frame}} \left(0, \frac{\vec{k}}{|\vec{k}|}\right)$$

$$\varepsilon(k,3)\cdot\varepsilon(k,3) = \frac{(k - n(k\cdot n))(k - n(k\cdot n))}{(k\cdot n)^2 - k^2}$$

$$= \frac{k\cdot k + n\cdot n(k\cdot n)^2 - 2n\cdot k(k\cdot n)}{(k\cdot n)^2 - k^2}$$

$$= \frac{-(k\cdot n)^2 + k^2}{(k\cdot n)^2 - k^2} = -1$$

$$k\cdot\varepsilon(k,3) = \frac{k\cdot(k - n(k\cdot n))}{((k\cdot n)^2 - k^2)^{1/2}} = \frac{k^2 - (k\cdot n)^2}{((k\cdot n)^2 - k^2)^{1/2}} = -((k\cdot n)^2 - k^2)^{1/2}$$

— $\lambda=0$

$$\varepsilon(k,4) = n$$

— Completeness relation

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda}\, \varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) = g_{\mu\nu}$$

— Isolate transverse mode, 横波

$$\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda) = -g_{\mu\nu} + n_\mu n_\nu - \frac{[k_\mu - n_\mu(k\cdot n)][k_\nu - n_\nu(k\cdot n)]}{(k\cdot n)^2 - k^2}$$

$$= -g_{\mu\nu} - \frac{-((k\cdot n)^2 - k^2)n_\mu n_\nu + k_\mu k_\nu + n_\mu n_\nu (k\cdot n)^2 - k_\mu n_\nu (k\cdot n) - k_\nu n_\mu (k\cdot n)}{(k\cdot n)^2 - k^2}$$

$$= -g_{\mu\nu} - \frac{k_\mu k_\nu + n_\mu n_\nu k^2 - (k_\mu n_\nu + k_\nu n_\mu)(k\cdot n)}{(k\cdot n)^2 - k^2}$$

$$(k^2=0,\ \text{virtual photons}) = -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot n)^2} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}$$

Quantization of photon field Maxwell场的量子化.

Hamiltonian

- Feynman Gauge

Maxwell 场的 lagrangian density 是：

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

现在改写为 (j=0 时)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi(\partial_\sigma A^\sigma)^2 \quad (\xi=1,\ \text{Feynman Gauge})$$

—— 提取有意义项：

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}(\partial_\sigma A^\sigma)(\partial_\lambda A^\lambda)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\nu A_\mu)(\partial^\mu A^\nu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\partial^\mu\left((\partial_\nu A_\mu)A^\nu - (\partial_\nu A^\nu)A_\mu\right) \underbrace{- \frac{1}{2}(\partial^\mu\partial_\nu A_\mu)A^\nu + \frac{1}{2}(\partial^\mu\partial_\nu A^\nu)A_\mu}_{0}$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\partial^\mu\left((\partial_\nu A_\mu)A^\nu - (\partial_\nu A^\nu)A_\mu\right)$$

$$\int d^4x\,\mathcal{L} = \int d^4x\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right)$$

只有第一项有意义！

最终！

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)$$

- Hamiltonian Formalism！

$$L = \int d^3x\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right) = L[A^\mu, \partial_0 A^\mu]$$

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\mu)} = \frac{\partial\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right)}{\partial(\partial_0 A^\mu)} = -\partial^0 A_\mu = -\partial_0 A_\mu = -\dot{A}_\mu$$

$$H = \int d^3x(\pi_\mu\dot{A}^\mu - \mathcal{L}) = \int d^3x\left(-\partial^0 A_\mu\cdot\partial_0 A^\mu + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right)$$

$$= \int d^3x\left(-\frac{1}{2}\pi^\mu\pi_\mu + \frac{1}{2}(\partial_k A_\nu)(\partial^k A^\nu)\right) \qquad k=1,2,3.$$

$$= \int d^3x\left(-\frac{1}{2}\pi^0\pi_0 + \frac{1}{2}\vec{\pi}\cdot\vec{\pi} - \frac{1}{2}(\partial_k A_\nu)(\partial_k A^\nu)\right)$$

$$= \int d^3x\left(-\frac{1}{2}\pi^0\pi^0 + \frac{1}{2}\vec{\pi}\cdot\vec{\pi} - \frac{1}{2}(\nabla A^0)\cdot(\nabla A^0) + \sum_{i=1}^{3}\frac{1}{2}(\nabla A^i)(\nabla A^i)\right)$$

$$= \int d^3x\left(\sum_{i=1}^{3}\frac{1}{2}\left[(\pi^i)^2 + (\nabla A^i)^2\right] - \frac{1}{2}\left[\pi^0\pi^0 + (\nabla A^0)\cdot(\nabla A^0)\right]\right)$$

$$\Big\}\ \pi^\mu = -\dot{A}^\mu$$

$$= \int d^3x\left(\sum_{i=1}^{3}\frac{1}{2}\left[(\dot{A}^i)^2 + (\nabla A^i)\cdot(\nabla A^i)\right] - \frac{1}{2}\left[(\dot{A}^0)^2 + (\nabla A^0)^2\right]\right)$$

$$= H[\pi^\mu, A^\mu]$$

Fourier Decomposition.

- Field operator 可被展开为：

$$A^{\mu}(x) = \int \frac{d^3k}{\sqrt{2W_k(2\pi)^3}} \sum_{\lambda=0}^{3} \left(a_{k,\lambda}\, \epsilon^{\mu}(k,\lambda) e^{-ik\cdot x} + a_{k,\lambda}^{+} \quad e^{ik\cdot x} \right)$$

— Conjugate field：

$$\pi^{\mu} = - \dot{A}^{\mu}$$

$$= i \int \frac{d^3k}{\sqrt{2W_k(2\pi)^3}} \cdot W_k \sum_{\lambda=0}^{3} \left(a_{k,\lambda}\, \epsilon^{\mu}(k,\lambda)\, e^{-ik\cdot x} - a_{k,\lambda}^{+}\, \epsilon^{\mu}(k,\lambda)\, e^{ik\cdot x} \right)$$

ETCR

- ETCR 定义为:

$$[A^\mu(x,t), \pi^\nu(x',t)] = i g^{\mu\nu} \delta^{(3)}(x-x') \qquad \pi^\mu = -\partial^0 A^\mu$$

$$[A^\mu(x,t), \dot{A}^\nu(x',t)] = -i g^{\mu\nu} \delta^{(3)}(x-x') \quad ; \quad \text{others} = 0$$

—— Lorentz Gauge 不可能满足! (Lorentz Gauge 是在说: $\partial_\mu A^\mu = 0$)

$$[\partial_\mu A^\mu(x,t), A^\nu(x',t)] = [\partial_0 A^0 + \nabla\cdot\vec{A}, A^\nu]$$

$$= [\partial_0 A^0(x,t), A^\nu(x',t)] + \nabla\cdot[\vec{A}(x,t), A^\nu(x',t)]$$

$$= -[\pi^0(x,t), A^\nu(x',t)]$$

$$= -g^{\nu 0} \delta^{(3)}(x-x') \neq 0$$

$\partial_\mu A^\mu(x,t) \neq 0$!

- ETCR for creation and annihilation operator.

Fourier decomposition of Field operator:

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} \sum_{\lambda=0}^{3} \left(a_{k,\lambda} \varepsilon^\mu(k,\lambda) e^{-ik\cdot x} + a_{k,\lambda}^\dagger \varepsilon^\mu(k,\lambda) e^{ik\cdot x}\right)$$

由 proca 场中的推导:

$$\begin{cases} \varepsilon^\mu(k,\lambda)\varepsilon_\mu(k,\lambda') = g_{\lambda\lambda'} \\ (A(x), A'(x)) = i\int d^3x\, A^{\mu *}(x) \overleftrightarrow{\partial_0} A'_\mu(x) \\ A \overleftrightarrow{\partial_0} B = A(\partial_0 B) - (\partial_0 A) B \end{cases}$$

$$\begin{cases} (A(k,\lambda), A(x)) = g_{\lambda\lambda} a_{k,\lambda} \\ (A^*(k,\lambda), A(x)) = -g_{\lambda\lambda} a_{k,\lambda}^\dagger \end{cases}$$

写为:

$$a_{k,\lambda} = i g_{\lambda\lambda} \int d^3x\, A^{\mu *}(k,\lambda) \overleftrightarrow{\partial_0} A_\mu(x)$$

$$a_{k,\lambda}^\dagger = -i g_{\lambda\lambda} \int d^3x\, A^\mu(k,\lambda) \overleftrightarrow{\partial_0} A_\mu(x)$$

近一步:

$$a_{k,\lambda} = i g_{\lambda\lambda} \int d^3x \left(A^{\mu *}(k,\lambda)\partial_0 A_\mu(x) - \partial_0 A^{\mu *}(k,\lambda) A_\mu(x)\right)$$

$$A^\mu(k,\lambda) = \frac{1}{\sqrt{2\omega_k (2\pi)^3}} \varepsilon^\mu(k,\lambda) e^{-ik\cdot x}$$

$$\Downarrow$$

$$\partial_0 A^{\mu *}(k,\lambda) = \frac{i\omega_k}{\sqrt{2\omega_k (2\pi)^3}} \varepsilon^\mu(k,\lambda) e^{ik\cdot x}$$

$$= i g_{\lambda\lambda} \int d^3x \left(A^{\mu *}(k,\lambda) \dot{A}_\mu(x) - i\omega_k A^{\mu *}(k,\lambda) A_\mu(x)\right)$$

$$= i g_{\lambda\lambda} \int d^3x\, A^{\mu *}(k,\lambda)\left(\dot{A}_\mu(x) - i\omega_k A_\mu(x)\right) \qquad -(1)$$

$$a_{k,\lambda}^\dagger = -i g_{\lambda\lambda} \int d^3x\, A^\mu(k,\lambda) \overleftrightarrow{\partial_0} A_\mu(x)$$

$$= -i g_{\lambda\lambda} \int d^3x\, A^\mu(k,\lambda)\left(\dot{A}_\mu(x) + i\omega_k A_\mu(x)\right) \qquad -(2)$$

Commutation Between Creation & annhilation operator :

$$[a_{k,\lambda}\ ,\ a^{+}_{k',\lambda'}] = [-i g_{\lambda\lambda}\int d^3x\, A^{\mu *}(k,\lambda)(\dot{A}_\mu(x) - i\omega_k A_\mu(x)),$$
$$-i g_{\lambda'\lambda'}\int d^3x' A^{\mu}(k',\lambda')(\dot{A}_\mu(x') + i\omega_{k'} A_\mu(x'))]$$
$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'}\int d^3x\, d^3x' \cdot A^{\mu *}(k,\lambda,x)\, A^{\nu}(k',\lambda',x')$$
$$[\dot{A}_\mu(x) - i\omega_k A_\mu(x),\ \dot{A}_\nu(x') + i\omega_{k'} A_\nu(x')]$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \cdot \int d^3x\, d^3x'\ A^{\mu *}(k,\lambda,x)\, A^{\nu}(k',\lambda',x')$$
$$(i\omega_{k'}[\dot{A}_\mu(x), A_\nu(x')] + i\omega_k[\dot{A}_\nu(x'), A_\mu(x)])$$

commutation relation :

$$[A^{\mu}(x,t)\ ,\ \dot{A}^{\nu}(x',t)] = -i g^{\mu\nu}\delta^{(3)}(x-x') \quad ; \quad \text{others} = 0$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'}\int d^3x\, d^3x'\ A^{\mu *}(k,\lambda,x)\, A^{\nu}(k',\lambda',x')$$
$$(-i\omega_{k'}(-i g_{\nu\mu})\delta^{(3)}(x-x') + i\omega_k\cdot(i g_{\mu\nu})\delta^{(3)}(x-x'))$$

$$A^{\mu}(k,\lambda,x) = \frac{1}{\sqrt{2\omega_k(2\pi)^3}}\,\varepsilon^{\mu}(k,\lambda)\, e^{-ik\cdot x}$$
$$A^{\nu}(k',\lambda',x') = \frac{1}{\sqrt{2\omega_{k'}(2\pi)^3}}\,\varepsilon^{\nu}(k',\lambda')\, e^{-ik'\cdot x'}$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'}\int d^3x\, d^3x' \cdot \frac{1}{\sqrt{2\omega_k(2\pi)^3}}\,\varepsilon^{\mu}(k,\lambda)\, e^{ik\cdot x}\,\frac{1}{\sqrt{2\omega_{k'}(2\pi)^3}}\,\varepsilon^{\nu}(k',\lambda')\, e^{-ik'\cdot x'}$$
$$(-\omega_{k'} g_{\nu\mu} - \omega_k g_{\mu\nu})\,\delta^{(3)}(x-x')$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \cdot \int d^3x\, \frac{1}{\sqrt{2\omega_k}\ \sqrt{2\omega_{k'}}} \cdot \frac{1}{(2\pi)^3} \cdot \varepsilon^{\mu}(k,\lambda)\,\varepsilon^{\nu}(k',\lambda')\, e^{i(k-k')\cdot x}$$
$$(-\omega_{k'} - \omega_k)\, g_{\mu\nu}$$

$$= -g_{\lambda\lambda}\, g_{\lambda'\lambda'}\ g_{\mu\nu}\ \varepsilon^{\mu}(k,\lambda)\,\varepsilon^{\nu}(k,\lambda')\,\delta^{(3)}(k-k')$$
$$= -g_{\lambda\lambda}\, g_{\lambda'\lambda'}\,\varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k,\lambda)\,\delta^{(3)}(k-k')$$

$$\varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k,\lambda') = g_{\lambda,\lambda'}$$

$$= -\ g_{\lambda\lambda'}\,\delta^{(3)}(k-k')$$

physical quantity 物理量.

- Hamiltonian 按前文的 hamilton formalism：

$$H = \int d^3x \left(\sum_{i=1}^{3} \frac{1}{2} [(\dot{A}^i)^2 + (\nabla A^i)\cdot(\nabla A^i)] - \frac{1}{2}[(\dot{A}^0)^2 + (\nabla A^0)^2] \right)$$

Normal ordering (creation to left)

$$H = \int d^3k \, w_k \left(\sum_{\lambda=1}^{3} a_{k\lambda}^{\dagger} a_{k\lambda} - a_{k0}^{\dagger} a_{k0} \right)$$

- Momentum :

$$P = \int d^3k \, \vec{k} \left(\sum_{\lambda=1}^{3} a_{k\lambda}^{\dagger} a_{k\lambda} - a_{k0}^{\dagger} a_{k0} \right)$$

physical state.

○ 为什么是 creation & annihilation operator :

由于: Hamiltonian

$$H = \int d^3k\, w_k \left(\sum_{\lambda=1}^{3} a^{\dagger}_{k\lambda} a_{k\lambda} - a^{\dagger}_{k0} a_{k0} \right)$$

Commutation relation :

$$[a_{k,\lambda}, a^{\dagger}_{k',\lambda'}] = -g_{\lambda\lambda'} \delta^{(3)}(k-k')$$

于是:

$$[a_{k,\lambda}, H] = [a_{k,\lambda}, -\int d^3k'\, w_{k'} (\sum_{\lambda'=0}^{3} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} g_{\lambda'\lambda'}) \quad]$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} \cdot [a_{k,\lambda}, a^{\dagger}_{k'\lambda'} a_{k'\lambda'}] g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} (a_{k,\lambda} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} - a^{\dagger}_{k'\lambda'} a_{k,\lambda} a_{k',\lambda'} + a^{\dagger}_{k',\lambda'} a_{k,\lambda} a_{k',\lambda'} - a^{\dagger}_{k',\lambda'} a_{k',\lambda'} a_{k,\lambda}) g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} [a_{k,\lambda}, a^{\dagger}_{k',\lambda'}] a_{k',\lambda'} g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} (-g_{\lambda\lambda'}) \delta^{(3)}(k-k') a_{k'\lambda'} g_{\lambda'\lambda'}$$

$$= w_k a_{k,\lambda}$$

$$[a^{\dagger}_{k,\lambda}, H] = [a^{\dagger}_{k,\lambda}, -\int d^3k'\, w_{k'} (\sum_{\lambda'=0}^{3} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} g_{\lambda'\lambda'}) \quad]$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} \cdot [a^{\dagger}_{k,\lambda}, a^{\dagger}_{k'\lambda'} a_{k'\lambda'}] g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} (a^{\dagger}_{k,\lambda} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} - a^{\dagger}_{k'\lambda'} a^{\dagger}_{k,\lambda} a_{k',\lambda'} + a^{\dagger}_{k',\lambda'} a^{\dagger}_{k,\lambda} a_{k',\lambda'} - a^{\dagger}_{k',\lambda'} a_{k',\lambda'} a^{\dagger}_{k,\lambda}) g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} (-a^{\dagger}_{k',\lambda'}) \underset{\sim\sim\sim}{[a_{k',\lambda'}, a^{\dagger}_{k,\lambda}]} g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'} \int d^3k'\, w_{k'} (-a^{\dagger}_{k',\lambda'}) (-g_{\lambda\lambda'} \delta^{(3)}(k-k')) g_{\lambda'\lambda'}$$

$$= -w_k a^{\dagger}_{k,\lambda}$$

可见,若有 H 的本征态: (则 $a^{\dagger}_{k,\lambda}$ 将其变为 $E+w_k$ 的本征态, $a_{k,\lambda}$ 将其变为 $E-w_k$ 的本征态)

$$H|E\rangle = E|E\rangle$$

则:

$$H a_{k,\lambda}|E\rangle = a_{k,\lambda} H|E\rangle + [H, a_{k,\lambda}]|E\rangle$$

$$= E a_{k,\lambda}|E\rangle - w_k a_{k,\lambda}|E\rangle$$

$$= (E - w_k) a_{k,\lambda}|E\rangle$$

$$H a^{\dagger}_{k,\lambda}|E\rangle = a^{\dagger}_{k,\lambda} H|E\rangle + [H, a^{\dagger}_{k,\lambda}]|E\rangle$$

$$= (E + w_k) a^{\dagger}_{k,\lambda}|E\rangle$$

◦ 定义 physical state.

—— 真空态定义, 真空态定义为 $a_{k,\lambda}|0\rangle = 0$! (能量最低态)

则, $a^+_{k,\lambda}|0\rangle$ 是有 $\omega_{k,\lambda}$ 能量的本征态!

激发态本征态内积:

$$\langle 0|a_{k,\lambda}a^+_{k,\lambda}|0\rangle = \langle 0|a^+_{k,\lambda}a_{k,\lambda} + [a_{k,\lambda}, a^+_{k,\lambda}]|0\rangle$$

$$[a_{k,\lambda}, a^+_{k',\lambda'}] = -g_{\lambda\lambda'}\delta^{(3)}(k-k')$$

$$= \langle 0|a^+_{k,\lambda}a_{k,\lambda} - g_{\lambda\lambda}\delta^{(3)}(0)|0\rangle$$

$$= -g_{\lambda\lambda}\delta^{(3)}(0)\langle 0|0\rangle$$

$$= -g_{\lambda\lambda}\delta^{(3)}(0)$$

—— 解决 $\delta^{(3)}(0)$ 的存在.

改变定义: 在k附近有非零值的 Function.

$$\begin{cases} |1_{k,\lambda}\rangle = \int d^3k' F_k(k') a^+_{k',\lambda}|0\rangle \\ \int d^3k' |F(k')|^2 = 1 \end{cases}$$

$$\langle 1_{k,\lambda}|1_{k,\lambda}\rangle = \int d^3k'' d^3k' F^*_k(k'') F_k(k')\left(-g_{\lambda\lambda}\delta^{(3)}(k'-k'')\right)$$

$$= -g_{\lambda\lambda}\langle 0|0\rangle$$

$$= -g_{\lambda\lambda}$$

—— 生成 $n_{k,\lambda}$ 个 k,λ 粒子.

$$\int d^3k' d^3k'' d^3k''' d^3k'''' \langle 0| F_k(k''') F_k(k'''') a_{k''',\lambda} a_{k'''',\lambda} F_k(k') F_k(k'') a^+_{k',\lambda} a^+_{k'',\lambda}|0\rangle$$

$$[a_{k,\lambda}, a^+_{k',\lambda'}] = -g_{\lambda\lambda'}\delta^{(3)}(k-k')$$

$$= \int d^3k' d^3k'' d^3k''' d^3k'''' F_k(k') F_k(k'') F_k(k''') F_k(k'''') \cdot \langle 0|a_{k''',\lambda} a_{k'''',\lambda} a^+_{k',\lambda} a^+_{k'',\lambda}|0\rangle$$

$$= \int d^3k' d^3k'' d^3k''' d^3k'''' F_k(k') F_k(k'') F_k(k''') F_k(k'''') \cdot \left(\langle 0|a_{k''',\lambda} a^+_{k',\lambda} a_{k'''',\lambda} a^+_{k'',\lambda}|0\rangle - \langle 0|a_{k''',\lambda} a^+_{k'',\lambda}|0\rangle g_{\lambda\lambda}\delta^{(3)}(k'-k'''')\right)$$

$$= \int d^3k' d^3k'' d^3k''' d^3k'''' F_k(k') F_k(k'') F_k(k''') F_k(k'''') \cdot \left(\langle 0|a_{k''',\lambda} a^+_{k',\lambda} a^+_{k'',\lambda} a_{k'''',\lambda}|0\rangle - \langle 0|a^+_{k'',\lambda} a_{k''',\lambda}|0\rangle g_{\lambda\lambda}\delta^{(3)}(k'-k'''') - \langle 0|a_{k''',\lambda} a^+_{k',\lambda}\rangle g_{\lambda\lambda}\delta^{(3)}(k''-k'''') + \langle 0|0\rangle g_{\lambda\lambda'} g_{\lambda\lambda'}\delta^{(3)}(k'-k'''')\delta^{(3)}(k''-k''')\right)$$

$$= \int d^3k' d^3k'' d^3k''' d^3k'''' F_k(k') F_k(k'') F_k(k''') F_k(k'''') \cdot \left(\langle 0|a_{k''',\lambda} a^+_{k',\lambda} a^+_{k'',\lambda} a_{k'''',\lambda}|0\rangle - \langle 0|a^+_{k'',\lambda} a_{k''',\lambda}|0\rangle g_{\lambda\lambda}\delta^{(3)}(k'-k'''') + \langle 0|0\rangle g_{\lambda\lambda} g_{\lambda\lambda}\delta^{(3)}(k''-k'''')\delta^{(3)}(k'''-k') + \langle 0|0\rangle g_{\lambda\lambda'} g_{\lambda\lambda'}\delta^{(3)}(k'-k'''')\delta^{(3)}(k''-k''')\right)$$

$$= 2 g_{\lambda\lambda} \cdot g_{\lambda\lambda}.$$

—— 定义.

$$|n_{k,\lambda}\rangle = \frac{1}{\sqrt{n_{k,\lambda}!}}(a^+_{k,\lambda})^{n_{k,\lambda}}|0\rangle \; ; \longrightarrow \begin{cases} a_{k,\lambda}|n_{k,\lambda}\rangle = (-g_{\lambda\lambda})\sqrt{n_{k,\lambda}}\,|n_{k,\lambda}-1\rangle \\ a^+_{k,\lambda}|n_{k,\lambda}\rangle = \sqrt{n_{k,\lambda}+1}\cdot|n_{k,\lambda}+1\rangle \end{cases}$$

$$\langle n_{k,\lambda} | n_{k,\lambda} \rangle = (g_{\lambda\lambda})^{n_{k,\lambda}} (\delta^{(3)}(0))^{n_{k,\lambda}}$$

—— 能量

$$\langle n_{k,\lambda} | H | n_{k,\lambda} \rangle = n_{k,\lambda} W_{k,\lambda} \langle n_{k,\lambda} | n_{k,\lambda} \rangle .$$

$$= n_{k,\lambda} W_{k,\lambda} \cdot (g_{\lambda\lambda})^{n_{k,\lambda}} (\delta^{(3)}(0))^{n_{k,\lambda}}$$

—— 粒子数

$$\hat{n}_{k,\lambda} = a^{+}_{k,\lambda} a_{k,\lambda}$$

$$\hat{n}_{k,\lambda} | n_{k,\lambda} \rangle = a^{+}_{k,\lambda} a_{k,\lambda} | n_{k,\lambda} \rangle$$

$$= a^{+}_{k,\lambda} (-g_{\lambda\lambda}) \sqrt{n_{k,\lambda}} | n_{k,\lambda} - 1 \rangle$$

$$= (-g_{\lambda\lambda}) \cdot n_{k,\lambda} | n_{k,\lambda} - 1 \rangle$$

$$\langle n_{k,\lambda} | \hat{n}_{k,\lambda} | n_{k,\lambda} \rangle = (-g_{\lambda\lambda}) n_{k,\lambda} \langle n_{k,\lambda} | n_{k,\lambda} \rangle$$

$$= (-g_{\lambda\lambda})^{n_{k,\lambda}+1} \cdot n_{k,\lambda} (\delta^{(3)}(0))^{n_{k,\lambda}}$$

Gupta – Bleuler Method.

- Require :

$$\langle \Phi | \partial_\mu A^\mu(x) | \Phi \rangle = 0$$

stronger :

(positive frequency field, annihilation op)

Gurantees

$$\partial_\mu A^{\mu(+)}(x) | \Phi \rangle = 0$$

$$\langle \Phi | \partial_\mu A^{\mu(-)}(x) = 0$$

$$\int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} e^{-ik\cdot x} \sum_\lambda a_{k,\lambda} k_\mu \varepsilon^\mu(k,\lambda) | \Phi \rangle = 0$$

由于 :

$\vec{k}$, $\vec{\varepsilon}(k,2)$, $\vec{\varepsilon}(k,1)$

$$\varepsilon(k,1) = (0, \vec{\varepsilon}(k,1))$$

$$\varepsilon(k,2) = (0, \vec{\varepsilon}(k,2))$$

$$\varepsilon(k,3) = \frac{k - n(k\cdot n)}{((k\cdot n)^2 - k^2)^{1/2}} \qquad (n = (1,0,0,0) \text{ In special frame!})$$

$$\varepsilon(k,0) = n$$

$(k^2=0)$ $k\cdot\varepsilon(k,1) = k\cdot\varepsilon(k,2) = 0 \qquad k\cdot\varepsilon(k,3) = -(k\cdot n) \qquad k\cdot\varepsilon(k,0) = k\cdot n$

$$\int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} \cdot e^{-ik\cdot x} \left(a_{k,0}\, k\cdot\varepsilon(k,0) + a_{k,3}\, k\cdot\varepsilon(k,3) \right) | \Phi \rangle = 0$$

$$\int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} e^{-ik\cdot x} (n\cdot k)(a_{k,0} - a_{k,3}) | \Phi \rangle = 0$$

$$L_k | \Phi \rangle = 0$$

$$(a_{k,0} - a_{k,3}) | \Phi \rangle = 0$$

—— Expectation values of numbers of longitudinal $(\lambda=3)$ and scalar $(\lambda=0)$ photons are equal.

$$\langle \Phi | a^\dagger_{k,0} a_{k,0} | \Phi \rangle = \langle \Phi | a^\dagger_{k,3} a_{k,3} | \Phi \rangle$$

$$\langle \Phi | a^\dagger_{k,0} = \langle \Phi | a^\dagger_{k,3}$$

$$a_{k,0} | \Phi \rangle = a_{k,3} | \Phi \rangle$$

—— All states from transverse state

$$| \Phi_c \rangle = R_c | \Phi_T \rangle$$

Transverse photons only

$$R_c = 1 + \int d^3k\, c(k) L^\dagger_k + \int d^3k\, d^3k'\, c(k,k') L^\dagger_k L^\dagger_{k'} + \cdots$$

$$L_k = a_{k,0} - a_{k,3}$$

—— state from transverse satisfies Gupta – Bleuler relation.

$$[L_k, L^\dagger_{k'}] = [a_{k,0} - a_{k,3}, a^\dagger_{k',0} - a^\dagger_{k',3}] = [a_{k,0}, a^\dagger_{k',0}] + [a_{k,3}, a^\dagger_{k',3}] = 0$$

$$[L_k, R_c] = 0$$

$$L_k R_c | \Phi_T \rangle = R_c L_k | \Phi_T \rangle = 0$$

— pseudo photons do not contributes to norm.

$$\langle \Phi_{C'}' | \Phi_C \rangle = \langle \Phi_{T'}' | R_{C'}^{\dagger} R_C | \Phi_T \rangle = \langle \Phi_{T'}' | R_C R_{C'}^{\dagger} | \Phi_T \rangle = \langle \Phi_{T'} | \Phi_T \rangle$$

$$[R_C, R_{C'}^{\dagger}] = 0$$

Feynman propagator:

○ 不分解的 propagator:

类比 proca field 的 propagator:

$$i D_F^{\mu\nu}(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\sum_{\lambda=0}^{3} (-g_{\lambda\lambda}) \varepsilon^{\mu}(k,\lambda) \varepsilon^{\nu}(k,\lambda) \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

completeness:

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda} \varepsilon^{\mu}(k,\lambda) \varepsilon^{\nu}(k,\lambda) = g^{\mu\nu}$$

$$= \int \frac{d^4k}{(2\pi)^4} (-g^{\mu\nu}) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$= i \int \frac{d^4k}{(2\pi)^4} D_F^{\mu\nu}(k) e^{-ik\cdot(x-y)}$$

$$D_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2+i\varepsilon}$$

○ Decomposition propagator

$$D_{F\mu\nu}(k) = \frac{1}{k^2+i\varepsilon} \left(\sum_{\lambda=1}^{2} \varepsilon_{\mu}(k,\lambda) \varepsilon_{\nu}(k,\lambda) + \varepsilon_{\mu}(k,3) \varepsilon_{\nu}(k,3) - \varepsilon_{\mu}(k,0) \varepsilon_{\nu}(k,0) \right)$$

$$= \frac{1}{k^2+i\varepsilon} \left(\sum_{\lambda=1}^{2} \varepsilon_{\mu}(k,\lambda) \varepsilon_{\nu}(k,\lambda) + \frac{(k_\mu - n_\mu(k\cdot n))(k_\nu - n_\nu(k\cdot n))}{(k\cdot n)^2 - k^2} - n_\mu n_\nu \right)$$

$$= \frac{1}{k^2+i\varepsilon} \Big(\underbrace{\sum_{\lambda=1}^{2} \varepsilon_{\mu}(k,\lambda) \varepsilon_{\nu}(k,\lambda)}_{\text{trans part}} + \underbrace{\frac{k^2 n_\mu n_\nu}{(k\cdot n)^2 - k^2}}_{\text{coul part}} + \underbrace{\frac{k_\mu k_\nu - (k_\mu n_\nu + k_\nu n_\mu)(k\cdot n)}{(k\cdot n)^2 - k^2}}_{\text{resid part!}} \Big)$$

—— coul part: ($n=(1,0,0,0)$, In special Lorentz frame 时)

$$D_{F\mu\nu}(\text{coul})(k) = \frac{\delta_{\mu 0}\delta_{\nu 0}}{|\vec{k}|^2}$$

$$D_{F\mu\nu}(\text{coul})(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \cdot \frac{\delta_{\mu 0}\delta_{\nu 0}}{|\vec{k}|^2}$$

$$= \delta(x_0-y_0)\, \delta_{\mu 0}\delta_{\nu 0} \int \frac{d^3k}{(2\pi)^3} \frac{e^{+i\vec{k}\cdot(\vec{x}-\vec{y})}}{|\vec{k}|^2}$$

$$= \delta_{\mu 0}\delta_{\nu 0} \frac{\delta(x_0-y_0)}{4\pi|\vec{x}-\vec{y}|}$$ ← 用了奇点的性质, Cauchy 积分.

第五章 Proca 场的量子化

Proca Equation

- Lagrangian:

real value A^μ $\quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$

charged field $\quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu^* A^\mu - j_\mu A^\mu$

— Not Gauge Invariant:

$$A^\mu A_\mu \to (A^\mu + \partial^\mu \Lambda)(A_\mu + \partial_\mu \Lambda) = A^\mu A_\mu + 2(\partial^\mu \Lambda) A_\mu + (\partial^\mu \Lambda)(\partial_\mu \Lambda)$$

- Euler-lagrange equation

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}(2 \cdot \partial^\mu A^\nu - 2\partial^\nu A^\mu)$$

$$= -F^{\mu\nu}$$

E-L Equation:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0$$

$$m^2 A^\nu - j^\nu + \partial_\mu F^{\mu\nu} = 0$$

$$m^2 A^\nu - j^\nu + \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

$$(\partial_\mu \partial^\mu) A^\nu + m^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) = j^\nu \longrightarrow \begin{cases} \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = j^\nu \\ \partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu \end{cases}$$

Proca Equation $\quad \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) + m^2 A^\nu = j^\nu$

— 性质: Automatically satisfies Lorentz condition!

$$\partial_\nu(\Box A^\nu) - \partial_\nu \partial^\nu (\partial_\mu A^\mu) + m^2 \partial_\nu A^\nu = \partial_\nu j^\nu \longrightarrow (\Box + m^2) A^\nu = j^\nu$$

$$\partial_\nu A^\nu = \frac{1}{m^2} \partial_\nu j^\nu$$

Conserve-current!

$$\partial_\nu A^\nu = 0$$

- Energy-momentum tensor:

$$\Theta^{\mu\nu} = -\mathcal{L} g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \cdot (\partial^\nu A_\sigma)$$

$$= -\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu\right) g^{\mu\nu} - F^{\mu\sigma} \cdot \partial^\nu A_\sigma$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\sigma} \cdot \partial^\nu A_\sigma - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma$$

—— Modified Momentum-Energy Tensor:

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma(\chi^{\sigma\mu\nu}) \quad \leftarrow \text{其中 } \chi^{\sigma\mu\nu} = -\chi^{\mu\sigma\nu} \quad (\text{使得 } \textstyle\int d^3x\, \Theta^{0\nu} \text{不变})$$

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma(F^{\mu\sigma} A^\nu)$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\sigma} \partial^\nu A_\sigma - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma + \partial_\sigma(F^{\mu\sigma} A^\nu)$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\sigma} \partial^\nu A_\sigma - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma$$
$$+ \partial_\sigma (F^{\mu\sigma}) A^\nu + F^{\mu\sigma} \partial_\sigma (A^\nu)$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\sigma} F^\nu{}_\sigma - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma$$
$$+ \partial_\sigma (F^{\mu\sigma}) A^\nu$$

Proca Equation:
$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\sigma} F_\sigma{}^\nu - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma - j^\mu A^\nu$$
$$+ m^2 A^\mu A^\nu$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\sigma} F_\sigma{}^\nu - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + m^2 A^\mu A^\nu + g^{\mu\nu} j_\sigma A^\sigma$$
$$- j^\mu A^\nu$$

—— Modified Energy and momentum density

$$w = T^{00} = \frac{1}{2} (B^2 + E^2) + \frac{1}{2} m^2 (A_0^2 + \vec{A}^2) - \vec{j} \cdot \vec{A}$$
$$\vec{p} = E \times B - m^2 A^0 \vec{A} - j^0 \vec{A}$$

- Angular momentum : 和 Maxwell same!

plane wave Expansion — proca field!

∘ Proca Equation: $\left\{\begin{array}{l}(\Box+m^2)A^\nu = j^\nu \\ \partial_\nu A^\nu = 0\end{array}\right\} \to j=0$时: $\left\{\begin{array}{l}(\Box+m^2)A^\nu = 0 \\ \partial_\nu A^\nu = 0\end{array}\right.$

特定动量解: $k=(\sqrt{m^2+\vec{k}^2},\ \vec{k}) \to A^\mu(k,x,\lambda) = N_k\cdot \exp(-i k_\mu x^\mu)\cdot \varepsilon^\mu(k,\lambda)$

$$\left\{\begin{array}{ll}((-i)^2 k^\mu k_\mu + m^2) = 0 & (\text{自动满足}) \\ k_\mu \varepsilon^\mu = 0 & \end{array}\right.$$

当然, $A^{\mu *}(k,x,\lambda)$ 也是解!

∘ General form of field mode.

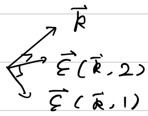
$$A^\mu(\vec{k},x,\lambda) = N_{\vec{k}}\cdot \exp(-i(\omega_k t - \vec{k}\cdot\vec{x}))\cdot \varepsilon^\mu(\vec{k},\lambda)$$

$$\omega_{\vec{k}} = \sqrt{m^2+|\vec{k}|^2}$$

$$\varepsilon^\mu(\vec{k},\lambda)\cdot \varepsilon_\mu(\vec{k},\lambda') = g_{\lambda,\lambda'}$$

— $\lambda=1$ spacelike $\varepsilon(\vec{k},1) = (0,\ \vec{\varepsilon}(\vec{k},1))$

— $\lambda=2$ spacelike $\varepsilon(\vec{k},2) = (0,\ \vec{\varepsilon}(\vec{k},2))$



— $\lambda=3$ spacelike $k^\mu \varepsilon_\mu(\vec{k},3)=0$

Normalization condition: $\varepsilon^\mu(\vec{k},\lambda)\varepsilon_\mu(\vec{k},\lambda') = g_{\lambda\lambda'}$ ⟶ 空间指标和前两者正交 说明在空间指标上, 和 $\vec{k}$ 同方向!

$$\varepsilon(\vec{k},3) = \left(\frac{|\vec{k}|}{m},\ \frac{\vec{k}}{|\vec{k}|}\frac{k_0}{m}\right)$$

↓ Normalization condition satisfied:

$$\varepsilon^\mu(\vec{k},3)\varepsilon_\mu(\vec{k},3) = \frac{|\vec{k}|^2}{m^2} - \frac{|\vec{k}|^2}{|\vec{k}|^2}\frac{k_0^2}{m^2}$$

$$= \frac{|\vec{k}|^2}{m^2} - \frac{k_0^2}{m^2} \qquad k^0=\sqrt{|\vec{k}|^2+m^2}$$

$$= -1 = g_{33}$$

— $\lambda=0$ Timelike $\varepsilon(\vec{k},0) = \frac{k}{m} = \frac{1}{m}(k^0,\vec{k})$ ⟶ 但不满足 $k_\mu\varepsilon^\mu=0$!

— Complete relation:

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda}\,\varepsilon_\mu(\vec{k},\lambda)\varepsilon_\nu(\vec{k},\lambda) = g_{\mu\nu}$$

$$\sum_{\lambda=1}^{3} \varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) = -\left(g_{\mu\nu} - \frac{1}{m^2}k_\mu k_\nu\right)$$

proca - field quantization

Hamiltonian formalism

- Hamiltonian formalism.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu$$

$L = \int d^3x\, \mathcal{L}(A^\mu, \partial_0 A^\mu)$

$L = L[A^\mu, \partial_0 A^\mu]$

$$\pi^\mu = \frac{\delta L}{\delta \dot{A}_\mu} = \frac{\delta L}{\delta(\partial_0 A_\mu)} = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\mu)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$$

$$\Big\}\ \frac{\partial(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial^\mu A^\nu)} = -F_{\mu\nu}$$

$$= -F^{0\mu}$$

$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$

$\partial_\mu F^{\mu 0} + m^2 A^0 = 0$

$-\nabla\cdot\vec{E} + m^2 A^0 = 0$

$A^0 = \frac{1}{m^2}\vec{\nabla}\cdot\vec{E}$

↑ dependent variable!

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \to F^{0\mu} = (0, -E^1, -E^2, -E^3)$$

$$\pi^\mu = (0, E^1, E^2, E^3)$$

— Hamiltonian density

$$\mathcal{H} = \pi_\mu \dot{A}^\mu - \mathcal{L} = -F_{0\mu}\dot{A}^\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + j_\mu A^\mu$$

$$= -\vec{E}\cdot\dot{\vec{A}} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + j_\mu A^\mu$$

$$\Big\}\ F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2) \qquad j = 0$$

$$= -\vec{E}\cdot\dot{\vec{A}} - \frac{1}{2}(E^2 - B^2) - \frac{1}{2}m^2(A_0^2 - \vec{A}^2)$$

$$\vec{E}\cdot\vec{\nabla}A_0 = \vec{\nabla}\cdot(\vec{E}A_0) - A_0(\vec{\nabla}\cdot\vec{E})$$

$$\Big\}\ \vec{E} = -\vec{\nabla}A_0 - \partial_0\vec{A}$$

$$= \vec{\nabla}\cdot(\vec{E}A_0) + A_0\,\vec{\nabla}\cdot(\vec{\nabla}A_0 + \partial_0\vec{A})$$

$$= \vec{\nabla}\cdot(\vec{E}A_0) + A_0(\nabla^2 A_0 + \partial_0\vec{\nabla}\cdot\vec{A})$$

$$\Big\}\ \partial_\nu A^\nu = 0$$

$$= \vec{\nabla}\cdot(\vec{E}A_0) + (\nabla^2 A_0 - \partial_0^2 A_0)A_0$$

$$\Big\}\ (\Box + m^2)A^\nu = j^\nu$$

$$= \vec{\nabla}\cdot(\vec{E}A_0) + m^2 A_0^2$$

$$\Big\}\ \partial_0\vec{A} = -\vec{\nabla}A_0 - \vec{E}$$

$$= -\vec{E}\cdot(-\vec{\nabla}A_0 - \vec{E}) + \frac{1}{2}(B^2 - E^2 + m^2\vec{A}^2) - \frac{1}{2}m^2 A_0^2$$

$$= \vec{\nabla}\cdot(\vec{E}A_0) + m^2 A_0^2 + \frac{1}{2}(B^2 + E^2 + m^2\vec{A}^2) - \frac{1}{2}m^2 A_0^2$$

$$= \frac{1}{2}(B^2 + E^2 + m^2\vec{A}^2 + m^2 A_0^2) + \vec{\nabla}\cdot(\vec{E}\cdot A_0)$$

$$= \frac{1}{2}(E^2 + (\nabla\times A)^2 + m^2\vec{A}^2 + m^2 A_0^2) + \vec{\nabla}\cdot(\vec{E}\cdot A_0)$$

— Hamiltonian; H

$$= \int d^3x\,\mathcal{H} = \int d^3x \left(\frac{1}{2}(E^2 + (\nabla\times A)^2 + m^2\vec{A}^2 + m^2 A_0^2) + \vec{\nabla}\cdot(\vec{E}\cdot A_0)\right)$$

$$= \int d^3x\, \frac{1}{2}(E^2 + (\nabla\times\vec{A})^2 + m^2\vec{A}^2 + \frac{1}{m^2}(\nabla\cdot E)^2)$$

$$= H[\vec{A}, \vec{E}]$$

$A_0 = \frac{1}{m^2}\nabla\cdot E$

$\frac{1}{2}\frac{1}{m^2}(\nabla\cdot E)^2$

↑ leads to 0.

Equal Time commutation relation.

- ETCR: $[A^i(\vec{x},t), E^j(\vec{x}',t)] = -i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{x}')$

  $[A^i(\vec{x},t), A^j(\vec{x}',t)] = 0$

  $[E^i(\vec{x},t), E^j(\vec{x}',t)] = 0$

- Heisenberg equation of motion:

— EOM of A $\quad \frac{d\vec{A}}{dt} = \frac{1}{i\hbar}[\vec{A}, H]$

$$i\frac{d\vec{A}}{dt} = [\vec{A}, H]$$

$$= [\vec{A}(x,t), \int d^3x' \frac{1}{2}(E^2 + (\nabla\times\vec{A})^2 + m^2\vec{A}^2 + \frac{1}{m^2}(\nabla\cdot E)^2)]$$

$$= \int d^3x' \frac{1}{2}[\vec{A}(x,t), E^2(x',t)] + \int d^3x' \frac{1}{2}\frac{1}{m^2}[\vec{A}(x,t), (\nabla\cdot\vec{E}(x',t))^2]$$

$$[A^i(\vec{x},t), E^j(\vec{x}',t)] = -i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{x}')$$

$$\Downarrow$$

$$[A^i(\vec{x},t), E^2(x',t)] = A^i E^i(x',t)E^i(x',t) - E^i E^i A^i$$

$$= A^i E^i E^i - E^i A^i E^i + E^i A^i E^i - E^i E^i A^i$$

$$= [A^i(\vec{x},t), E^i(\vec{x}',t)]E^i(\vec{x}',t) + E^i(\vec{x},t)[A^i(\vec{x},t), E^i(\vec{x}',t)]$$

$$= -i2\delta^{(3)}(x-x')E^i(\vec{x},t)$$

$$[\vec{A}(\vec{x},t), E^2(\vec{x}',t)] = -2i\delta^{(3)}(x-x')\vec{E}(\vec{x}',t)$$

$$[A^i(\vec{x},t), (\nabla\cdot\vec{E}(\vec{x}',t))^2] = A^i(\nabla\cdot E)(\nabla\cdot E) - (\nabla\cdot E)(\nabla\cdot E)A^i$$

$$= A^i(\nabla\cdot E)(\nabla\cdot E) - (\nabla\cdot E)A^i(\nabla\cdot E) + (\nabla\cdot E)A^i(\nabla\cdot E) - (\nabla\cdot E)(\nabla\cdot E)A^i$$

$$= [A^i(\vec{x},t), \nabla\cdot E(x',t)]\nabla\cdot E(x',t) + (\nabla\cdot E(x',t))\cdot[A^i(\vec{x},t), \nabla\cdot E(x',t)]$$

$$= \nabla\cdot[A^i(\vec{x},t), \vec{E}(x',t)]\nabla\cdot E(x',t) + (\nabla\cdot E(x',t))\nabla[A^i(x,t), \vec{E}(x',t)]$$

$$= 2\partial_i(-i\delta^{(3)}(x-x'))\nabla\cdot E(x',t)$$

$$= -2i[\partial_i\delta^{(3)}(x-x')]\nabla\cdot E(x',t)$$

$$[\vec{A}(x,t), (\nabla\cdot E(x',t))^2] = -2i(\vec{\nabla}\delta^{(3)}(x-x'))\nabla\cdot E(x',t)$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')\vec{E}(\vec{x}',t) - \frac{1}{m^2}i\cdot\vec{\nabla}\delta^{(3)}(x-x')\ \nabla\cdot E(x',t)\right)$$

$$= -i\vec{E}(x,t) + \frac{i}{m^2}\vec{\nabla}(\nabla\cdot E(x,t))$$

$$\frac{d\vec{A}}{dt} = -\vec{E}(x,t) + \frac{1}{m^2}\vec{\nabla}(\nabla\cdot E(x,t))$$

—— EOM of $\vec{E}$:

$$i\frac{d\vec{E}}{dt} = [\vec{E}, H]$$

$$= [\vec{E}(x,t), \int d^3x' \frac{1}{2}(E^2 + (\nabla\times\vec{A})^2 + m^2\vec{A}^2 + \frac{1}{m^2}(\nabla\cdot E)^2)]$$

$$= \frac{1}{2}\int d^3x' [\vec{E}(x,t), (\nabla\times\vec{A})^2] + \frac{m^2}{2}\int d^3x' [\vec{E}(x,t), \vec{A}^2(x',t)]$$

$$= \int d^3x' \left(\frac{1}{2}[\vec{E}(x,t), (\nabla\times\vec{A})^2] + \frac{m^2}{2}[\vec{E}(x,t), \vec{A}^2(x',t)]\right)$$

Consider: $[E^1(x,t), (\nabla\times\vec{A}(x',t))^2]$

$$= [E^1(x,t), (\nabla\times\vec{A}(x',t))_y^2 + (\nabla\times\vec{A}(x',t))_z^2]$$

$$= [E^1(x,t), (\partial_z A^1(x',t) - \partial_x A^3(x',t))^2 + (\partial_x A^2(x',t) - \partial_y A^1(x',t))^2]$$

$$= [E^1(x,t), \partial_3' A^1(x',t)\cdot\partial_3' A^1(x',t) - 2\partial_3' A^1(x',t)\partial_1' A^3(x',t)]$$

$$+[E^1(x,t), \partial_2' A^1(x',t)\partial_2' A^1(x',t) - 2\partial_1' A^2(x',t)\partial_2' A^1(x',t)]$$

$$= 2\partial_3' A^1(x',t)\cdot\partial_3'\cdot[E^1(x,t), A^1(x',t)]$$

$$-2\partial_1' A^3(x',t)\,\partial_3'\,[E^1(x,t), A^1(x',t)]$$

$$+2\partial_2' A^1(x',t)\partial_2'\cdot[E^1(x,t), A^1(x',t)]$$

$$-2\partial_1' A^2(x',t)\partial_2'\,[E^1(x,t), A^1(x',t)]$$

$$= 2i\partial_3' A^1(x',t)\,\partial_3'\,\delta^{(3)}(x-x')$$

$$-2i\partial_1' A^3(x',t)\partial_3'\,\delta^{(3)}(x-x')$$

$$+2i\partial_2' A^1(x',t)\partial_2'\,\delta^{(3)}(x-x')$$

$$-2i\partial_1' A^2(x',t)\partial_2'\,\delta^{(3)}(x-x')$$

$$i\frac{dE^1}{dt} = \int d^3x' \Big(\frac{1}{2}\cdot(2i\partial_3' A^1(x',t)\partial_3'\delta^{(3)}(x-x') - 2i\partial_1' A^3(x',t)\partial_3'\delta^{(3)}(x-x')$$

$$+2i\partial_2' A^1(x',t)\partial_2'\delta^{(3)}(x-x') - 2i\partial_1' A^2(x',t)\partial_2'\delta^{(3)}(x-x'))$$

$$+\frac{m^2}{2}2i\cdot A^1(x',t)\cdot\delta^{(3)}(x-x')\Big)$$

$$= \int d^3x' \Big(-i(\partial_3'^2 A^1(x',t))\cdot\delta^{(3)}(x-x') - i\cdot(\partial_2'^2 A^1(x',t))\cdot\delta^{(3)}(x-x')$$

$$-i(\partial_1'^2 A^1(x',t))\delta^{(3)}(x-x') + i(\partial_1'^2 A^1(x',t))\delta^{(3)}(x-x')$$

$$+i(\partial_1'\partial_2' A^2(x',t))\delta^{(3)}(x-x') + i(\partial_1'\partial_3' A^3(x',t))\delta^{(3)}(x-x')$$

$$+ m^2\cdot i\cdot A^1(x',t)\delta^{(3)}(x-x')\Big)$$

$$= -i(\partial_1^2+\partial_2^2+\partial_3^2)A^1(x,t) + i\partial_1(\partial_1 A^1(x,t) + \partial_2 A^2(x,t) + \partial_3 A^3(x,t))$$

$$+ i m^2 A^1(x,t)$$

$$\frac{d\vec{E}}{dt} = -\nabla^2\vec{A}(x,t) + \nabla(\nabla\cdot\vec{A}(x,t)) + m^2\vec{A}(x,t)$$

Fourier Decomposition — Proca Field.

- fixed-momentum solution of proca field:

$$A^{\mu}(\vec{k}, x, \lambda) = N_{\vec{k}} \cdot \exp(-i(w_k t - \vec{k}\cdot\vec{x})) \cdot \varepsilon^{\mu}(\vec{k}, \lambda)$$

normalisation factor
$N_k = (2W_k(2\pi)^3)^{-1/2}$
$\lambda = 1, 2, 3$
$k = (\sqrt{|\vec{k}|^2 + m^2}, \vec{k})$

用 fixed-momentum solution linear Combine 为 解: (Neutral)

$$A^{\mu}(x) = \int d^3k \cdot \sum_{\lambda=1}^{3} \left(a_{k,\lambda} A^{\mu}(k,\lambda,x) + a^{+}_{k,\lambda} A^{\mu *}(k,\lambda,x) \right) \longleftarrow \text{为了 Unitary}$$

$$= \int d^3k \sum_{\lambda=1}^{3} \left(a_{k,\lambda}\, \varepsilon^{\mu}(k,\lambda) e^{-ik\cdot x} + a^{+}_{k,\lambda}\, \varepsilon^{\mu}(k,\lambda) e^{ik\cdot x} \right) N_k$$

For charged field (一般不考虑)

$$A^{\mu}(x) = \int d^3k \sum_{k=1}^{3} \left(a_{k,\lambda} \varepsilon^{\mu}(k,\lambda) e^{-ik\cdot x} + b^{+}_{k,\lambda} \varepsilon^{\mu}(k,\lambda) e^{-ik\cdot x} \right) N_k$$

—— vector potential

$$\vec{A}(x) = \int d^3k \sum_{\lambda=1}^{3} \left(a_{k,\lambda} e^{-ik\cdot x} + a^{+}_{k,\lambda} e^{ik\cdot x} \right) \vec{\varepsilon}(k,\lambda)\, N_k$$

—— Electro field strength tensor

$$E = -\frac{\partial A}{\partial t} - \nabla A^0$$

$$\vec{E}(x) = -\int d^3k \sum_{\lambda=1}^{3} \left(-iW_k a_{k,\lambda} e^{-ik\cdot x} + iW_k a^{+}_{k,\lambda} e^{ik\cdot x} \right) \vec{\varepsilon}(k,\lambda)\, N_k$$
$$- \int d^3k \sum_{\lambda=1}^{3} \left(a_{k,\lambda} e^{-ik\cdot x} - a^{+}_{k,\lambda} e^{ik\cdot x} \right) i\vec{k}\cdot \varepsilon^0(k,\lambda)\, N_k$$

$$= \int d^3k \cdot i \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \sum_{\lambda=1}^{3} W_k \left(\vec{\varepsilon}(k,\lambda) - \frac{\vec{k}}{W_k} \varepsilon^0(k,\lambda) \right) \left(a_{k,\lambda} e^{-ik\cdot x} - a^{+}_{k,\lambda} e^{ik\cdot x} \right)$$

Introducing:

$$\vec{\tilde{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{\vec{k}}{W_k} \varepsilon^0(k,\lambda)$$

Transverserality relation:

$$W_k \varepsilon^0 - \vec{k}\cdot\varepsilon = 0$$

$$= \vec{\varepsilon}(k,\lambda) - \frac{\vec{k}}{W_k^2} \left(\vec{k}\cdot\vec{\varepsilon}(k,\lambda) \right)$$

Property:

$$\vec{k}\cdot\vec{\tilde{\varepsilon}}(k,\lambda) = \left(1 - \frac{|\vec{k}|^2}{W_k^2}\right)\left(\vec{k}\cdot\vec{\varepsilon}(k,\lambda)\right) = \frac{m^2}{W_k^2}\, \vec{k}\cdot\vec{\varepsilon}(k,\lambda)$$

$$\vec{\tilde{\varepsilon}}(k,\lambda) = f_\lambda \vec{\varepsilon}(k,\lambda) \qquad f_\lambda = \begin{cases} 1 & \lambda = 1,2 \\ m^2/W_k^2 & \lambda = 3 \end{cases}$$

∘ Project out creation & annihilation operator.

—— scalar product of Two proca field:

definition $(A(x), A'(x)) = i\int d^3x \; A^{\mu *}(x) \overleftrightarrow{\partial_0} A'_\mu(x)$ $\quad\quad (A \overleftrightarrow{\partial_0} B = A(\partial_0 B) - (\partial_0 A) B)$

Fixed-Momentum-solution:

$$A^\mu(\vec{k}, x, \lambda) = N_{\vec{k}} \cdot \exp(-i(w_k t - \vec{k}\cdot\vec{x})) \cdot \varepsilon^\mu(\vec{k}, \lambda)$$ 不是算符,是场.

$$(A(k', \lambda'), A(k,\lambda)) = i\int d^3x \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \varepsilon^\mu(k',\lambda')\,\varepsilon_\mu(k,\lambda) \cdot \left(e^{ik'\cdot x} \overleftrightarrow{\partial_0} e^{-ik\cdot x}\right)$$

$$= i\int d^3x \,(e^{i(k'\cdot x - k\cdot x)})(-iW_k - iW_{k'}) \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^\mu(k',\lambda')\,\varepsilon_\mu(k,\lambda)$$

$$= i\cdot(2\pi)^3 \cdot \delta^{(3)}(k'-k)\cdot(-2iW_k)\frac{1}{2W_k(2\pi)^3}\varepsilon^\mu(k',\lambda')\,\varepsilon_\mu(k,\lambda)$$

$$= \delta^{(3)}(k-k')\,\varepsilon^\mu(k,\lambda')\,\varepsilon_\mu(k,\lambda)$$

$$= \delta^{(3)}(k-k')\cdot g_{\lambda\lambda'}$$

$$(A^*(k',\lambda'), A^*(k,\lambda)) = i\int d^3x \frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\frac{1}{\sqrt{2W_k(2\pi)^3}}\varepsilon^\mu(k',\lambda')\,\varepsilon_\mu(k,\lambda)\left(e^{-ik'\cdot x}\overleftrightarrow{\partial_0} e^{ik\cdot x}\right)$$

$$= i\int d^3x\,(iW_k + iW_{k'})\,e^{-ik'\cdot x + ik\cdot x}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\frac{1}{\sqrt{2W_k(2\pi)^3}}\varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda)$$

$$= -\;2W_k\cdot(2\pi)^3\cdot\delta^{(3)}(k-k')\frac{1}{2W_k(2\pi)^3}\,\varepsilon^\mu(k,\lambda')\,\varepsilon_\mu(k,\lambda)$$

$$= -\;\delta^{(3)}(k-k')\,\varepsilon^\mu(k,\lambda')\,\varepsilon_\mu(k,\lambda)$$

$$= -\;\delta^{(3)}(k-k')\;g_{\lambda\lambda'}$$

$$(A^*(k',\lambda'), A(k,\lambda)) = i\int d^3x\frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\frac{1}{\sqrt{2W_k(2\pi)^3}}\varepsilon^\mu(k',\lambda')\,\varepsilon_\mu(k,\lambda)\left(e^{-ik'\cdot x}\overleftrightarrow{\partial_0} e^{-ik\cdot x}\right)$$

$$= i\int d^3x\cdot(-iW_k + iW_{k'})\cdot e^{-i(k\cdot x + k'\cdot x)}\ldots$$

$$= 0$$

$$= (A(k',\lambda'), A^*(k,\lambda))$$

—— Project creation and annihilation op

$(A(k,\lambda), A(x)) =$ $\quad A^\mu(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}}\cdot\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x}$

$$\int d^3k\frac{1}{\sqrt{2W_k(2\pi)^3}}\sum_{\lambda=1}^{3}\left(a_{k,\lambda}\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x} + a^\dagger_{k,\lambda}\,\varepsilon^\mu(k,\lambda)\,e^{ik\cdot x}\right)$$

$$= i\int d^3x\, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\, e^{+ik\cdot x} \overleftrightarrow{\partial_0} \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}}$$
$$\sum_{\lambda'=1}^{3} \left(a_{k',\lambda'} \varepsilon_{\mu}(k',\lambda')\, e^{-ik'\cdot x} + a^{\dagger}_{k',\lambda'} \varepsilon_{\mu}(k',\lambda')\, e^{ik'\cdot x}\right)$$

$$= i\sum_{\lambda'=1}^{3}\int d^3x\, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k',\lambda')$$
$$\Big[e^{ik\cdot x}\cdot\left(-iW_{k'}\, a_{k',\lambda'}\, e^{-ik'\cdot x} + iW_{k'}\, a^{\dagger}_{k',\lambda'}\, e^{ik'\cdot x}\right)$$
$$- iW_k\, e^{ik\cdot x}\left(a_{k',\lambda'}\, e^{-ik'\cdot x} + a^{\dagger}_{k',\lambda'}\, e^{ik'\cdot x}\right)\Big]$$

$$= i\sum_{\lambda'=1}^{3}\int d^3x\, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k',\lambda')$$
$$e^{ik\cdot x}\Big[a_{k',\lambda'}\, e^{-ik'\cdot x}\,(-iW_{k'} - iW_k) + a^{\dagger}_{k',\lambda'}\, e^{ik'\cdot x}(-iW_{k'} - iW_k)\Big]$$

$$= i\sum_{\lambda'=1}^{3}\int d^3k'(2\pi)^3 \frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k',\lambda')$$
$$a_{k',\lambda'}\,\delta^{(3)}(k-k')\,(-2iW_k)$$
$$= \sum_{\lambda'=1}^{3} (2\pi)^3 \frac{1}{2W_k(2\pi)^3}\, \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k,\lambda')\, a_{k,\lambda'}\,(2W_k)$$

$$= \sum_{\lambda'=1}^{3} g_{\lambda,\lambda'}\, a_{k,\lambda'}$$
$$= -a_{k,\lambda} \quad (g_{\lambda,\lambda} = -1 \text{ for } \lambda \neq 0)$$

这只是个场，但 $A(x)$ 是 operator!

$$(A^{*}(k,\lambda), A(x)) = \quad \begin{cases} A^{\mu *}(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \varepsilon^{\mu}(k,\lambda)\, e^{+ik\cdot x} \\ A(x) = \int d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \sum_{\lambda=1}^{3} \left(a_{k,\lambda}\, \varepsilon^{\mu}(k,\lambda)\, e^{-ik\cdot x} + a^{\dagger}_{k,\lambda}\, \varepsilon^{\mu}(k,\lambda)\, e^{ik\cdot x}\right) \end{cases}$$

$$= i\int d^3x\, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\, e^{-ik\cdot x} \overleftrightarrow{\partial_0} \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}}$$
$$\sum_{\lambda'=1}^{3} \left(a_{k',\lambda'} \varepsilon_{\mu}(k',\lambda')\, e^{-ik'\cdot x} + a_{k',\lambda'}\, e^{ik'\cdot x}\right)$$

$$= i\sum_{\lambda'=1}^{3}\int d^3x\, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k',\lambda')$$
$$\Big[e^{-ik\cdot x}\cdot\left(-iW_{k'}\, a_{k',\lambda'}\, e^{-ik'\cdot x} + iW_{k'}\, a^{\dagger}_{k',\lambda'}\, e^{ik'\cdot x}\right)$$
$$+ iW_k\, e^{-ik\cdot x}\left(a_{k',\lambda'}\, e^{+ik'\cdot x} + a^{\dagger}_{k',\lambda'}\, e^{-ik'\cdot x}\right)\Big]$$

$$= i\sum_{\lambda'=1}^{3}\int d^3x\, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k',\lambda')$$
$$\Big(e^{-ik\cdot x}\, a_{k',\lambda'}\, e^{-ik'\cdot x}(-iW_{k'} + iW_k)$$
$$+ e^{-ik\cdot x}\, a^{\dagger}_{k',\lambda'}\, e^{ik\cdot x}\,(iW_{k'} + iW_k)\Big)$$

$$= i\sum_{\lambda'=1}^{3}\int d^3k'\,(2\pi)^3 \frac{1}{2W_k\cdot(2\pi)^3}\cdot \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k,\lambda')\cdot(2iW_k)\, a^{\dagger}_{k,\lambda'}\,\delta^{(3)}(k-k')$$

$$= -\sum_{\lambda'=1}^{3} \varepsilon^{\mu}(k,\lambda)\,\varepsilon_{\mu}(k,\lambda')\, a^{\dagger}_{k,\lambda'} = a^{\dagger}_{k,\lambda}$$

Commutation of creation and annihilation operator.

- $a_{k,\lambda}$; $a^\dagger_{k,\lambda}$ 的另一种表达(不同于上页):

—— $a_{k,\lambda} = -(A(k,\lambda), A(x))$

$$= -i\int d^3x\, A^{\mu *}(k,\lambda)\overleftrightarrow{\partial_0} A_\mu(x)$$

$$A^\mu(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}}\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x}$$

$$= -i\int d^3x\,\left(A^{\mu *}(k,\lambda)\,\partial_0 A_\mu(x) - (\partial_0 A^{\mu *}(k,\lambda))\,A_\mu(x)\right)$$

$$= -i\int d^3x\,\left(A^{\mu *}(k,\lambda)\,\partial_0 A_\mu(x) - iW_k A^{\mu *}(k,\lambda)\,A_\mu(x)\right)$$

$$= -i\int d^3x\,\frac{1}{\sqrt{2W_k(2\pi)^3}}\,\varepsilon^\mu(k,\lambda)\,e^{ik\cdot x}\left(\partial_0 A_\mu(x) - iW_k A_\mu(x)\right)$$

proca Equation: $\partial_\mu A^\mu = 0 \rightarrow \partial_0 A^0 = -\nabla\cdot\vec{A}$

Definition of A: $\vec{E} = -\partial_0\vec{A} - \vec{\nabla}A_0$

$$= -i\int d^3x\,\frac{1}{\sqrt{2W_k(2\pi)^3}}\,e^{ik\cdot x}\left(\varepsilon^0\partial_0 A_0 - \vec{\varepsilon}\cdot\partial_0\vec{A} - iW_k\varepsilon^0 A_0 + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

$$= -i\int d^3x\,\frac{1}{\sqrt{2W_k(2\pi)^3}}\,e^{ik\cdot x}\left(\underline{-\varepsilon^0\nabla\cdot\vec{A}} + \vec{\varepsilon}\cdot\vec{E} + \underline{\vec{\varepsilon}\cdot\vec{\nabla}A_0} - iW_k\varepsilon^0 A_0 + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

对橙色部分用 Integral by parts!

$$= -i\int d^3x\,\frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(-i\varepsilon^0\vec{k}\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + \underline{i\vec{\varepsilon}\cdot\vec{k}A_0 - iW_k\varepsilon^0 A_0} + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

$k^\mu = (W_k, \vec{k})$, $W_k = \sqrt{|\vec{k}|^2 + m^2}$

$k^\mu\varepsilon_\mu = 0$ (Fixed momentum - proca solution, 性质)

$$= -i\int d^3x\,\frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(-i\varepsilon^0\vec{k}\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

$$= \int d^3x\,\frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left((W_k\vec{\varepsilon} - \varepsilon^0\vec{k})\cdot\vec{A} - i\,\vec{\varepsilon}\cdot\vec{E}\right)$$

$$\tilde{\vec{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{\varepsilon^0}{W_k}\vec{k}$$

$$= \int d^3x\,\frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(W_k\,\tilde{\vec{\varepsilon}}(k,\lambda)\,\vec{A}(x) - i\,\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)\right)$$

—— $a^\dagger_{k,\lambda} = +(A^*(k,\lambda), A(x))$

$$= +i\int d^3x\,A^\mu(k,\lambda)\overleftrightarrow{\partial_0}A_\mu(x)$$

$$A^\mu(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}}\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x}$$

$$= i\int d^3x\left(A^\mu(k,\lambda)\,\partial_0 A_\mu(x) - \partial_0 A^\mu(k,\lambda)\,A_\mu(x)\right)$$

$$= i\int d^3x\,A^\mu(k,\lambda)\left(\partial_0 A_\mu(x) + iW_k A_\mu(x)\right)$$

$$= i\int d^3x \frac{1}{\sqrt{2\omega_k(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\, e^{-ik\cdot x}(\partial_0 A_\mu(x) + i\omega_k A_\mu(x))$$

} proca eq: $\partial_\mu A^\mu \to \partial_0 A_0 = -\vec{\nabla}\cdot\vec{A}$
} Definition relation A & E: $\partial_0\vec{A} = -\vec{E} - \nabla A_0$

$$= i\int d^3x \frac{1}{\sqrt{2\omega_k(2\pi)^3}} e^{-ik\cdot x}(-\varepsilon^0\nabla\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + \vec{\varepsilon}\cdot\nabla A_0$$
$$+ i\omega_k\varepsilon^0 A_0 - i\omega_k\vec{\varepsilon}\cdot\vec{A})$$

} 分部积分: $\varepsilon^0\nabla\cdot\vec{A} \to -i\vec{k}\cdot\vec{A}\ \varepsilon^0$
$\vec{\varepsilon}\cdot\vec{\nabla}A_0 \to -i\vec{k}\cdot\vec{\varepsilon}\ A_0$

$$= i\int d^3x \frac{1}{\sqrt{2\omega_k(2\pi)^3}} e^{-ik\cdot x}(+i\vec{k}\cdot\vec{A}\varepsilon^0 + \vec{\varepsilon}\cdot\vec{E} - i\vec{k}\cdot\vec{\varepsilon}A_0$$
$$+ i\omega_k\varepsilon^0 A_0 - i\omega_k\vec{\varepsilon}\cdot\vec{A})$$

} $k^\mu\varepsilon_\mu = 0$

$$= i\int d^3x \frac{1}{\sqrt{2\omega_k(2\pi)^3}} e^{-ik\cdot x}(i\vec{k}\cdot\vec{A}\varepsilon^0 + \vec{\varepsilon}\cdot\vec{E} - i\omega_k\vec{\varepsilon}\cdot\vec{A})$$

$$= \int d^3x \frac{1}{\sqrt{2\omega_k(2\pi)^3}} e^{-ik\cdot x}\Big(\underbrace{(\omega_k\vec{\varepsilon} - \vec{k}\varepsilon^0)}_{=\omega_k\tilde{\vec{\varepsilon}}}\cdot\vec{A} + i\ \vec{\varepsilon}\cdot\vec{E}\Big)$$

———— 综上:

$$a_{k,\lambda} = \int d^3x \frac{e^{ik\cdot x}}{\sqrt{2\omega_k(2\pi)^3}}\left(\omega_k\tilde{\vec{\varepsilon}}(k,\lambda)\vec{A}(x) - i\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)\right)$$

$$a^\dagger_{k,\lambda} = \int d^3x \frac{e^{-ik\cdot x}}{\sqrt{2\omega_k(2\pi)^3}}\left(\omega_k\tilde{\vec{\varepsilon}}(k,\lambda)\vec{A}(x) + i\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)\right)$$

$$\tilde{\vec{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{\varepsilon^0(k,\lambda)}{\omega_k}\vec{k}$$

- Commutation of creation & anni

$$[a_{k',\lambda'}, a^\dagger_{k,\lambda}] = \int d^3x\, d^3x' \frac{1}{\sqrt{2\omega_{k'}(2\pi)^3}}\frac{1}{\sqrt{2\omega_k(2\pi)^3}} e^{ik'\cdot x'}e^{-ik\cdot x}$$
$$[\omega_{k'}\tilde{\vec{\varepsilon}}(k',\lambda')\cdot\vec{A}(x') - i\vec{\varepsilon}(k',\lambda')\cdot\vec{E}(x'),$$
$$\omega_k\tilde{\vec{\varepsilon}}(k,\lambda)\cdot\vec{A}(x) + i\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)]$$

$$= \int d^3x\, d^3x' \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{ik'\cdot x'} e^{-ik\cdot x}$$

$$\Big([W_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{A}(x'), i\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)] \to pt1$$

$$+ [-i\vec{\varepsilon}(k',\lambda')\cdot\vec{E}(x'), W_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{A}(x)] \Big) \to pt2$$

$$pt1 = iW_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{\varepsilon}(k,\lambda)(-i\delta^{(3)}(x-x'))$$

$$pt2 = iW_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k',\lambda')(-i\delta^{(3)}(x-x'))$$

$$= \int d^3x\, d^3x' \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{ik'\cdot x'} e^{-ik\cdot x} \delta^{(3)}(x-x')$$

$$\Big(W_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{\varepsilon}(k,\lambda) + W_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k',\lambda') \Big)$$

$$= \int d^3x \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{ik'\cdot x} e^{-ik\cdot x}$$

$$\Big(W_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{\varepsilon}(k,\lambda) + W_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k',\lambda') \Big)$$

$$= \frac{1}{\sqrt{2W_k}\sqrt{2W_{k'}}} \cdot \delta^{(3)}(k-k')(\sim)$$

$$= \frac{1}{2W_k} W_k \Big(\vec{\tilde{\varepsilon}}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) + \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k,\lambda') \Big) \delta^{(3)}(k-k')$$

$$\vec{\tilde{\varepsilon}}(k,\lambda') = \vec{\varepsilon}(k,\lambda') - \frac{1}{W_k}\varepsilon^0(k,\lambda')\vec{k}$$

$$\vec{\tilde{\varepsilon}}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) = \Big(\vec{\varepsilon}(k,\lambda') - \frac{\varepsilon^0(k,\lambda')}{W_k}\vec{k}\Big)\cdot\vec{\varepsilon}(k,\lambda)$$

$$= \vec{\varepsilon}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) - \frac{\varepsilon^0(k,\lambda')}{W_k} \underbrace{\vec{k}\cdot\vec{\varepsilon}(k,\lambda)}_{= W_k \varepsilon^0(k,\lambda)}$$

$$= \vec{\varepsilon}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) - \varepsilon^0(k,\lambda')\varepsilon^0(k,\lambda)$$

$$= -\varepsilon(k,\lambda')\cdot\varepsilon(k,\lambda)$$

$$= -g_{\lambda,\lambda'}$$

$$= \delta_{\lambda,\lambda'}$$

$$= \delta^{(3)}(k-k')\,\delta_{\lambda,\lambda'}$$

Commutator of Field A

∘ $$A^\mu(x) = \sum_{\lambda=1}^{3} \int \frac{d^3k}{\sqrt{2W_k(2\pi)^3}} \varepsilon^\mu(k,\lambda) \cdot \left(a_{k,\lambda} e^{-ik\cdot x} + a^\dagger_{k,\lambda} e^{ik\cdot x} \right)$$

$\Downarrow$

$$[A^\mu(x), A^\nu(y)] = \int \frac{d^3k'}{\sqrt{2W_{k'}(2\pi)^3}} \int \frac{d^3k}{\sqrt{2W_k(2\pi)^3}} \sum_{\lambda,\lambda'=1}^{3} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda) \Big([a_{k',\lambda'}, a^\dagger_{k,\lambda}] e^{-i(k'\cdot x - k\cdot y)}$$
$$+ [a^\dagger_{k',\lambda'}, a_{k,\lambda}] e^{i(k'\cdot x - ky)} \Big)$$

$$[a_{k,\lambda}, a^\dagger_{k',\lambda'}] = i\delta_{\lambda,\lambda'}\delta^{(3)}(k-k')$$

$$= \int \frac{d^3k}{2W_k(2\pi)^3} \sum_{\lambda,\lambda'=1}^{3} \varepsilon^\mu(k,\lambda')\varepsilon^\nu(k,\lambda) \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

$$\sum_{\lambda,\lambda'=1}^{3} \varepsilon^\mu(k,\lambda')\varepsilon^\nu(k,\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}$$

$$= \int \frac{d^3k}{2W_k(2\pi)^3} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

$$= \left(-g^{\mu\nu} - \frac{1}{m^2}\partial^\mu\partial^\nu\right) \int \frac{d^3k}{2W_k(2\pi)^3} \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

Invariant Pauli–Jordan Func:

$$i\Delta(x-y) = -i\int \frac{d^3k}{(2\pi)^3} \frac{\sin(k\cdot(x-y))}{W_k}$$

$$[A^\mu(x), A^\nu(y)] = -i\left(g^{\mu\nu} + \frac{1}{m^2}\partial^\mu\partial^\nu\right)\Delta(x-y)$$

Feynman Propogator

- Feynman propogator 定义为:

$$i\Delta_F^{\mu\nu}(x-y) := \langle 0|T(A^\mu(x) A^\nu(y))|0\rangle$$

$$\left\{ A^\mu(x) = \sum_{\lambda'=1}^{3} \int \frac{d^3k'}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\left(a_{k',\lambda'} e^{-ik'\cdot x} + a^\dagger_{k',\lambda'} e^{ik'\cdot x}\right)\right.$$

$$= \langle 0| \sum_{\lambda,\lambda'=1}^{3} \int d^3k' d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$(a_{k',\lambda'} e^{-ik'\cdot x} + a^\dagger_{k',\lambda'} e^{ik'\cdot x})(a_{k,\lambda} e^{-ik\cdot y} + a^\dagger_{k,\lambda} e^{ik\cdot y})|0\rangle$$

$$= \sum_{\lambda,\lambda'=1}^{3} \int d^3k' d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$\Big[e^{-ik'\cdot x} e^{ik\cdot y} \langle 0| a_{k',\lambda'} a^\dagger_{k,\lambda}|0\rangle \Theta(x_0-y_0)$$
$$+ e^{ik'\cdot x} e^{-ik\cdot y} \langle 0| a_{k,\lambda} a^\dagger_{k',\lambda'}|0\rangle \Theta(y_0-x_0) \Big]$$

$$\left\{ \begin{aligned} &\text{Creation / annihilation commutation} \\ &[a_{k',\lambda'}, a^\dagger_{k,\lambda}] = \delta^{(3)}(k-k')\delta_{\lambda,\lambda'} \\ &\langle 0| a_{k',\lambda'} a^\dagger_{k,\lambda}|0\rangle = \langle 0| \delta^{(3)}(k-k')\delta_{\lambda,\lambda'} + a^\dagger_{k,\lambda} a_{k',\lambda'}|0\rangle = \delta^{(3)}(k-k')\delta_{\lambda,\lambda'} \end{aligned}\right.$$

$$= \sum_{\lambda,\lambda'=1}^{3} \int d^3k' d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$\Big[e^{-ik'\cdot x} e^{ik\cdot y} \Theta(x_0-y_0)\delta^{(3)}(k-k')$$
$$+ e^{ik'\cdot x} e^{-ik\cdot y} \Theta(y_0-x_0)\delta^{(3)}(k-k') \Big]\delta_{\lambda,\lambda'}$$

$$= \sum_{\lambda=1}^{3} \int d^3k \frac{1}{2W_k(2\pi)^3} \varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda)$$
$$\left(e^{-ik\cdot(x-y)}\Theta(x_0-y_0) + e^{ik\cdot(x-y)}\Theta(y_0-x_0)\right)$$

$$\left\{ \sum_{\lambda=1}^{3} \varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda) = -\left(g^{\mu\nu} - \frac{1}{m^2}k^\mu k^\nu\right)\right.$$

$$= -\int d^3k \frac{1}{2W_k(2\pi)^3}\left(g^{\mu\nu} - \frac{1}{m^2}k^\mu k^\nu\right)\left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)}\right)$$

-

$$i\partial_x^\mu\, i\partial_x^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)}\right)$$
$$= k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)}\right)$$
$$+ g^{\mu 0}\cdot[i\partial_x^0\Theta(x_0-y_0)][i\partial_x^\nu e^{-ik\cdot(x-y)}] + g^{\mu 0}[i\partial_x^0\Theta(y_0-x_0)][i\partial_x^\nu e^{ik(x-y)}]$$
$$+ g^{\nu 0}[i\partial_x^0\Theta(x_0-y_0)][i\partial_x^\mu e^{-ik\cdot(x-y)}] + g^{\nu 0}[i\partial_x^0\Theta(y_0-x_0)][i\partial_x^\mu e^{ik\cdot(x-y)}]$$
$$+ g^{\mu 0}g^{\nu 0}[i\partial_x^0\, i\partial_x^0\Theta(x_0-y_0)] e^{-ik\cdot(x-y)} + g^{\mu 0}g^{\nu 0}[i\partial_x^0\, i\partial_x^0\Theta(y_0-x_0)] e^{ik(x-y)}$$

$-\partial_x^0\,\delta$ $\qquad\qquad$ $-\partial_x^0$

$$
\begin{aligned}
= & \; k^\mu k^\nu \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right) \\
& + g^{\mu 0} \left[i\,\delta(x_0-y_0) \right] \left[k^\nu e^{-ik\cdot(x-y)} \right] + g^{\mu 0} \left[-i\,\delta(x_0-y_0) \right] \left[-k^\nu e^{ik(x-y)} \right] \\
& + g^{\nu 0} \left[i\,\delta(x_0-y_0) \right] \left[k^\mu e^{-ik\cdot(x-y)} \right] + g^{\nu 0} \left[-i\,\delta(x_0-y_0) \right] \left[-k^\mu e^{ik\cdot(x-y)} \right] \\
& + g^{\mu 0} g^{\nu 0} \left[-\delta'(x_0-y_0) \right] e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0} \left[+\delta'(x_0-y_0) \right] e^{ik\cdot(x-y)}
\end{aligned}
$$

$$
\begin{aligned}
= & \; k^\mu k^\nu \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right) \\
& + i k^\nu g^{\mu 0}\, \delta(x_0-y_0)\, e^{-ik\cdot(x-y)} + i k^\nu g^{\mu 0}\, \delta(x_0-y_0)\, e^{ik(x-y)} \\
& + i k^\mu g^{\nu 0}\, \delta(x_0-y_0)\, e^{-ik\cdot(x-y)} + i k^\mu g^{\nu 0}\, \delta(x_0-y_0)\, e^{ik\cdot(x-y)} \\
& - g^{\mu 0} g^{\nu 0}\, \delta'(x_0-y_0)\, e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0}\, \delta'(x_0-y_0)\, e^{ik\cdot(x-y)}
\end{aligned}
$$

于是:

$$
\begin{aligned}
& k^\mu k^\nu \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right) \\
& = i\partial_x^\mu\, i\partial_x^\nu \left(\Theta(x_0-y_0)\, e^{-ik(x-y)} + \Theta(y_0-x_0)\, e^{ik(x-y)} \right) \\
& \quad - i\left(k^\nu g^{\mu 0} + k^\mu g^{\nu 0} \right) \delta(x_0-y_0) \left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right) \\
& \quad + g^{\mu 0} g^{\nu 0}\, \delta'(x_0-y_0)\, e^{-ik\cdot(x-y)} - g^{\mu 0} g^{\nu 0}\, \delta'(x_0-y_0)\, e^{ik\cdot(x-y)}
\end{aligned}
$$

$$
\begin{aligned}
& i\Delta_F^{\mu\nu}(x-y) \\
& = -\int d^3k\, \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} k^\mu k^\nu \right) \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right) \\
& = -\int d^3k\, \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} (i\partial_x^\mu)(i\partial_x^\nu) \right) \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right) \\
& \quad + \frac{1}{m^2} \int d^3k\, \frac{1}{2\omega_k (2\pi)^3} \Big[-i\left(k^\nu g^{\mu 0} + k^\mu g^{\nu 0} \right) \delta(x_0-y_0) \left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right) \\
& \qquad\qquad + g^{\mu 0} g^{\nu 0}\, \delta'(x_0-y_0)\, e^{-ik\cdot(x-y)} - g^{\mu 0} g^{\nu 0}\, \delta'(x_0-y_0)\, e^{ik\cdot(x-y)} \Big]
\end{aligned}
$$

$\int d^3k \cdot f(k)$ (k的奇func) $= 0$

$\delta'(x-x_0) f(x) \to -\delta(x-x_0) f'(x)$

$$
\begin{aligned}
& = -\int d^3k\, \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} (i\partial_x^\mu)(i\partial_x^\nu) \right) \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right) \\
& \quad + \frac{1}{m^2} \int d^3k\, \frac{1}{2\omega_k (2\pi)^3} \Big[+i\omega_k\, g^{\mu 0} g^{\nu 0}\, \delta(x_0-y_0)\, e^{-ik\cdot(x-y)} \\
& \qquad\qquad + i\omega_k\, g^{\mu 0} g^{\nu 0}\, \delta(x_0-y_0)\, e^{ik(x-y)} \Big]
\end{aligned}
$$

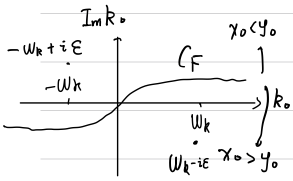
$$
= -\left(g^{\mu\nu} + \frac{1}{m^2} \partial_x^\mu \partial_x^\nu \right) i\Delta_F(x-y) + \frac{i}{m^2} g^{\mu 0} g^{\nu 0}\, \delta^{(4)}(x-y)
$$

$$
i\Delta_F(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right)
$$

∘ Momentum Feynman propogator :

$$\nearrow k = (\omega_k, \vec{k})$$

$$i\Delta_F(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\Theta(x_0 - y_0) \cdot e^{-ik\cdot(x-y)} + \Theta(y_0 - x_0)\, e^{ik\cdot(x-y)} \right)$$



$$= \int \frac{d^3k}{2\omega_k (2\pi)^3} e^{-i\omega_k(x_0-y_0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \quad (x_0 > y_0) \rightarrow = \int d^3k \left[\int_{-\infty-i\varepsilon}^{+\infty+i\varepsilon} -\frac{1}{2\pi i} \frac{dk_0 \cdot e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)} \right] \frac{1}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$+ \int \frac{d^3k}{2\omega_k (2\pi)^3} e^{-i\omega_k(x_0-y_0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \quad (y_0 > x_0)$$

$$\downarrow$$

$$= \int d^3k \left[\int_{-\infty-i\varepsilon}^{+\infty+i\varepsilon} \frac{-1}{2\pi i} \frac{dk_0 \cdot e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)} \right] \frac{1}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= \frac{i}{(2\pi)^4} \int_{C_F} \frac{d^4k}{(k_0-\omega_k)(k_0+\omega_k)} e^{-ik\cdot(x-y)} \qquad \} \ \omega_k \rightarrow \omega_k - i\varepsilon$$

$$= \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 - \omega_k^2 + i\varepsilon} e^{-ik\cdot(x-y)}$$

$$\Big\} \ (k-\omega_k)(k+\omega_k) \rightarrow k^2 - (\omega_k - i\varepsilon)^2$$

$$= k^2 - \omega_k^2 + \varepsilon^2 + 2i\omega_k\varepsilon$$

$$\downarrow$$

$$\underline{k^2 - \omega_k^2 + i\varepsilon}$$

∘ $$i\Delta_F^{\mu\nu}(x-y) = -\left(g^{\mu\nu} + \frac{1}{m^2}\partial_x^\mu \partial_x^\nu\right) i\Delta_F(x-y) + \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)$$

$$= -\left(g^{\mu\nu} + \frac{1}{m^2}\partial_x^\mu \partial_x^\nu\right) \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 - \omega_k^2 + i\varepsilon} \cdot e^{-ik\cdot(x-y)} + \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \int d^4k \frac{1}{(2\pi)^4} e^{-ik\cdot(x-y)}$$

$$= \int \frac{i\,d^4k}{(2\pi)^4} \cdot \left[-\frac{\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - \omega_k^2 + i\varepsilon} + \frac{1}{m^2} g^{\mu 0} g^{\nu 0} \right]$$

$$\Delta_F^{\mu\nu}(k) = -\frac{\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - \omega_k^2 + i\varepsilon} + \frac{1}{m^2} g^{\mu 0} g^{\nu 0}$$

Feynman Propogator

o Feynman propogator 定义为：

$$i\Delta_F^{\mu\nu}(x-y) := \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle$$

$$\left\{ A^\mu(x) = \sum_{\lambda'=1}^{3}\int \frac{d^3k'}{\sqrt{2W_{k'}(2\pi)^3}}\,\varepsilon^\mu(k',\lambda')\left(a_{k',\lambda'}e^{-ik'\cdot x} + a^\dagger_{k',\lambda'}e^{ik'\cdot x}\right)\right.$$

$$= \langle 0|\sum_{\lambda,\lambda'=1}^{3}\int d^3k'\,d^3k\,\frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\,\varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$(a_{k',\lambda'}e^{-ik'\cdot x} + a^\dagger_{k',\lambda'}e^{ik'\cdot x})(a_{k,\lambda}e^{-ik\cdot y} + a^\dagger_{k,\lambda}e^{ik\cdot y})|0\rangle$$

$$= \sum_{\lambda,\lambda'=1}^{3}\int d^3k'\,d^3k\,\frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\,\varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$\Big[\, e^{-ik'\cdot x}e^{ik\cdot y}\langle 0|a_{k',\lambda'}a^\dagger_{k,\lambda}|0\rangle\,\Theta(x_0-y_0)$$

$$+\, e^{ik'\cdot x}e^{-ik\cdot y}\langle 0|a_{k,\lambda}a^\dagger_{k',\lambda'}|0\rangle\,\Theta(y_0-x_0)\,\Big]$$

creation / annihilation commutation

$$[a_{k',\lambda'}, a^\dagger_{k,\lambda}] = \delta^{(3)}(k-k')\delta_{\lambda,\lambda'}$$

$$\langle 0|a_{k',\lambda'}a^\dagger_{k,\lambda}|0\rangle = \langle 0|\delta^{(3)}(k-k')\delta_{\lambda,\lambda'} + a^\dagger_{k,\lambda}a_{k',\lambda'}|0\rangle = \delta^{(3)}(k-k')\delta_{\lambda,\lambda'}$$

$$= \sum_{\lambda,\lambda'=1}^{3}\int d^3k'\,d^3k\,\frac{1}{\sqrt{2W_k(2\pi)^3}}\frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\,\varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$\Big[\, e^{-ik'\cdot x}e^{ik\cdot y}\,\Theta(x_0-y_0)\,\delta^{(3)}(k-k')$$

$$+\, e^{ik'\cdot x}e^{-ik\cdot y}\,\Theta(y_0-x_0)\,\delta^{(3)}(k-k')\,\Big]\delta_{\lambda,\lambda'}$$

$$= \sum_{\lambda=1}^{3}\int d^3k\,\frac{1}{2W_k(2\pi)^3}\,\varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda)$$

$$\left(e^{-ik\cdot(x-y)}\,\Theta(x_0-y_0) + e^{ik\cdot(x-y)}\,\Theta(y_0-x_0)\right)$$

$$\left\{ \sum_{\lambda=1}^{3}\varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda) = -\left(g^{\mu\nu} - \frac{1}{m^2}k^\mu k^\nu\right)\right.$$

$$= -\int d^3k\,\frac{1}{2W_k(2\pi)^3}\left(g^{\mu\nu} - \frac{1}{m^2}k^\mu k^\nu\right)\left(\Theta(x_0-y_0)\,e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\,e^{ik\cdot(x-y)}\right)$$

o

$$i\partial_x^\mu\, i\partial_x^\nu\left(\Theta(x_0-y_0)e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)e^{ik\cdot(x-y)}\right)$$

$$= k^\mu k^\nu\left(\Theta(x_0-y_0)e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)e^{ik\cdot(x-y)}\right)$$

$$+ g^{\mu 0}\left[i\partial_x^0\Theta(x_0-y_0)\right]\left[i\partial_x^\nu e^{-ik\cdot(x-y)}\right] + g^{\mu 0}\left[i\partial_x^0\Theta(y_0-x_0)\right]\left[i\partial_x^\nu e^{ik(x-y)}\right]$$

$$+ g^{\nu 0}\left[i\partial_x^0\Theta(x_0-y_0)\right]\left[i\partial_x^\mu e^{-ik\cdot(x-y)}\right] + g^{\nu 0}\left[i\partial_x^0\Theta(y_0-x_0)\right]\left[i\partial_x^\mu e^{ik\cdot(x-y)}\right]$$

$$+ g^{\mu 0}g^{\nu 0}\left[i\partial_x^0\, i\partial_x^0\,\Theta(x_0-y_0)\right]e^{-ik\cdot(x-y)} + g^{\mu 0}g^{\nu 0}\left[i\partial_x^0\, i\partial_x^0\,\Theta(y_0-x_0)\right]e^{ik\cdot(x-y)}$$

$-\partial_x^i \delta$ $\qquad\qquad$ $-\partial_x^0$

$$= k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$
$$+ g^{\mu 0} \left[i\,\delta(x_0-y_0) \right] \left[k^\nu e^{-ik\cdot(x-y)} \right] + g^{\mu 0} \left[-i\delta(x_0-y_0) \right] \left[-k^\nu e^{ik(x-y)} \right]$$
$$+ g^{\nu 0} \left[i\,\delta(x_0-y_0) \right] \left[k^\mu e^{-ik\cdot(x-y)} \right] + g^{\nu 0} \left[-i\delta(x_0-y_0) \right] \left[-k^\mu e^{ik\cdot(x-y)} \right]$$
$$+ g^{\mu 0} g^{\nu 0} \left[-\delta'(x_0-y_0) \right] e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0} \left[+\delta'(x_0-y_0) \right] e^{ik\cdot(x-y)}$$

$$= k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$
$$+ ik^\nu g^{\mu 0} \delta(x_0-y_0) e^{-ik\cdot(x-y)} + ik^\nu g^{\mu 0} \delta(x_0-y_0) e^{ik(x-y)}$$
$$+ ik^\mu g^{\nu 0} \delta(x_0-y_0) e^{-ik\cdot(x-y)} + ik^\mu g^{\nu 0} \delta(x_0-y_0) e^{ik\cdot(x-y)}$$
$$- g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{ik\cdot(x-y)}$$

于是:

$$k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$
$$= i\partial_x^\mu\, i\partial_x^\nu \left(\Theta(x_0-y_0) e^{-ik(x-y)} + \Theta(y_0-x_0) e^{ik(x-y)} \right)$$
$$- i\left(k^\nu g^{\mu 0} + k^\mu g^{\nu 0} \right) \delta(x_0-y_0) \left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right)$$
$$+ g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{-ik\cdot(x-y)} - g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{ik\cdot(x-y)}$$

$$i\Delta_F^{\mu\nu}(x-y)$$
$$= -\int d^3k \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} k^\mu k^\nu \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$= -\int d^3k \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} (i\partial_x^\mu)(i\partial_x^\nu) \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$+ \frac{1}{m^2} \int d^3k \frac{1}{2\omega_k (2\pi)^3} \Big[-i\left(k^\nu g^{\mu 0} + k^\mu g^{\nu 0} \right) \delta(x_0-y_0) \left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right)$$
$$+ g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{-ik\cdot(x-y)} - g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{ik\cdot(x-y)} \Big]$$

$\int d^3k \cdot f(k)$ (k的奇 func) $=0$

$\delta'(x-x_0) f(x) \to -\delta(x-x_0) f'(x)$

$$= -\int d^3k \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} (i\partial_x^\mu)(i\partial_x^\nu) \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$+ \frac{1}{m^2} \int d^3k \frac{1}{2\omega_k (2\pi)^3} \Big[+i\omega_k\, g^{\mu 0} g^{\nu 0} \delta(x_0-y_0) e^{-ik\cdot(x-y)}$$
$$+ i\omega_k\, g^{\mu 0} g^{\nu 0} \delta(x_0-y_0) e^{ik(x-y)} \Big]$$

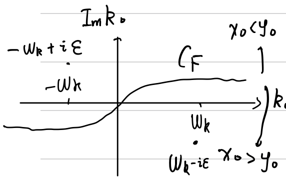
$$= -\left(g^{\mu\nu} + \frac{1}{m^2} \partial_x^\mu \partial_x^\nu \right) i\Delta_F(x-y) + \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)$$

$$i\Delta_F(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

○ Momentum Feynman propogator :

$$\nearrow k = (\omega_k, \vec{k})$$

$$i\Delta_F(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\Theta(x_0 - y_0) \cdot e^{-ik\cdot(x-y)} + \Theta(y_0 - x_0)\, e^{ik\cdot(x-y)} \right)$$



$$= \int \frac{d^3k}{2\omega_k (2\pi)^3} e^{-i\omega_k(x_0-y_0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \quad (x_0 > y_0) \rightarrow = \int d^3k \left[\int_{-\infty - i\varepsilon}^{+\infty + i\varepsilon} -\frac{1}{2\pi i} \frac{dk_0 \cdot e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)} \right] \frac{1}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$+ \int \frac{d^3k}{2\omega_k (2\pi)^3} e^{-i\omega_k(x_0-y_0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \quad (y_0 > x_0)$$

$$\downarrow$$

$$= \int d^3k \left[\int_{-\infty - i\varepsilon}^{+\infty + i\varepsilon} \frac{-1}{2\pi i} \frac{dk_0 \cdot e^{-ik_0(x_0-y_0)}}{(k_0-\omega_k)(k_0+\omega_k)} \right] \frac{1}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= \frac{i}{(2\pi)^4} \int_{C_F} \frac{d^4k}{(k_0-\omega_k)(k_0+\omega_k)} e^{-ik\cdot(x-y)} \qquad \} \ \omega_k \to \omega_k - i\varepsilon$$

$$= \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 - \omega_k^2 + i\varepsilon} e^{-ik\cdot(x-y)}$$

$$(k-\omega_k)(k+\omega_k) \to k^2 - (\omega_k - i\varepsilon)^2 = k^2 - \omega_k^2 + \varepsilon^2 + 2i\omega_k\varepsilon \to \underline{k^2 - \omega_k^2 + i\varepsilon}$$

○ $$i\Delta_F^{\mu\nu}(x-y) = -\left(g^{\mu\nu} + \frac{1}{m^2}\partial_x^\mu \partial_x^\nu\right) i\Delta_F(x-y) + \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)$$

$$= -\left(g^{\mu\nu} + \frac{1}{m^2}\partial_x^\mu \partial_x^\nu\right) \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 - \omega_k^2 + i\varepsilon} \cdot e^{-ik\cdot(x-y)} + \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \int d^4k \frac{1}{(2\pi)^4} e^{-ik\cdot(x-y)}$$

$$= \int \frac{i\, d^4k}{(2\pi)^4} \cdot \left[-\frac{\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - \omega_k^2 + i\varepsilon} + \frac{1}{m^2} g^{\mu 0} g^{\nu 0} \right]$$

$$\Delta_F^{\mu\nu}(k) = -\frac{\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - \omega_k^2 + i\varepsilon} + \frac{1}{m^2} g^{\mu 0} g^{\nu 0}$$

第六章 相互作用量子场论

6.1 S 矩阵

用相互作用绘景下的演化算符定义 S 算符

在相互作用绘景下, Hamiltonian:

$$H^I = H_0^I + H_1^I \tag{6.1}$$

态矢量演化用演化算符决定:

$$|\alpha, t\rangle^I = U(t, t_0)|\alpha, t_0\rangle^I \tag{6.2}$$

其中, 演化算符满足:

$$i\hbar\partial_t U(t, t_0) = H_1^I U(t, t_0) \tag{6.3}$$

考虑算符 H_0^I 的本征态 $|\Phi_1\rangle \cdots |\Phi_i\rangle \cdots$, 如果在开始时刻系统处在态矢量

$$|\Phi_i\rangle \tag{6.4}$$

经过了时间演化，它处在态矢量:

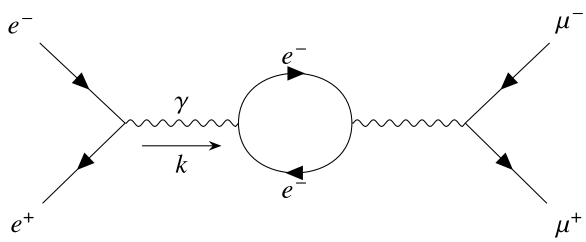
$$U(t, t_0)|\Phi_i\rangle \tag{6.5}$$

从初态演化到末态概率:

$$S_{fi} = \langle\Phi_f|U(t, t_0)|\Phi_i\rangle \tag{6.6}$$

S 算符定义为:

$$S = \lim_{t\to+\infty, t_0\to-\infty} U(t, t_0) \tag{6.7}$$



这个图随便画的，没有什么意义。

第七章 量子 Poincare 变换

7.1 量子 Poincare 变换

Poincare 变换是这个:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \tag{7.1}$$

量子 Poincare 变化是这个: 他是由于坐标的 Poincare 变换诱导出的态矢变换。这个算符一定是幺正的, 这是为了保证态矢的内积在量子 Poincare 变化下不变。

$$|\Psi'\rangle = U(\Lambda, a)|\Psi\rangle \tag{7.2}$$

由于两次 Poincare 变换 (Λ_1, a_1), (Λ_2, a_2) 叠加到一起相当于 $(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$ 于是就可以找到逆变换可以这样写:

$$U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1}a) \qquad (\Lambda^{-1})^{\mu}_{\nu} = \Lambda^{\mu}_{\nu} \tag{7.3}$$

量子 Lorentz 变换, 量子时空平移变换的生成元算符是这样定义的。

$$U(1+\omega, \epsilon) = 1 - \frac{i}{2}J^{\mu\nu}\omega_{\mu\nu} - iP^{\mu}\epsilon_{\mu} \tag{7.4}$$

从一阶展开的角度理解他 $J^{\mu\nu} = 2i\frac{\partial U}{\partial\omega_{\mu\nu}}$, $P^{\mu} = i\frac{\partial U}{\partial\epsilon_{\mu}}$.
构造一个式子。

$$U^{-1}(\Lambda, a)U(1+\omega, \epsilon)U(\Lambda, a) \tag{7.5}$$

这个式子可以用上面提到的 Poincare 变化的叠加性质来把三个算符写成一个最终的算符 U, 然后再展开, 会有三项。当然也可以先把中间的算符做一阶展开。然后就写成一个有三项的式子。
既然这两种方法得到的结果理应是一样的, 那么就会有一个性质.(详细计算没写, 在中山大学余钊焕的讲义里面)

$$\begin{aligned} &U^{-1}(\Lambda, a)J^{\mu\nu}U(\Lambda, a) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma} + \Lambda^{\mu}{}_{\rho}a^{\nu}P^{\rho} - \Lambda^{\nu}{}_{\rho}a^{\mu}P^{\rho} \\ &U^{-1}(\Lambda, a)P^{\mu}U(\Lambda, a) = \Lambda^{\mu}_{\nu}P^{\nu} \end{aligned} \tag{7.6}$$

7.2 Lorentz 代数

利用量子 Poincare 变换中生成元算符的性质:

$$U^{-1}(\Lambda, a)J^{\mu\nu}U(\Lambda, a) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma} + \Lambda^{\mu}{}_{\rho}a^{\nu}P^{\rho} - \Lambda^{\nu}{}_{\rho}a^{\mu}P^{\rho} \tag{7.7}$$

如果 Λ 可以展开成 $1+\omega$. 那么带入上面的式子, 可以得到生成元算符的对易关系:

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= i\Big(g^{\nu\rho}J^{\mu\sigma} - (\mu \leftrightarrow \nu)\Big) - (\rho \leftrightarrow \sigma) \\ [J^{\mu\nu}, P^{\rho}] &= i\Big(g^{\nu\rho}P^{\mu} - g^{\mu\rho}P^{\nu}\Big) \end{aligned} \tag{7.8}$$

是10 维 Poincare 群的 Lie 代数 回顾量子 Poincare 变换的生成元算符的定义过程。如果我们说 ω 是反对称的，那么 J 也就是反对称的矩阵, 这样他只有 6 个独立变量。下面我们在讨论 Lorentz 代数的时候只考虑 J.
我们说这六个独立变量张成了一个线性空间，并且由于对易子的性质，我们说这个空间中的矢量是对乘法封闭的。。这叫做 Lorentz 代数.... 好像这个对易子叫做 Lie 括号, 对于 Lie 括号运算封闭的就是 Lie 代数. 上面的对易子或者叫做 Lie 括号满足的条件叫做 Lorentz 代数关系。
所以我们接下来做的事情是找到一套满足 Lorentz 代数关系的生成元算符... 希望没有理解错。。。

如果做变量代换会找到 SO(3) 群的代数关系。

变量代换是这样做的

$$
\begin{aligned}
&J^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \quad K^i = J^{0i} \\
&\theta^i = -\frac{1}{2}\epsilon^{ijk}\omega_{jk} \quad \xi^i = -\omega_{0i} \\
&J = (J^{23}, J^{31}, J^{12}) \quad K = (J^{01}, J^{02}, J^{03}) \\
&\theta = (-\omega^{23}, -\omega^{31}, -\omega^{12}) \quad \xi = (-\omega^{01}, -\omega^{02}, -\omega^{03})
\end{aligned}
\tag{7.9}
$$

做了变换之后: 注意, 我们这里没有时空平移变换。

$$
U(1+\omega) = 1 + i\theta J + i\xi K \tag{7.10}
$$

并且生成元算符有对易关系:

$$
\begin{aligned}
[J^i, J^j] &= i\epsilon^{ijk}J^k \\
[J^k, K^j] &= i\epsilon^{ijk}K^k \\
[K^i, K^j] &= -i\epsilon^{ijk}J^k
\end{aligned}
\tag{7.11}
$$

需要注意的是，如果我们现在就只考虑有三个分量的 J. 他的李括号是这样的.

$$
[J^i, J^j] = i\epsilon^{ijk}J^k \tag{7.12}
$$

这恰好是 SO(3) 群的代数关系。SO(3) 群 (三维空间转动群) 代数是 Lorentz 代数的子代数。

7.3 Poincare 代数

考虑量子 Poincare 变换中的生成元算符 (全部 10 个独立的生成元) 的性质:

$$
\begin{aligned}
&U^{-1}(\Lambda, a)J^{\mu\nu}U(\Lambda, a) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma} + \Lambda^{\mu}{}_{\rho}a^{\nu}P^{\rho} - \Lambda^{\nu}{}_{\rho}a^{\mu}P^{\rho} \\
&U^{-1}(\Lambda, a)P^{\mu}U(\Lambda, a) = \Lambda^{\mu}_{\nu}P^{\nu}
\end{aligned}
\tag{7.13}
$$

这样, 得到对易关系。

$$
\begin{aligned}
&[J^{\mu\nu}, J^{\rho\sigma}] = i\Big(g^{\nu\rho}J^{\mu\sigma} - (\mu \leftrightarrow \nu)\Big) - (\rho \leftrightarrow \sigma) \\
&[J^{\mu\nu}, P^{\rho}] = i\Big(g^{\nu\rho}P^{\mu} - g^{\mu\rho}P^{\nu}\Big) \\
&[P^{\mu}, P^{\nu}] = 0
\end{aligned}
\tag{7.14}
$$

是 10 维 Poincare 群的 Lie 代数。Lorentz 代数是 Poincare 代数的子代数。
如果按照 Lorentz 的变量代换，并且把 P 叫做 $(H, \vec{P})$. 这样代数关系叫做:

$$
\begin{aligned}
&[P^i, J^j] = -i\epsilon^{ijk}P^k \\
&[P^i, K^j] = i\delta^{ij}H \\
&[H, K^i] = iP^i \\
&[H, J^i] = [H, P^i] = [P^i, P^j] = 0
\end{aligned}
\tag{7.15}
$$

7.4 粒子

这里学的很不明白, 感觉主要是群论学的不好。
考虑一个粒子态 $|\Psi_\sigma(p^\mu)\rangle$, 满足本征值的定义式子:$P|\Psi_\sigma(p)\rangle = p|\Psi_\sigma(p)\rangle$.
用量子 Lorentz 变换作用到这个态上 $PU(\Lambda)|\Psi_\sigma(p)\rangle = U(\Lambda)\ U^{-1}(\Lambda)PU(\Lambda)|\Psi_\sigma(p)\rangle$. 由量子 Poincare 变化的生成元的性质. 这个式子是等于 $\Lambda^{\mu}{}_{\nu}p^{\nu}U(\Lambda)|\Psi_\sigma(p)\rangle$。于是我们发现量子 Lorentz 变换算符作用在态矢上面之后的本

征值的变化恰好等于原来动量本征值的 Lorentz 变换。所以我们说.

$$U(\Lambda)|\Psi_\sigma(p)\rangle = \Sigma_{\sigma'} C_{\sigma'\ \sigma}(\Lambda, p)|\Psi_{\sigma'}(\Lambda p)\rangle \tag{7.16}$$

然后定义态矢。定义方法是通过固有保时 Lorentz 变换。用一个基矢量 k，做固有保时 Lorentz 变换 $V(p)$, 满足 $p = V(p)k$.

$$|\Psi_\sigma(p)\rangle \triangleq N(p)U[V(p)]|\Psi_\sigma(k)\rangle \tag{7.17}$$

把态矢的量子 Lorentz 变换也用这个基矢量 k 表示:

$$\begin{aligned} U(\Lambda)|\Psi_\sigma(p)\rangle =& N(p)\ U(V(\Lambda p))U^{-1}(V(\Lambda p))\ U(\Lambda)U[V(p)]|\Psi_\sigma(k)\rangle \\ =& N(p)U(V(\Lambda p))U(V^{-1}(\Lambda p)\Lambda V(p))|\Psi_\sigma(k)\rangle \\ =& N(p)U(V(\Lambda p))U(W)|\Psi_\sigma(k)\rangle \end{aligned} \tag{7.18}$$

其中

$$W^\mu_\nu = [V^{-1}(\Lambda p)\Lambda V(p)]^\mu_\nu \tag{7.19}$$

我们注意到 W 有一个性质，就是把 k 在 Lorentz 变换后得到的矢量还是 k。
我们说 {W} 构成标准动量 k 对应的小群。
如果我们说

$$U(W)|\Psi_\sigma(k)\rangle = \Sigma_{\sigma'} D_{\sigma'\ \sigma}(W)|\Psi_{\sigma'}(k)\rangle \tag{7.20}$$

那么这个 D 矩阵有性质:

$$U(W1)U(W2)|\Psi_\sigma(k)\rangle = D(W1)D(W2)|\Psi_\sigma(k)\rangle = D(W1W2)|\Psi_\sigma(k)\rangle \tag{7.21}$$

上面的式子省略掉了求和和指标，反正意思就是说.

$$D_{\sigma'\ \sigma}(W_1W_2) = D_{\sigma'\ \sigma''}(W_1)D_{\sigma''\sigma}(W_2) \tag{7.22}$$

这个式子叫做同态关系. 我们说 {D(W)} 构成了小群的线性表示

现在回到量子 Lorentz 变换作用在态矢的式子。我们用那个引入的态矢的定义, 和 k 对应的态矢量的变化来写他。首先我们考虑 p 在进行 Lorentz 变换后的值作为本征值的态矢量:

$$|\Psi_{\sigma'}(\Lambda p)\rangle = N(\Lambda p)U[V(\Lambda p)]|\Psi_{\sigma'}(k)\rangle \tag{7.23}$$

然后带入上面说到的量子 Lorentz 变化的式子中:

$$\begin{aligned} & U(\Lambda)|\Psi_\sigma(p)\rangle \\ &= N(p)U(V(\Lambda p))U(V^{-1}(\Lambda p)\Lambda V(p))|\Psi_\sigma(k)\rangle \\ &= N(p)U(V(\Lambda p))D_{\sigma'\ \sigma}\big(V^{-1}(\Lambda p)\Lambda V(p)\big)|\Psi_{\sigma'}(k)\rangle \\ &= \frac{N(p)}{N(\Lambda p)}\Sigma_{\sigma'} D_{\sigma'\ \sigma}\big(V^{-1}(\Lambda p)\Lambda V(p)\big)|\Psi_{\sigma'}(\Lambda p)\rangle \end{aligned} \tag{7.24}$$

于是有了系数公式

$$C_{\sigma'\sigma}(\Lambda, p) = \frac{N(p)}{N(\Lambda p)} D_{\sigma'\sigma}(V^{-1}(\Lambda p)\Lambda V(p)) \tag{7.25}$$

7.5 量子场的 Lorentz 变换, 量子场的自旋讨论

Lorentz 变换的量子意义: 经典来看，Lorentz 变换 (Lorentz 不变性体现在作用量不变)

$$\begin{cases} S = \int d^4x \mathcal{L}\left(\phi(x), \partial^\mu\phi(x)\right) \\ S' = \int d^4x' \mathcal{L}\left(\phi'(x'), \partial^\mu\phi' \circ (x')\right) \end{cases} \tag{7.26}$$

量子力学的角度，做用量是算符，Lorentz 变换是一种幺正变换:

$$U^{-1}(\Lambda)SU(\Lambda)=\int d^4x\mathcal{L}\left(U^{-1}(\Lambda)\phi(x)U(\Lambda),U^{-1}(\Lambda)\partial^\mu\phi\circ(x)U(\Lambda)\right) \tag{7.27}$$

合适的量子 Lorentz 变换应该和经典相吻合:

$$U^{-1}(\Lambda)\phi(x)U(\Lambda)=\phi'\circ(x) \tag{7.28}$$

$$U^{-1}(\Lambda)\partial^\mu\phi\circ(x)U(\Lambda)=\partial^\mu\phi'\circ(x) \tag{7.29}$$

我们要求量子标量场有 Lorentz 变换不变性。

$$\phi'(x')=\phi(x)\Rightarrow\phi'(x)=\phi(\Lambda^{-1}x) \tag{7.30}$$

如果把标量场理解为算符, 在 Lorentz 变换下, 算符的变换是:

$$\phi(x)\to U(\Lambda)^{-1}\phi(x)U(\Lambda)=\phi'(x)=\phi(\Lambda^{-1}x) \tag{7.31}$$

同样的, 如果把场对时空的偏导数也认为是 Hilbert 空间里的算符. 他们同样会被量子 Lorentz 变换。

$$\partial^\mu\phi(x)\to U^{-1}(\Lambda)\partial^\mu\phi(x)U(\Lambda) \tag{7.32}$$

这个变换后:

$$U^{-1}(\Lambda)\partial^\mu\phi(x)U(\Lambda)=\partial^\mu\phi'(x)=\frac{\partial}{\partial x_\mu}\phi(\Lambda_\nu^{\ \mu}x^\nu)=\frac{\partial}{\partial x_\mu}\phi(\Lambda^\beta_{\ \alpha}x_\beta)=\Lambda^\mu_{\ \nu}(\partial^\nu\phi)(\Lambda^\beta_{\ \nu}x_\beta) \tag{7.33}$$

标量场的 Lagrangian 中含有的项就是 ϕ 和 $\partial\phi$. 标量场的拉格朗日量做量子 Lorentz 变换的规则就是前面的两个式子。

推广 Heisenberg Field operator evolution 在 Heisenberg picture 下，场算符的演化可以写为:

$$\left\{\exp\left(\tfrac{-1}{i\hbar}H(t-t_0)\right)\phi(\vec{x},t_0)\exp\left(\tfrac{1}{i\hbar}H(t-t_0)\right)=\phi(\vec{x},t)\right. \tag{7.34}$$

按照前面说的理论，Poincare 量子变换的微元形式是:

$$\begin{cases}U(\Lambda,a) &=1-\frac{i}{2}w_{\mu\nu}J^{\mu\nu}-iP^\mu a_\mu\\ U^{-1}(\Lambda,a) &=U(\Lambda^{-1},-\Lambda^{-1}a)\end{cases} \tag{7.35}$$

按照前面一段的结论:

$$\left\{U(\Lambda)^{-1}\phi(x)U(\Lambda)=\phi'(x')=\phi(\Lambda x)\right. \tag{7.36}$$

把它变成 Poincare 变换:

$$\left\{U(\Lambda,\epsilon)^{-1}\phi(x)U(\Lambda,\epsilon)=\phi'(x)=\phi(\Lambda x+\epsilon)\right. \tag{7.37}$$

当然，没有 Lorentz 变换，只有时空平移变换的情况下:

$$\left(1+iP^\mu\epsilon_\mu\right)\phi(x)\left(1-iP^\mu\epsilon_\mu\right)=\phi(x+\epsilon) \tag{7.38}$$

经过反复量子平移操作 (可以理解为一种 Heisenberg evolution 的推广, 为了方便，把 hbar 又加了进去):

$$\exp\left(\frac{-1}{i\hbar}P\cdot(x_2-x_1)\right)\phi(x_1)\exp\left(\frac{1}{i\hbar}P\cdot(x_2-x_1)\right)=\phi(x_2) \tag{7.39}$$

如何说明标量场没有自旋: 利用前面定义的产生元和生成元算符 (注意我们这里吧变换后的场写成了 x)P85 余钊焕

$$\begin{aligned}U(\Lambda)^{-1}\phi(x)U(\Lambda)&=(1+\frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta})\phi(x)(1-\frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta})\\&=\phi(x)-\frac{i}{2}\omega_{\mu\nu}[\phi(x),J^{\mu\nu}]\end{aligned} \tag{7.40}$$

同时，我们注意到:

$$\phi(\Lambda^{-1}x)=\phi(x)-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x) \tag{7.41}$$

其中的 L 只是一个算符罢了，上面的式子其实也就是一个 tylor 展开罢了。L 具体是这样的

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{7.42}$$

于是由于要求量子矢量场在 lorentz 变化下不变，那么就要求有这个性质.

$$L^{\mu\nu}\phi(x) = [\phi(x), J^{\mu\nu}] \tag{7.43}$$

有个东西叫做对偶矢量, 我先不知道他是什么意思，反正。是这样定义的的

$$L^i = i\frac{1}{2}\epsilon^{ijk}L^{jk} \tag{7.44}$$

发现这个恰好是角动量算符。(一般用 ijk 就是只有 123 循环)

$$L = -ix \times \nabla \tag{7.45}$$

然后就有

$$\vec{L}\ \phi = [\phi(x), \vec{J}] \tag{7.46}$$

这个式子叫做总角动量算符 J 生成了轨道角动量算符 L。
综上, 标量场是没有自旋的。

说明矢量场是有自旋的 和标量场不一样, 我们要求矢量场在 Lorentz 变化下满足:

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x) \tag{7.47}$$

如果这样要求, 拉格朗日量密度在 Lorentz 变化下就是不变的。(那拉格朗日量不就变了... 算了没搞懂)。和标量场是一个套路，我们把左边展开成生成元算符，右边吧 Λ 展开，也把 A 泰勒展开。对了,x' 应该写成 x, x 应该写成 $\Lambda^{-1}x$。我们定义一个新的矩阵叫做 $\mathcal{J}^{\mu\nu}$.

$$(\mathcal{J}^{\mu\nu})^\alpha_\beta = i(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta) \tag{7.48}$$

他的意义在于展开 Lorentz 变化... 就离谱.... 它是 Lorentz 群本身表示的生成元算符。

$$\Lambda^\alpha_\beta = \delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta \tag{7.49}$$

于是，接着我们展开左右两边, 一边带入生成元算符, 一边泰勒展开。然后就有了

$$[A^\mu, J^{\rho\sigma}] = L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu_\nu A^\nu(x) \tag{7.50}$$

同样的，$\mathcal{J}$ 的对耦三维矢量是

$$\mathcal{J}^i = i\epsilon^{ijk}\mathcal{J}^{jk} \tag{7.51}$$

于是用对耦三维矢量来表示上面的式子

$$[A^\mu, \vec{J}] = \vec{L}A^\mu(x) + (\vec{\mathcal{J}})^\mu_\nu A^\nu(x) \tag{7.52}$$

而且这个 $\mathcal{J}$ 是满足 SU(2) 的生成元算符的代数关系的。而且也满足 Lorentz 代数关系。所以说他是 SU(2) 的某个线性表示的生成元算符
并且, 比较神奇的是 $(\mathcal{J}^2)^\mu_\nu = 2\delta^\mu_\nu$ 所以说他的本征值是 2。然后有人说 SU(2) 群的线性表示生成元的本征值一定是 S(S+1)... 我也不知道是为啥，也许不是这样，也许是我根本就记错了。。。所以说他的自旋就是 1。量子矢量场的自旋是 1! ! .....
也说 $\mathcal{J}$ 是 lorentz 群矢量表示的生成元。

用 Lorentz 变换其他表示做场的变换 Lorentz 变换群有表示

$$L_A^B(\Lambda) = \delta_A^B - \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^B. \tag{7.53}$$

在经典场中，考虑变化

$$L_A^B(\Lambda)\varphi_B(x) = \varphi'_A(x') = \varphi'_A(\Lambda x). \tag{7.54}$$

在这个场的变化以及坐标变化下，可以证明场的作用量是不变的。对于量子场，量子场的变化写为 (这里都没有考虑平移变换)

$$\begin{aligned} U^{-1}(\Lambda)\varphi_A(x')U(\Lambda) &= \varphi'_A(x'), \\ U^{-1}(\Lambda)\varphi_A(x')U(\Lambda) &= L_A^{\ B}(\Lambda)\varphi_B(x), \\ U^{-1}(\Lambda)\varphi_A(x)U(\Lambda) &= L_A^{\ B}(\Lambda)\varphi_B(\Lambda^{-1}x). \end{aligned} \tag{7.55}$$

考虑到没有平移时的 Lorentz 变换对应的变换算符; 以及 Lorentz 群表示的生成算符; 以及 taylor 一阶展开

$$\begin{cases} U(\Lambda) = \mathbb{I} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}, \\ L_A^{\ B}(\Lambda) = \delta_A^{\ B} - \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^{\ B}, \\ \varphi(\Lambda^{-1}x) = \varphi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\varphi(x). \end{cases} \tag{7.56}$$

带入到对算符的量子 Lorentz 变化的条件中得到

$$\begin{aligned} \left(\mathbb{I} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\varphi_A(x)\left(\mathbb{I} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) &= \left(\delta_A^{\ B} - \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^{\ B}\right)\left(\varphi_B(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\varphi_B(x)\right), \\ \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\varphi_A(x) - \varphi_A(x)\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} &= -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\varphi_A(x) - \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^{\ B}\varphi_B(x), \\ [\varphi_A(x), J^{\mu\nu}] &= L^{\mu\nu}\varphi_A(x) + (S^{\mu\nu})_A^{\ B}\varphi_B(x). \end{aligned} \tag{7.57}$$

这个式子说明 Lorentz 群表示的生成元算符 S 和角动量有关系。

第八章 路径积分

8.1 量子力学的路径积分方法

路径积分的起点是 Heisenberg 坐标算符 $\hat{q}(t)$ 本征态的内积:$\langle q',t'|q,t\rangle$ 这个内积有一个名字叫做 Feynman kernel. Feynman kernel 的一个重要的性质在于他相当于一个薛定谔方程的解。

$$\psi(q',t') = \langle q',t'|\Psi\rangle_H = \int dq\langle q',t'|q,t\rangle\langle q,t|\Psi\rangle_H = \int dq\langle q',t'|q,t\rangle\psi(q,t) \tag{8.1}$$

上面的式子中的波函数的内积表达式用到了这个性质:

$$\begin{cases} \text{Schrodinger picture: } |\alpha,t\rangle = U(t,t_0)|\alpha,t_0\rangle \quad Q|q\rangle = q|q\rangle \quad \psi(q,t) = \langle q|\alpha,t\rangle \\ \text{Heisenberg picture: } |\alpha\rangle = |\alpha,t_0\rangle \quad Q(t) = U^\dagger(t,t_0)QU(t,t_0) \quad Q(t)U^\dagger(t,t_0)|q\rangle = qU^\dagger(t,t_0)|q\rangle \ \Rightarrow\ |q,t\rangle = U^\dagger(t,t_0)|q\rangle \\ \text{In all: } \langle q,t|\alpha\rangle = \langle q|\alpha,t\rangle = \psi(q,t) \end{cases} \tag{8.2}$$

实际上 Feynman kernel 可以写为泛函积分的表达形式:

$$\begin{aligned} &\langle q',t'|q,t\rangle = \int \mathcal{D}q\mathcal{D}p\exp[\frac{i}{\hbar}\int_t^{t'} d\tau(p\dot{q} - H(p,q))] \\ &\int \Pi_{n=1}^{N-1} dq_n \to \int \mathcal{D}q \\ &\int \Pi_{n=0}^{N-1} dp_n\frac{1}{2\pi\hbar} \to \int \mathcal{D}p \end{aligned} \tag{8.3}$$

Feynman kernel 泛函积分表达式的推导, 以及 q_n 和 p_n 的定义:

在 t' 和时间 t 之间插入 N-1 个点, 实际上相当于把时间分成了 N 小份。然后利用 Heisenberg 绘景下的坐标本征态的归一化关系。

$$\langle q',t'|q,t\rangle = \int dq_{N-1}...dq_1\langle q',t'|q_{N-1},t_{N-1}\rangle\langle q_{N-1},t_{N-1}|q_{N-2},t_{N-2}\rangle\langle q_{N-2},t_{N-2}|...q_1,t_1\rangle\langle q_1,t_1|q,t\rangle \tag{8.4}$$

考虑:(严格来说，其中的 H 是 schrodinger 绘景中的算符)

$$\langle q_{n+1},t_{n+1}|q_n,t_n\rangle = \langle q_{n+1}|e^{i\hat{H}(t_n-t_{n+1})/\hbar}|q_n\rangle \tag{8.5}$$

然后需要用到泰勒展开，将指数展开到一阶项:

$$\langle q_{n+1},t_{n+1}|q_n,t_n\rangle = \langle q_{n+1}|1 + i\hat{H}(t_n - t_{n+1})/\hbar|q_n\rangle \tag{8.6}$$

插入 Schrodinger 绘景中的动量本征态算符, 如果 $\hat{H}$ 中的 $\hat{p}$ 和 $\hat{q}$ 都是独立的 (如果是独立的, 那么就可以把 $\hat{p}$ 和 $\hat{q}$ 分开来写, 如果不是独立的, 就要涉及到 weyl's operator ordering, 这个后面再说)

$$\begin{aligned} \langle q_{n+1},t_{n+1}|q_n,t_n\rangle = \langle q_{n+1}|1 - \frac{i(t_{n+1}-t_n)}{\hbar}\hat{H}|q_n\rangle \\ &= \int\frac{dp_n}{2\pi\hbar}\langle q_{n+1}|p_n\rangle\langle p_n|1 - \frac{i(t_{n+1}-t_n)}{\hbar}\hat{H}|q_n\rangle \\ &= \int\frac{dp_n}{2\pi\hbar}\langle q_{n+1}|p_n\rangle\langle p_n|q_n\rangle(1 - \frac{i(t_{n+1}-t_n)}{\hbar}H(p_n,q_n)) \\ &= \int\frac{dp_n}{2\pi\hbar}e^{i\frac{p_n}{\hbar}(q_{n+1}-q_n)}(1 - \frac{i(t_{n+1}-t_n)}{\hbar}H(p_n,q_n)) \end{aligned} \tag{8.7}$$

按照上面的展开方法，Feynman Kernel 于是可以写成:

$$\begin{aligned}\langle q',t'|q,t\rangle &= \int dq_{N-1}...dq_1 \int dp_{N-1}...dp_0(\frac{1}{2\pi\hbar})^{N-1}\ \Pi_{n=0}^{n=N-1} e^{i\frac{p_n}{\hbar}(q_{n+1}-q_n)}(1-\frac{i(t_{n+1}-t_n)}{\hbar}H(p_n,q_n)) \\ &= \int dq_{N-1}...dq_1 \int dp_N...dp_0(\frac{1}{2\pi\hbar})^{N-1}\ e^{i\Sigma_{n=0}^{n=N-1}\frac{p_n}{\hbar}(q_{n+1}-q_n)}\Pi_{n=0}^{n=N-1}(1-\frac{i(t_{n+1}-t_n)}{\hbar}H(p_n,q_n))\end{aligned} \tag{8.8}$$

有一个数学上的式子可以把连乘转换为 e 指数上面的求和算符,

$$lim_{N\to+\infty}\Pi_{n=0}^{n=N-1}(1+x_n/N) = \exp(lim_{N\to+\infty}\Sigma_{n=0}^{n=N}x_n/N) \tag{8.9}$$

这样,Feynman kernel 就可以写为路径积分的形式:

$$\langle q',t'|q,t\rangle = \int dq_{N-1}...dq_1 \int dp_{N-1}...dp_0(\frac{1}{2\pi\hbar})^{N-1}\ e^{\frac{i}{\hbar}\int_t^{t'}d\tau(p(\tau)\dot{q}(\tau)-H(p(\tau),q(\tau)))} \tag{8.10}$$

用泛函积分的语言表达就是:

$$\begin{aligned}&\langle q',t'|q,t\rangle = \int \mathcal{D}q\mathcal{D}p\exp[\frac{i}{\hbar}\int_t^{t'}d\tau(p\dot{q}-H(p,q))] \\ &\int \Pi_{n=1}^{N-1}dq_n \to \int \mathcal{D}q \\ &\int \Pi_{n=0}^{N-1}dp_n\frac{1}{2\pi\hbar} \to \int \mathcal{D}p\end{aligned} \tag{8.11}$$

8.1.1 Feynman's Path Integral

Feynman's Path Integral 指动能项是动量的平方项时, 路径积分可以简化，并且可以看到和 Action 作用量之间的联系。

观察上面的8.7 ，我们在这里把他写为:

$$\langle q_{n+1},t_{n+1}|q_n,t_n\rangle = \int \frac{dp_n}{2\pi\hbar}e^{i\frac{p_n}{\hbar}(q_{n+1}-q_n)}e^{-\frac{i(t_{n+1}-t_n)}{\hbar}H(p_n,q_n)} \tag{8.12}$$

我们考虑哈密顿量中的动量是二次型的形式:

$$H(p,q) = \frac{p^2}{2m}+V(q) \tag{8.13}$$

于是可以改写 Feynman kernel 的计算:

$$\begin{aligned}\langle q_{n+1},t_{n+1}|q_n,t_n\rangle &= \int \frac{dp_n}{2\pi\hbar}e^{-\frac{i}{2m\hbar}(t_{n+1}-t_n)(p_n^2-2\frac{m(q_{n+1}-q_n)p_n}{t_{n+1}-t_n})}e^{-\frac{i(t_{n+1}-t_n)}{\hbar}V(q_n)} \\ &= \int \frac{dp_n}{2\pi\hbar}e^{-\frac{i}{2m\hbar}(t_{n+1}-t_n)(p_n-\frac{m(q_{n+1}-q_n)}{t_{n+1}-t_n})^2+\frac{i}{2m\hbar}(t_{n+1}-t_n)m^2(\frac{q_{n+1}-q_n}{t_{n+1}-t_n})^2}e^{-\frac{i(t_{n+1}-t_n)}{\hbar}V(q_n)} \\ &=\frac{1}{2\pi\hbar}\sqrt{\frac{\pi\ 2m\hbar}{i(t_{n+1}-t_n)}}e^{\frac{i}{\hbar}(t_{n+1}-t_n)(\frac{\dot{q}^2}{2m}-V(q))}\end{aligned} \tag{8.14}$$

于是 Feynman kernel 就写为了 Feynman Path Integral 的形式:

$$\begin{aligned}\langle q',t'|q,t\rangle = (\frac{2i\pi\hbar\epsilon}{m})^{-(N-1)/2}\int dq_{N-1}...dq_1\ e^{\frac{i}{\hbar}\int_t^{t'}d\tau(\frac{\dot{q}^2}{2m}-V(q))} \\ = \mathcal{N}\int \mathcal{D}qe^{\frac{i}{\hbar}W(q,\dot{q})}\end{aligned} \tag{8.15}$$

其中 W 就可以理解为作用量。

8.1.2 多个粒子的 Feynman Kernel 和多个粒子的 Feynman Kernel 的 Feynman 路径积分表达式

首先考虑多个粒子的 Feynman Kernel, 和单粒子一样，我们给中间插入了 N 个时间间隔。...emmm 这个排版好乱....

$$\begin{aligned}&\langle q_1', q_2'...q_D', t'|q_1, q_2...q_D, t\rangle = \\ &\int dq_{1(1)}...dq_{D(1)}dq_{1(2)}...dq_{D(2)}......dq_{1(N-1)}...dq_{D(N-1)} \\ &\langle q_1', q_2'...q_D', t'|q_{1(N-1)}...q_{D(N-1)}, t_{N-1}\rangle\langle q_{1(N-1)}...q_{D(N-1)}, t_{N-1}|...\langle q_{1(1)}...q_{D(1)}, t_1|q_1, q_2...q_D, t\rangle\end{aligned} \tag{8.16}$$

和单粒子的 Feynman kernel 一样，我们关注, 其中时间相邻的两项之间的 Heisenberg 坐标本征态之间的内积:

$$\begin{aligned}\langle q_{1(n+1)}...t_{n+1}|q_{1(n)}...t_n\rangle &= \langle q_{1(n+1)}...|e^{-i\hat{H}(t_{n+1}-t_n)/\hbar}|q_{1(n)}...\rangle \\ &= \int dp_{1(n)}dp_{2(n)}...(\frac{1}{2\pi\hbar})^D\langle q_{1(n+1)}...|p_{1(n)}...\rangle\langle p_{1(n)}...|e^{-i\hat{H}(t_{n+1}-t_n)/\hbar}|q_{1(n)}...\rangle \\ &= \int dp_{1(n)}dp_{2(n)}...(\frac{1}{2\pi\hbar})^D\langle q_{1(n+1)}...|p_{1(n)}...\rangle\langle p_{1(n)}...|q_{1(n)}...\rangle e^{-iH(p_{(n)},q_{(n)})(t_{n+1}-t_n)/\hbar} \\ &= \int dp_{1(n)}...(\frac{1}{2\pi\hbar})^D e^{\frac{i}{\hbar}[\Sigma_{\alpha=1}^{\alpha=D}p_{\alpha(n)}(q_{\alpha(n+1)}-q_{\alpha(n)})-H(p_n,q_n)(t_{n+1}-t_n)]}\end{aligned} \tag{8.17}$$

这样，Feynman Kernel 可以写成泛函积分的形式:

$$\begin{aligned}\langle q_1'...t'|q_1...t\rangle &= \int \mathcal{D}q\mathcal{D}p e^{\frac{i}{\hbar}\int d\tau(\Sigma_{\alpha=1}^{D}p_\alpha\dot{q}_\alpha - H(p(\tau),q(\tau)))} \\ \mathcal{D}q &= dq_{1(1)}...dq_{D(1)}.....dq_{1(N-1)}...dq_{D(N-1)} \\ \mathcal{D}p &= (\frac{1}{2\pi\hbar})^{D\,N}dp_{1(0)}...dp_{D(0)}......dp_{1(N-1)}...dp_{D(N-1)}\end{aligned} \tag{8.18}$$

当哈密顿量中的动量是二次型时, 可以把 Path Integral 写为坐标的路径积分形式。得到多粒子 Feynman Kernel 的 Feynman 路径积分表达式.

结论是:(ϵ 是" 路径" 中插入时间片段的长度)

如果系统的拉格朗日量可以写为:

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + b(q)^T\dot{q} - V(q) \tag{8.19}$$

那么按照 legendre 变换，可以用速度和位置定义一个动量:

$$p_\alpha = \frac{\partial L}{\partial \dot{q_\alpha}} = M_{\alpha\beta}\dot{q}_\beta + b_\alpha \tag{8.20}$$

继而可以用动量和坐标表示速度，然后得到一个 Hamilton 量，或者说 Hamilton 算符。

总之，计算后的 Feynman kernel 是下面的形式。Grenier 书里讲的比较详细。

$$\langle q', t'|q, t\rangle = \lim_{N\to+\infty}(i2\pi\epsilon\hbar)^{-\frac{ND}{2}}\int \Pi_{\alpha=1}^{\alpha=D}\Pi_{n=1}^{n=N-1}dq_{\alpha n}\exp(\frac{i}{\hbar}\int_t^{t'}(\frac{1}{2}\dot{q}^T M\dot{q} + b^T\dot{q} - V(q) - \frac{i\hbar}{2}\delta(0)TrM(q))) \tag{8.21}$$

计算上面的式子时候用到了变量代换，和多维高斯积分 Gauss Integration 的技巧。先不详细写过程了。

这里后面算一下，我不是很清楚 $\delta 0$ 是怎么出来的, 不过这个问题好像不是很紧急

8.1.3 用 Time ordered Product 表示 N-Point Function

n-Point Function 是这样定义的:

$$\begin{aligned}\langle 0|T[\hat{q}(t_1)\hat{q}(t_2)\hat{q}(t_3)...]|0\rangle &= \lim_{t'\to\infty,t\to-\infty}\frac{\langle q', t'|T[\hat{q}(t_1)\hat{q}(t_2)\hat{q}(t_3)...]|q, t\rangle}{\langle q', t'|q, t\rangle} \\ &= \lim_{t'\to\infty,t\to-\infty}\frac{\int \mathcal{D}q\, q(t_1)q(t_2)q(t_3)...\exp[\frac{i}{\hbar}\int_-^+ L(q,\dot{q})dt]}{\int \mathcal{D}q\exp[\frac{i}{\hbar}\int_-^+ L(q,\dot{q})dt]}\end{aligned} \tag{8.22}$$

下面是关于这个式子的推倒 (Grenier P356)

证明

$$
\begin{aligned}
&\langle q',t'|T[\hat{q}(t_1)\hat{q}(t_2)\hat{q}(t_3)...]|q,t\rangle\\
&=\Sigma_{n,n'}\langle q',t'|n'\rangle\langle n'|T[\hat{q}(t_1)\hat{q}(t_2)]|n\rangle\langle n|q,t\rangle\\
&=\Sigma_{n,n'}\langle q'|e^{-i\hat{H}t'/\hbar}|n'\rangle\langle n'|T[\hat{q}(t_1)\hat{q}(t_2)]|n\rangle\langle n|e^{i\hat{H}t/\hbar}|q\rangle\\
&=\Sigma_{n,n'}e^{-iE_{n'}t'/\hbar}e^{iE_nt/\hbar}\langle q'|n'\rangle\langle n'|T[\hat{q}(t_1)\hat{q}(t_2)]|n\rangle\langle n|q\rangle\\
&=\Sigma_{n,n'}e^{-i(E_{n'}t'-E_nt)/\hbar}\langle q'|n'\rangle\langle n|q\rangle\langle n'|T[\hat{q}(t_1)\hat{q}(t_2)]|n\rangle
\end{aligned}
$$

现在说 t' 和 t 分别有一个趋近,t'$\to +\infty$, $t \to -\infty$ (8.23)

之后需要定义 $t' \to \tau' e^{-i\delta}, t \to \tau e^{-i\delta}$

与此同时, $\tau \to -\infty$, $\tau' \to +\infty$与此同时让 $\delta = \pi/2$, 也就是,$t' = -i\tau'$, $t = -i\tau$

这样, 上面式子的第一项就变成了

$e^{-(E_{n'}\tau'-E_n\tau)/\hbar}$

为了让这一项不要小的离谱，只好让 n 和 n' 变成 0

于是得到了等式

$$
\langle q',t'|T[\hat{q}(t_1)\hat{q}(t_2)\hat{q}(t_3)...]|q,t\rangle = e^{-i(E_0t'-E_0t)/\hbar}\langle q'|0\rangle\langle 0|q\rangle\langle 0|T[\hat{q}(t_1)\hat{q}(t_2)...]|0\rangle \tag{8.24}
$$

实际上这个 R.H.S 的前面三项可以写成:

$$
\begin{aligned}
&\langle q|0\rangle\langle 0|q\rangle e^{-i(E_0t'-E_0t)/\hbar}\\
&=\langle q|e^{-i\hat{H}t'/\hbar}|0\rangle\langle 0|e^{i\hat{H}t/\hbar}|q\rangle\\
&=\Sigma_n\langle q|e^{-i\hat{H}t'/\hbar}|n\rangle\langle n|e^{i\hat{H}t/\hbar}|q\rangle \quad \text{这一步也挺奇怪的，应该也是用到了时间扭动} \qquad (8.25)\\
&=\Sigma_n\langle q,t'|n\rangle\langle n|q,t\rangle\\
&=\langle q,t'|q,t\rangle
\end{aligned}
$$

所以说:

$$
\langle q',t'|T[\hat{q}(t_1)\hat{q}(t_2)\hat{q}(t_3)...]|q,t\rangle = \langle q',t'|q,t\rangle\langle 0|T[\hat{q}(t_1)\hat{q}(t_2)...]|0\rangle \tag{8.26}
$$

证毕

上面的推导中，暗含了 Hamiltonian 不含时间，于是能量本征态是定态。同时，我们采取的操作是让时间中含有一个负的虚部，他同理于 $H \to H(1-i\epsilon)$(是 Srednicki 中的做法)。

8.1.4 Vacuum Persistence Amplitude W[J] 或者叫做 vacuum-vacuum amplitude

W[J] 是这样定义的:

$$
\begin{aligned}
W[J] \triangleq \langle 0|0\rangle_J &= \lim_{t'\to -i\tau', t\to -i\tau}|_{\tau\to+\infty\tau'\to-\infty}\frac{\langle q't'|q,t\rangle_J}{\langle q',t'|q,t\rangle}\\
&= \mathcal{N}'\int \mathcal{D}q\exp[\frac{i}{\hbar}\int_-^+ dt(L(q,\dot{q})+J(t)q)]
\end{aligned} \tag{8.27}
$$

同样的，计算过程给省略了。。。$\mathcal{N}'$ 需要满足的条件是 $W[0]=1$

可以用 vacuum-vacuum amplitude 来计算 N-point Function:

$$
\langle 0|T[\hat{q}(t_1)\hat{q}(t_2)\hat{q}(t_3)...]|0\rangle = (\frac{\hbar}{i})^n\frac{\delta W[J]}{\delta J(t_1)...\delta J(t_n)} \tag{8.28}
$$

这里是在 Grenier P356

8.2 场论中的路径积分方法

8.2.1 vacuum-vacuum transition functional 和 Green functions(ie. 相当于 n-point Function)

$$W[J] = \mathcal{N}' \int \mathcal{D}\phi \exp[\frac{i}{\hbar} \int_{-}^{+} d^4x(L(\phi, \dot{\phi}) + J(t,x)\phi)] \tag{8.29}$$

$$\langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)...]|0\rangle = (\frac{\hbar}{i})^n \frac{\delta W[J]}{\delta J(x_1)...\delta J(x_n)} \tag{8.30}$$

需要知道的是上面的坐标是四维的。

8.2.2 利用 Wick Rotation 得到 Euclidian Field Theory, Propagator

Wick Rotation 有正逆。$x \to x_E$ 和 $p_E \to p$ 用的是正向的，另外两个是逆向的。有一些性质比如:$x_E^2 = -x^2$, $p_E^2 = -p^2$, $p_E \cdot x_E = Et + \vec{p} \cdot \vec{x}$

p375 12.3 The Feynman Propagator

第一个是把时间坐标变到虚数。首先定义一个变量 x_4

$$x_4 = ix_0 = it \tag{8.31}$$

在取了这个变换后,如果一个式子里面有对时空的积分，然后我们想把对空间部分的积分化为 x_4 的积分，这样时空体积元就变成了 $-id^4x_E$。这里的 x_E 就是 Euclidian 欧几里得空间中的向量,写成分量的形式就是 (x_1, x_2, x_3, x_4) 但是注意，这个时候的 x_4 的积分范围实际上本来应该是从 $-i\infty$ 到 $+i\infty$。但是 Wick Rotation 这个操作告诉我们，x_4 的积分范围是 $-\infty \to +\infty$。虽然很没有道理，但是就是这样的。

第二个是在能动量空间中把能量做了一个变量替换

$$p_4 = -ip_0 = -iE \tag{8.32}$$

对于动量的 WickRotation。生成一个 Euclidian momentum. $p_E = (p_1, p_2, p_3, p_4)$. 如果一个式子中如果有对于能动量空间的积分，那么能动量体积元应该有一个变化从 d^4p 变成 id^4p_E。而且在变换后，p_4 的积分范围应该是从 $+i\infty$ 到 $-\infty$. 但是 Wick Rotation 告诉我们，p_4 的积分范围是 $-\infty \to +\infty$

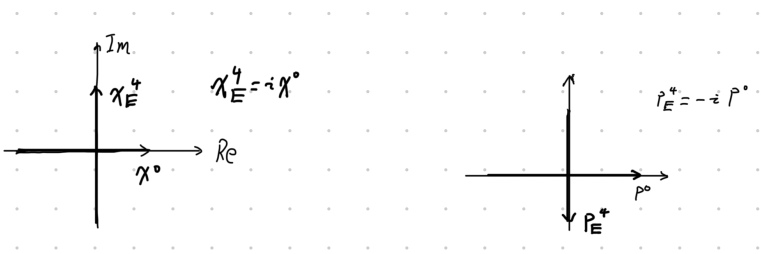


图 8.1: WickRotation

有了这个知识，我们可以写出来 Euclidian vacuum functional. 实际上就是 vacuum-vacuum transition functional 做了一个 wick rotation 的结果。

$$W[J] = \mathcal{N}' \int \mathcal{D}\phi \exp[\frac{i}{\hbar} \int_{-}^{+} d^4x(L(\phi, \dot{\phi}) + J(t,x)\phi)] \tag{8.33}$$

$$\begin{cases} x_E^4 = ix^0 \rightarrow x^0 = -ix_E^4 \\ d^4x_E = id^4x \rightarrow d^4x = -id^4x_E \\ WickRotation : x_E \text{ 的积分区间从 } (-i\infty, +\infty) \text{ 变为 } (-\infty, +\infty) \\ \text{其他函数的变换写为:} \phi(x_E) = \phi(\vec{x}, -ix_E^4) \end{cases} \tag{8.34}$$

$$W_E[J] = \mathcal{N}_E \int \mathcal{D}\phi \exp[+\frac{1}{\hbar}\int_{-}^{+} d^4x_E(L(\phi, i\frac{\partial\phi}{\partial x_4}) + J\phi)] \tag{8.35}$$

如果考虑把拉格朗日量的具体形式带入:

$$\begin{cases} \mathcal{L} = \frac{1}{2}\hbar^2\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x^\mu} - \frac{1}{2}m^2c^2\phi^2 \end{cases} \tag{8.36}$$

$$W_E[J] = \mathcal{N}_E \int \mathcal{D}\phi \exp[-\frac{1}{\hbar}\int_{-}^{+} d^4x_E(\frac{1}{2}(\hbar^2\partial_{E\mu}\phi\partial_{E\mu}\phi + m^2\phi^2) + V(\phi) - J\phi)] \tag{8.37}$$

接下说场的相互作用可以忽略的情况，就是说 $V(\phi) = 0$, 在这个情况下一般式子上面会带一个 0. 比如说我们考虑 Euclidian Vacuum Functional

$$W_0^E[J] = \mathcal{N}_E \int \mathcal{D}\phi \exp[-\int_{-}^{+} d^4x_E(\frac{1}{2}\frac{1}{\hbar}(\hbar^2\partial_{E\mu}\phi\partial_{E\mu}\phi + m^2\phi^2) - \frac{1}{\hbar}J\phi)] \tag{8.38}$$

现在我们知道 D 维高斯积分 Gauss integration. (上面的形式很像，实际上他是无穷维的 Gauss Integration.)

$$\int d^Dv \exp[-\frac{1}{2}v^TAv + \rho^Tv] = (2\pi)^{D/2}\exp[-\frac{1}{2}Tr\ln A]\exp[\frac{1}{2}\rho^TA^{-1}\rho] \tag{8.39}$$

无穷维的线性代数是这样的

$$A(x_E', x_E) = \frac{1}{\hbar}(\hbar^2\partial_{E\mu}'\partial_{E\mu} + m^2)\delta^{(4)}(x_E' - x_E) \tag{8.40}$$

稍微说一下这个 $A(x_E', x_E)$. 他里面的微分算符是作用在 delta 函数上面的。$\int(\partial_x\delta(x))\phi(x)dx = \Delta[\delta(x)\phi(x)]|_{-}^{+} - \int\phi'(x)\delta(x)dx$. 这样，他就满足性质:

$$\int d^4x_E'\ d^4x_E\phi(x_E')A(x_E', x_E)\phi(x_E) = \int d^4x_E\ \frac{1}{2}\frac{1}{\hbar}(\hbar^2\partial_{E\mu}\phi\partial_{E\mu}\phi + m^2\phi^2) \tag{8.41}$$

并且，观察发现。

$$\rho = \frac{J(x_E)}{\hbar} \tag{8.42}$$

这样，如果忽略掉常数部分 (毕竟他们总会被归一化掉), Euclidian Vacuum Functional 就可以写成:

$$W_0^E[J] = \exp[\int d^4x_E'd^4x_E\frac{1}{2}\frac{J(x_E')}{\hbar}A^{-1}(x_E', x_E)\frac{J(x_E)}{\hbar}] \tag{8.43}$$

现在说一个数学上的技巧。考虑一个无穷维的矩阵 A. 定义一个多项式

$$f(A) = \Sigma_nC_nA^n \tag{8.44}$$

把这个式子写开就涉及到所说的无穷维矩阵之间的乘法了。

$$f(A) = C_0 + C_1A(x', x) + \int dx_1C_2A(x', x_1)A(x_1, x)... \tag{8.45}$$

然后如果说 A(x',x) 只和 x'-x 有关系，那么他的傅立叶展开表达式是这个样子

$$A(x', x) = \int\frac{dk}{2\pi}\exp[ik(x' - x)]\tilde{A}(k) \tag{8.46}$$

这样的话我们发现

$$\begin{aligned}\int dx_1 A(x',x_1)A(x_1,x) &= \int dx_1 \frac{dk}{2\pi}\frac{dk'}{2\pi}\exp[ik(x'-x_1)]\exp[ik'(x_1-x)]\tilde{A}(k)\tilde{A}(k') \\ &= \int \frac{dk}{2\pi}\exp[ik(x'-x)](\tilde{A}(k))^2\end{aligned} \tag{8.47}$$

这件事情告诉我们

$$f(A) = \int \frac{dk}{2\pi}\exp[ik(x'-x)]f(\tilde{A}(k)) \tag{8.48}$$

通过直接计算，也能得到 $A^{-1} = \int \frac{dk}{2\pi}exp[ik(x'-x)]\tilde{A}(k)^{-1}$

现在回到拉格朗日量中无穷维矩阵 A 的表达式:

$$\begin{aligned}A(x'_E,x_E) &= \frac{1}{\hbar}(\hbar^2\partial'_{E\mu}\partial_{E\mu}+m^2)\int \frac{d^4p_E}{(2\pi\hbar)^4}\exp[ip_E(x'_E-x_E)/\hbar] \\ &= \int \frac{d^4p_E}{(2\pi\hbar)^4}\frac{1}{\hbar}(p_E^2+m^2)\exp[ip_E(x'_E-x_E)/\hbar]\end{aligned} \tag{8.49}$$

上面的式子其实可以看出 A 在做完傅立叶变换后的结果。简单地对傅立叶变换后的结果取一个倒数。

$$A^{-1}(p_E) = \frac{\hbar}{p_E^2+m^2} \tag{8.50}$$

$$A^{-1}(x'_E,x_E) = \int d^4p_E \frac{1}{(2\pi\hbar)^4}\frac{\hbar}{p_E^2+m^2}\exp[ip_E(x'_E-x_E)/\hbar] \tag{8.51}$$

于是:

$$\begin{aligned}&\int d^4x'_E d^4x_E \frac{1}{2}\frac{J(x'_E)}{\hbar}A^{-1}(x'_E,x_E)\frac{J(x_E)}{\hbar} \\ &= \int d^4x'_E d^4x_E \frac{1}{2}\frac{J(x'_E)}{\hbar}\int d^4p_E\frac{1}{(2\pi\hbar)^4}\frac{\hbar}{p_E^2+m^2}\exp[ip_E(x'_E-x_E)/\hbar]\frac{J(x_E)}{\hbar}\end{aligned} \tag{8.52}$$

可以定义 Euclidean Feynman Propagator 为:

$$\begin{aligned}\Delta_F^E(x'_E-x_E) &= \frac{1}{\hbar}A^{-1}(x'_E,x_E) \\ &= \int \frac{d^4p_E}{(2\pi\hbar)^4}\frac{1}{p_E^2+m^2}\exp[ip_E(x'_E-x_E)/\hbar]\end{aligned} \tag{8.53}$$

$$\text{上式} = \int d^4x'_E d^4x_E \frac{1}{2\hbar}J(x'_E)\ \Delta_F^E(x'_E-x_E)\ J(x_E) \tag{8.54}$$

认为现在的积分是 WickRotation 后的结果，想要还原成 WickRotation 前的结果。产生 Feynman Propagator:(下面式子的等式左边为什么有一个 i 呢，可以把它理解为定义。就是我定义他是 $i\Delta_F$)

$$\begin{cases} p_E^4 = -ip_0 \longrightarrow d^4p_E = -id^4p \quad p_E^2 = -p^2 \\ x_E^4 = ix^0 \longrightarrow d^4x_E = id^4x \\ p_E x_E = -p\cdot x \\ \text{积分范围还原为 Rotation 前: } p_E^4 : (-\infty,+\infty) \longrightarrow (+i\infty,-i\infty) \Rightarrow p^0 : (-\infty,+\infty) \\ \text{积分范围还原为 Rotation 前: } x_E^4 : (-\infty,+\infty) \longrightarrow (-i\infty,+i\infty) \Rightarrow x^0 : (-\infty,+\infty) \end{cases} \tag{8.55}$$

$$\text{上式} = -\int d^4x' d^4x \frac{1}{2}\frac{J(x')}{\hbar}\int id^4p\frac{1}{(2\pi\hbar)^4}\frac{\hbar}{-p^2+m^2}\exp[-ip(x'-x)/\hbar]\frac{J(x)}{\hbar} \tag{8.56}$$

可以定义

$$\begin{cases} i\Delta_F(x'-x) & = -i\int \frac{d^4p}{(2\pi\hbar)^4}\frac{1}{-p^2+m^2}\exp[-ip(x'-x)/\hbar] \\ & = i\int \frac{d^4p}{(2\pi\hbar)^4}\frac{1}{p^2-m^2}\exp[-ip(x'-x)/\hbar] \end{cases} \tag{8.57}$$

$$\text{上式} = -\int d^4x'd^4x\frac{i}{2\hbar}J(x')\Delta_F(x'-x)J(x) \tag{8.58}$$

这样，Vacuum Functional 就有 Euclidian 和 Minkovski 两种写法

$$\begin{aligned} W_0^E[J] &= \exp[\int d^4x'_Ed^4x_E\frac{1}{2\hbar}J(x'_E)\Delta_F^E(x'_E-x_E)J(x_E)] \\ W_0[J] &= \exp[-\int d^4x'd^4x\frac{i}{2\hbar}J(x')\Delta_F(x'-x)J(x)] \end{aligned} \tag{8.59}$$

8.2.3 直接做动量变换的方法得到真空生成函数 (不用上一部分说的 wick 转动)

这一子节用的是 Srednicki 的记号。需要计算的函数是:

$$Z_0[J] = \int \mathcal{D}\varphi \exp\left[i\int d^4x\,(\mathcal{L}_0+J\varphi)\right] \tag{8.60}$$

Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 \tag{8.61}$$

关键在于计算积分:

$$S_0 = \int d^4x\left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + J\varphi\right) \tag{8.62}$$

Lorentz 变化:

$$\begin{cases} \varphi(k) = \int d^4xe^{ikx}\varphi(x) \\ \varphi(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\varphi(k) \\ \partial^\mu\varphi(x) = \int \frac{d^4k}{(2\pi)^4}(-ik^\mu)e^{-ikx}\varphi(k) \\ \partial_\mu\varphi(x) = \int \frac{d^4k}{(2\pi)^4}(-ik^\mu)e^{-ikx}\varphi(k) \end{cases} \tag{8.63}$$

$$\begin{aligned} S_0 &= \int d^4xd^4kd^4k'\left(\frac{1}{2}\frac{1}{(2\pi)^8}(-k^\mu k'_\mu)e^{-i(k+k')x}\varphi(k)\varphi(k') - \frac{1}{2}m^2\frac{1}{(2\pi)^8}\varphi(k)\varphi(k')e^{-i(k+k')x} + \frac{1}{(2\pi)^8}J(k)\varphi(k')e^{-i(k+k')x}\right) \\ &= \int d^4kd^4k'\left(\frac{1}{2}\frac{1}{(2\pi)^4}(-k^\mu k'_\mu)\delta^{(4)}(k+k')\varphi(k)\varphi(k') - \frac{1}{2}m^2\frac{1}{(2\pi)^4}\varphi(k)\varphi(k')\delta^{(4)}(k+k') + \frac{1}{(2\pi)^4}J(k)\varphi(k')\delta^{(4)}(k+k')\right) \\ &= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(-\varphi(k)(-k^2+m^2)\varphi(-k) + J(k)\varphi(-k) + J(-k)\varphi(k)\right) \end{aligned} \tag{8.64}$$

积分变化:

$$\begin{cases} \chi(k) = \varphi(k) + \frac{J(k)}{k^2-m^2} \end{cases} \tag{8.65}$$

$$\begin{aligned} S_0 &= \\ & \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(-\left(\chi(k)-\frac{J(k)}{k^2-m^2}\right)(-k^2+m^2)\left(\chi(-k)-\frac{J(-k)}{k^2-m^2}\right) + J(k)\left(\chi(-k)-\frac{J(-k)}{k^2-m^2}\right) + J(-k)\varphi(k)\left(\chi(k)-\frac{J(k)}{k^2-m^2}\right)\right) \\ &= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\Big((k^2-m^2)\chi(k)\chi(-k) + J(k)J(-k)\frac{1}{k^2-m^2} - \chi(k)J(-k) - J(k)\chi(-k) \\ &\quad + J(k)\chi(-k) + J(-k)\chi(k) - \frac{J(k)J(-k)}{k^2-m^2} - \frac{J(-k)J(k)}{k^2-m^2}\Big) \\ &= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(-\frac{J(k)J(-k)}{k^2-m^2} + (k^2-m^2)\chi(k)\chi(-k)\right) \end{aligned} \tag{8.66}$$

于是:

$$Z_0[J] \propto \int \mathcal{D}\chi \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left(-\frac{J(k)J(-k)}{k^2-m^2}+(k^2-m^2)\chi(k)\chi(-k)\right)\right] \tag{8.67}$$

归一化条件认为:

$$\left\{Z_0[0]=1\right. \tag{8.68}$$

于是得到结果

$$Z_0[J] = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left(-\frac{J(k)J(-k)}{k^2-m^2}\right)\right] \tag{8.69}$$

定义 (Feynman Propagator)，$H \to H(1-i\epsilon)$ 导致了 $m^2 \to m^2 - i\epsilon$ (这是在联系多点关联函数和 Feynman Kernel 时用到的性质。)

$$\left\{\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon}\right. \tag{8.70}$$

直接计算:

$$\begin{aligned}
\frac{i}{2}\int d^4xd^4x'J(x)\Delta(x-x')J(x') &= \frac{i}{2}\int d^4xd^4x'd^4kd^4k'd^4k''\frac{1}{(2\pi)^{12}}J(k)e^{-ikx}\Delta(k')e^{-ik'(x-x')}J(k'')e^{-ik''x'} \\
&= \frac{i}{2}\int d^4kd^4k'd^4k''\frac{1}{(2\pi)^4}J(k)\Delta(k')J(k'')\delta^{(4)}(k+k')\delta^{(4)}(k''-k') \\
&= \frac{i}{2}\int d^4k\frac{1}{(2\pi)^4}J(k)\Delta(k)J(-k)
\end{aligned} \tag{8.71}$$

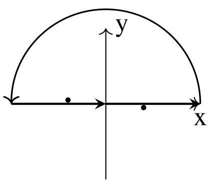
于是:

$$Z_0[J] = \exp\left[\frac{i}{2}\int d^4xd^4x'J(x)\Delta(x-x')J(x')\right] \tag{8.72}$$

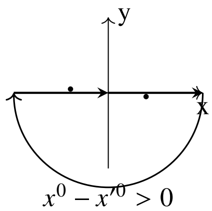
$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} \tag{8.73}$$

传播子的 Bessel 积分表达　考虑传播子的形式:

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} &= \int dk^0 \int \frac{d^3k}{(2\pi)^4}\frac{e^{-ik^0(x^0-x'^0)}}{-k^2+m^2-i\epsilon}e^{i\vec{k}(\vec{x}-\vec{x}')} \\
&= \theta(x^0-x'^0)\int dk^0\int\frac{d^3k}{(2\pi)^4}\frac{e^{-ik^0(x^0-x'^0)}}{-\left(k^0-\sqrt{m^2+\boldsymbol{k}^2}+i\epsilon'\right)\left(k^0+\sqrt{m^2+\boldsymbol{k}^2}-i\epsilon'\right)}e^{i\vec{k}(\vec{x}-\vec{x}')} \\
&+\theta(x'^0-x^0)\int dk^0\int\frac{d^3k}{(2\pi)^4}\frac{e^{-ik^0(x^0-x'^0)}}{-\left(k^0-\sqrt{m^2+\boldsymbol{k}^2}+i\epsilon'\right)\left(k^0+\sqrt{m^2+\boldsymbol{k}^2}-i\epsilon'\right)}e^{i\vec{k}(\vec{x}-\vec{x}')}
\end{aligned} \tag{8.74}$$



$x^0 - x'^0 < 0$



$x^0 - x'^0 > 0$

按照上面的积分路径，将积分范围扩展为了一个闭合围道。用留数定理，得到:

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} &= \theta(x^0-x'^0)\int\frac{d^3k}{(2\pi)^4}(-i2\pi)\frac{e^{-i\omega(x^0-x'^0)}}{-2\omega}e^{i\vec{k}(\vec{x}-\vec{x}')} \\
&+\theta(x'^0-x^0)\int\frac{d^3k}{(2\pi)^4}(+i2\pi)\frac{e^{+i\omega(x^0-x'^0)}}{2\omega}e^{i\vec{k}(\vec{x}-\vec{x}')} \\
&= \theta(x^0-x'^0)\int i\,d\tilde{k}e^{-ikx}+\theta(x'^0-x^0)\int i\,d\tilde{k}e^{+ikx}
\end{aligned} \tag{8.75}$$

其中

$$\left\{d\tilde{k} = \frac{d^3k}{(2\pi)^3(2\omega)}, \quad k = (\omega, \vec{k}) = (\sqrt{\vec{k}^2 + m^2}, \vec{k}) \right. \tag{8.76}$$

总的来说:

$$\Delta(x - x') = \theta(x^0 - x'^0) \int i\, d\tilde{k} e^{-ik(x-x')} + \theta(x'^0 - x^0) \int i\, d\tilde{k} e^{+ik(x-x')} \tag{8.77}$$

8.2.4 有相互作用场的 Generating Function 和 Vacuum Functional

首先定义有相互作用场的 Vacuum Functional.
我们首先写出他的拉格朗日量:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = \frac{\hbar^2}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - gV(\phi) \tag{8.78}$$

$$\begin{aligned}
W[J] &= \mathcal{N}\int \mathcal{D}\phi \exp[\frac{i}{\hbar}\int d^4x(\mathcal{L}_0 + \mathcal{L}_{int} + J\phi)] \\
&= \mathcal{N}\int \mathcal{D}\phi \exp[-\int d^4x\frac{i}{\hbar}gV(\phi)] \exp[\frac{i}{\hbar}\int d^4x(\mathcal{L}_0 + J\phi)] \\
&= \mathcal{N}\exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]\int \mathcal{D}\phi \exp[\frac{i}{\hbar}\int d^4x(\mathcal{L}_0 + J\phi)] \\
&= \mathcal{N}\exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]W_0[J]
\end{aligned} \tag{8.79}$$

归一化系数是为了当 J 为 0 的时候可以把 Vacuum Functional 变成 1。

$$\begin{aligned}
\mathcal{N}^{-1} &= \exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]\int \mathcal{D}\phi \exp[\frac{i}{\hbar}\int d^4x(\mathcal{L}_0 + J\phi)]|_{J=0} \\
&= \exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]W_0[J]|_{J=0}
\end{aligned} \tag{8.80}$$

接下来定义 n-Point Green Function

$$G^{(n)}(x_1, x_2, x_3, x_4 ...) = (\frac{\hbar}{i})^n \frac{\delta^n W[J]}{\delta J(x_1)...\delta J(x_n)} \tag{8.81}$$

8.2.5 Feynman 图

反正就是把 W[J] 展开了，然后用图像表示积分，同样的，也把 n-point Function G 展开并且用图像表示了。。。还是找个例子演示一下这个方法画 Feynman 图吧.... 标量场的 Feynman 图当然。
需要知道，这样生成的 Green Function 是没有真空图像的，这一点是因为 $W[J]|_{J=0} = 1$。由于他的归一化条件, 使得他不能有真空图, 不然就成无穷大了。(这一点在 Schwinger-Dyson Function 的方法中也能看出来)
(Grenier P384 12.5 Generating function for interacting fields)

问题的核心在于有相互作用场的 Vacuum Functional, 其中 $W_0[J]$ 是真空, 没有相互作用场的 Vacuum Functional.

$$W[J] = \mathcal{N}\exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]W_0[J] \tag{8.82}$$

注意上面这个式子的 R.H.S 的后面两项, 可以对他们进行小量展开

$$\exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]W_0[J] = W_0[J](1 + gu_1[J] + g^2u_2[J]...) \tag{8.83}$$

实际上, 直接展开就可以得到相应的展开泛函。

$$u_1[J] = W_0[J]^{-1}(-\int d^4x\frac{i}{\hbar}V(\frac{\hbar\delta}{i\delta J(x)}))W_0[J] \tag{8.84}$$

$$u_2[J] = \frac{1}{2}W_0[J]^{-1}(-\int d^4x\frac{i}{\hbar}V(\frac{\hbar\delta}{i\delta J(x)}))^2W_0[J] \tag{8.85}$$

这样, 我们就可以进一步展开有 Interaction Field 的 Vacuum Functional.

$$W[J]=W_0[J](1+gw_1[J]+g^2w_2[J]...)=\frac{W_0[J](1+gu_1[J]+g^2u_2[J]...)}{W_0[0](1+gu_1[0]+g^2u_2[0]...)} \tag{8.86}$$

当然,$W_0[0]=1$

对应的, 可以写出 Vacuum Functional 的小量形式:

$$W[J]=W_0[J](1+gu_1[J]+g^2u_2[J]...)\left(1-(gu_1[0]+g^2u_2[0]...)+(gu_1[0]+g^2u_2[0]...)^2...\right) \tag{8.87}$$

把上面的式子展开到二阶:

$$W[J]=W_0[J]\left(1+g\,(u_1[J]-u_1[0])+g^2\,(\,(u_1[0]-u_1[J])u_1[0]+u_2[J]-u_2[0])\right) \tag{8.88}$$

写的再明确一点就是

$$w_1[J]=u_1[J]-u_1[0] \tag{8.89}$$

$$w_2[J]=(u_1[0]-u_1[J])u_1[0]+u_2[J]-u_2[0] \tag{8.90}$$

通过前面无穷维高斯积分的性质, 把 non Interaction Vacuum Functional 写成了这个形式

$$W_0[J]=\exp[-\int d^4x'd^4x\frac{i}{2\hbar}J(x')\Delta_F(x'-x)J(x)] \tag{8.91}$$

并且 vacuum propagator 可以写成这个形式

$$\mathcal{N}\exp[-\int d^4x\frac{i}{\hbar}gV(\frac{\hbar\delta}{i\delta J(x)})]W_0[J] \tag{8.92}$$

然后我们计算上面式子 Vacuum Propagator 的把 e 指数展开到一阶后对 non-interacting Vacuum Propagator 的作用效果是这样的。

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$$\begin{aligned}&(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})^4 exp[-\int d^4x'd^4x\frac{i}{2\hbar}J(x')\Delta_F(x'-x)J(x)]\\&=\left(3(\frac{\hbar}{i})^2\Delta_F(0)\Delta_F(0)-6\frac{\hbar}{i}\left(\int d^4x'J(x')\Delta_F(x'-x)\right)^2\Delta_F(0)+\left(\int d^4x'J(x')\Delta_F(x'-x)\right)^4\right)exp[...]\end{aligned} \tag{8.93}$$

然后我们考虑这个。

$$-\int d^4x\frac{i}{\hbar}g\frac{1}{4!}(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})^4W_0[J] \tag{8.94}$$

实际上，这个东西最后是这样的:

$$\int d^4x\, g\frac{1}{4!}\left(-3\frac{\hbar}{i}\Delta_F(0)\Delta_F(0)+6\Big(\int d^4x'J(x')\Delta_F(x'-x)\Big)^2\Delta_F(0)-\frac{i}{\hbar}\Big(\int d^4x'J(x')\Delta_F(x'-x)\Big)^4\right)W_0[J] \tag{8.95}$$

实际上, 已经发现了,$u_1[J]$ 就是上面的式子除以 $W_0[J]$. 刚好把最后一项给除掉了。

按照前面的说法, 于是可以定义 $w_1[J]=u_1[J]-u_0[J]$，这样刚好可以减去 Feynman Propagator 在 0 的取值。

$$w_1[J]=\int d^4x\, g\frac{1}{4!}\left(6\Big(\int d^4x'J(x')\Delta_F(x'-x)\Big)^2\Delta_F(0)-\frac{i}{\hbar}\Big(\int d^4x'J(x')\Delta_F(x'-x)\Big)^4\right) \tag{8.96}$$

这个式子一般这么写 (一般会提出来一个因子)。

$$\begin{aligned}&-\int d^4x\; g\;\frac{i}{\hbar}\;\frac{1}{4!}\left(6i\hbar\Big(\int d^4x_1J(x_1)\Delta_F(x_1-x)\Big)\Big(\int d^4x_2J(x_2)\Delta_F(x_2-x)\Big)\Delta_F(0)+\right.\\&\left.\Big(\int d^4x_1J(x_1)\Delta_F(x_1-x)\Big)\Big(\int d^4x_2J(x')\Delta_F(x_2-x)\Big)\Big(\int d^4x_3J(x_3)\Delta_F(x_3-x)\Big)\Big(\int d^4x_4J(x_4)\Delta_F(x_4-x)\Big)\right)\end{aligned} \tag{8.97}$$

当然, 如果想要用它来表示W[J]的话就是这样的:

$$\Big(1-\int d^4x\ g\ \frac{i}{\hbar}\ \frac{1}{4!}\Big(6i\hbar\Big(\int d^4x_1 J(x_1)\Delta_F(x_1-x)\Big)\Big(\int d^4x_2 J(x_2)\Delta_F(x_2-x)\Big)\Delta_F(0)+$$
$$\Big(\int d^4x_1 J(x_1)\Delta_F(x_1-x)\Big)\Big(\int d^4x_2 J(x')\Delta_F(x_2-x)\Big)\Big(\int d^4x_3 J(x_3)\Delta_F(x_3-x)\Big)\Big(\int d^4x_4 J(x_4)\Delta_F(x_4-x)\Big) \tag{8.98}$$
$$\dots\Big)exp[-\int d^4x' d^4x\frac{i}{2\hbar}J(x')\Delta_F(x'-x)J(x)]$$

如果用图来表示他是这样的。

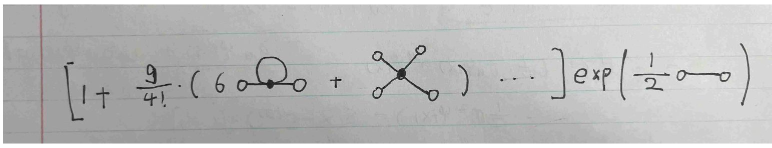


图 8.2: phi4 理论

然后我们可以总结一下标量场的 Feynman 规则是这样的。

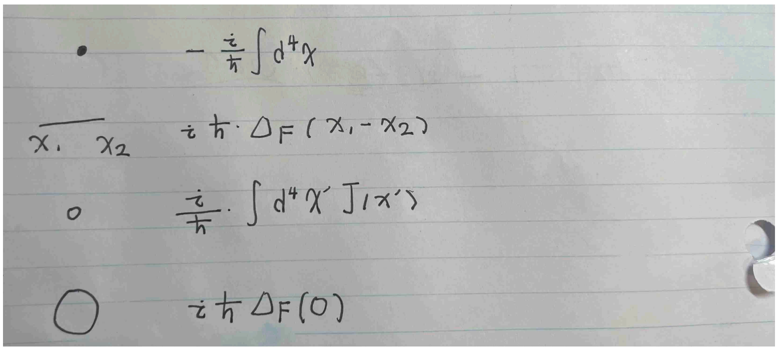


图 8.3: 标量场的 Feynman Rules

然后根据泛函求导可以算出 n-Point Function:

我们从这一步开始就看着 Feynman 图计算了。计算方法是考虑到:

$$(\frac{\hbar}{i})\frac{\delta}{\delta J(x_1)}(\frac{i}{\hbar}\int d^4x' J(x')\dots) = \int d^4x'\delta(x'-x_1)\dots = \dots|_{x_1} \tag{8.99}$$

按照这个规则，我们分别画出 2-point Function，4-point Function 的在 J=0 时也不为零的项:

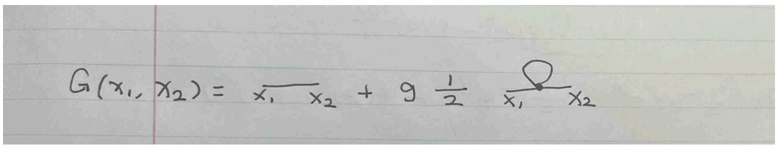


图 8.4: 2-Point Function

这个图片的位置好丑啊。。

8.2.6 动量空间的 Feynman 图

动量空间的 Green 函数 (Green function in momentum space) 的定义是 (这个来自于 GrenierP391,12.6Green Function in momentum space)

$$G^{(n)}(p_1,p_2\dots p_n)(2\pi\hbar)^4\delta^4(p_1+\dots p_n) = \int d^4x_1\dots d^4x_n e^{i(p_1x_1\dots p_nx_n)}G^{(n)}(x_1\dots x_n) \tag{8.100}$$

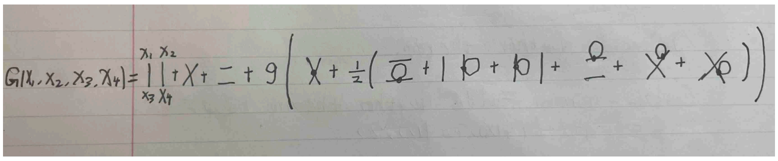


图 8.5: 4-Point Function

在 ϕ^4 理论中的 2-point Function。把这个 2-point Function 展开到一阶，他都是 connected。具体来讲。用图像表示他就是8.4。用公式表示他就是:

$$G(x_1,x_2)=i\hbar\Delta_F(x_1-x_2)+\frac{1}{2}g\int\frac{1}{i\hbar}d^4x(i\hbar)\Delta_F(x_1-x)(i\hbar)\Delta_F(x-x_2)(i\hbar)\Delta_F(x-x) \tag{8.101}$$

然后按照求动量空间的 Green Function 的方法，先观察 0 阶项。我们要求的积分是:

$$\int d^4x_1d^4x_2e^{i(p_1x_1+p_2x_2)/\hbar}(i\hbar)\Delta_F(x_1-x_2) \tag{8.102}$$

我们的方法是变量代换。具体来说:$x=x_1-x_2\quad X=\frac{1}{2}(x_1+x_2)$ 与 $p=\frac{1}{2}(p_1-p_2)\quad P=p_1+p_2$。
先考虑积分的变换。因为从 x_1 x_2 变成了 x X。会涉及到一个 Jacobi 行列式:

$$det\frac{\partial(x_1,x_2)}{\partial(x,X)} \tag{8.103}$$

有变量代换的规定，可以得到:

$$x_1=X+\frac{1}{2}x\quad x_2=X-\frac{1}{2}x \tag{8.104}$$

得到的 Jacobi 矩阵会是一个 8×8 的矩阵。(不是很确定, jacobi 矩阵的行指标 i 应该对应的分子的第 i 个元素。)

$$\begin{bmatrix}\frac{1}{2}I_{4\times4} & 1I_{4\times4}\\ -\frac{1}{2}I_{4\times4} & 1I_{4\times4}\end{bmatrix} \tag{8.105}$$

然后他的行列式是 1 对吧。然后回到积分 8.102。并且能够注意到 $p_1x_1+p_2x_2=PX+px$。所以这个积分变成了:

$$\int d^4xd^4Xe^{i(px+PX)/\hbar}i\hbar\Delta_F(x) \tag{8.106}$$

对 X 的积分会得到一个 $\int d^4Xe^{iPX/\hbar}=(2\pi\hbar)^4\delta(P)=(2\pi\hbar)^4\delta(p_1+p_2)$ 剩下的部分会得到:

$$\int d^4xe^{ipx}(i\hbar)\Delta_F(x)=\frac{i\hbar}{p^2-m^2} \tag{8.107}$$

于是这个积分是:

$$\begin{aligned}\int d^4x_1d^4x_2e^{i(p_1x_1+p_2x_2)/\hbar}(i\hbar)\Delta_F(x_1-x_2)=\int d^4Xe^{iPX/\hbar}\int d^4xe^{ipx}(i\hbar)\Delta_F(x)&=\frac{i\hbar}{p^2-m^2}(2\pi\hbar)^4\delta(P)\\&=i\hbar\Delta_F(\frac{p_1-p_2}{2})(2\pi\hbar)^4\delta^4(p_1+p_2)\end{aligned} \tag{8.108}$$

于是这一部分对于 Momentum Green Function 的贡献就是

$$i\hbar\Delta_F(\frac{p_1-p_2}{2}) \tag{8.109}$$

然后考虑含有 g 的一阶项。具体来讲, 这个积分是:

$$\int d^4x_1d^4x_2d^4x\frac{1}{2}g(i\hbar)^2e^{i\left(p_1(x_1-x)+p_2(x_2-x)+(p_1+p_2)x\right)/\hbar}\Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(0) \tag{8.110}$$

上面对于 x_1 和 x_2 的积分可以得到动量空间的 Feynman 传播子 (就是费恩曼传播子的傅立叶变换)。对于 x 的积分会生成一个要求动量守恒的东西。
具体来说上面的积分在计算完之后得到的结果会是:

$$\frac{1}{2}g(i\hbar)\Delta_F(p_1)(i\hbar)\Delta_F(p_2)\int\frac{d^4k}{(2\pi\hbar)^4}\frac{1}{i\hbar}\frac{i\hbar}{k^2-m^2}(2\pi\hbar)^4\delta^4(p_1+p_2) \tag{8.111}$$

所以说 Momentum 空间的 Green Function 对于两点会是:

$$i\hbar\Delta_F(\frac{p_1-p_2}{2})+\frac{1}{2}g(i\hbar)\Delta_F(p_1)(i\hbar)\Delta_F(p_2)\int\frac{d^4k}{(2\pi\hbar)^4}\frac{1}{i\hbar}(i\hbar)\Delta_F(k) \tag{8.112}$$

所以说动量空间的 Feynman 规则是

- 对于每一个顶点要有 $\frac{1}{i\hbar}$
- 对于每一个顶点要求动量守恒。而且会加入一个因子 g。
- 对于内线要有积分 $\frac{d^4k}{(2\pi\hbar)^4}$
- 对于一条动量线, 代表的值是 $i\hbar\Delta_F(p)$

但是对于两个点直接连的这种情况其实挺奇怪的。因为他会出来一个 $\Delta_F(\frac{p_1-p_2}{2})$。他是没有相互作用时候的动量空间的 Green Function。对于有四个点的动量空间的 Green Function, 没有相互作用时候，我们知道他是两个点两两相连的。比如说 1-2，3-4 链接。那么积分就会是这样

$$\int d^4x_1d^4x_2e^{i(p_1x_1+p_2x_2)/\hbar}(i\hbar)\Delta(x_1-x_2)\int d^4x_3d^4x_4e^{i(p_3x_3+p_4x_4)/\hbar}(i\hbar)(x_3-x_4) \tag{8.113}$$

这个积分会导致:

$$(i\hbar)^2\Delta_F(\frac{p_1-p_2}{2})\Delta_F(\frac{p_3-p_4}{2})(2\pi\hbar)^8\delta(p_1+p_2)\delta(p_3+p_4) \tag{8.114}$$

然后利用 delta 函数的性质:$\delta(x)\delta(y)=\delta(x+y)\delta(x-y)$ 于是上面的结果在经过这个操作后就可以提出来 $(2\pi\hbar)^4\delta(p_1+p_2+p_3+p_4)$ 这样的因子。剩下的就是对四点动量空间的格林函数的贡献。

$$(i\hbar)^2(2\pi\hbar)^4\delta^4(p_1+p_2-p_3-p_4)\Delta_F(\frac{p_1-p_2}{2})\Delta_F(\frac{p_3-p_4}{2}) \tag{8.115}$$

这个式子挺奇怪的。找不到太多规律，但是对于 n 点没有相互作用的贡献都可以这样一步步算出来。

8.2.7 irreducible Diagram 和 Effective Action(有效作用量)

需要知道的事这里的 W[J] 是 Connected Generating Function. 和前面的 W[J] 是不一样的。Grenier 的书里面这个东西叫做 Z[J], 然后他的定义方式是 $e^{\frac{i}{\hbar}Z[J]}=W[J]$。然后 Z[J] 作为 Connected Generating Function, 他的生成方式是:

$$\begin{aligned} Z[J] &= \Sigma_n\int dx_1^4...dx_n^4\frac{1}{n!}(\frac{\hbar}{i})^{n-1}\frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0}(\frac{i}{\hbar})^{n-1}J(x_1)...J(x_n) \\ &= \Sigma_n\int dx_1^4...dx_n^4 G_c(x_1,...x_n)(\frac{i}{\hbar})^{n-1}J(x_1)...J(x_n) \end{aligned} \tag{8.116}$$

上面式子中的 $G_c(x_1...x_n)$ 就叫做 n-point connected Green Function。为什么叫做这个呢，是因为他和普通的 Green Function 相比较起来发现他只有普通 Generating Function 中的相互之间有联系 (不能分割成几个部分)。

然后定义有效作用量。为什么是有效作用量呢，是因为如果他作为 Generating Function, 那么算出的图是 n-point irreducible Chart. 为什么是不可约呢，就是说这个图里不能通过剪断一个内线来变成没有链接的 Green Function。所以说是不可约的。

$$\Gamma[\phi]=-W[J]+\int J(x)\phi(x)d^4x$$

in which:

$$\phi(x,J]=\frac{\delta W[J]}{\delta J(x)}$$

上面定义的 ϕ 就出现了一个细节问题。如果 W[J] 的图里有一个只有一个端点含有 J。那么对 J(x) 求泛函导数之后这个图里就没有 J 了。于是这个图在 J=0 时仍然是 x 的函数。所以 J=0 时的场就叫做平均场 $\bar{\phi}$

From the definition of effective action.

$$\begin{aligned} \frac{\delta\Gamma[\phi]}{\delta\phi(x)} &= -\int d^4y\frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y,\phi]}{\delta\phi(x)}+\int d^4y\frac{\delta J(y,\phi]}{\delta\phi(x)}\phi(y)+J(x) \\ &= -\int d^4y\phi(y)\frac{\delta J(y,\phi]}{\delta\phi(x)}+\int d^4y\phi(y)\frac{\delta J(y,\phi]}{\delta\phi(x)}+J(x) \\ &= J(x,\phi] \end{aligned} \tag{8.117}$$

As defined:

$$\bar{\phi}(x) = \frac{\delta W[J]}{\delta J(x)}|_{J=0} = \phi(x, J = 0]$$

我们知道 J=0。也就是 $\phi = \bar{\phi}$ 时候, 不可约生成函数是 0。

$$\Gamma[\bar{\phi}] = \Gamma[\phi[J = 0]] = 0$$

为了让生成函数是 ϕ 的函数。并且要可以展开，那么。(零阶是 0)

$$\hat{\Gamma}[\phi] = \Gamma[\phi + \bar{\phi}] - \Gamma[\bar{\phi}]$$

In this case:

$$\frac{\delta\hat{\Gamma}[\phi]}{\delta\phi(x)} = J(x, \phi + \bar{\phi}]$$

So:

$$\frac{\delta\hat{\Gamma}[\phi]}{\delta\phi(x)}|_{\phi=0} = J(x, \bar{\phi}] = 0$$

So, $\hat{\Gamma}$ can be expanded without the first two orders:

$$\hat{\Gamma}[\phi] = \Sigma_{n=2} \int dx_1^4 ... dx_n^4 \frac{1}{n!}(i\hbar)^{n-1} \frac{\delta^n \hat{\Gamma}}{\delta\phi(x_1)...\delta\phi(x_n)}|_{\phi=0} (\frac{1}{i\hbar})^{n-1} \phi(x_1)...\phi(x_n) \tag{8.118}$$

Also, as we always do. we expand $W[J]$ (connected generation function) to series:

$$W[J] = \Sigma_n \int dx_1^4 ... dx_n^4 \frac{1}{n!}(\frac{\hbar}{i})^{n-1} \frac{\delta^n W[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0} (\frac{i}{\hbar})^{n-1} J(x_1)...J(x_n)$$

n-point irreducible diagram is defined by:

$$\hat{\Gamma}^{(n)}(x_1, \cdots, x_n) = (i\hbar)^{n-1} \frac{\delta^n \hat{\Gamma}}{\delta\phi(x_1)...\delta\phi(x_n)}|_{\phi=0} \tag{8.119}$$

In which n-point connected function:

$$W^{(n)}(x_1, x_2 ... x_n) = (\frac{\hbar}{i})^{n-1} \frac{\delta^n W[J]}{\delta J(x_1)...\delta J(x_n)}$$

Consider:

$$\frac{\delta\phi(x_1)}{\delta\phi(x_2)} = \delta(x_1 - x_2)$$

Which means:

$$\int d^4y \frac{\delta\phi(x_1, J]}{\delta J(y)} \frac{\delta J(y, \phi]}{\delta\phi(x_2)} = \delta(x_1 - x_2)$$

As we know:

$$\phi(x_1, J] = \frac{\delta W[J]}{\delta J(x_1)} \quad \frac{\delta\hat{\Gamma}[\phi - \bar{\phi}]}{\delta\phi(y)} = J(y, \phi]$$

By insertion, we obtain:

$$\int d^4y \frac{\delta^2 W[J]}{\delta J(y)\delta J(x_1)} \frac{\delta^2\hat{\Gamma}[\phi - \bar{\phi}]}{\delta\phi(y)\delta\phi(x_2)} = \delta^4(x_1 - x_2) \tag{8.120}$$

apply $\frac{\delta}{\delta J(x_3)}$ to the above equation 8.120. change the notation of the variable $y \to y_1, x_2 \to y_2$ and noticed that $\frac{\delta}{\delta J(x_3)} = \int d^4y_3 \frac{\delta^2 W[J]}{\delta J(y_3)\delta J(x_3)} \frac{\delta}{\delta\phi(y_3)}$

$$\begin{aligned} &\int d^4y_1 \frac{\delta^3 W[J]}{\delta J(y_1)\delta J(x_1)\delta J(x_3)} \frac{\delta^2\hat{\Gamma}[\phi - \hat{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)} \\ &+ \int d^4y_1 d^4y_3 \frac{\delta^2 W[J]}{\delta J(y_1)\delta J(x_1)} \frac{\delta^3\hat{\Gamma}[\phi - \bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)} \frac{\delta^2 W[J]}{\delta J(y_3)\delta J(x_3)} = 0 \end{aligned} \tag{8.121}$$

Then we apply integration $\int d^4y_2 \frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)}$ to the formula above. then we attain:

$$\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = -\int d^4y_1 d^4y_2 d^4y_3 \frac{\delta^2 W[J]}{\delta J(y_1)\delta J(x_1)} \frac{\delta^2 W[J]}{\delta J(y_3)\delta J(x_3)} \frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)} \frac{\delta^3\hat{\Gamma}[\phi - \bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)} \tag{8.122}$$

Which means: (When $J=0, \phi=\bar{\phi}$)

$$W^{(3)}(x_1,x_2,x_3)=\int\frac{d^4y_1}{i\hbar}\frac{d^4y_2}{i\hbar}\frac{d^4y_3}{i\hbar}W^{(2)}(y_1,x_1)W^{(2)}(y_2,x_2)W^{(2)}(y_3,x_3)\hat{\Gamma}^{(3)}(y_1,y_2,y_3) \tag{8.123}$$

Using the equation 8.122 . we apply $\frac{\delta}{\delta J(x_4)}$. Then we can have:

$$\begin{aligned}
&\frac{\delta^4W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\\
=&-\int d^4y_1d^4y_2d^4y_3\frac{\delta^3W[J]}{\delta J(y_1)\delta J(x_1)\delta J(x_4)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)}\\
&-\int d^4y_1d^4y_2d^4y_3\frac{\delta^2W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^3W[J]}{\delta J(y_3)\delta J(x_3)\delta J(x_4)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)}\\
&-\int d^4y_1d^4y_2d^4y_3\frac{\delta^2W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^3W[J]}{\delta J(y_2)\delta J(x_2)\delta J(x_4)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)}\\
&-\int d^4y_1d^4y_2d^4y_3d^4y_4\frac{\delta^2W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^2W[J]}{\delta J(y_4)\delta J(x_4)}\\
&\cdot\frac{\delta^4\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)\delta\phi(y_4)}
\end{aligned} \tag{8.124}$$

Insert equation 8.122 to the equation above to substitute the 3-point connected function. For example, the first term would be:

$$\begin{aligned}
&-\int d^4y_1d^4y_2d^4y_3\frac{\delta^3W[J]}{\delta J(y_1)\delta J(x_1)\delta J(x_4)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)}\\
=&\int d^4y_1d^4y_2d^4y_3\;d^4y_4d^4y_5d^4y_6\frac{\delta^2W[J]}{\delta J(y_4)\delta J(y_1)}\frac{\delta^2W[J]}{\delta J(y_5)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_6)\delta J(x_4)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_4)\delta\phi(y_5)\delta\phi(y_6)}\\
&\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)}
\end{aligned} \tag{8.125}$$

in this case all the term can be written as:

$$\begin{aligned}
&\frac{\delta^4W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\\
&=\int d^4y_1d^4y_2d^4y_3\;d^4y_4d^4y_5d^4y_6\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)}\frac{\delta^3\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_4)\delta\phi(y_5)\delta\phi(y_6)}\\
&\left(\frac{\delta^2W[J]}{\delta J(y_4)\delta J(y_1)}\frac{\delta^2W[J]}{\delta J(y_5)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_6)\delta J(x_4)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\right.\\
&+\frac{\delta^2W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_4)\delta J(y_3)}\frac{\delta^2W[J]}{\delta J(y_5)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_6)\delta J(x_4)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\\
&\left.+\frac{\delta^2W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_4)\delta J(y_2)}\frac{\delta^2W[J]}{\delta J(y_5)\delta J(x_2)}\frac{\delta^2W[J]}{\delta J(y_6)\delta J(x_4)}\right)\\
&-\int d^4y_1d^4y_2d^4y_3d^4y_4\frac{\delta^2W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2W[J]}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^2W[J]}{\delta J(y_4)\delta J(x_4)}\\
&\cdot\frac{\delta^4\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)\delta\phi(y_4)}
\end{aligned} \tag{8.126}$$

In all from 8.126and 8.123 . we can have the graphic representation: 可以看图:8.7, 8.6 (这个是 phi3 场的例子)

当然，如果平均场是 0，也就是 $\bar{\phi}=0$ 的时候，不可约图像的生成函数可以不用这么定义，他直接就是 $\Gamma[\phi]$

在很多重整化理论中，需要计算的是几个点的不可约 Green Function。计算他们的步骤就是先计算 Connected Green Function. 然后首先利用公式8.120。进行 Fourier 变换之后可以解两点 irreducible Green Function。然后利用图 8.7，先通过 3-Point Connected Green Function 计算 3-Point irreducible Green Function. 再计算四点的。

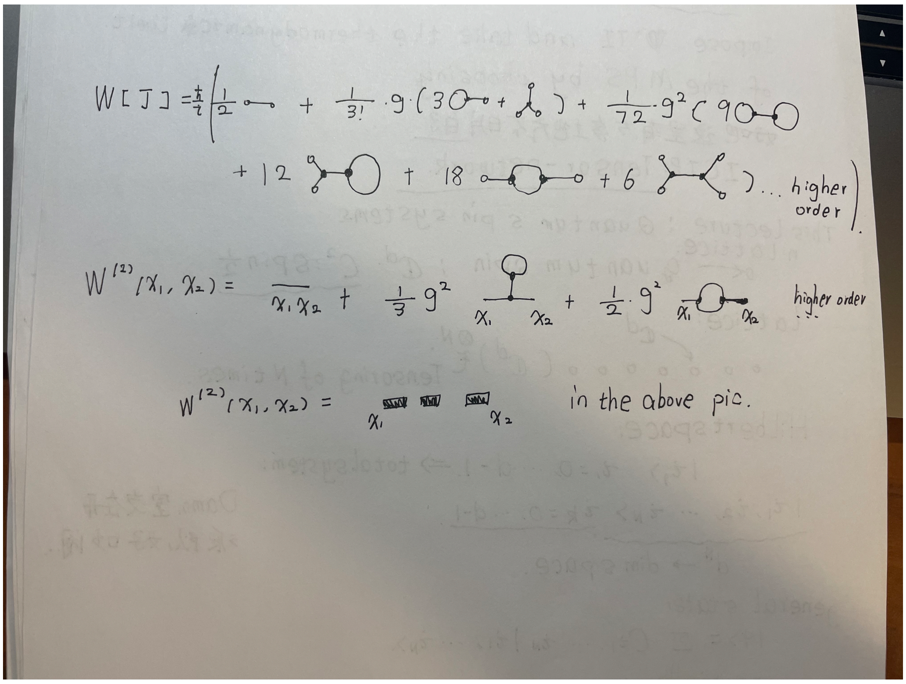


图 8.6: w2

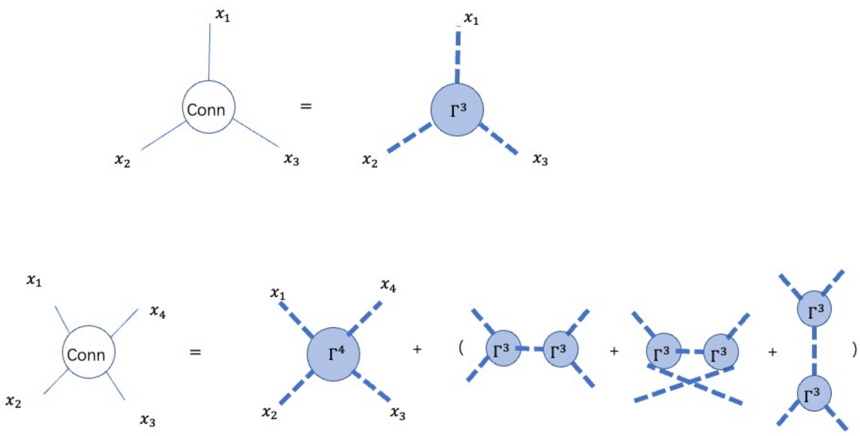


图 8.7: w3w4

当然也可以通过图片直接去观察 Connected Green Function 中的哪里是不可约的。
好，现在直接到 phi4 理论。之前把 Green Function 的 Generating Function 图表示过。这个是图像：8.2
但是在一般重整化中，是算到了二阶。所以有必要写一下二阶的 Generating Function. 图太多了，不知道对不对，先不放出来了。

第九章 标量场

9.1 生成函数与 Counter term

总述 标量场的 Lagrangian 写为:

$$\mathcal{L} = \frac{1}{2}Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}Z_m m^2 \varphi^2 + \frac{1}{6}Z_g g \varphi^3 + Y\varphi \tag{9.1}$$

其中:

$$Z \sim 1 + O(g^2) \quad Y \sim O(g) \tag{9.2}$$

确定其中系数的方式:

1 m: Mass of particle

2 g: Require cross depend on g in particular way(不理解)

3 Normalized by: $\langle 0|\varphi(x)|0\rangle = 0$ $\langle k|\varphi(x)|0\rangle e^{ikx}$

Ignore Counterterm 求 Generating Function Lagrangian 分类:

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}m^2\varphi^2 \\
\mathcal{L}_1 &= \frac{1}{6}Z_g g \varphi^3 + \mathcal{L}_{ct} \\
\mathcal{L}_{ct} &= \frac{1}{2}(Z_\varphi - 1)\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2 + Y\varphi
\end{aligned} \tag{9.3}$$

Generating Function:

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\varphi \exp\left[i\int d^4x\,(\mathcal{L}_0 + \mathcal{L}_1 + J\varphi)\right] \\
&= \exp\left[i\int d^4x \mathcal{L}_1(\frac{1}{i}\frac{\delta}{\delta J(x)})\right]\int \mathcal{D}\varphi \exp\left[i\int d^4x\,(\mathcal{L}_0 + J\varphi)\right] \\
&\propto \exp\left[i\int d^4x \mathcal{L}_1(\frac{1}{i}\frac{\delta}{\delta J(x)})\right]\exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right]
\end{aligned} \tag{9.4}$$

Ignore Counterterm:

$$\begin{aligned}
Z_1[J] &\propto \exp\left[i\frac{1}{6}Z_g g\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})^3\right]\exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right] \\
&= \sum_{V=0}^{+\infty}\frac{1}{V!}\left(i\frac{1}{6}Z_g g\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})^3\right)^V \sum_{P=0}^{+\infty}\frac{1}{P!}\left(\frac{i}{2}\int d^4y d^4z J(y)\Delta(y-z)J(z)\right)^P \\
&= \sum_{V=0}^{+\infty}\frac{1}{V!}\left(i\frac{1}{6}Z_g g\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})^3\right)^V \sum_{P=0}^{+\infty}\frac{1}{P!}\left(\frac{1}{2}\int d^4y d^4z iJ(y)\frac{1}{i}\Delta(y-z)iJ(z)\right)^P
\end{aligned} \tag{9.5}$$

Vertex, Propagator, External Source 对于上面计算 Generating Functional 中展开系数 V、P 对应的项，在经历了 3V 个泛函求导之后，剩下的 External source 数量 E 是:

$$E = 2P - 3V \tag{9.6}$$

这一项的相位系数是 (这个系数包含在了 Feynman 图表示的积分中了):

$$\text{Phase Factor} = (i)^{V+P-3V} = (i)^{V+E-P} \tag{9.7}$$

Feynman Rules 用图像来表示积分，表示方法为– line:$\frac{1}{i}\Delta(x-y)$; Source:$\int d^4x\, iJ(x)$; Vertex: $\int d^4x\, iZ_g g$

Symmetry Factor (For connected diagram) 对于 Generating Functional，它的系数是。不过由于有很多图它的计算结果相同，所以这个系数会被消掉大部分，只剩下一个 $\frac{1}{S}$，S 是 Symmetry Factor。

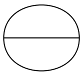
$$\frac{1}{V!}\frac{1}{(3!)^V}\frac{1}{P!}\frac{1}{(2!)^P} \tag{9.8}$$

观察 Generating Function，每个 Vertex 中的三个腿可以相互交换，形成 $(3!)^V$ 个对称性；每个 Vertex 可以互相交换，形成 $V!$ 个对称性；每个 Propagator 中的两个端点可以相互交换，形成 $(2!)^P$ 个对称性；每个 Propagator 互相可以交换形成 $P!$ 个对称性。这些交换生成的图像的数值结果是一样的，将这些对称性乘以系数得到:

$$\frac{1}{V!}\frac{1}{(3!)^V}\frac{1}{P!}\frac{1}{(2!)^P}\times(3!)^V V!(2!)^P P! = 1 \tag{9.9}$$

接下来我们考虑 Symmetry Factor, 也就是重复计算的对称性。

我们从一个例子出发:



在这个 Feynman 图中，同时交换 3 个 Propagator 的左右 source term 得到的效果和同时交换左右两个 Vertex 的效果是相同的；三个 Propagator 互相交换位置 (有 3! 种方式) 和左右 Vertex 同时交换相应 functional derivative 的顺序的效果是相同的。也就是，有 12 种对称性算重复了。于是:

$$S = \frac{1}{2\times 3!} = \frac{1}{12}. \tag{9.10}$$

Use Connected Diagram represent Generating functional 上面讨论的 Symmetry Factor 是针对 Connected Diagram 的。实际上的 Generating function 有些 non-connected 项是可以用 connected diagram 乘在一起表示。我们将 connected diagram 用 $\{1,2,3,\cdots,I,\cdots\}$ 表示，相应的 diagram 表示的积分写为 C_I。(C_I 是包含 Symmetry Factor 的)

现在考虑一个 Generating Function 中的一项，他可以写成 n_I 个 C_I 图相乘。($I = 1,2,\cdots$)

$$D = \Pi_{I=1}^{\infty}\frac{1}{S_I}(C_I)^{n_I} \tag{9.11}$$

对于其中的 Symmetry Factor, 它是因为同时交换不同图中的 Vertex 和同时交换不同图中的 Propagator 得到的结果相同，所以这一部分对称性算重复了。于是:

$$S_I = n_I! \quad D = \Pi_{I=1}^{+\infty}\frac{1}{n_I!}(C_I)^{n_I} \tag{9.12}$$

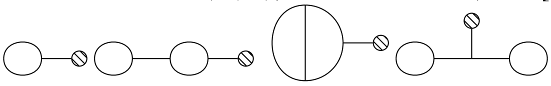
用 connected diagram 计算 Generating Functional:

$$\begin{aligned}
Z_1[J] &= \sum_{\{n_I\}}\left(\Pi_{I=1}^{+\infty}\frac{1}{n_I!}(C_I)^{n_I}\right)\\
&= \sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\cdots\frac{1}{n_1!}(C_1)^{n_1}\frac{1}{n_2!}(C_2)^{n_2}\cdots\\
&= \left(\sum_{n_1=1}^{+\infty}\frac{1}{n_1!}(C_1)^{n_1}\right)\left(\sum_{n_2=1}^{+\infty}\frac{1}{n_2!}(C_2)^{n_2}\right)\cdots\\
&= \Pi_{I=1}^{\infty}\left(\sum_{n_I=1}^{+\infty}\frac{1}{n_I!}(C_I)^{n_I}\right)\\
&= \exp(\sum_{I=1}^{\infty}C_I)
\end{aligned} \tag{9.13}$$

由于归一化条件，$Z_1[0] = 1$，所有的 Vacuum Diagram（也就是没有 External Source）的都会被归一化除掉，所以用 $C_{I'}$ 标记 non-Vacuum Connected Diagram, Generating Functional 可以写为:

$$Z_1[J] = \exp(\sum_{I'}C_{I'}) \tag{9.14}$$

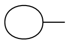
Y-Counterterm 在没有 Y-Counterterm 时，$Z_1[J]$ 中包含了 Single-Source Term。也就是 $E=1$。比如说:



利用生成函数，求 $\langle 0|\varphi(x)|0\rangle$。

$$\langle 0|\varphi(x)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x)}Z_1[J] \tag{9.15}$$

考虑关于 g 的最低阶项 ($Z_g \sim 1+O(g^2)$)，得到的结果用 Feynman 图表示为:

$\langle 0|\varphi(x)|0\rangle =$  x $= \frac{1}{2}\int d^4y ig\,\frac{1}{i}\Delta(y-y)\frac{1}{i}\Delta(y-x)+O(g^3)$ (9.16)

这样的结果和条件 $\langle 0|\varphi(x)|0\rangle = 0$ 不相符。所以需要引入 $Y\varphi$ 的 counterterm。

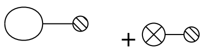
此时:

$$Z[J]\propto \exp\left[iY\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})\right]\exp\left[i\frac{1}{6}Z_g g\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})^3\right]\exp\left[\frac{i}{2}\int d^4xd^4x'J(x)\Delta(x-x')J(x')\right] \tag{9.17}$$

Counterterm 会生成一种新的节点（它只连结一个 Propagator），表示为:

$$\int d^4xiY = \otimes \tag{9.18}$$

用含有这个节点的 Feynman Diagram 来消除含有一个 Source 的项 (只考虑 g 的一阶):

 $= 0$

$$\int d^4x'\left(iY+\frac{1}{2}\frac{1}{i}\Delta(0)ig\right)\frac{1}{i}\Delta(x'-x) = 0 \tag{9.19}$$

$$Y = i\frac{1}{2}g\Delta(0)$$

Ignore Tadpoles 如果一个 Connected Diagram 中有一部分，它与其他部分通过一个 propagator 相连接，并且它没有 source 项（这一部分叫做 tadpole）。于是在 $Z_1[J]$ 中一定有这样的 tadpole 连接了一个 Source。但是通过 counterterm，一定有一个和他相反的项。与他相反的项也可以将 Source 替换为其他的子图。所以这个 Connected Diagram 总是不用考虑的。

另一个 Counterterm 考虑 Lagrangian 中剩下的 Counterterm:

$$\begin{aligned} S &\sim i\int d^4x\left(\frac{1}{2}(Z_\varphi-1)\partial^\mu\varphi\partial_\mu\varphi-\frac{1}{2}(Z_m-1)m^2\varphi^2\right)\\ &= i\int d^4x\left(-\frac{1}{2}(Z_\varphi-1)\varphi\partial^\mu\partial_\mu\varphi-\frac{1}{2}(Z_m-1)m^2\varphi^2\right)\end{aligned} \tag{9.20}$$

他们写为泛函导数形式 (利用分部积分)

$$\exp\left[\int id^4x-\frac{i}{2}\left((Z_\varphi-1)(\frac{\delta}{i\delta J(x)})\partial^\mu\partial_\mu(i\frac{\delta}{\delta J(x)})+(Z_m-1)m^2(\frac{\delta}{i\delta J(x)})^2\right)\right] \tag{9.21}$$

这一项会生成一个两点 Vertex。(2 因为对称性消掉了) 它可以表示为:

$$-i\int d^4x\left((Z_\varphi-1)\partial^2+(Z_m-1)m^2\right) \tag{9.22}$$

Noticed that partial 作用在一个 propagator 而不是两个。

总结 生成函数可以表示为:

$$Z[J] = \exp\left[\sum_{I'}C_{I'}\right] \tag{9.23}$$

其中 $C_{I'}$ 是 Connected Diagram with symmetry factor，并且没有 Tadpoles。同时 $C_{I'}$ 中需要有由 Counterterm 引起的两腿 vertex。

9.2 树图散射振幅

LSZ Reduction Formula under new normalize condition 标量场的 LSZ Reduction formula 在 Greiner 书中写为 (ipad 上有推导):

$$\begin{aligned}\langle f|i\rangle &= \langle 0|Ta_{qm}(+\infty)\cdots a_{q1}(+\infty)a^\dagger_{p1}(-\infty)\cdots a^\dagger_{pn}|0\rangle \\ &= (i)^{n+m}\int d^4x_1 u_{p1}(x_1)(\partial^2_{x1}+m^2)\cdots\int d^4y_1 u^*_{q1}(y_1)(\partial^2_{y1}+m^2)\cdots\langle 0|T\varphi(x_1)\cdots\varphi(x_n)\varphi(y_1)\cdots\varphi(y_m)|0\rangle\end{aligned} \tag{9.24}$$

其中,

$$u_{\vec{p}}(x) = \frac{1}{\sqrt{2w_p(2\pi)^3}}e^{-i(px)} \quad p = (w_p, \vec{p}) \tag{9.25}$$

在 Greiner 的书中,归一化条件是 $\langle i|i\rangle = \delta^{(3)}(p_1-p_1)\cdots$。现在改变它的归一化条件为 $\langle i|i\rangle = 2w_{p1}\cdots 2w_{pn}\delta^{(3)}(p_1-p_1)\cdots$ 定义新的态矢量 $|i\rangle \to \sqrt{2w_{p1}\cdots 2w_{pn}}|i\rangle$ 相应的 LSZ Reduction Formula 也就变成了:

$$\begin{aligned}\langle f|i\rangle &= \sqrt{2w_{p1}\cdots 2w_{pn}}\sqrt{2w_{q1}\cdots 2w_{qm}}\langle 0|Ta_{qm}(+\infty)\cdots a_{q1}(+\infty)a^\dagger_{p1}(-\infty)\cdots a^\dagger_{pn}|0\rangle \\ &= (i)^{n+m}\int d^4x_1 e^{-ip_1x_1}(\partial^2_{x1}+m^2)\cdots\int d^4y_1 e^{iq_1y_1}(\partial^2_{y1}+m^2)\cdots\langle 0|T\varphi(x_1)\cdots\varphi(x_n)\varphi(y_1)\cdots\varphi(y_m)|0\rangle\end{aligned} \tag{9.26}$$

只用 Fully Connected Part 算散射振幅 生成函数写为:

$$Z[J] = \exp\left[\sum_{I'} C_{I'}\right] = \exp(iW[J]) \tag{9.27}$$

两点关联函数:

$$\begin{aligned}\langle 0|\varphi(x_1)\varphi(x_2)|0\rangle &= \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}Z[J] \\ &= \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}iW[J] + \frac{\delta}{i\delta J(x_1)}iW[J]\frac{\delta}{i\delta J(x_2)}iW[J]\end{aligned} \tag{9.28}$$

由于有 Counterterm Y 的存在，后面的一项是 0。四点关联函数:

$$\begin{aligned}&\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}Z[J] \\ &= \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\left(\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}iW[J] + \frac{\delta}{i\delta J(x_3)}iW[J]\frac{\delta}{i\delta J(x_4)}iW[J]\right)Z[J] \\ &= \frac{\delta}{i\delta J(x_1)}\left(\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}iW[J] + \frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}iW[J]\frac{\delta}{i\delta J(x_4)}iW[J] + \frac{\delta}{i\delta J(x_3)}iW[J]\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_4)}iW[J]\right)Z[J] \\ &+ \frac{\delta}{i\delta J(x_1)}\left(\frac{\delta}{i\delta J(x_2)}iW[J]\right)\left(\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}iW[J] + \frac{\delta}{i\delta J(x_3)}iW[J]\frac{\delta}{i\delta J(x_4)}iW[J]\right)Z[J] \\ &= \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}iW[J] + \left(\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}iW[J]\right)\left(\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}iW[J]\right) \\ &+ \left(\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_3)}iW[J]\right)\left(\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_4)}iW[J]\right) + \left(\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_4)}iW[J]\right)\left(\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}iW[J]\right)\Big|_{J=0}\end{aligned} \tag{9.29}$$

前一项是一个有四个 Source 的 connect Feynman 图对 S matrix 的贡献，后面这些项都是 Non-connect part。考虑其中一个 Non-connect part 比如:

$$\left(\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}iW[J]\right)\left(\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}iW[J]\right) = F(x_1-x_2)F(x_3-x_4) \tag{9.30}$$

考虑这一项经过 LSZ reduction formula 对散射振幅的贡献: (f 是 F 经过微分算符作用后的函数，具体形式不重要)

$$\begin{aligned}&(i)^{2+2}\int d^4x_1 e^{-ip_1x_1}(\partial^2_{x1}+m^2)\int d^4x_2 e^{-ip_2x_2}(\partial^2_{x2}+m^2)\int d^4x_3 e^{+ip_3x_3}(\partial^2_{x3}+m^2)\int d^4x_4 e^{+ip_4x_4}(\partial^2_{x4}+m^2)F(x_1-x_2)F(x_3-x_4) \\ &= \int d^4x_1 e^{-ip_1x_1}\int d^4x_2 e^{-ip_2x_2}\int d^4x_3 e^{+ip_3x_3}\int d^4x_4 e^{+ip_4x_4} f(x_1-x_2)f(x_3-x_4)\end{aligned} \tag{9.31}$$

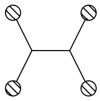
积分变换:

$$\begin{cases} u = \frac{1}{2}(x_1 + x_2) \\ v = \frac{1}{2}(x_1 - x_2) \\ y = \frac{1}{2}(x_3 + x_4) \\ z = \frac{1}{2}(x_3 - x_4) \end{cases} \tag{9.32}$$

$$\begin{aligned} \text{原式} &= \int d^4u d^4v d^4y d^4z \det\left(\frac{\partial x_1, x_2}{\partial u, v}\right) \det\left(\frac{\partial x_3, x_4}{\partial y, z}\right) \sim \\ &= 4\int d^4u d^4v d^4y d^4z e^{-i(p_1+p_2)u - i(p_1-p_2)v} e^{+i(p_3+p_4)y + i(p_3-p_4)z} f(2v) f(2z) \\ &= \delta^{(4)}(p_1 - p_2)\delta^{(4)}(p_3 - p_4) \sim \end{aligned} \tag{9.33}$$

他是没有意义的，因为相当于没有散射。同理，经过具体计算，发现 Non-connect 项都没有贡献，所以只有 Fully Connected Part 对散射振幅有贡献。

最低阶 2-2 散射的树图散射振幅计算 考虑 Generating Function 中的 Connected-4Source-NoTadpole Diagram In Lowest Order.

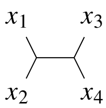


$$\approx O(g^2) \qquad S = \frac{1}{2\times 2\times 2} = \frac{1}{8} \tag{9.34}$$

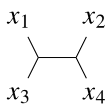
用这部分 Generating Functional 求 4 点关联函数:

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)} iW[J]$$

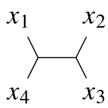
$$= (2\times 2\times 2)\frac{1}{8}$$



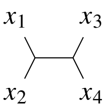
$$+ (2\times 2\times 2)\frac{1}{8}$$



$$+ (2\times 2\times 2)\frac{1}{8} \tag{9.35}$$



其中，$2\times 2\times 2$ 来自于对不同的 source 求导。(Symmetry Factor 被消掉是树图的共同特点) 一般来说将入射写在左边，出射写在右边:



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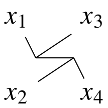

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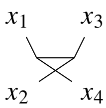

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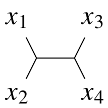

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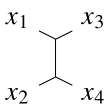

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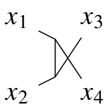


$$\begin{aligned} &= \int d^4y d^4z ig\, ig \left(\frac{1}{i}\Delta(x_1 - y)\frac{1}{i}\Delta(x_2 - y)\frac{1}{i}\Delta(y - z)\frac{1}{i}\Delta(z - x_3)\frac{1}{i}\Delta(z - x_4) \sim\right) \\ &= i\int d^4y d^4z \;\; g^2\Big(\Delta(x_1 - y)\Delta(x_2 - y)\Delta(y - z)\Delta(z - x_3)\Delta(z - x_4) + \Delta(x_1 - y)\Delta(x_2 - z)\Delta(y - z)\Delta(y - x_3)\Delta(z - x_4) \\ &\quad + \Delta(x_1 - y)\Delta(x_2 - z)\Delta(y - z)\Delta(z - x_3)\Delta(y - x_4)\Big) \end{aligned} \tag{9.36}$$

考虑 LSZ Formula 以及传播子的具体表达:

$$\begin{cases} \langle f|i\rangle = (i)^{n+m}\int d^4x_1 e^{-ip_1x_1}(\partial_{x1}^2 + m^2)\cdots\int d^4y_1 e^{iq_1y_1}(\partial_{y1}^2 + m^2)\cdots\langle 0|T\varphi(x_1)\cdots\varphi(x_n)\varphi(y_1)\cdots\varphi(y_m)|0\rangle \\ \Delta(x - x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} \\ (\partial_x^2 + m^2)\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4}e^{-ik(x-x')} = \delta^{(4)}(x - x') \end{cases} \tag{9.37}$$

于是:

$$
\begin{aligned}
\langle f|i\rangle =& i\int d^4yd^4zd^4x_1d^4x_2d^4x_3d^4x_4d^4k\ g^2e^{-ip_1x_1-ip_2x_2+ip_3x_3+ip_4x_4}\\
&\frac{1}{(2\pi)^4}\frac{e^{-ik(y-z)}}{-k^2+m^2}\Big(\delta^{(4)}(x_1-y)\delta^{(4)}(x_2-y)\delta^{(4)}(z-x_3)\delta^{(4)}(z-x_4)+\delta^{(4)}(x_1-y)\delta^{(4)}(x_2-z)\delta^{(4)}(y-x_3)\delta^{(4)}(z-x_4)\\
&+\delta^{(4)}(x_1-y)\delta^{(4)}(x_2-z)\delta^{(4)}(x_3-z)\delta^{(4)}(x_4-y)\Big)\\
=& i\int d^4yd^4zd^4k\ g^2\frac{1}{(2\pi)^4}\frac{e^{-ik(y-z)}}{-k^2+m^2}\left(e^{-i(p_1+p_2)y+i(p_3+p_4)z}+e^{-i(p_1-p_3)y+i(-p_2+p_4)z}+e^{-i(p_1-p_4)y+i(-p_2+p_3)z}\right)\\
=& i\int d^4k\ g^2(2\pi)^4\frac{1}{-k^2+m^2}\Big(\delta^{(4)}(p_1+p_2+k)\delta^{(4)}(p_3+p_4+k)\\
&+\delta^{(4)}(p_1-p_3+k)\delta^{(4)}(-p_2+p_4+k)+\delta^{(4)}(p_1-p_4+k)\delta^{(4)}(-p_2+p_3+k)\Big)\\
=& i\ g^2(2\pi)^4\Big(\frac{1}{-(p_1+p_2)^2+m^2}\delta^{(4)}(p_1+p_2-p_3-p_4)+\frac{1}{-(p_1-p_3)^2+m^2}\delta^{(4)}(p_1-p_3+p_2-p_4)\\
&+\frac{1}{-(p_1-p_4)^2+m^2}\delta^{(4)}(p_1-p_4+p_2-p_3)\Big)\\
=& i\ g^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\left(\frac{1}{-(p_1+p_2)^2+m^2}+\frac{1}{-(p_1-p_3)^2+m^2}+\frac{1}{-(p_1-p_4)^2+m^2}\right)\\
=& (2\pi)^4\delta^{(4)}(p_{in}-p_{out})i\mathcal{T}
\end{aligned}
\tag{9.38}
$$

确实可以发现 $\langle f|i\rangle$ 以及 $\mathcal{T}$ 是 Lorentz Invariant 的。δ 函数暗含了能量守恒。

动量空间的 Feynman 规则 先画图，考虑 Generating Functional 的 Symmetry Factor，进而考虑 n-point functional 的 Symmetry Factor。左侧是入射侧，动量从外流向节点；最右侧是出射侧，动量从节点流向外面。这些动量都是自己标记的，标记为 $k_1\cdots k_{1'}\cdots$ 之类的。每一个节点都是 4 动量守恒的。总结起来，可以是下面这些关键点。

(1) 每个最外侧的 propagator 当作 1

(2) 每个内侧动量是 k 的传播子当作 $\frac{-i}{-k^2+m^2-i\epsilon}=\frac{1}{i}\Delta(k^2)$

(3) 每个节点贡献 iZ_gg

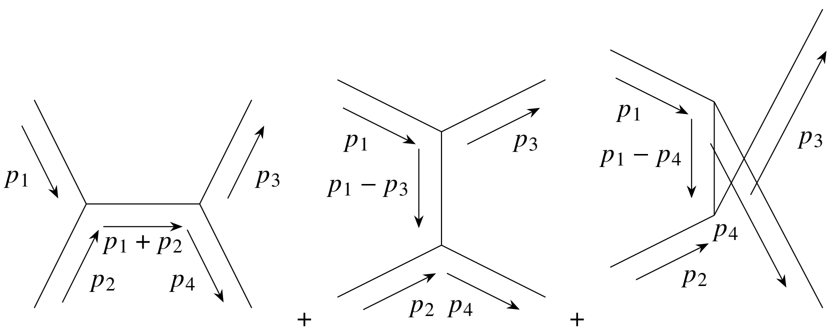
(4) 每个没有确定的动量 l 贡献积分:$\frac{d^4l}{(2\pi)^4}$

(5) Leftover Symmetry 可以通过圈图贡献直接计算。具体来说，当有圈图时，如果有改变 propagator 和 vertex 导致的结果相同，那么就会贡献 Leftover symmetry。(不用从 Generating Functional 开始算了)

(6) 需要考虑 Counterterm Vertex 他是:$-i\left(-(Z_\varphi-1)k^2+(Z_m-1)m^2\right)$

(7) 将所有图计算完之后的结果就是 $i\mathcal{T}$

利用 Feynman Rule In Momentum space，可以将上面的树图贡献的 $i\mathcal{T}$ 写为:



(9.39)

9.3 散射截面

Lorentz 不变相空间 Lorentz 变换后的相空间:

$$dp'_x dp'_y dp'_z \tag{9.40}$$

Lorentz 变换:

$$\begin{cases} p'_x = p_x \quad p'_y = p_y \quad p'_z = \gamma(p_z - \frac{\beta}{c}E) \quad E' = \gamma(E - \beta c p_z) \quad E = \sqrt{m^2c^4 + p^2c^2} \end{cases} \tag{9.41}$$

$$\begin{aligned} \text{上式} &= dp_x dp_y dp_z \gamma(1 - \beta/c\frac{\partial E}{\partial p_z}) \\ &= dp_x dp_y dp_z \gamma(1 - \beta c\frac{p_z}{E}) \\ &= dp_x dp_y dp_z \frac{E'}{E} \end{aligned} \tag{9.42}$$

也就是:

$$\frac{d^3p}{(2\pi)^3 2E'} = \frac{d^3p}{(2\pi)^3 2E} \tag{9.43}$$

它定义为 Lorentz 不变相空间

散射截面的定义 考虑两个粒子的散射，他们散射后变成了 k 个出射粒子。
两个粒子的速度分别是 v_1 与 v_2, 如果说两个粒子的体积都估计为 V ，同时他们的相对速度是 $|\vec{v}_1 - \vec{v}_2|$。这样相当于一个粒子用这个相对速度去撞击另一个粒子。既然一个粒子占据的体积是 V。那么相当于有一个入射流 (单位时间单位面积入射的粒子数量)

$$\Phi = \frac{1}{V}|\vec{v}_1 - \vec{v}_2| \tag{9.44}$$

然后考虑一个原子物理里面的微分截面的东西。其实也就是入射粒子流中的一个小小面积元 $d\sigma$, 在时间 T 之后, 认为这个面积元对应的 $p_j\ dp_j\ (j \in \{1, 2...k\})$，出射状态，就是说给定了一个出射状态, 当然，这个状态是怎么说，无穷小的。一个状态元。就和原子物理里面出射到一个球面角元是一样的。然后在这个状态元的粒子数量是 dP, 当然他是小于 1 的, 这个时候应该理解为概率
所以上面的话就描述的是这个式子。

$$d\sigma\frac{1}{V}|\vec{v}_1 - \vec{v}_2|T = dP \tag{9.45}$$

也就是说：

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|}dP \tag{9.46}$$

一般来说，散射截面是在实验室参考系中定义的。在实验室参考系中，一般将靶粒子 2 固定，此时，相对速度是 $\frac{|\vec{p}_1|_{FT}}{(E_1)_{FT}}$。(FT means fixed target)

$$d\sigma = \frac{V}{T}\frac{(E_1)_{FT}}{|\vec{p}_1|_{FT}}dP \tag{9.47}$$

然后, 我们从量子的角度考虑 dP.

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\Pi_j \frac{V}{(2\pi)^3}d^3\vec{p}_j \tag{9.48}$$

至于这里为什么加入了一个体积 V (注意连乘符号后面的所有东西都是要连乘的...)，可以理解成一维动量量子化后的结果是 $p_i = \frac{2\pi}{L}i$
接下来考虑粒子态。

$$\sqrt{(2\pi)^3\ 2w_k}a_k^\dagger|0\rangle = |k\rangle \tag{9.49}$$

同时, 湮灭算符和产生算符有对易关系

$$[a_p, a_p^\dagger] = \delta(\vec{p} - \vec{q}) \tag{9.50}$$

这样, 动量为 p 的单粒子态和自己的内积就是 (动量空间的 delta 函数可以用坐标空间的积分表示)

$$\langle p|p\rangle = 2\omega_p (2\pi)^3 \delta^{(3)}(0) = 2\omega_p V \tag{9.51}$$

于是

$$\langle i|i\rangle = (2E_1V)(2E_2V) \tag{9.52}$$

$$\langle f|f\rangle = \Pi_j (2E_jV) \tag{9.53}$$

S 就是相互作用绘景里面的时间演化算符。我们说 S 可以展开成平庸的部分和演化的部分。

$$S = 1 + i\mathcal{M} = 1 + (2\pi)^4 \delta^{(4)}(\Sigma p) i\mathcal{T} \tag{9.54}$$

这个演化项作用在初态, 再和末态做内积后:

$$|\langle f|S|i\rangle|^2 = \delta^{(4)}(0)\delta^{(4)}(\Sigma p)(2\pi)^8 |\langle f|\mathcal{T}|i\rangle|^2 = VT\delta^{(4)}(\Sigma p)(2\pi)^8 |\langle f|\mathcal{T}|i\rangle|^2 \tag{9.55}$$

于是:

$$\begin{aligned} dP &= \frac{\delta^{(4)}(0)\delta^{(4)}(\Sigma p)(2\pi)^8 |\langle f|\mathcal{T}|i\rangle|^2}{(2E_1V)(2E_2V)\Pi_j(2E_jV)} \Pi_j \frac{V}{(2\pi)^3} d^3\vec{p}_j \\ &= \frac{T}{V}\frac{1}{(2E_1)(2E_2)} |\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4 \delta^{(4)}(\Sigma p)\Pi_j \frac{d^3 p_j}{(2\pi)^3}\frac{1}{2E_j} \end{aligned} \tag{9.56}$$

于是

$$d\sigma = \frac{(E_1)_{FT}}{(2E_1)(2E_2)|\vec{p}_1|_{FT}} |\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4 \delta^{(4)}(\Sigma p)\Pi_j \frac{d^3 p_j}{(2\pi)^3}\frac{1}{2E_j} \tag{9.57}$$

其中后面的那个叫洛伦兹不变相空间体元。(Lorentz invariant phase spave-LIPS)

$$dLIPS_{n'}(p_1 + p_2) = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_j p_j)\Pi_j \frac{d^3 p_j}{(2\pi)^3}\frac{1}{2E_j} \tag{9.58}$$

为了使得计算出的 dP 中含有的 T/V 能和 $d\sigma$ 的定义式中的 V/T 能消掉，要求计算出射粒子也是要在 Fixed-Target Frame。此时:

$$d\sigma = \frac{1}{4(E_2)_{FT}|\vec{p}_1|_{FT}} |\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4 \delta^{(4)}(\Sigma p)\Pi_j \frac{d^3 p_j}{(2\pi)^3}\frac{1}{2E_j} \tag{9.59}$$

后面的这些项都是 Lorentz-Invariant 的。

Mandelstam 变量　Mandelstam 变量定义为:(使用了能动量守恒的性质)

$$\begin{aligned} s &= (k_1 + k_2)^2 = (k_{1'} + k_{2'})^2 \\ t &= (k_1 - k_{1'})^2 = (k_2 - k_{2'})^2 \\ u &= (k_1 - k_2')^2 = (k_2 - k_{1'})^2 \end{aligned} \tag{9.60}$$

它们三个并不是完全互相独立的。满足约束关系

$$s + t + u = m_1^2 + m_2^2 + m_{1'}^2 + m_{2'}^2 \tag{9.61}$$

它的意义在与，它是一个 Lorentz 不变量。同时，他可以用来表示其他的量。比如，在 Fixed-Target Frame 中，$|\vec{k}_2| = 0$。

$$s = (E_1 + m_2)^2 - |\vec{k}_1|^2 \quad E_1 = \sqrt{m_1^2 + |\vec{k}_1|^2} \tag{9.62}$$

可以得到:

$$|\vec{k}_1|_{FT} = \frac{1}{2m_2}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \tag{9.63}$$

又比如在 Center-of-Mass 参考系中:

$$s = (E_1 + E_2)^2 \quad E_1 = \sqrt{m_1^2 + |\vec{k}_1|^2} \quad E_2 = \sqrt{m_2^2 + |\vec{k}_1|^2} \tag{9.64}$$

可以得到:

$$|\vec{k}_1|_{CM} = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \tag{9.65}$$

类似的，对于出射粒子:

$$s = (E_{1'} + E_{2'})^2 \quad E_{1'} = \sqrt{m_{1'}^2 + |\vec{k}_{1'}|^2} \quad E_{2'} = \sqrt{m_{2'}^2 + |\vec{k}_{1'}|^2} \tag{9.66}$$

可以得到:

$$|\vec{k}_{1'}|_{CM} = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_{1'}^2 + m_{2'}^2)s + (m_{1'}^2 - m_{2'}^2)^2} \tag{9.67}$$

对于 Mandelstam 变量 t:

$$t = (k_1 - k_{1'})^2 = (E_1 - E_{1'})^2 - (\vec{k}_1 - \vec{k}_{1'})^2 = m_1^2 + m_{1'}^2 - 2E_1E_{1'} + 2|\vec{k}_1||\vec{k}_{1'}|\cos\theta \tag{9.68}$$

如果是在 CM 系，$t \propto \cos(\theta)$。

2-2 散射的散射截面表达 对于 2-2 散射问题，

$$d\sigma = \frac{1}{4(E_2)_{FT}|\vec{p}_1|_{FT}}|\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4\delta^{(4)}(\Sigma p)\Pi_j\frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j} \tag{9.69}$$

利用性质:

$$\left\{(E_2)_{FT}|\vec{p}_1|_{FT} = m_2|\vec{p}_1|_{FT} = \sqrt{s}|\vec{p}_1|_{CM}\right. \tag{9.70}$$

$$d\sigma = \frac{1}{4\sqrt{s}|\vec{p}_1|_{CM}}|\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4\delta^{(4)}(p - p_{1'} - p_{2'})\frac{d^3p_{1'}d^3p_{2'}}{(2\pi)^6}\frac{1}{2E_{1'}2E_{2'}} \tag{9.71}$$

由于除了第一项，后面的都是 Lorentz Invariant 的，可以在任何参考系计算。这里在 Center of mass 参考系计算。首先计算后面的部分:

$$\begin{aligned}&(2\pi)^4\delta^{(4)}(p - p_{1'} - p_{2'})\frac{d^3p_{1'}d^3p_{2'}}{(2\pi)^6}\frac{1}{2E_{1'}2E_{2'}}\\ &= \frac{1}{(2\pi)^2}\delta(\sqrt{s} - E_{1'} - E_{2'})\delta^{(3)}(0 - p_{1'} - p_{2'})d^3p_{1'}d^3p_{2'}\frac{1}{2E_{1'}2E_{2'}}\end{aligned} \tag{9.72}$$

对 $p_{2'}$ 积分:

$$\begin{aligned}\text{上式} &= \frac{1}{(2\pi)^2}\delta(\sqrt{s} - E_{1'} - E_{2'})\frac{d^3p_{1'}}{2E_{1'}2E_{2'}} \quad E_{1'} = \sqrt{\vec{p_{1'}}^2 + m_{1'}^2} \quad E_{2'} = \sqrt{\vec{p_{1'}}^2 + m_{2'}^2}\\ &= \frac{1}{(2\pi)^2}\frac{1}{2(E_{1'})_{CM}2(E_{2'})_{CM}}\delta(\sqrt{s} - E_{1'} - E_{2'})|p_{1'}|^2 d|p_{1'}|d\Omega_{CM}\end{aligned} \tag{9.73}$$

delta 函数有性质:

$$\left\{\int dx\delta(f(x)) = \delta(f(x))df(x)\frac{1}{\frac{df(x)}{dx}}\right. \tag{9.74}$$

于是经过积分，筛选出能动量守恒解出的 $|p_{1'}|_{CM}$:

$$\begin{aligned}\text{上式} &= \frac{1}{(2\pi)^2}\frac{1}{2(E_{1'})_{CM}2(E_{2'})_{CM}}|\vec{p}_{1'}|_{CM}^2 d\Omega_{CM}\frac{1}{|\frac{d}{d|p_{1'}|}(\sqrt{s} - E_{1'} - E_{2'})|}\\ &= \frac{1}{(2\pi)^2}\frac{1}{2(E_{1'})_{CM}2(E_{2'})_{CM}}|\vec{p}_{1'}|_{CM}^2 d\Omega_{CM}\frac{1}{|\vec{p_{1'}}|_{CM}\left(\frac{1}{E_{1'}} + \frac{1}{E_{2'}}\right)}\\ &= \frac{1}{(2\pi)^2}\frac{1}{2(E_{1'})_{CM}2(E_{2'})_{CM}}|\vec{p}_{1'}|_{CM}^2 d\Omega_{CM}\frac{1}{|\vec{p_{1'}}|_{CM}\left(\frac{\sqrt{s}}{E_{2'}E_{1'}}\right)}\\ &= \frac{1}{(2\pi)^2}\frac{1}{4\sqrt{s}|\vec{p}_{1'}|_{CM}}|\vec{p}_{1'}|_{CM}^2 d\Omega_{CM}\end{aligned} \tag{9.75}$$

于是:

$$\begin{aligned} d\sigma &= \frac{1}{4\sqrt{s}|\vec{p}_1|_{CM}}|\langle f|\mathcal{T}|i\rangle|^2 \quad \frac{1}{(2\pi)^2}\frac{1}{4\sqrt{s}|\vec{p}_{1'}|_{CM}}|\vec{p}_{1'}|^2_{CM}d\Omega_{CM} \\ &= \frac{|\vec{p}_{1'}|_{CM}}{64\pi^2 s|\vec{p}_1|_{CM}}|\langle f|\mathcal{T}|i\rangle|^2 d\Omega_{CM} \end{aligned} \tag{9.76}$$

微分散射截面:

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{|\vec{p}_{1'}|_{CM}}{64\pi^2 s|\vec{p}_1|_{CM}}|\langle f|\mathcal{T}|i\rangle|^2 \tag{9.77}$$

有时候，采用 t 作为变量，由于:

$$\begin{cases} t &= m_1^2 + m_{1'}^2 - 2E_1E_{1'} + 2|\vec{p}_1||\vec{p}_{1'}|\cos\theta \\ dt &= 2|\vec{p}_1||\vec{p}_{1'}|d\cos\theta \quad \text{只有在 CM 系可以这样算，否则 } \vec{p}_{1'}| \text{ 也是 } \theta \text{ 的函数} \\ &= -2|\vec{p}_1|_{CM}|\vec{p}_{1'}|_{CM}\frac{1}{2\pi}d\Omega_{CM} \end{cases} \tag{9.78}$$

于是:

$$\begin{aligned} \frac{d\sigma}{dt} &= -\frac{\pi}{|\vec{p}_1|_{CM}|\vec{p}_{1'}|_{CM}}\frac{|\vec{p}_{1'}|_{CM}}{64\pi^2 s|\vec{p}_1|_{CM}}|\langle f|\mathcal{T}|i\rangle|^2 \\ &= -\frac{1}{64\pi s|\vec{p}_1|^2_{CM}}|\langle f|\mathcal{T}|i\rangle|^2 \end{aligned} \tag{9.79}$$

对称因子 如果出射的粒子 j 有 n_j 个与他相同。这些粒子有 $n_j!$ 种排布顺序导致的散射结果结果都是相同的。也就是只用 $\frac{1}{n_i!}$ 的散射截面就可以得到同样的散射结果。

$$\sigma \to \frac{1}{S}\sigma \quad S = \Pi_j \frac{1}{n_j!} \tag{9.80}$$

$\varphi^3$2-2 树图散射最低阶对散射截面的贡献 ... 懒得写了，在 Srednicki 上面

衰变概率 这部分好像是有点问题，LSZ formula 可能本来不适用，如果直接认为 LSZ formula 仍然适用。需要改变的是

$$\langle i|i\rangle = 2E_1V\,2E_2V \to \langle i|i\rangle = 2E_1V \tag{9.81}$$

原来两个粒子碰撞时候:

$$\begin{aligned} dP &= \frac{\delta^{(4)}(0)\delta^{(4)}(\Sigma p)(2\pi)^8|\langle f|\mathcal{T}|i\rangle|^2}{(2E_1V)(2E_2V)\Pi_j(2E_jV)}\Pi_j\frac{V}{(2\pi)^3}d^3\vec{p}_j \\ &= \frac{T}{V}\frac{1}{(2E_1)(2E_2)}|\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4\delta^{(4)}(\Sigma p)\Pi_j\frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j} \end{aligned} \tag{9.82}$$

现在改写为:

$$dP = \frac{T}{1}\frac{1}{(2E_1)}|\langle f|\mathcal{T}|i\rangle|^2 \quad (2\pi)^4\delta^{(4)}(\Sigma p)\Pi_j\frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j} \tag{9.83}$$

$$\frac{dP}{T} = \Gamma = \frac{1}{2E_1}|\langle f|\mathcal{T}|i\rangle|^2 dLIPS_{n'}(p_1) \tag{9.84}$$

$1/E_1$ 和时间膨胀有关系

9.4 维度分析

Lagrangian 写为:

$$\mathcal{L} = \frac{1}{2}Z_\varphi\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}Z_m m^2\varphi^2 + \frac{1}{n!}Z_g g\varphi^n + Y\varphi \tag{9.85}$$

自然单位制下，$\hbar = c = 1$。所有单位都可以用质量单位表示。考虑自然单位的定义。

$$[\hbar] = [m][L]^2[T]^{-1} \quad [c] = [L][T]^{-1} \tag{9.86}$$

得到:

$$[m] = [m] \quad [L] = [T] = [m]^{-1} \tag{9.87}$$

由于作用量的单位是 1（作用量出现在 e 指数上），观察 Lagrangian 中第一项

$$[L]^d\ [\varphi]^2[L]^{-2} = 1 \Rightarrow [\varphi] = [L]^{(2-d)/2} = [m]^{(d-2)/2} \tag{9.88}$$

g 的单位（也是通过作用量的单位为 1 得到）：

$$[L]^d[g][\varphi]^n = 1 \Rightarrow [m]^{-d}[g][m]^{n(d-2)/2} = 1 \Rightarrow [g] = [m]^{d-\frac{1}{2}(d-2)n} \tag{9.89}$$

发现有特殊点:$d = 4, n = 3 \Rightarrow [g] = 1$

9.5 Lehmann-Kallen form of exact propagator

场和真空归一化条件 在 Poincare 变换章节中讲到推广的 Heisenberg Equation of motion (7.39)。

$$\left\{\exp\left(-\frac{1}{i\hbar}P\cdot(x_2-x_1)\right)\phi(x_1)\exp\left(\frac{1}{i\hbar}P\cdot(x_2-x_1)\right) = \phi(x_2) \tag{9.90}$$

考虑内积式:

$$\langle 0|\varphi(x)|0\rangle \tag{9.91}$$

利用量子场的平移关系:

$$\langle 0|\varphi(x)|0\rangle = \langle 0|\exp(iP\cdot x)\,\varphi(0)\exp(-iP\cdot x)\,|0\rangle \tag{9.92}$$

对于上面的式子，$|0\rangle$ 代表 $t\to\pm\infty$ 时候的真空。四维动量 P 是场的动量。同时也是 Poincare 变化的生成元算符。通过 Poincare 代数生成元的对易关系，P 之间是相互对易的，意味着它们在 Heisenberg Pic 中是不随时间变化的。同时，知道在 $t\to\pm\infty$ 时，场可以按照生成湮灭算符展开，此时，能动量算符 P 也可以用产生湮灭算符展开。也就是 $P|0\rangle = 0$。于是:

$$\begin{aligned}\langle 0|\varphi(x)|0\rangle &= \langle 0|\exp(iP\cdot(x-x_0))\,\varphi(x_0)\exp(-iP\cdot(x-x_0))\,|0\rangle \\ &= \langle 0|\varphi(x_0)|0\rangle.\end{aligned} \tag{9.93}$$

实际上 $\langle 0|\varphi(x)|0\rangle$ 是一个与 x 没有关系的量，它对于所有的 x 都相等。

特别的，要求计算 $\langle 0|\varphi(x)|0\rangle|_{t\to\infty}$ 时场可以用产生湮灭算符展开。此时 $\langle 0|\varphi(x)|0\rangle = 0$。于是:

$$\langle 0|\varphi(x)|0\rangle = 0. \tag{9.94}$$

场和动量本征态归一化条件 考虑:

$$\langle p|\varphi(x)|0\rangle. \tag{9.95}$$

这个式子中的 $|0\rangle$ 就是在 $t\to+\infty$ 时定义的自由场的真空态，而 $|p\rangle$ 是单粒子能动量的本征态，满足归一化关系（我对单粒子的理解是，它是在 $t\to+\infty$ 的自由场的 Fock 空间中定义的单粒子态）：

$$\langle p|p'\rangle = (2\pi)^{d-1}2w_{\boldsymbol{p}}\delta^{(d-1)}(\vec{p}-\vec{p}'), \quad \int\frac{d^{d-1}p}{(2\pi)^{d-1}2w_{\boldsymbol{p}}}|p\rangle\langle p| = \mathbb{I}_1. \tag{9.96}$$

其中 I_1 是单粒子子空间单位算符。使用 Poincare 代数导出的场算符的平移关系:

$$\langle p|\varphi(x)|0\rangle = \langle p|\exp(+iP\cdot(x-x_0))\,\varphi(x_0)\exp(-iP\cdot(x-x_0))\,|0\rangle. \tag{9.97}$$

真空态是能动量算符的本征态，本征值是 0。考虑到这一点，

$$\langle p|\varphi(x)|0\rangle = \exp(+ip\cdot(x-x_0))\,\langle p|\varphi(x_0)|0\rangle. \tag{9.98}$$

对于上式，取 $t_{x_0}\to\infty$，此时场可以用自由场展开。要求（$|p\rangle$ 是单粒子态）：

$$\langle p|\varphi(x_0)|0\rangle = \exp(ip\cdot x_0)\,. \tag{9.99}$$

于是:

$$\langle p|\varphi(x)|0\rangle = \exp(+ip\cdot x)\times 1. \tag{9.100}$$

小结归一化条件 量子化条件要求:

$$\langle p|\varphi(x)|0\rangle = \exp(+ip\cdot x)\,,\quad \langle 0|\varphi(x)|0\rangle = 0. \tag{9.101}$$

多粒子态 多粒子态定义为:

$$|p,n\rangle,\quad P|p,n\rangle = p|p,n\rangle,\quad \langle p,n|p',n'\rangle = \delta_{n,n'}(2\pi)^{d-1}2w_{\boldsymbol{p}}\delta^{(d-1)}(\vec{p}-\vec{p}'),\quad \sum_n\int\frac{d^{d-1}p}{(2\pi)^{d-1}2w_{\boldsymbol{p}}}|p,n\rangle\langle p,n| = \mathbb{I}_{\mathcal{H}/\mathcal{I}_1}. \tag{9.102}$$

我对多粒子的理解是，在 $t\to+\infty$ 时自由场生成的 Fock 空间中定义的多粒子态。单用他们的总动量 p 是不能描述状态的，n 中包含了粒子的数量和他们的相对速度性质。所以：

$$w_{\boldsymbol{p}} = \sqrt{M^2+\boldsymbol{p}^2},\quad M\geq 2m. \tag{9.103}$$

利用 Poincare 代数平移性质:

$$\left\{\exp\left(-\tfrac{1}{i\hbar}P\cdot(x_2-x_1)\right)\phi(x_1)\exp\left(\tfrac{1}{i\hbar}P\cdot(x_2-x_1)\right) = \phi(x_2).\right. \tag{9.104}$$

多粒子态的内积性质

$$\begin{aligned}\langle p,n|\varphi(x)|0\rangle &= \langle p,n|\exp(+iP\cdot x)\,\varphi(0)\exp(-iP\cdot x)\,|0\rangle\\ &= \exp(+ip\cdot x)\,\langle p,n|\varphi(0)|0\rangle\end{aligned} \tag{9.105}$$

因为这一项没有办法在平移到自由场状态时找到普遍规律，所以就先写为这个形式。(如果简单的将多粒子态考虑为 $t\to+\infty$ 时候的粒子态，那么它和真空态与场算符的缩并一定是 0，不过实际上不是，Srednicki 40 页说了，以后再整理吧)

真实场真空内积关系 考虑真实场和真空的内积关系:

$$\langle 0|\varphi(x)\varphi(y)|0\rangle. \tag{9.106}$$

考虑由所有本征态组成的单位算符：

$$\left\{\mathbb{I} = |0\rangle\langle 0| + \int d\tilde{p}|p\rangle\langle p| + \sum\nolimits_n\int d\tilde{p}|p,n\rangle\langle p,n|.\right. \tag{9.107}$$

将它插入到内积关系中：

$$\begin{aligned}\langle 0|\varphi(x)\varphi(y)|0\rangle =&\langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle\\ &+\int d\tilde{p}\langle 0|\varphi(x)|p\rangle\langle p|\varphi(y)|0\rangle\\ &+\sum_n\int d\tilde{p}\langle 0|\varphi(x)|p,n\rangle\langle p,n|\varphi(y)|0\rangle.\end{aligned} \tag{9.108}$$

利用粒子态的内积性质:

$$\begin{cases}\langle p,n|\varphi(x)|0\rangle = \exp(+ip\cdot x)\,\langle p,n|\varphi(0)|0\rangle\\ \langle p|\varphi(x)|0\rangle = \exp(+ip\cdot x)\,,\quad \langle 0|\varphi(x)|0\rangle = 0.\end{cases} \tag{9.109}$$

$$\begin{aligned}\langle 0|\varphi(x)\varphi(y)|0\rangle &= \int d\tilde{p}e^{-ip(x-y)} + \sum_n\int d\tilde{p}e^{-ip(x-y)}\langle 0|\varphi(0)|p,n\rangle\langle p,n|\varphi(0)|0\rangle\\ &= \int d\tilde{p}e^{-ip(x-y)} + \int_{4m^2}^{\infty}ds\int d\tilde{p}e^{-i\sqrt{s+\boldsymbol{p}^2}(x^0-y^0)+i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}\rho(s).\end{aligned} \tag{9.110}$$

上面式子中的 s 表示了多粒子态粒子数以及相对运动速度。在固定 s 后，多粒子态的能量可以写为 $\sqrt{s+\boldsymbol{p}^2}$ $s \geq (2m)^2$。

真实传播子:

$$\langle 0|T\ \varphi(x)\varphi(y)|0\rangle = \theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle. \tag{9.111}$$

考虑到计算自由场传播子时候的性质以及一些积分围道的巧妙选取 (8.2.3):

$$\left\{\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} = \theta(x^0-x'^0)\int i\ d\tilde{k}e^{-ik(x-x')} + \theta(x'^0-x^0)\int i\ d\tilde{k}e^{+ik(x-x')}.\right. \tag{9.112}$$

而真实传播子可以写为:

$$\begin{aligned}\langle 0|T\ \varphi(x)\varphi(y)|0\rangle &= \theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle \\ &= \theta(x^0-y^0)\int d\tilde{p}e^{-ip(x-y)} + \theta(y^0-x^0)\int d\tilde{p}e^{-ip(y-x)} \\ &\quad + \int_{4m^2}^{\infty} ds\rho(s)\int\left(\theta(x^0-y^0)d\tilde{p}e^{-i\sqrt{s+\boldsymbol{p}^2}(x^0-y^0)+i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} + \theta(y^0-x^0)d\tilde{p}e^{-i\sqrt{s+\boldsymbol{p}^2}(y^0-x^0)+i\boldsymbol{p}(\boldsymbol{y}-\boldsymbol{x})}\right) \\ &= \frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} ds\rho(s)\frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik(x-x')}}{-k^2+s-i\epsilon}.\end{aligned} \tag{9.113}$$

考虑真实传播子的定义以及动量空间中的传播子:

$$\left\{i\langle 0|T\ \varphi(x)\varphi(y)|0\rangle = \boldsymbol{\Delta}(\boldsymbol{x}-\boldsymbol{y}) = \int\frac{d^dk}{(2\pi)^d}\boldsymbol{\Delta}(k^2)\right. \tag{9.114}$$

于是，真实传播子以及真实传播子在动量空间中的函数为:

$$\begin{aligned}\boldsymbol{\Delta}(\boldsymbol{x}-\boldsymbol{y}) &= \int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} ds\rho(s)\int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik(x-x')}}{-k^2+s-i\epsilon}, \\ \boldsymbol{\Delta}(k^2) &= \frac{e^{-ik(x-x')}}{-k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} ds\rho(s)\frac{e^{-ik(x-x')}}{-k^2+s-i\epsilon}\end{aligned} \tag{9.115}$$

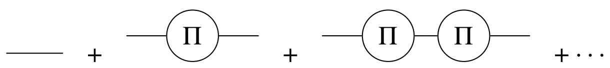
重要的性质 Lehmann-Kallen 真实传播子告诉我们真实传播子在动量空间中存在奇点 $k^2 = m^2$。并且留数是 1。

9.6 Loop Correction to Propagator

1 Particle Irreducible diagram and Real propagator 定义 Real propagator 为 (Generation Func: $Z[J] = \exp iW[J]$, $iW[J]$ is Connected Diagram without Tadpoles With two-point counterterm):

$$\frac{1}{i}\boldsymbol{\Delta}(x_1-x_2) = \langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = \frac{1}{i\delta J(x_1)}\frac{1}{i\delta J(x_2)}iW[J]|_{J=0} \tag{9.116}$$

Real Propagator 总可以表示成几何级数的形式 (观察得到)



(9.117)

中间的圈叫做"1 particle Irreducible diagram"。如图, Real Propagator 在动量空间可以写为 (它并不是 LSZ Reduction formula 得到的):

$$\begin{aligned}&\frac{1}{i}\Delta(k^2) + \frac{1}{i}\Delta(k^2)i\Pi(k^2)\frac{1}{i}\Delta(k^2) + \cdots \\ &= \frac{1}{i}\Delta(k^2)\frac{1}{1-\Pi(k^2)\Delta(k^2)}\end{aligned} \tag{9.118}$$

考虑动量空间传播子的表达式:

$$\left\{\frac{1}{i}\Delta(k^2) = \frac{-i}{-k^2+m^2-i\epsilon}\right. \tag{9.119}$$

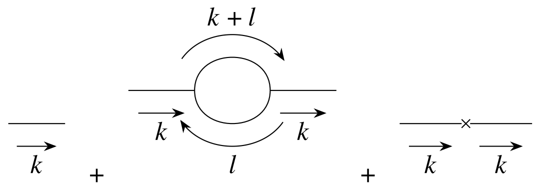
$$\text{上式} = \frac{1}{i}\frac{1}{-k^2+m^2-i\epsilon-\Pi(k^2)} \tag{9.120}$$

按照 lehmann-kallen form propagator 中的推导（这个推导一直看不明白），1-PI 图应该满足性质，这个是用来确

定 Counterterm 的。

$$\Pi(m^2) = 0 \quad \Pi'(m^2) = 0 \tag{9.121}$$

1-PI in lowest order 直接在动量图里考虑 Real Propagator。考虑 2 阶修正，Real Propagator 可以表示为:

$\frac{1}{i}\boldsymbol{\Delta}(k^2) =$  + higher order (9.122)

具体用积分表示为 (第二个图上下 propagator 交换与左右 vertex 同时交换腿的结果相同所以有一个 symmetry factor 1/2):

$$\frac{1}{i}\boldsymbol{\Delta}(k^2) = \frac{1}{i}\Delta(k^2) + \frac{1}{i}\Delta(k^2)\left[i\Pi(k^2)\right]\frac{1}{i}\Delta(k^2) + O(g^4) \tag{9.123}$$

其中

$$i\Pi(k^2) = \frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^dl}{(2\pi)^d}\Delta\left((l+k)^2\right)\Delta\left(l^2\right) - i\left(-(Z_\varphi - 1)k^2 + (Z_m - 1)m^2\right) + O(g^4) \tag{9.124}$$

有时将 Counter term 表示为:

$$A = (Z_\varphi - 1) \quad B = (Z_m - 1) \tag{9.125}$$

按照前文，1-PI diagram 的最低阶为 (**d<4 时收敛，使用维度正规化拓展到 6 维**):

$$i\Pi(k^2) = \frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^dl}{(2\pi)^d}\Delta\left((l+k)^2\right)\Delta\left(l^2\right) - i\left(-(Z_\varphi - 1)k^2 + (Z_m - 1)m^2\right) + O(g^4) \tag{9.126}$$

我们现在计算它在约束条件 $\Pi(m^2) = 0 \;\; \Pi'(m^2) = 0$ 下的值。

Feynman 参数法 Feynman Formula:

$$\begin{aligned}\frac{1}{A_1A_2\cdots A_n} &= \int dF_n(x_1A_1 + \cdots x_nA_n)^{-n} \\ \int dF_n &= \int_0^1 (n-1)!dx_1\cdots dx_n\delta(x_1 + \cdots x_n - 1) \\ \int dF_n &= 1\end{aligned} \tag{9.127}$$

对 $n = 2$ 时候，

$$\begin{aligned}\frac{1}{A_1A_2} &= \int_0^1 dx_1dx_2\delta(x_1 + x_2 - 1)\frac{1}{(x_1A_1 + x_2A_2)^2} \\ &= \int_0^1 dx_1\frac{1}{(x_1A_1 + (1-x_1)A_2)^2}\end{aligned} \tag{9.128}$$

在这个例子中 ($m^2 - i\epsilon$ 简写为 m^2):

$$\begin{aligned}\Delta\left((k+l)^2\right)\Delta\left(l^2\right) &= \frac{1}{(-l^2 + m^2)\left(-(l+k)^2 + m^2\right)} \\ &= \int_0^1 dx\frac{1}{\left[x\left(-(l+k)^2 + m^2\right) + (1-x)(-l^2 + m^2)\right]^2} \\ &= \int_0^1 dx\frac{1}{\left[-(l+kx)^2 + m^2 - x(1-x)k^2\right]^2}\end{aligned} \tag{9.129}$$

取 (D^2 中的 k^2 如果很大的，D^2 是有可能小于 0 的, 我觉得解释这一点的方法是认为 k^2 总是在 m^2 附近的):

$$q = l + kx \quad D^2 = m^2 - x(1-x)k^2 - i\epsilon \;\; (D^2 > 0) \;\; D \sim \sqrt{m^2 - x(1-x)k^2 - i\epsilon'} \tag{9.130}$$

需要考虑的积分（先只考虑 1PI 中的第一项）变为:

$$\frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^dl}{(2\pi)^d}\Delta\left((l+k)^2\right)\Delta\left(l^2\right)=\frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int\frac{d^dl}{(2\pi)^d}\int_0^1 dx\frac{1}{\left[-q^2+D^2\right]^2} \tag{9.131}$$

做积分变换:

$$dl\to dq \tag{9.132}$$

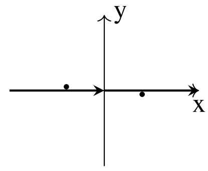
于是:

$$\begin{aligned}\text{上式}&=\frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{1}{\left[-q^2+D^2\right]^2}\\&=\frac{1}{2}(g)^2\int_0^1 dx\int\frac{d^{d-1}qdq^0}{(2\pi)^d}\frac{1}{\left[-(q^0-\sqrt{|\vec{q}|^2+D^2})(q^0+\sqrt{|\vec{q}|^2+D^2})\right]^2}\end{aligned} \tag{9.133}$$

其中:

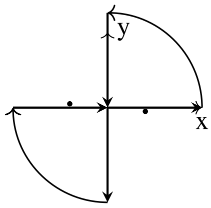
$$\sqrt{|\vec{q}|^2+D^2}\sim\sqrt{|\vec{q}|^2+m^2-x(1-x)k^2}-i\epsilon' \tag{9.134}$$

Wick Rotation 如果先对 q^0 积分，它的积分围道是:



(9.135)

现在把积分扩展到复平面，首先考虑如果进行如下围道的积分:



(9.136)

因为它不经过任何奇点，所以它的积分值是 0，而圆周部分的积分，如果在无穷远处也是 0 的话，就可以把 $-\infty\to+\infty$ 的积分转化为 $-i\infty\to+i\infty$ 的积分。

$$\int_{-\infty}^{+\infty}dq^0\sim=\int_{-i\infty}^{+i\infty}dq^0\sim \tag{9.137}$$

为了方便，一般定义新变量:

$$q^0=iq^d\to\int_{-i\infty}^{+i\infty}dq^0=i\int_{-\infty}^{+\infty}dq^d\quad;\quad q^2=(q^0)^2-|\vec{q}|^2=-|q^d|^2-|\vec{q}|^2=-|q^{(d)}|^2 \tag{9.138}$$

将这个操作用到上面的积分（把 $q^{(d)}$ 理解为 minkowski 空间中的向量, 后面就不写这个上指标了。）:

$$\begin{aligned}\text{上式}&=\frac{1}{2}(ig)^2\left(\frac{1}{i}\right)^2\int_0^1 dx\int\frac{d^{(d-1)}qdq^0}{(2\pi)^d}\frac{1}{\left[-q^2+D^2\right]^2}\\&=i\frac{1}{2}(g)^2\int_0^1 dx\int\frac{d^{(d-1)}qdq^d}{(2\pi)^d}\frac{1}{\left[(q^{(d)})^2+D^2\right]^2}\\&=i\frac{1}{2}(g)^2\int_0^1 dx\int\frac{d^dq^{(d)}}{(2\pi)^d}\frac{1}{\left[(q^{(d)})^2+D^2\right]^2}\end{aligned} \tag{9.139}$$

利用 Gamma Function 积分 积分公式写为 (式子中所有向量都是 Minkowski 空间的):

$$\int\frac{d^dq}{(2\pi)^d}\frac{(q^2)^a}{(q^2+D)^b}=\frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)}D^{-(b-a-d/2)} \tag{9.140}$$

对于所要考虑的积分，$a=0,\ b=2$, 重新定义了一下 D, $D^2 \to D$

$$
\begin{aligned}
\text{上式} &= i\frac{1}{2}(g)^2\int_0^1 dx\int\frac{d^d q^{(d)}}{(2\pi)^d}\frac{1}{\left[(q^{(d)})^2+D^2\right]^2} \\
&= i\frac{1}{2}(g)^2\int_0^1 dx\frac{\Gamma(2-\frac{1}{2}d)\Gamma(\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(2)\Gamma(\frac{1}{2}d)}D^{-(2-d/2)} \\
D &= m^2-x(1-x)k^2
\end{aligned}
\tag{9.141}
$$

根据维度分析的结论，在自然单位制下 $[g]=[m]^{d-\frac{1}{2}(d-2)n}$。对于这个问题，$n=3$，于是 $[g]=[m]^{3-\frac{1}{2}d}$ 取 $d=6-\epsilon$。$[g]=[m]^{\epsilon/2}$。(这种 $\epsilon\to 0$ 的方法是 Dimensional Regularization) 为了让 g 是一个没有量纲的数：

$$
g \to g\tilde{\mu}^{\epsilon/2} \quad [\tilde{\mu}]=[m]
\tag{9.142}
$$

于是

$$
\begin{aligned}
\text{上式} &= i\frac{1}{2}(g\tilde{\mu}^{\epsilon/2})^2\int_0^1 dx\frac{\Gamma(-1+\frac{\epsilon}{2})}{(4\pi)^3(4\pi)^{-\frac{\epsilon}{2}}}D^{1-\frac{\epsilon}{2}} \\
&= i\frac{1}{2}(g)^2\frac{\Gamma(-1+\frac{\epsilon}{2})}{(4\pi)^3}\int_0^1 dxD\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2}
\end{aligned}
\tag{9.143}
$$

Gamma Function 具有性质 (非负整数 n 和小量 x):

$$
\begin{cases}
\Gamma(n+1) &= n! \\
\Gamma(n+\frac{1}{2}) &= \frac{(2n)!}{n!2^n}\sqrt{\pi} \\
\Gamma(-n+x) &= \frac{(-1)^n}{n!}\left[\frac{1}{x}-\gamma+\sum_{k=1}^n k^{-1}+O(x)\right] \\
\gamma &= 0.5772
\end{cases}
\tag{9.144}
$$

按照这个性质进行展开 ($A^{\frac{\epsilon}{2}}=\exp[\frac{\epsilon}{2}\ln(A)]\sim 1+\frac{\epsilon}{2}\ln(A)$):

$$
\begin{aligned}
\text{上式} &= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}(\frac{2}{\epsilon}-\gamma+1)\int_0^1 dxD\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)+O(\epsilon^2)\right) \\
&= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx\left((\frac{2}{\epsilon}+1)D-\gamma D+D\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)+O(\epsilon)\right) \\
&= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx\left((\frac{2}{\epsilon}+1)D+D\ln\left(\frac{4\pi\tilde{\mu}^2}{e^\gamma D}\right)+O(\epsilon)\right)
\end{aligned}
\tag{9.145}
$$

考虑到 D 的表达式，并定义新的变量:

$$
\begin{cases}
D &= m^2-x(1-x)k^2 \to \int dxD = m^2-\frac{1}{2}k^2+\frac{1}{3}k^2 = m^2-\frac{1}{6}k^2 \\
\mu &= \sqrt{4\pi}e^{\gamma/2}\tilde{\mu}
\end{cases}
\tag{9.146}
$$

$$
\text{上式} = -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\left((\frac{2}{\epsilon}+1)(m^2-\frac{1}{6}k^2)+\int_0^1 dxD\ln\left(\frac{\mu^2}{D}\right)\right)
\tag{9.147}
$$

1-PI in lowest order.

$$
\begin{aligned}
i\Pi(k^2) &= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\left((\frac{2}{\epsilon}+1)(m^2-\frac{1}{6}k^2)+\int_0^1 dxD\ln\left(\frac{\mu^2}{D}\right)\right)-i\left(-(Z_\varphi-1)k^2+(Z_m-1)m^2\right) \\
&= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\left((\frac{2}{\epsilon}+1)(m^2-\frac{1}{6}k^2)+\int_0^1 dxD\ln\left(\frac{\mu^2}{D}\right)\right)-i\left(-Ak^2+Bm^2\right) \\
&= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\left((\frac{2}{\epsilon}+1)(m^2-\frac{1}{6}k^2)+\int_0^1 dxD\ln\left(\frac{m^2}{D}\right)+\int_0^1 dx2D\ln\left(\frac{\mu}{m}\right)\right)-i\left(-Ak^2+Bm^2\right) \\
&= -i\frac{1}{2}\frac{g^2}{(4\pi)^3}\left((\frac{2}{\epsilon}+1)(m^2-\frac{1}{6}k^2)+\int_0^1 dxD\ln\left(\frac{m^2}{D}\right)+2(m^2-\frac{1}{6}k^2)\ln\left(\frac{\mu}{m}\right)\right)-i\left(-Ak^2+Bm^2\right) \\
&= i\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dxD\ln\left(\frac{D}{m^2}\right)+i\left(\frac{1}{6}\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]+A\right)k^2-i\left(\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]+B\right)m^2
\end{aligned}
\tag{9.148}
$$

取系数 A,B 分别为:

$$\begin{cases} A = -\frac{1}{6}\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]-\frac{1}{6}\frac{g^2}{(4\pi)^3}k_A \\ B = -\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]-\frac{g^2}{(4\pi)^3}k_B \end{cases} \tag{9.149}$$

$$i\Pi(k^2) = i\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx D\ln\left(\frac{D}{m^2}\right)+i\frac{g^2}{(4\pi)^3}\left(-\frac{1}{6}k_A k^2+k_B m^2\right) \tag{9.150}$$

将系数写为:

$$\begin{cases}\alpha = \frac{g^2}{(4\pi)^3}\end{cases} \tag{9.151}$$

$$i\Pi(k^2) = i\frac{1}{2}\alpha\int_0^1 dx D\ln\left(\frac{D}{m^2}\right)+i\alpha\left(-\frac{1}{6}k_A k^2+k_B m^2\right) \tag{9.152}$$

为了满足条件:$\Pi(m^2)=0\quad\Pi'(m^2)=0$。直接计算发现可以这样给定系数 k_A, k_B

$$i\Pi(k^2) = i\frac{1}{2}\alpha\int_0^1 dx D\ln\left(\frac{D}{D_0}\right)-i\alpha\left(-\frac{1}{12}k^2+\frac{1}{12}m^2\right) \tag{9.153}$$

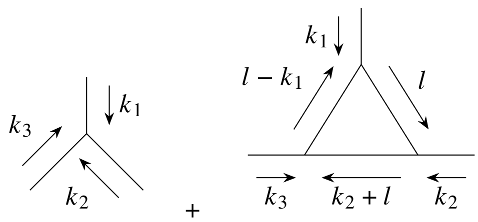
其中

$$\begin{cases}D = m^2-x(1-x)k^2 \quad D_0 = m^2-x(1-x)m^2 \quad \alpha = \frac{g^2}{(4\pi)^3}\end{cases} \tag{9.154}$$

9.7 Loop Correction to the vertex

这里也是 φ^3 场，直接在动量空间中考虑。对于 Exact 3-point vertex function $iV(k_1,k_2,k_3)$. 它定义为 sum of one-particle irreducible diagrams with 3 external lines。直接考虑 Exact 3-point vertex function 在动量表示中的最低阶修正:

$$iV(k_1,k_2,k_3) =$$



$$+O(g^5) \tag{9.155}$$

用式子表达上面的图为

$$iV(k_1,k_2,k_3) = iZ_g g+(ig)^3(\frac{1}{i})^3\int\frac{d^d l}{(2\pi)^d}\Delta\left((l-k_1)^2\right)\Delta\left((l+k_2)^2\right)\Delta\left(l^2\right) \tag{9.156}$$

Feynman 参数法 考虑积分中的传播子相乘:

$$\begin{aligned}\Delta\left((l-k_1)^2\right)\Delta\left((l+k_2)^2\right)\Delta\left(l^2\right) &= \frac{1}{-(l-k_1)^2+m^2-i\epsilon}\,\frac{1}{-(l+k_2)^2+m^2-i\epsilon}\,\frac{1}{-(l)^2+m^2-i\epsilon} \\ &= \int dF_3\left[-x_1(l-k_1)^2-x_2(l+k_2)^2-x_3 l^2+m^2\right]^{-3}\end{aligned} \tag{9.157}$$

其中

$$\int dF_3 = dx_1 dx_2 dx_3\delta(x_1+x_2+x_3-1) \tag{9.158}$$

于是:

$$
\begin{aligned}
-x_1(l-k_1)^2-x_2(l+k_2)^2-x_3l^2+m^2 &= -x_1(l^2+k_1^2-2lk_1)-x_2(l^2+k_2^2+2lk_2)-(1-x_1-x_2)l^2+m^2 \\
&= -l^2+2l(x_1k_1-x_2k_2)-x_1k_1^2-x_2k_2^2+m^2 \\
&= -(l-(x_1k_1-x_2k_2))^2+(x_1k_1-x_2k_2)^2-x_1k_1^2-x_2k_2^2+m^2 \\
&= -(l-(x_1k_1-x_2k_2))^2-x_1(1-x_1)k_1^2-x_2(1-x_2)k_2^2-2x_1x_2k_1k_2+m^2 \\
&= -q^2+D
\end{aligned}
\tag{9.159}
$$

$$
\begin{aligned}
D &= -x_1(1-x_1)k_1^2-x_2(1-x_2)k_2^2-2x_1x_2k_1k_2+m^2 \\
&= -x_1(1-x_1-x_2)k_1^2-x_2(1-x_2-x_1)k_2^2-x_1x_2k_1^2-x_1x_2k_2^2-2x_1x_2k_1k_2+m^2 \\
&= -x_1(1-x_1-x_2)k_1^2-x_2(1-x_2-x_1)k_2^2-x_1x_2(k_1+k_2)^2+m^2 \\
&= -x_1x_3k_1^2-x_2x_3k_2^2-x_1x_2k_3^2+m^2 \\
q &= l-(x_1k_1-x_2k_2)
\end{aligned}
\tag{9.160}
$$

经过 Feynman 参数法后，Vertex Function 修正为 (**d<6 时才收敛，采用维度正规化逼近 6 维**):

$$
\begin{aligned}
iV(k_1,k_2,k_3) &= iZ_g g+(ig)^3(\frac{1}{i})^3\int dF_3\frac{d^dl}{(2\pi)^d}\frac{1}{\left[-q^2+D\right]^3} \\
&= iZ_g g+(g)^3\int dF_3\frac{d^dq}{(2\pi)^d}\frac{1}{\left[-q^2+D\right]^3}
\end{aligned}
\tag{9.161}
$$

Wick Rotation 如果考虑到虚数部分，

$$
D=-x_1x_3k_1^2-x_2x_3k_2^2-x_1x_2k_3^2+m^2-i\epsilon \tag{9.162}
$$

和上一节一样，可以使用 Wick Rotation。

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}dq \to \int_{-i\infty}^{+i\infty}dq \\
&q^0=iq^d \to \int_{-i\infty}^{+i\infty}dq^0=i\int_{-\infty}^{+\infty}dq^d \quad q^2=(q^0)^2-|\vec{q}|^2=-|q^{(d)}|^2
\end{aligned}
\tag{9.163}
$$

使用 Wick Rotation 之后，积分变为 Euclidian 空间中的积分:

$$
iV(k_1,k_2,k_3)=iZ_g g+i(g)^3\int dF_3\frac{d^dq}{(2\pi)^d}\frac{1}{\left[q^2+D\right]^3} \tag{9.164}
$$

Gamma Function 积分 Gamma Function 积分公式写为 (其中所有向量都是 Minkowski 空间中的):

$$
\int\frac{d^dq}{(2\pi)^d}\frac{(q^2)^a}{(q^2+D)^b}=\frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)}D^{-(b-a-d/2)} \tag{9.165}
$$

在这个问题中，$a=0, b=3$.

$$
\begin{aligned}
\frac{d^dq}{(2\pi)^d}\frac{1}{\left[q^2+D\right]^3} &= \frac{\Gamma(3-\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(3)}D^{-(3-\frac{d}{2})} \\
&= \frac{\Gamma(3-\frac{1}{2}d)}{2(4\pi)^{d/2}}D^{-(3-\frac{d}{2})}
\end{aligned}
\tag{9.166}
$$

在 $d=6-\epsilon$ 处考虑。ϵ 是小量。并且改写 $g\to g\tilde{\mu}^{\epsilon/2}$，改写后的 g 是一个无量纲的量。此时，Vertex Function 写为:

$$\begin{aligned}
iV(k_1,k_2,k_3) &= iZ_g g\tilde{\mu}^{\epsilon/2} + i(g)^3(\tilde{\mu})^{3\epsilon/2}\int dF_3 \frac{\Gamma(3-\frac{1}{2}d)}{2(4\pi)^{d/2}} D^{-(3-\frac{d}{2})} \\
&= iZ_g g\tilde{\mu}^{\epsilon/2} + i(g)^3(\tilde{\mu})^{3\epsilon/2}\int dF_3 \frac{\Gamma(\frac{\epsilon}{2})}{2(4\pi)^3(4\pi)^{-\frac{\epsilon}{2}}} D^{-(3-\frac{d}{2})} \\
&= iZ_g g\tilde{\mu}^{\epsilon/2} + i(g)^3(\tilde{\mu})^{3\epsilon/2}\int dF_3 \frac{\Gamma(\frac{\epsilon}{2})}{2(4\pi)^3(4\pi)^{-\frac{\epsilon}{2}}} D^{-\frac{\epsilon}{2}} \\
&= iZ_g g\tilde{\mu}^{\epsilon/2} + i(g)^3\int dF_3 \frac{\Gamma(\frac{\epsilon}{2})}{2(4\pi)^3}\left(\frac{4\pi\tilde{\mu}^3}{D}\right)^{\frac{\epsilon}{2}}
\end{aligned} \tag{9.167}$$

Gamma Function 具有性质 (非负整数 n 和小量 x):

$$\begin{cases}
\Gamma(n+1) &= n! \\
\Gamma(n+\frac{1}{2}) &= \frac{(2n)!}{n!2^n}\sqrt{\pi} \\
\Gamma(-n+x) &= \frac{(-1)^n}{n!}\left[\frac{1}{x}-\gamma+\sum_{k=1}^{n}k^{-1}+O(x)\right] \\
\gamma &= 0.5772
\end{cases} \tag{9.168}$$

$$\Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon}-\gamma+O(\epsilon) \tag{9.169}$$

于是 ($A^x\sim\exp(x\ln(A))\sim 1+x\ln(A)$):

$$\begin{aligned}
iV(k_1,k_2,k_3) &= iZ_g g\tilde{\mu}^{\epsilon/2} + i(g)^3\int dF_3 \frac{\Gamma(\frac{\epsilon}{2})}{2(4\pi)^3}\left(\frac{4\pi\tilde{\mu}^3}{D}\right)^{\frac{\epsilon}{2}} \\
&= iZ_g g\tilde{\mu}^{\epsilon/2} + i(g)^3(\frac{2}{\epsilon}-\gamma+O(\epsilon))\int dF_3 \frac{1}{2(4\pi)^3}\left(\frac{4\pi\tilde{\mu}^3}{D}\right)^{\frac{\epsilon}{2}} \\
&= ig\tilde{\mu}^{\epsilon/2}\left(Z_g+\frac{1}{2}g^2(\frac{2}{\epsilon}-\gamma+O(\epsilon))\int dF_3\frac{1}{(4\pi)^3}\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\frac{\epsilon}{2}}\right) \\
&= ig\tilde{\mu}^{\epsilon/2}\left(Z_g+\frac{1}{2}(g)^2(\frac{2}{\epsilon}-\gamma+O(\epsilon))\int dF_3\frac{1}{(4\pi)^3}\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)+O(\epsilon^2)\right)\right) \\
&= ig\tilde{\mu}^{\epsilon/2}\left(Z_g+\frac{1}{2}\frac{(g)^2}{(4\pi)^3}(\frac{2}{\epsilon}-\int dF_3\gamma+O(\epsilon))\left(1+\int dF_3\frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right)+O(\epsilon^2)\right)\right)
\end{aligned} \tag{9.170}$$

$$iV(k_1,k_2,k_3) = ig\tilde{\mu}^{\epsilon/2}\left(Z_g+\frac{1}{2}\frac{(g)^2}{(4\pi)^3}\left(\frac{2}{\epsilon}+\int dF_3\ln\left(\frac{4\pi\tilde{\mu}^2}{e^{\gamma}D}\right)+O(\epsilon)\right)\right) \tag{9.171}$$

Counter Term 取 Counter term 以及系数 g 中含有的质量项分别为:

$$Z_g = 1+C \qquad \frac{4\pi\tilde{\mu}^2}{e^{\gamma}} = \mu^2 \tag{9.172}$$

$$\begin{aligned}
\frac{iV(k_1,k_2,k_3)}{ig\tilde{\mu}^{\epsilon/2}} &= 1+C+\frac{1}{2}\frac{(g)^2}{(4\pi)^3}\left(\frac{2}{\epsilon}+\int dF_3\ln\left(\frac{4\pi\tilde{\mu}^2}{e^{\gamma}D}\right)\right) \\
&= 1+C+\frac{(g)^2}{(4\pi)^3}\left(\frac{1}{\epsilon}+\ln(\frac{\mu}{m})\right)-\frac{1}{2}\frac{(g)^2}{(4\pi)^3}\int dF_3\ln(\frac{D}{m^2})
\end{aligned} \tag{9.173}$$

取 C 为:

$$C = -\frac{(g)^2}{(4\pi)^3}\left(\frac{1}{\epsilon}+\ln(\frac{\mu}{m})+K_C\right) \tag{9.174}$$

于是:

$$\frac{iV(k_1,k_2,k_3)}{ig\tilde{\mu}^{\epsilon/2}} = 1-\frac{(g)^2}{(4\pi)^3}K_C-\frac{1}{2}\frac{(g)^2}{(4\pi)^3}\int dF_3\ln(\frac{D}{m^2}) \tag{9.175}$$

并且发现式子与 g 中的质量项 μ 没有关系了。

归一化条件写为

$$iV_3(0,0,0) = ig\tilde{\mu}^{\epsilon/2} \tag{9.176}$$

注意到:

$$D = -x_1x_3k_1^2 - x_2x_3k_2^2 - x_1x_2k_3^2 + m^2 \to \underline{k_1 = k_2 = k_3 = 0} \to D = m^2 \tag{9.177}$$

可以取

$$K_C = 0 \tag{9.178}$$

于是:

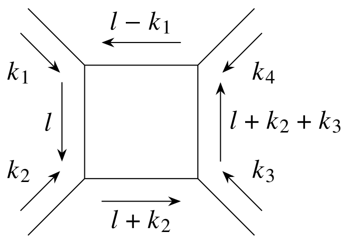
$$\frac{iV(k_1,k_2,k_3)}{ig\tilde{\mu}^{\epsilon/2}} = 1 - \frac{1}{2}\frac{(g)^2}{(4\pi)^3}\int dF_3 \ln(\frac{D}{m^2}) \tag{9.179}$$

9.8 Other 1PI vertices

在计算圈图对 vertices 修正时，方法是先考虑有三个外腿的 1PI diagram。在计算 Feynman 图时不考虑外腿。同样的方法可以用来计算圈图对 n-point vertex 的修正。其中，n-point vertex 定义为 $iV_n(k_1,\cdots,k_n)$。

将刚才提到的计算方法用到计算 4-point Vertex $iV_4(k_1,k_2,k_3,k_4)$ 上。

画出 1-PI，4 外腿图 如果仅考虑最低阶修正，图画为:



(9.180)

另外两种不同的构型对 4-point vertex 也有贡献。

计算 4-point vertex 按照第一段说的方法，直接写出:

$$iV_4 = (i)^4 g^4 \int \frac{d^6 l}{(2\pi)^6}\frac{1}{i^4}\Delta\left((l-k_1)^2\right)\Delta\left((l+k_2)^2\right)\Delta\left((l+k_2+k_3)^2\right)\Delta\left(l^2\right) + (k_3\leftrightarrow k_2) + (k_3 \leftrightarrow k_4) + O(g^6) \tag{9.181}$$

Feynman 参数法:

$$\begin{aligned} &\Delta\left((l-k_1)^2\right)\Delta\left((l+k_2)^2\right)\Delta\left((l+k_2+k_3)^2\right)\Delta\left(l^2\right) \\ &= \int dF_4 \left[-x_1(l-k_1)^2 - x_2(l+k_2)^2 - x_3(l+k_2+k_3)^2 - x_4 l^2 + m^2\right]^{-4} \\ &= \int dF_4 \left[q^2 + D_{1234}\right]^{-4}. \end{aligned} \tag{9.182}$$

其中 (我直接抄书了，好懒啊我):

$$\begin{cases} D_{1234} = -x_1x_4k_1^2 - x_2x_4k_2^2 - x_2x_3k_3^2 - x_1x_3k_4^2 - x_1x_2(k_1+k_2)^2 - x_3x_4(k_3+k_4)^2 + m^2 \\ q = l - x_1k_1 + x_2k_2 + x_3(k_2+k_3) \end{cases} \tag{9.183}$$

先用 Wick Rotation 再用 Gamma Function 积分 (自己没算，好懒啊我):

$$\left\{\int \frac{d^6 q}{(2\pi)^6}\frac{1}{(-q^2+D)^4} = \frac{i}{6(4\pi)^3 D}\right. \tag{9.184}$$

总的来说:

$$V_4 = \frac{g^4}{6(4\pi)^3}\int dF_4\left(\frac{1}{D_{1234}} + \frac{1}{D_{1324}} + \frac{1}{D_{1423}}\right) + O(g^6) \tag{9.185}$$

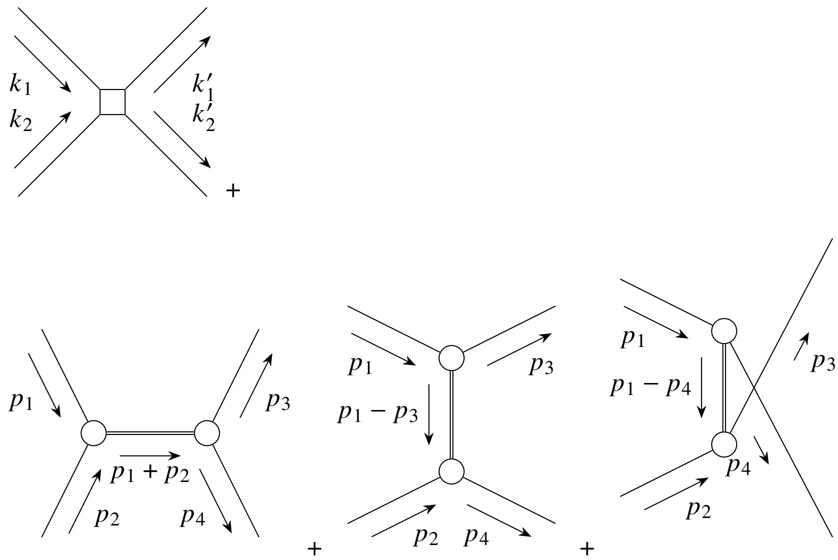
9.9 skeleton expansion

对于有 E 个外腿的图。可以通过 Skeleton expansion 的方法直接求出散射振幅。

- 根据归一化条件 $\Pi(p^2) = 0$ $\Pi'(p^2) = 0$ 以及 $V_3(0,0,0) = g$ 逐级确定 Counter term。
- 计算出真实传播子以及真实 3 点 vertex iV_3
- 首先计算 n point vertex。$(4 \le n \le E)$。(画出有 n 个外腿的 1-PI 图，计算他们对 n-point Vertex 的贡献。注意，画 n-point 图时不要考虑圈图对传播子的修正以及圈图对 vertex 的修正。)
- 计算 n point vertex 时，将传播子当作真实传播子算，将 vertex 当作真实 vertex 算。
- 画有 E 个外腿的树图，其中含有 Vertex $iV_3 \cdot iV_n$。并且他们之间的传播子是真实传播子。

2 particle elastic scattering at one loop 之前在介绍用 LSZ reduction formula 计算散射振幅并且引入了动量空间的 Feynman 图时，计算了 φ^3 弹性散射（出射粒子数目 = 入射粒子数目）的最低阶树图修正。现在想用 Skeleton 展开的方法求它的低阶修正（圈图修正）。

按照 Skelton expansion 的方法，现在有 $E = 4$ 个外腿，需要考虑的 n-point vertex 有 iV_3, iV_4。按照 Skeleton expansion 的方法，直接画出对 2 粒子弹性散射有贡献的图。



(9.186)

按照计算树图散射振幅的 Symmetry factor 性质，他们的 Symmetry factor 都是 1。按照 Skeleton expansion 的方法，图中的方框 vertex 和圆圈 vertex 分别是 4-point vertex iV_4，以及 3-point vertex iV_3。(上面有的写成 p 了，不想改了，反正他们都是 k 实际上，我好懒)

利用 Mandelstam 变量:

$$\begin{cases} s = (k_1 + k_2)^2 = (k'_1 + k'_2)^2, \\ t = (k_1 - k'_1)^2 = (k_2 - k'_2)^2, \\ u = (k_1 - k'_2)^2 = (k_2 - k'_1)^2, \\ s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 = 4m^2 \quad (\varphi^3 \ \text{theory}). \end{cases} \tag{9.187}$$

之前计算过的 n-point Vertex (n=3 or 4)，以及真实传播子 $\boldsymbol{\Delta}$:

$$\begin{cases} \boldsymbol{\Delta}(s) & = \frac{1}{-s+m^2-i\epsilon-\Pi(s)}, \\ i\Pi(s) & = i\frac{1}{2}\alpha \int_0^1 dxD \ln\left(\frac{D}{D_0}\right) - i\alpha\left(-\frac{1}{12}s + \frac{1}{12}m^2\right), \\ \frac{iV(k_1,k_2,k_3)}{ig\tilde{\mu}^{\epsilon/2}} & = 1 - \frac{1}{2}\frac{(g)^2}{(4\pi)^3}\int dF_3 \ln\left(\frac{D_3}{m^2}\right), \\ V_4 & = \frac{g^4}{6(4\pi)^3}\int dF_4 \left(\frac{1}{D_{1234}} + \frac{1}{D_{1324}} + \frac{1}{D_{1423}}\right) + O(g^6). \end{cases} \tag{9.188}$$

其中（推导 n-point vertex 时将他们的动量都当做了入射动量，不过这里入射和出射是分开的，需要注意）:

$$
\begin{cases}
D = m^2 - x(1-x)s \quad D_0 = m^2 - x(1-x)m^2 \quad \alpha = \frac{g^2}{(4\pi)^3}, \\
D_3 = -x_1x_3k_1^2 - x_2x_3k_2^2 - x_1x_2k_3^2 + m^2, \\
D_{1234} = -x_1x_4k_1^2 - x_2x_4k_2^2 - x_2x_3k_3^2 - x_1x_3k_4^2 - x_1x_2(k_1+k_2)^2 - x_3x_4(k_3+k_4)^2 + m^2
\end{cases}
\tag{9.189}
$$

观察发现，V_3 如果有两个入射，一个出射，并且入射的都是 on-shell 的。那么它的值之和出射的量有关。对于上面的 D_3，取 $k_1+k_2=-k_3$。

$$
\begin{aligned}
D_3 &= -x_1x_3k_1^2 - x_2x_3k_2^2 - x_1x_2k_3^2 + m^2, \\
&= (1 - x_3(x_1+x_2))\, m^2 - x_1x_2k_3^2, \\
&= (1 - x_3(x_1+x_2))\, m^2 - x_1x_2s^2, \\
&= D_3(s).
\end{aligned}
\tag{9.190}
$$

同样的，对于 4-point vertex, 如果入射的全都是 on-shell(满足能动量方程)，那么它也可以写为简单的形式:

$$
D_{1234} = -x_1x_2s - x_3x_4t + (1 - (x_1+x_2)(x_3+x_4))\, m^2. \tag{9.191}
$$

同理:

$$
\begin{aligned}
D_{1324} &= -x_1x_2t - x_3x_4u + (1 - (x_1+x_2)(x_3+x_4))\, m^2, \\
D_{1423} &= -x_1x_2u - x_3x_4s + (1 - (x_1+x_2)(x_3+x_4))\, m^2.
\end{aligned}
\tag{9.192}
$$

之后将这三个 4-point vertex 中的 D 函数写为。

$$
D_4(s,t) = -x_1x_2s - x_3x_4t + (1 - (x_1+x_2)(x_3+x_4))\, m^2. \tag{9.193}
$$

并且，使用 Feynman 参数法的积分是:

$$
\begin{aligned}
\int dF_n &= (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(x_1 + \cdots x_{n-1} - 1) f(x), \\
&= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots x_{n-2}} dx_{n-1} f(x)|_{x_n = 1-x_1\cdots-x_{n-1}}.
\end{aligned}
\tag{9.194}
$$

直接写出散射振幅:

$$
i\mathcal{T}_{1-loop} = (iV_3(s))^2 \frac{1}{i}\mathbf{\Delta}(s) + (iV_3(t))^2 \frac{1}{i}\mathbf{\Delta}(t) + (iV_3(u))^2 \frac{1}{i}\mathbf{\Delta}(u) + iV_4(s,t,u). \tag{9.195}
$$

散射振幅不容易直接计算，取高能近似来计算。具体来说就是取各个出入射动量比质量 m 大很多。类似于将静质量设置为 0。(计算直接抄的 srednicki，我好懒)

$$
\begin{aligned}
\Pi(s) &= -\frac{1}{2}\alpha s \int_0^1 dx x(1-x) \left[\ln\left(\frac{-s}{m^2}\right) + \ln\left(\frac{x(1-x)}{1-x(1-x)}\right)\right] + \frac{1}{12}\alpha s, \\
&= -\frac{1}{12}\alpha s \left[\ln(\frac{-s}{m^2}) + 3 - \pi\sqrt{3}\right].
\end{aligned}
\tag{9.196}
$$

$$
\begin{aligned}
\mathbf{\Delta}(s) &= \frac{1}{-s - \Pi(s)}, \\
&= -\frac{1}{s}\left(1 + \frac{1}{12}\alpha\left[\ln\left(-s/m^2\right) + 3 - \pi\sqrt{3}\right]\right) + O(\alpha^2).
\end{aligned}
\tag{9.197}
$$

$$
\begin{aligned}
V_3(s)/g &= 1 - \frac{1}{2}\alpha \int dF_3 \left[\ln(\frac{-s}{m^2}) + \ln(x_1x_2)\right], \\
&= 1 - \frac{1}{2}\left[\ln(\frac{-s}{m^2}) - 3\right].
\end{aligned}
\tag{9.198}
$$

$$
\begin{aligned}
\int \frac{dF_4}{D_4(s,t)} &= -\frac{3}{s+t}\left(\pi^2 + [\ln(s/t)]^2\right), \\
&= \frac{3}{u}\left(\pi^2 + [\ln(s/t)]^2\right).
\end{aligned}
\tag{9.199}
$$

最终散射振幅的计算结果是:

$$\mathcal{T}_{1-loop} = g^2\left[F(s,t,u)+F(t,u,s)+F(u,s,t)\right], \tag{9.200}$$

其中

$$F(s,t,u) = -\frac{1}{s}\left(1-\frac{11}{12}\alpha\left[\ln(-s/m^2)+c\right]-\frac{1}{2}\alpha\left[\ln(t/u)\right]^2\right), \tag{9.201}$$

其中 $c=(6\pi^2+\pi\sqrt{3}-39)/11=2.33s$ 。

在上面的计算中，用到了性质 $s>0,\ t<0,\ u<0$。

9.10 Modified minimal subtraction

重新考虑确定 Counter term 前的 Real propagator 中的 1-PI 项

$$i\Pi(k^2) = i\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dxD\ln\left(\frac{D}{m^2}\right)+i\left(\frac{1}{6}\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]+A\right)k^2-i\left(\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]+B\right)m^2. \tag{9.202}$$

其中

$$D=-x(1-x)k^2+m^2. \tag{9.203}$$

考虑到 Real propagator 的性质，用来确定 Counter term 的条件一般是 $\Pi(m^2)=0,\quad \Pi'(m^2)=0$。但是在 $m=0$ 时，关于一阶导数的条件是没办法满足的，因为计算 1-PI 项时会遇到 ill-defined 积分。这个现象的深层原因是用 Lehmann-Callan Form Real-propagator 确定真实传播子时如果 $m=0$, $4m^2$ 和 m^2 会接在一起。最终导致奇点并不是 $k^2=m^2$, 并且留数并不是 1。

如果不采用原来的正规化条件，在确定 Counter term 后，认为真实传播子有了奇点 m_{ph}^2, 同时在这个地方的留数是 R。现在重新考虑用 LSZ reduction formula 计算 S matrix 以及引出散射振幅的过程。在推导 LSZ reduction formula 时，用的是没有 Counter-term 的场。但是却把它用在了有 Counter-term 的情况下。从物理直觉上，为了让 LSZ formula 能够生成和之前一样的散射振幅形式需要对他进行修改。

- 为了让出入射粒子保持静质量是 m_{ph} 的 on-shell 状态，$\partial^2+m^2\to\partial^2+m_{ph}^2$。
- 为了让 Real propagator 在奇点的留数是 1，在计算 n-point function 时 $\varphi\to R^{-1/2}\varphi$。

将这些操作叫做 modified minimal subtraction scheme。在这个 scheme 下，散射振幅的变化是:

- 每个外腿不再是 1，而是 $R^{1/2}$ ，这是因为计算场缩并时引入了 $R^{-1/2}$，结合奇点的留数 R，最终得到。
- 内部的传播子仍然是 $\frac{1}{i}\frac{1}{-k^2+m^2}$, 这里的质量是 Lagrangian 中的质量。
- 每一个 Vertex 仍然是 $iZ_g g$，并且也要考虑 2-point vertex 引起的 counter term。

$\overline{MS}$ Scheme for φ^3 到现在，依然没有说 Counterterm 应该怎么选取。选取规则就是，只要一个发散项，不要常数项。对于 φ^3theory

$$i\Pi(k^2) = i\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dxD\ln\left(\frac{D}{m^2}\right)+i\left(\frac{1}{6}\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]+A\right)k^2-i\left(\frac{g^2}{(4\pi)^3}\left[\frac{1}{\epsilon}+\frac{1}{2}+\ln(\frac{\mu}{m})\right]+B\right)m^2. \tag{9.204}$$

选取最少的项抵消发散:

$$\begin{aligned} A &= -\frac{1}{6}\alpha\frac{1}{\epsilon}, \\ B &= -\alpha\frac{1}{\epsilon}. \end{aligned} \tag{9.205}$$

同样的，对于 3-point Vertex

$$\frac{iV(k_1,k_2,k_3)}{ig\tilde{\mu}^{\epsilon/2}} = 1+C+\frac{(g)^2}{(4\pi)^3}\left(\frac{1}{\epsilon}+\ln(\frac{\mu}{m})\right)-\frac{1}{2}\frac{(g)^2}{(4\pi)^3}\int dF_3\ln(\frac{D}{m^2}). \tag{9.206}$$

也是选择最少的抵消项

$$C=-\alpha\frac{1}{\epsilon}. \tag{9.207}$$

在用了这些抵消项之后，1-PI diagram 以及 3-Vertex 写为:

$$\begin{aligned}\Pi_{\bar{MS}}(k^2) &= -\frac{1}{12}\alpha(-k^2+6m^2)+\frac{1}{2}\alpha\int_0^1 dxD\ln\left(D/\mu^2\right)+O(\alpha^2),\\ V_{3,\overline{MS}} &= g\left[1-\frac{1}{2}\int dF_3\ln\left(D_3/\mu^2\right)+O(\alpha^2)\right].\end{aligned} \tag{9.208}$$

其中 $D=-x(1-x)k^2+m^2$, $D_3=xyk_1^2+yzk_2^2+zxk_3^2+m^2$。

物理质量以及留数 物理质量指的是 Real propagator 出现奇点的地方。

$$-m_{ph}^2+m^2-\Pi_{\overline{MS}}(m_{ph}^2)=0. \tag{9.209}$$

近似解写为

$$m_{ph}^2=m^2-\Pi_{\overline{MS}}(m^2)+O(\alpha^2). \tag{9.210}$$

结果是

$$m_{ph}^2=m^2\left[1+\frac{5}{12}\alpha\left(\ln(\mu^2/m^2)+c'\right)+O(\alpha^2)\right], \tag{9.211}$$

其中 $c'=(34-3\pi\sqrt{3})/15=1.18$。

对于真实传播子，它在 $k^2=m_{ph}^2$ 奇点处的留数是

$$R^{-1}=1+\Pi'(m_{ph}^2). \tag{9.212}$$

计算得到

$$R^{-1}=1+\frac{1}{12}\alpha\left(\ln(\mu^2/m^2)+c''\right)+O(\alpha^2)., \tag{9.213}$$

其中 $c''=(17-3\pi\sqrt{3})/3=0.23$。

散射振幅 在这个 Scheme 下计算散射振幅得到

$$\mathcal{T}=R^2\mathcal{T}_0\left[1-\frac{11}{12}\alpha\left(\ln(s/\mu^2)+O(m^0)\right)+O(\alpha^2)\right]. \tag{9.214}$$

再考虑解决红外发散时引入的系数

$$|\mathcal{T}|^2_{obs}=|\mathcal{T}|^2\left[1+\frac{1}{3}\alpha\left(\ln(\delta^2 s/m^2)+O(m^0)\right)+O(\alpha^2)\right]. \tag{9.215}$$

其中 $\mathcal{T}_0$ 是没有任何修正的树图散射振幅。

红外发散 这一部分没有看的很明白，但是核心思想是，当粒子的质量很小时，有时候出射粒子并不是一个，但是探测器却把他们当做了一个，导致的结果是（其中 δ 表示了仪器的分辨率）

$$|\mathcal{T}|^2_{obs}=|\mathcal{T}_0|^2\left[1-\alpha\left(\frac{3}{2}\ln(s/m^2)+\frac{1}{3}\ln(1/\delta^2)+O(m^0)\right)+O(\alpha^2)\right]. \tag{9.216}$$

观测量不含修正 μ 需要保证观测量 $|\mathcal{T}|^2_{obs}$ 不随 μ 变化

$$\ln|\mathcal{T}|^2_{obs}=C_1+2\ln\alpha+3\alpha(\ln\mu+C_2)+O(\alpha^2). \tag{9.217}$$

其中 C_1 以及 C_2 都与 μ 以及 α 独立。需要观测量与 μ 无关，于是对它求导。

$$\begin{aligned}0 &= \frac{d}{d\ln\mu}\ln|\mathcal{T}|^2_{obs},\\ &= \frac{2}{\alpha}\frac{d\alpha}{d\ln\mu}+3\alpha+O(\alpha^2).\end{aligned} \tag{9.218}$$

最终得到:

$$\begin{aligned}\frac{d\alpha}{d\ln\mu} &= -\frac{3}{2}\alpha^2 + O(\alpha^3),\\ \alpha(\mu_2) &= \frac{\alpha(\mu_1)}{1+\frac{3}{2}\alpha(\mu_1)\ln(\mu_2/\mu_1)}.\end{aligned} \tag{9.219}$$

类似的，物理质量也不能和 μ 有关系

$$\frac{dm}{d\ln\mu} = (-\frac{5}{12}\alpha + O(\alpha^2))m. \tag{9.220}$$

第十章 Spin 1/2

10.1 Lorentz Transformation

Classical 群论中提到 Lorentz Transformation 有 Left-handed spinor representation

$$\left\{\Lambda = \exp\left(t^i \tilde{J}_i + s^i \tilde{K}_i\right) \mapsto R_L(\Lambda) = R_{(1/2,0)}(\Lambda) = \exp\left[\tfrac{1}{2}(-s^i - it^i)\sigma_i\right].\right. \tag{10.1}$$

考虑在表示空间中定义的二维向量场 $\varphi_a(x)$。由量子 Poincare 变换部分讲到的，对于经典场，Lorentz 不变的意思是

$$\begin{cases} S = \int d^4x \mathcal{L}\left(\varphi_a(x), \partial^\mu \varphi_a(x)\right), \\ S' = \int d^4x' \mathcal{L}\left(\phi'_a(x'), \partial^\mu \phi'_a \circ (x')\right). \end{cases} \tag{10.2}$$

其中，场满足变换条件

$$\phi'_a(x') = R(\Lambda)_a^{\ b} \phi_b(x),\ \partial'^\mu \phi'_a(x') = \Lambda^\mu_{\ \nu} R(\Lambda)_a^{\ b} \partial^\nu \phi_b(x). \tag{10.3}$$

其中的 R 是 Lorentz 群的表示。这里应该是 Left-handed spinor representation。

Lorentz Invariant Lagrangian 由 Group Theory, Spinor Representation 中有 Lorentz 不变的项 Weyl Mass 以及按照 Lorentz Transformation 变化的项

$$\begin{cases} \left(\chi_L^c\right)^\dagger \psi_L, \quad \left(\chi_R^c\right)^\dagger \psi_R, \\ \chi_R^\dagger \bar{\sigma}_\mu \psi_R, \quad \chi_L^\dagger \sigma_\mu \psi_L, \\ \sigma_\mu = (\mathbb{I}, \sigma_i), \quad \bar{\sigma}_\mu = (\mathbb{I}, -\sigma_i). \end{cases} \tag{10.4}$$

为了和场论书中的符号相同，一般写为

$$\begin{cases} \left(\chi_L^c\right)^\dagger \psi_L, \quad \left(\chi_R^c\right)^\dagger \psi_R, \\ \chi_R^\dagger \sigma_\mu \psi_R, \quad \chi_L^\dagger \bar{\sigma}_\mu \psi_L, \\ \sigma^\mu = (\mathbb{I}, \sigma_i), \quad \bar{\sigma}^\mu = (\mathbb{I}, -\sigma_i). \end{cases} \tag{10.5}$$

构建 Lorentz Invariant Lagrangian 为

$$\mathcal{L}_D = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m\left(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R\right). \tag{10.6}$$

其中，m 是实数，后一项质量项是实数。前两个动量项也是实数，用第一项来说（用到了分部积分）

$$\left(i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R\right)^\dagger = -i\partial_\mu \psi_R^\dagger \sigma^\mu \psi_R = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R. \tag{10.7}$$

Dirac Spinor 有一种简单的写法

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{10.8}$$

这种写法下，质量项是

$$\mathcal{L}_{D,m} = -m\psi^\dagger \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \psi. \tag{10.9}$$

动能项是

$$\mathcal{L}_{D,k} = i\psi^\dagger \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \partial_\mu \psi. \tag{10.10}$$

方便的写法是定义 Gamma Matrix 为

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^0\gamma^\mu = \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}. \tag{10.11}$$

进一步定义

$$\bar{\psi} \equiv \psi^\dagger \gamma^0. \tag{10.12}$$

于是，Lagrangian 可以写为整齐的形式

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi. \tag{10.13}$$

Quantum 同样，由量子 Poincare 变换部分讲到的，对于量子场，量子 Lorentz 变换意义在于。

$$\begin{cases} U^{-1}(\Lambda)\varphi_a(x)U(\Lambda) = \varphi'_a(x) \\ U^{-1}(\Lambda)\partial^\mu \varphi_a(x)U(\Lambda) = \partial^\mu \varphi'_a \circ (x) \end{cases} \tag{10.14}$$

结合场满足的变换条件

$$\begin{aligned} U^{-1}(\Lambda)\varphi_a(x')U(\Lambda) =& R_L(\Lambda)_a^{\ b}\varphi_b(\Lambda^{-1}x') \\ U^{-1}(\Lambda)\partial^\mu \varphi_a(x')U(\Lambda) =& \Lambda^\mu_\nu R_L(\Lambda)_a^{\ b}\partial^\nu \varphi_b \circ (x) \end{aligned} \tag{10.15}$$

第十一章 矢量场

11.1 有质量矢量场的正则量子化

首先电磁场的能量动量张量是这个:

$$F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}=-F^{\nu\mu} \tag{11.1}$$

然后有质量矢量场的 Lagrangian 是这个:

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_{\mu}A^{\mu} \tag{11.2}$$

对应的 Euler-Lagrange 方程是这个:

$$\begin{aligned}\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})}&=\frac{\partial\mathcal{L}}{\partial_{A_{\nu}}}\\-\partial_{\mu}\partial^{\mu}A^{\nu}+\partial^{\nu}\partial_{\mu}A^{\mu}&=m^2A^{\nu}\end{aligned} \tag{11.3}$$

然后这个方程有一个名字叫做 Proca 方程。一般写成这个形式.

$$\partial_{\mu}F^{\mu\nu}+m^2A^{\nu}=0 \tag{11.4}$$

这个方程有一个直接的要求叫做Lorentz 条件，推导是对 Proca 方程去一个散度。

$$\partial_{\mu}\partial_{\nu}F^{\mu\nu}+m^2\partial_{\nu}A^{\nu}=0 \tag{11.5}$$

注意到 F 他是反对称的, 所以第一项就直接是 0 了。于是就有了 Lorentz 条件:

$$\partial_{\nu}A^{\nu}=0 \tag{11.6}$$

有了这个条件之后, Proca 方程就变成饿了 Klein-Gordan 方程。

$$(\partial^2+m^2)A^{\mu}(x)=0 \tag{11.7}$$

生成的动量算符是这样定义的:

$$\pi_{\mu}=\frac{\partial\mathcal{L}}{\partial(\partial^0A^{\mu})}=-F_{0\mu} \tag{11.8}$$

一般写成协变矢量:

$$\pi^{\mu}=\frac{\partial\mathcal{L}}{\partial(\partial^0A^{\mu})}=-F_0^{\ \mu} \tag{11.9}$$

由于 F 是反对称的,00 分量就是 0，于是其实只有空间分量的指标是有用的。

$$\vec{\pi}=-\dot{\vec{A}}-\vec{\nabla}A_0 \tag{11.10}$$

而场的量子化条件是这样的:

$$[A^i(\vec{x},t),\pi_j(\vec{y},t)]=i\delta^i_{\ j}\delta(\vec{x}-\vec{y})\quad[A^i(\vec{x},t),A^j(\vec{y},t)]=[\pi_i(\vec{x},t),\pi_j(\vec{y},t)]=0 \tag{11.11}$$

11.1.1 在极化矢量座作为基底下的展开 (实际上就是线性代数...)

考虑 4 个基底。在下面的式子里面, μ 代表着向量中的不同项。$\sigma=0,1,2,3$。代表着有四个基底。他们叫做极化矢量。

$$e^{\mu}(p,\sigma) \tag{11.12}$$

正交关系是这样的:

$$e^{\mu}(p,\sigma)e_{\mu}(p,\sigma')=g_{\sigma\sigma'} \tag{11.13}$$

完备性关系是这样的:

$$\Sigma_{\sigma=0}^{3}g_{\sigma\sigma}e_{\mu}(p,\sigma)e_{\nu}(p,\sigma)=g_{\mu\nu} \tag{11.14}$$

一个矢量的展开是这样的：

$$\Sigma_\sigma\left(g_{\sigma\sigma}V^\nu e_\nu(p,\sigma)\right)\, e_\mu(p,\sigma)=V_\mu \tag{11.15}$$

什么叫做完备性，就是任意一个矢量都可以展开到这个基底上，展开系数是基底和向量的内积。但是上面的式子里面多了一个 $g_{\sigma\sigma}$ 我只是想说这一项和正交关系是自洽的。因为如果把上面的矢量展开式左右同时用 $e^\mu(p,\sigma')$ 做内积的话，得到的结果是：

$$\Sigma_\sigma g_{\sigma\sigma}V^\nu e_\nu(p,\sigma)e_\mu(p,\sigma)e^\mu(p,\sigma')=V_\mu e^\mu(p,\sigma') \tag{11.16}$$

利用前面的正交性关系:

$$\Sigma_\sigma g_{\sigma\sigma}V^\nu e_\nu(p,\sigma)e_\mu(p,\sigma)e^\mu(p,\sigma')=\Sigma_\sigma g_{\sigma\sigma}g_{\sigma\sigma'}V^\nu e_\nu(p,\sigma) \tag{11.17}$$

当然如果是 Minkovski 空间的话，上面的式子当然就是 $V^\nu e_\nu(p,\sigma')$ 了。所以是自洽的。

然后我们具体考虑选取的四个极化矢量。为什么要满足四维横向条件因为波函数满足 Klein-Gordan 方程，所以解的形式是平面波。然后想让这个解满足 Lorentz 条件。于是就直接索性要求平面波展开的基底满足四维横向条件了。

先选择两个满足四维横向条件 ($e^\mu p_\mu=0$) 与三维横向条件 (三维部分和动量的三维部分点乘是 1) 的类时极化矢量。

$$|p_T|=\sqrt{(p^1)^2+(p^2)^2} \tag{11.18}$$

$$\begin{aligned}\vec{e}(p,1)&=\frac{1}{|\vec{p}||p_T|}(p^1p^3,p^2p^3,-|p_T|^2)\\ e^\mu(p,1)&=(0,\vec{e}(p,1))\end{aligned} \tag{11.19}$$

$$\begin{aligned}\vec{e}(p,2)&=\frac{1}{|p_T|}(-p^2,p^1,0)\\ e^\mu(p,2)&=(0,\vec{e}(p,2))\end{aligned} \tag{11.20}$$

然后再找一个满足四维横向条件与三维纵向条件的类空极化矢量：

$$e^\mu(p,3)=(\frac{|p|}{m},\frac{p^0\vec{p}}{m|p|}) \tag{11.21}$$

然后再找一个不满足四维横向条件，的类时极化矢量:

$$e^\mu(p,0)=\frac{p^\mu}{m}=\frac{1}{m}(p^0,\vec{p}) \tag{11.22}$$

反正这样构造出来的向量是满足正交关系:

$$e^\mu(p,\sigma)e_\mu(p,\sigma')=g_{\sigma\sigma'} \tag{11.23}$$

也是满足归一化关系的

$$\Sigma_{\sigma=0}^3 g_{\sigma\sigma}e_\mu(p,\sigma)e_\nu(p,\sigma)=g_{\mu\nu} \tag{11.24}$$

但是如果我们不要第零个极化矢量，因为他不满足四维横向条件，那么:

$$\Sigma_{\sigma=1}^3 e_\mu(p,\sigma)e_\nu(p,\sigma)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2} \tag{11.25}$$

于是我们构造三个极化矢量(坐极化-1, 右极化 +1, 纵向极化 0)

$$\begin{aligned}\epsilon^\mu(p,\pm)&=\frac{1}{\sqrt{2}}(\mp e^\mu(p,1)-ie^\mu(p,2))\\ e^\mu(p,0)&=e^\mu(p,3)\end{aligned} \tag{11.26}$$

他们满足正交条件和极化求和条件

$$\begin{aligned}\epsilon^*_\mu(p,\lambda)\epsilon^\mu(p,\lambda')&=-\delta_{\lambda\lambda'}\\ \Sigma_\lambda\epsilon^*_\mu(p,\lambda)\epsilon_\nu(p,\lambda)&=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\end{aligned} \tag{11.27}$$

直接吧这个定义的三个极化矢量带入矩阵方程就可以发现他恰好是螺旋度的本征向量。什么是螺旋度, 就是自旋角动量算符在动量方向上面的投影。具体来讲:($\lambda = -\ 0\ +$)

$$(\hat{p}\vec{\mathcal{J}})^{\mu}{}_{\nu}\epsilon^{\nu}(p,\lambda) = \lambda\epsilon^{\mu}(p,\lambda) \tag{11.28}$$

所以说极化矢量恰好是螺旋度的本征向量。

于是量子矢量场的平面波展开是:

$$A^{\mu}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Sigma_{\lambda=\pm,0}(\epsilon^{\mu}(p,\lambda)a_{p,\lambda}e^{-ipx} + \epsilon^{\mu *}(p,\lambda)a^{\dagger}_{p,\lambda}e^{ipx}) \tag{11.29}$$

如果把这个平面波展开直接带入到量子 Lorentz 变换生成元算符和场算符之间的对易关系的式子中，就可以得到湮灭算符和生成元算符的对易关系

$$[a_{p,\lambda}, \hat{p}\cdot\vec{J}] = \lambda a_{p,\lambda} \tag{11.30}$$

对他去共轭就得到了产生算符的对易关系，写开是这样的:

$$(\hat{p}\cdot\vec{J})a^{\dagger}_{p,\lambda} = a^{\dagger}_{p,\lambda}(\hat{p}\cdot\vec{J}) + \lambda a^{\dagger}_{p,\lambda} \tag{11.31}$$

在前面有一个小节叫做 (量子场的 Lorentz 变换)。得到了量子矢量场算符和量子 lorentz 变换生成元算符之间的对易关系

$$[A^{\mu}(x), \vec{J}] = \vec{L}A^{\mu}(x) + (\vec{\mathcal{J}})^{\mu}{}_{\nu}A^{\nu}(x) \tag{11.32}$$

基态时这样定义的:

$$a_{p,\lambda}|0\rangle = J|0\rangle = 0 \tag{11.33}$$

动量为 p, 螺旋度为 λ 的粒子态这样定义:

$$|p,\lambda\rangle = \sqrt{2E_p}a^{\dagger}_{p,\lambda}|0\rangle \tag{11.34}$$

这个粒子态有这样的性质—他的螺旋度恰好是 λ:

$$(\hat{p}\cdot\vec{J})|p,\lambda\rangle = \lambda|p,\lambda\rangle \tag{11.35}$$

如果要满足正则量子化条件, 得到生成湮灭算符之间的对易关系

$$\begin{gathered}[a_{p,\lambda}, a^{\dagger}_{q,\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(p-q) \\ [a_{p,\lambda}, a_{q,\lambda'}] = [a^{\dagger}_{p,\lambda}, a^{\dagger}_{q,\lambda'}] = 0\end{gathered} \tag{11.36}$$

系统总的哈密顿量 Hamiltonian, 后面的那一项一般叫做:

$$H = \Sigma_{\lambda=\pm,0}\int\frac{d^3\vec{p}}{(2\pi)^3}E_p a^{\dagger}_{p,\lambda}a_{p,\lambda} \ + \ (2\pi)^3\delta^{(3)}(0)\int\frac{d^3p}{(2\pi)^3}\frac{3}{2}E_p \tag{11.37}$$

11.2 没有质量的矢量场的正则量子化

我们先考虑规范对称性。电磁场, 或者说无质量矢量场的 Lagrangian 和 Euler-Lagrange 方程分别是

$$\begin{gathered}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ \partial_{\mu}F^{\mu\nu} = 0\end{gathered} \tag{11.38}$$

无质量矢量场是有规范对称性的。

如果定义一个新的矢量场，他和原来矢量场的关系是:

$$A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu}\chi(x) \tag{11.39}$$

于是用上面的式子完全可以用 $A'(x)$ 来表示 $A(x)$。于是可以用 A' 来表示拉格朗日量。然后发现拉格朗日量的形式是完全相同的。对应的 Euler-Lagrange 方程也是完全相同的。所以 A' 满足的方程和 A 是完全相同的。我们说这个叫做规范对称性。

然后我们用一个辅助场 (Auxiliary field) ξ 人为实现 Lorentz 规范。$\partial_{\mu}A^{\mu} = 0$ 考虑了辅助场之后的 Lagrangian 是这

样的:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{11.40}$$

这样对于辅助场的 Euler-Lagrange 方程就会得到一个 Lorentz 规范。所以说这个辅助场是 Gauge-Fixing Term. 按照这个 Lagrangian, 可以得到动量密度算符:(这里已经带入了 + - - -) 的度规。

$$\pi_i = -F_{0i} \quad \pi_0 = -\frac{1}{\xi}\partial_\mu A^\mu \tag{11.41}$$

然后我们假设规定量子化条件

$$[A^\mu(x,t), \pi_\nu(y,t)] = i\delta^\mu_\nu \delta^{(3)}(\vec{x}-\vec{y}) \;\; [A^\mu(x,t), A^\nu(y,t)] = [\pi_\mu(x,t), \pi_\nu(y,t)] = 0 \tag{11.42}$$

但是这样简单的规范固定和量子化程序不是自洽的。出现问题的点在于 (这个 π_0 就是用拉格朗日量算出来的动量密度算符)

$$[A^0(x,t), \partial_\mu A^\mu(y,t)] = [A^0(x,t), -\xi\pi_0(y,t)] = -\xi[A^0(x,t), \pi_0(y,t)] = -i\xi\delta^{(3)}(\vec{x}-\vec{y}) \tag{11.43}$$

这个不是 0。也就是说第一项里面的 $\partial_\mu A^\mu$ 不能是 0。然后引入一个 Feynman 规范，他是要求 $\xi = 1$ 于是得到的 Euler-Lagrange 方程叫做 D'alembert 方程:

$$\partial^2 A^\mu(x) = 0 \tag{11.44}$$

于是假设场算符有平面波解的形式:

$$A^\mu(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Sigma^3_{\sigma=0} e^\mu(p,\sigma)(b_{p,\sigma}e^{-ipx} + b^\dagger_{p,\sigma}e^{ipx}) \tag{11.45}$$

这样的话 Hamiltonian 算符是这样的:

$$H = \int \frac{d^3p}{(2\pi)^3}E_p(-b^\dagger_{p,0}b_{p,0} + \Sigma^3_{\sigma=1}b^\dagger_{p,\sigma}b_{p,\sigma}) + (2\pi)^3\delta^{(3)}(0)\int\frac{d^3p}{(2\pi)^3}2E_p \tag{11.46}$$

11.2.1 平面波展开

因为 $e(p,0)$ 和 $e(p,3)$ 的定义式中出现了静质 m。于是他们在定义光子时是不对的。于是要这样定义.

$$e^\mu(p,3) = (0, \frac{\vec{p}}{|p|}) \;\; e^\mu(p,0) = n^\mu = (1,0,0,0) \tag{11.47}$$

这样，完备性和正交性都可以保证

$$\Sigma^3_{\sigma=0} g_{\sigma\sigma} e_\mu(p,\sigma)e_\nu(p,\sigma) = g_{\mu\nu} \;\; e_\mu(p,\sigma)e^\mu(p,\sigma') = g_{\sigma\sigma'} \tag{11.48}$$

但是 0 和 3 (新定义的两个) 都不满足四维横向条件。如果把她们提了，完备性条件就变成了 (也许应该叫做极化求和条件):(我也不知道这时候还能不能叫做完备条件)。ϵ 的定义和有质量矢量场是一样的。

$$\Sigma^2_{\sigma=1} e_\mu(p,\sigma)e_\nu(p,\sigma) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} = \Sigma_{\lambda=\pm}\epsilon^*_\mu(p,\lambda)\epsilon_\nu(p,\lambda) \tag{11.49}$$

到此为止，再采用场算符的量子化条件，得到生成和湮灭算符之间的对易关系:

$$[b_{p,\sigma}, b^\dagger_{q,\sigma'}] = -(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(p-q) \;\; [b_{p,\sigma}, b_{q,\sigma'}] = [b^\dagger_{p,\sigma}, b^\dagger_{q,\sigma'}] = 0 \tag{11.50}$$

顺理成章的定义粒子态:

$$|p,\sigma\rangle\sqrt{2E_p}b^\dagger_{p,\sigma}|0\rangle \tag{11.51}$$

然后要指出一个 $\sigma = 0$ 时候会面临的问题。(虽然前面不是已经把 0 和 3 给淘汰掉了, 为什么还要继续搞)。首先粒子态的内积是这样的:

$$\langle q,\sigma'|p,\sigma\rangle = -2E_p(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\vec{p}-\vec{q}) \tag{11.52}$$

如果两个 σ 都是 1 的时候，会发现 g 就变成了 1。这个时候会发现他们的内积是小于 0 的。这个事情放到能量的测量值上会有更大的问题:

$$\langle p,0|p,0\rangle = (E_{vac} + E_p)\langle p,0|p,0\rangle < 0 \tag{11.53}$$

负能量问题就会出现。
为了规避这个事情，引入弱 Lorentz 条件。

$$\langle\Psi|\partial_\mu A^\mu(x)|\Psi\rangle = 0 \tag{11.54}$$

有一个更加强大的条件叫做Gupta-Bleuler条件。满足他就一定满足弱 Lorentz 条件

$$\partial_\mu A^{\mu(+)}(x)|\Psi\rangle = 0 \tag{11.55}$$

实际上把正能量解的平面波展开式直接带入。因为 $e^\mu(p,\sigma)$ 只在 $\sigma = 0$ 或者 3 的时候不满足四维横向条件。所以求四维三度的时候也只有这两个 σ 会有贡献。Gupta-Bleuler 条件意味着

$$(b_{p,0} - b_{p,3})|\Psi\rangle = 0 \tag{11.56}$$

于是:

$$\begin{aligned}\langle\Psi|H|\Psi\rangle &= \int \frac{d^3p}{(2\pi)^3} E_p\langle\Psi| - b^\dagger_{p,0}b_{p,0} + \Sigma^3_{\sigma=1}b^\dagger_{p,\sigma}b_{p,\sigma}|\Psi\rangle + E_{vac}\langle|\Psi|\Psi\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} E_p\Sigma^2_{\sigma=1}\langle\Psi|b^\dagger_{p,\sigma}b_{p,\sigma}|\Psi\rangle + E_{vac}\langle\Psi|\Psi\rangle\end{aligned} \tag{11.57}$$

动量也是一样的:

$$\begin{aligned}\langle\Psi|P|\Psi\rangle &= \int \frac{d^3p}{(2\pi)^3} p\langle\Psi| - b^\dagger_{p,0}b_{p,0} + \Sigma^3_{\sigma=1}b^\dagger_{p,\sigma}b_{p,\sigma}|\Psi\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} p\Sigma^2_{\sigma=1}\langle\Psi|b^\dagger_{p,\sigma}b_{p,\sigma}|\Psi\rangle\end{aligned} \tag{11.58}$$

也就是说只要态矢量满足了 Gupta-Bleuler 条件，那么就不会有能量有负号这样的事情。实际上 $|p,0\rangle$ 和 $|p,3\rangle$ 都是不满足的。

然后用 $\sigma = 1,2$ 的两个状态定义新的产生湮灭算符:

$$a_{p,\pm} = \frac{1}{\sqrt{2}}(\mp b_{p,1} + ib_{p,2}) \tag{11.59}$$

这里引入了 $\pi/2$ 的相位, 他和光学里面的圆偏振有很大的关系。但是具体的联系还不是很清楚。然后和有质量的矢量场一样的。定义极化矢量 $\epsilon^\mu(p,\lambda)$ 然后就有一些性质，首先是产生湮灭算符之间的对易关系:

$$\begin{aligned}&[a_{p,\lambda}, a^\dagger_{q,\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(p-q) \quad [a_{p,\lambda}, a_{q,\lambda'}] = [a^\dagger_{p,\lambda}, a^\dagger_{q,\lambda'}] = 0 \ \lambda,\lambda' = \pm \\ &[a_{p,\lambda}, b_{q,\sigma}] = [b_{q,\sigma}, a^\dagger_{q,\lambda}] = [a_{p,\lambda}, b_{q,\sigma}] = [a^\dagger_{p,\lambda}, b^\dagger_{q,\sigma}] = 0 \quad \lambda = \pm, \sigma = 0,3 \\ &[b_{p,\sigma}, b^\dagger_{q,\sigma'}] = -(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(p-q) \quad [b_{p,\sigma}, b_{q,\sigma'}] = [b^\dagger_{p,\sigma}, b^\dagger_{q,\sigma'}] = 0 \quad \sigma = 0,3\end{aligned} \tag{11.60}$$

同时有这样的关系

$$\begin{aligned}\Sigma^2_{\sigma=1}e^\mu(p,\sigma)b_{p,\sigma} &= \Sigma_{\lambda=\pm}\epsilon^\mu(p,\lambda)a_{p,\lambda} \\ \Sigma^2_{\sigma=1}e^\mu(p,\sigma)b^\dagger_{p,\sigma} &= \Sigma_{\lambda=\pm}\epsilon^{\mu *}(p,\lambda)a^\dagger_{p,\lambda}\end{aligned} \tag{11.61}$$

于是量子化后的量子矢量场是这样的:

$$\begin{aligned}A^\mu(x) = &\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Sigma_{\sigma=0,3}e^\mu(p,\sigma)(b_{p,\sigma}e^{-ipx} + b^\dagger_{p,\sigma}e^{ipx}) \\ &+ \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Sigma_{\lambda=\pm}\left(\epsilon^\mu(p,\lambda)a_{p,\lambda}e^{-ipx} + \epsilon^{\mu *}(p,\lambda)a^\dagger_{p,\lambda}e^{ipx}\right)\end{aligned} \tag{11.62}$$

书上说一行对应于非物理极化态，第二行对应于两种物理的圆极化态

第十二章 旋量场

Gamma 矩阵是满足反对易关系的一组矩阵:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{12.1}$$

如果从这个定义入手，当 $\mu = \nu$ 时，发现:

$$(\gamma^0)^2 = I \quad (\gamma^i)^2 = -I \tag{12.2}$$

上面的式子说明 γ^0 的本征值是 1 和 -1。γ^i 的本征值是 i 和 -i。这些值恰好是对角化矩阵之后对角元素的值。于是, 我们说γ^0 是厄米矩阵。γ^i 是反 Hermite 矩阵。也就是：

$$(\gamma^0)^\dagger = \gamma^0 \quad (\gamma^i)^\dagger = -\gamma^i \tag{12.3}$$

于是:

$$(\gamma^0)^\dagger\gamma^0 = (\gamma^0)^2 = 1 \quad (\gamma^i)^\dagger\gamma^i = -(\gamma^i)^2 = 1 \tag{12.4}$$

也就是说 gamma 矩阵都是幺正矩阵 (unitary)

然后定义一个反对称的 S 矩阵:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{12.5}$$

按照这个定义，S 是满足 Lorentz 代数生成元的算符的代数关系 (Lorentz 代数关系)。于是他可以是 Lorentz 群表示的生成元矩阵。用 S 生成的表示叫做旋量表示

这组表示生成Lorentz 群的旋量表示的固有保时向的有限变换。有限变换是这样表示的:$Y = -\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$

$$D(\Lambda) = exp(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}) = e^Y \tag{12.6}$$

然后说一个数学性质。定义一个对易算符:

$$[B, A^{(n)}] = [[B, A^{(n-1)}], A] \tag{12.7}$$

并且要求:

$$[B, A^0] = B \tag{12.8}$$

这样就有一个性质：

$$BA^k = \Sigma_{n=0}^{k}\frac{k!}{(k-n)!n!}A^{k-n}[B, A^{(n)}] \tag{12.9}$$

于是:

$$e^{-A}Be^A = \Sigma_{n=0}^{+\infty}\frac{1}{n!}[B, A^{(n)}] \tag{12.10}$$

考虑一个对易子:

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu \tag{12.11}$$

其中的 $\mathcal{J}$ 是 Lorentz 矢量表示的生成元算符。为什么会出现他呢，因为他也是用度规来定义的，所以在推导过程中出现他没有什么问题。

然后再考虑对易式子: 其中 $X = -\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}$

$$[\gamma^\mu, Y^{(n)}] = (X^n)^\mu{}_\nu\gamma^\nu \tag{12.12}$$

考虑 Lorentz 群旋量表示下的有限变换 $D(\Lambda) = e^Y$。于是

$$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \lambda^\mu{}_\nu\gamma^\nu \quad D^{-1}(\Lambda)S^{\mu\nu}D(\Lambda) = \lambda^\mu{}_\rho\lambda^\nu{}_\sigma S^{\rho\sigma} \tag{12.13}$$

定义

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{12.14}$$

有时候也写成这样:

$$\gamma^5 = -\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{12.15}$$

其中

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} -1 \ (\mu,\nu,\rho,\sigma)\text{是}(0,1,2,3)\text{的偶置换} \\ +1 \ (\mu,\nu,\rho,\sigma)\text{是}(0,1,2,3)\text{的奇置换} \\ 0 \ \text{其他} \end{cases} \tag{12.16}$$

说 ϵ 是 (0,4) 型张量。满足 Lorentz 变换

$$\epsilon_{\mu\nu\rho\sigma} \to \epsilon_{\alpha\beta\gamma\delta}(\Lambda^{-1})^\alpha_{\ \mu}(\Lambda^{-1})^\beta_{\ \nu}(\Lambda^{-1})^\gamma_{\ \rho}(\Lambda^{-1})^\delta_{\ \sigma} \tag{12.17}$$

下面说几个 γ^5 的性质。

$$(\gamma^5)^2 = 1 \tag{12.18}$$

$$(\gamma^5)^\dagger = \gamma^5 \tag{12.19}$$

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{12.20}$$

$$D^{-1}(\Lambda)\gamma^\mu\gamma^5 D(\Lambda) = D^{-1}(\Lambda)\gamma^\mu D(\Lambda)D^{-1}(\Lambda)\gamma^5 D(\Lambda) = \Lambda^\mu_{\ \nu}\gamma^\nu\gamma^5 \tag{12.21}$$

一般情况下，是不使用 S 矩阵这个符号，而使用的是 $\sigma^{\mu\nu} = 2S^{\mu\nu}$
用 γ^0 定义变换

$$D(P) = \xi\gamma^0 \tag{12.22}$$

需要要求 $|\xi|^2 = 1$。于是这个变换有这些性质:

$$D^\dagger(P)D(P) = \xi^*(\gamma^0)^\dagger\xi\gamma^0 = I \tag{12.23}$$

于是我们说:

$$D^{-1}(P) = \xi^*\gamma^0 = D^\dagger(P) \tag{12.24}$$

这样的变换和宇称变换也有联系。

$$D^{-1}(P)\gamma^\mu D(P) = \mathcal{P}^\mu_{\ \nu}\gamma^\nu \tag{12.25}$$

这个式子说明了 γ^μ 是极矢量

$$D^{-1}(P)\gamma^5 D(P) = -\gamma^5 \tag{12.26}$$

这个式子说明了 γ^5 是赝标量

$$D^{-1}(P)\gamma^\mu\gamma^5 D(P) = -\mathcal{P}^\mu_{\ \nu}\gamma^\nu\gamma^5 \tag{12.27}$$

这个式子说明了 $\gamma^\mu\gamma^5$ 是轴矢量

$$D^{-1}(P)\sigma^{\mu\nu}D(P) = \mathcal{P}^\mu_{\ \alpha}\mathcal{P}^\nu_{\ \beta}\sigma^{\alpha\beta} \tag{12.28}$$

然后我们考虑场算符的固有保时向 Lorentz 变换。被他作用的态矢量是 Dirac 旋量 (spinor)。

$$\psi'_a = D_{a,b}(\Lambda)\psi_b \quad \text{这个式子可以理解为经典场} \tag{12.29}$$

我们要求场算符的变化有这个性质:

$$\psi'_a(x') = U^{-1}(\Lambda)\psi_a(x')U(\Lambda) = D_{ab}(\Lambda)\psi_b(x) \tag{12.30}$$

其中的变换算符是:

$$D_{ab}(\Lambda) = \delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab} \tag{12.31}$$

于是上面的要求就写为了:

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x) \tag{12.32}$$

把旋量场的 Lorentz 变换带入，之后再对最右边的那一项做 Taylor 展开。这个等式可以得到性质:

$$[\psi(x), J^{\mu\nu}] = (L^{\mu\nu} + S^{\mu\nu})\psi(x) \tag{12.33}$$

这个式子说明旋量场是有自旋的。
现在定义一个特殊的 γ 矩阵。这个叫做 Weyl 表象。

$$\gamma^0 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \quad \gamma^i = \begin{bmatrix} & \sigma^i \\ -\sigma^i & \end{bmatrix} \tag{12.34}$$

可以定义一个元素是矩阵的四维矢量:

$$\sigma^\mu = (1, \sigma) \quad \bar{\sigma}^\mu = (1, -\sigma) \tag{12.35}$$

这样的定义下 γ 矩阵就是:

$$\gamma^\mu = \begin{bmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{bmatrix} \tag{12.36}$$

S 矩阵就是:

$$S^{\mu\nu} = \frac{i}{4}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix} \tag{12.37}$$

对他乘以一个 Levi-civita 符号，可以得到一个三维矢量

$$S^i = \frac{1}{2}\begin{bmatrix} \sigma^i & \\ & \sigma^i \end{bmatrix} \tag{12.38}$$

于是

$$S^2 = \frac{1}{2}(\frac{1}{2} + 1)I \tag{12.39}$$

我们说他描述的是自旋为 $\frac{1}{2}$ 的粒子。S 矩阵并不是厄米矩阵。这个事情导致。Lorentz 旋量表示的有限变换算符的逆不能简单地写成他的共轭。

$$D^\dagger(\Lambda) = (exp(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}))^\dagger = exp(\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger) \neq D^{-1}(\Lambda) \tag{12.40}$$

然后回到 S 矩阵。他有一个性质

$$(S^{\mu\nu})^\dagger\gamma^0 = \gamma^0 S^{\mu\nu} \tag{12.41}$$

可以推出这个结论:

$$D^\dagger(\Lambda)\gamma^0 = \gamma^0 D^{-1}(\Lambda) \tag{12.42}$$

然后，定义一个狄拉克共轭:

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \tag{12.43}$$

这样，在发生 Lorentz 变化后 Dirac 共轭的场算符就变成了:

$$\bar{\psi}'(x') = \bar{\psi}(x)D^{-1}(\Lambda) \tag{12.44}$$

于是这个是 Lorentz 不变的:

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x) \tag{12.45}$$

这种形式叫做旋量双线形。
可以构造出很多旋量双线形和他们的变化:

$$\bar{\psi}'(x')i\gamma^5\psi'(x') = \bar{\psi}(x)i\gamma^5\psi(x) \tag{12.46}$$

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \Lambda^\mu_{\ \nu}\bar{\psi}(x)\gamma^\nu\psi(x) \tag{12.47}$$

$$\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = \Lambda^\mu_{\ \nu}\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x) \tag{12.48}$$

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x) \tag{12.49}$$

现在考虑 Lagrangian:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{12.50}$$

带入 Euler-Lagrange 方程之后得到运动方程 (不知道为什么用 EL 方程处理的时候对 ψ 求偏导和 $\bar{\psi}$ 无关)

$$(i(\gamma^\mu)_{ab}\partial_\mu - m\delta_{ab})\psi_b(x) = 0 \tag{12.51}$$

对于这个方程，用 $(-i\gamma^\mu\partial_\mu - m)$ 左乘得到 Klein-Gordan 方程

$$(\partial^2 + m^2)\psi(x) = 0 \tag{12.52}$$

一般把态矢写成左手螺旋态 (left-hand-spinor)和右手螺旋态 (right-hand-spinor)的组合。

$$\psi = \begin{bmatrix}\eta_L \\ \eta_R\end{bmatrix} \tag{12.53}$$

他们满足的 Euler-Lagrange 方程叫做 Weyl equation.

$$\begin{aligned} i\bar{\sigma}^\mu\partial_\mu\eta_L - m\eta_R = 0 \\ i\sigma^\mu\partial_\mu\eta_R - m\eta_L = 0 \end{aligned} \tag{12.54}$$

既然场算符满足 Klein-gordan 方程。那么就用平面波来猜解。

$$\psi_a(x,k) = \omega(k^0,\vec{k})e^{-ikx} \tag{12.55}$$

这个带入原始的 Euler-Lagrange 方程中。会得到一个矩阵方程:

$$(k^0 - \gamma^0(\vec{k}\cdot\vec{\gamma}) - m\gamma^0)\omega(k^0,\vec{k}) = 0 \tag{12.56}$$

面对这个方程，要求系数行列式为 0:

$$det(k^0 - \gamma^0(\vec{k}\cdot\vec{\gamma}) - m\gamma^0) = 0 \tag{12.57}$$

一统计算之后发现这个矩阵的行列式等于

$$(E_k + k^0)^2(E_k - k^0)^2 = 0 \tag{12.58}$$

这个事情说明了有四个解。正能解和负能解分别有两个根。
正能解这么表示:

$$\omega^+(E_k,\vec{k},\sigma)exp\big(-i(E_kt - \vec{k}\cdot\vec{x})\big) = \quad \sigma = 1,2 \tag{12.59}$$

负能解这么表示:

$$\omega^{(-)}(-E_k,\vec{k},\sigma)exp\big(i(E_kt + \vec{k}\cdot\vec{x})\big) = \quad \sigma = 1,2 \tag{12.60}$$

一般会定义一个:

$$u(k,\sigma) = \omega^{(+)}(E_k,\vec{k},\sigma) \quad v(k,\sigma) = \omega^{(-)}(-E_k,-\vec{k},\sigma) \tag{12.61}$$

于是波函数写成展开的形式:

$$\psi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Sigma_{\sigma=1}^2(u(p,\sigma)c_{p,\sigma}e^{-ipx} + v(p,\sigma)d^\dagger_{p,\sigma}e^{ipx}) \tag{12.62}$$

铺垫了这么多。现在回到 Dirac 方程的解。
场算符的正能量解是:

$$\psi^{(+)}(x,p,\lambda) = u(p,\lambda)e^{-ipx} \quad \lambda = \pm \tag{12.63}$$

带入 Euler-Lagrange 方程可以得到:

$$\begin{aligned} (\not{p} - m)u(p,\lambda) = 0 \\ (\not{p} + m)v(p,\lambda) = 0 \end{aligned} \tag{12.64}$$

其中 $\not{p}$ 叫做 Dirac 斜线。
现在构造一个螺旋态，他是螺旋态算符的本征态。

$$\xi_-(p) = \frac{1}{\sqrt{2|p|(|p|+p^3)}}\begin{bmatrix} -p^1+ip^2 \\ |p|+p^3 \end{bmatrix} \quad \xi_+(p) = \frac{1}{\sqrt{2|p|(|p|+p^3)}}\begin{bmatrix} |p|+p^3 \\ -p^1+ip^2 \end{bmatrix} \tag{12.65}$$

螺旋态算符和他对应的螺旋态方程是:

$$(\hat{p}\cdot\sigma)\xi_\lambda(p) = \lambda\xi_\lambda(p) \tag{12.66}$$

螺旋度有正交归一关系

$$\begin{aligned} \xi_+^\dagger(p)\xi_+(p) &= 1 \\ \xi_-^\dagger(p)\xi_-(p) &= 1 \\ \xi_+^\dagger(p)\xi_-(p) &= 0 \end{aligned} \tag{12.67}$$

和完备性关系:

$$\Sigma_{\lambda=\pm}\xi_\lambda(p)\xi_\lambda^\dagger(p) = I \tag{12.68}$$

于是这样定义 u 和 v:

$$u(p,\lambda) = \begin{bmatrix} \omega_{-\lambda}(p)\xi_\lambda(p) \\ \omega_\lambda(p)\xi_\lambda(p) \end{bmatrix} v(p,\lambda) = \begin{bmatrix} \lambda\omega_\lambda(p)\xi_{-\lambda}(p) \\ -\lambda\omega_{-\lambda}(p)\xi_{-\lambda}(p) \end{bmatrix} \tag{12.69}$$

$$\omega_\lambda(p) = \sqrt{E_p+\lambda|p|} \tag{12.70}$$

这两个基底也是这个算符的本征向量:

$$\begin{aligned} (\hat{p}\cdot S)u(p,\lambda) &= \frac{\lambda}{2}u(p,\lambda) \\ (\hat{p}\cdot S)v(p,\lambda) &= -\frac{\lambda}{2}v(p,\lambda) \end{aligned} \tag{12.71}$$

u 和 v 满足正交的关系:

$$\begin{aligned} \bar{v}(p,\lambda)v(p,\lambda') &= 2m\delta_{\lambda\lambda'} \\ \bar{u}(p,\lambda)v(p,\lambda') &= 2m\delta_{\lambda\lambda'} \\ \bar{u}(p,\lambda)v(p,\lambda') = \bar{v}(p,\lambda)u(p,\lambda') &= 0 \\ v^\dagger(p,\lambda)v(p,\lambda') &= 2E_p\delta_{\lambda\lambda'} \\ u^\dagger(p,\lambda)u(p,\lambda') &= 2E_p\delta_{\lambda\lambda'} \\ u^\dagger(p,\lambda)v(p,\lambda') = v^\dagger(p,\lambda)u(p,\lambda') &= 0 \end{aligned} \tag{12.72}$$

现在再来量子化场。什么叫做等时反对易量子化呢:

$$\begin{aligned} \{\psi_a(x,t),\pi_b(y,t)\} &= i\delta_{ab}\delta^{(3)}(x-y) \\ \{\psi_a(x,t),\psi_b(y,t)\} = \{\pi_a(x,t),\pi_b(y,t)\} &= 0 \end{aligned} \tag{12.73}$$

什么是场的平面波解呢:

$$\psi(x,t)\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Sigma_{\lambda=\pm}(u(p,\lambda)a_{p,\lambda}e^{-ipx}+v(p,\lambda)b_{p,\lambda}^\dagger e^{ipx}) \tag{12.74}$$

这个平面波解酒对应着场的 Hamiltonian:

$$H = \Sigma_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}E_p(a_{p,\lambda}^\dagger a_{p,\lambda}-b_{p,\lambda}b_{p,\lambda}^\dagger) \tag{12.75}$$

然后带入了等时对易条件之后就有:

$$
\begin{aligned}
\{a_{p,\lambda}, a^{\dagger}_{q,\lambda}\} &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(p-q) \\
\{a_{p,\lambda}, a_{q,\lambda}\} &= \{a^{\dagger}_{p,\lambda}, a^{\dagger}_{q,\lambda}\} = 0 \\
\{b_{p,\lambda}, b^{\dagger}_{q,\lambda}\} &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(p-q) \\
\{b_{p,\lambda}, b_{q,\lambda}\} &= \{b^{\dagger}_{p,\lambda}, b^{\dagger}_{q,\lambda}\} = 0
\end{aligned} \tag{12.76}
$$

这时候, 场的 Hamiltonian:

$$
H = \Sigma_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_p (a^{\dagger}_{p,\lambda} a_{p,\lambda} + b^{\dagger}_{p,\lambda} b_{p,\lambda}) - (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} 2E_p \tag{12.77}
$$

接下来说一下 Lagrangian 中暗含的 U(1) 对称性。如果对场做一个变换:

$$
\psi'(x) = e^{iq\theta}\psi(x) \tag{12.78}
$$

发现 Lagrangian 是完全不会改变的。于是就可以用 Noether 定理。然后就有一个对应的 Noether 守恒荷。

$$
Q = \Sigma_{\sigma=\pm} \int \frac{d^3p}{(2\pi)^3} q (a^{\dagger}_{p,\lambda} a_{p,\lambda} - b^{\dagger}_{p,\lambda} b_{p,\lambda}) + 2\delta^{(3)}(0) q \int d^3p \tag{12.79}
$$

于是我们说 a 操作的是正粒子。b 操作的是反粒子。定义粒子态和粒子态的内积:

$$
\begin{aligned}
|p^+, \lambda\rangle &= \sqrt{2E_p} a^{\dagger}_{p,\lambda} |0\rangle \\
|p^-, \lambda\rangle &= \sqrt{2E_p} b^{\dagger}_{p,\lambda} |0\rangle \\
\langle q^+, \lambda | p^+, \lambda\rangle &= 2E_p (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(p-q) \\
\langle q^-, \lambda | p^-, \lambda\rangle &= 2E_p (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(p-q)
\end{aligned} \tag{12.80}
$$

前面说过的一个式子

$$
(L+S)\psi(x) = [\psi(x), J] \tag{12.81}
$$

把场的展开形式具体带入就会得到:

$$
(2\hat{p} \cdot J) a^{\dagger}_{p,\lambda} |0\rangle = \lambda a^{\dagger}_{p,\lambda} |0\rangle (2\hat{p} \cdot J) b^{\dagger}_{p,\lambda} |0\rangle = \lambda b^{\dagger}_{p,\lambda} |0\rangle \tag{12.82}
$$

第十三章 重整化 renormalisation

13.1 ϕ^4 理论的重整化

13.1.1 2-point Connected Feynman Diagram

首先需要关心 2-point Connected Feynman Diagram 的性质。在位置空间，他大概用 Feynman 图表示是: 在这

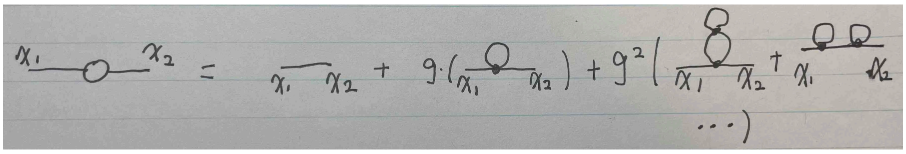


图 13.1: 2PointConnectedDiagramSpaceTime

个图里，没有写对称性因子。当然，这个图是用坐标空间的 Feynman 规则来表达成式子的。图:8.3

简单证明:$G_c(x_1, x_2)$ 只和 $x_2 - x_1$ 有关系。

这个严谨的证明不是很确定。但是如果只是取看上面的的图片，比如说第二项 (g 的一阶项)。他的式子里面有这么一项

$$\int \frac{1}{i\hbar} d^4x i\hbar\Delta(x_2 - x) i\hbar\Delta(x - x_1) i\hbar\Delta(0) \tag{13.1}$$

如果把 Feynman 传播子用 Fourier 展开，然后带入上面的式子，会有这样一项:

$$(i\hbar)^2\Delta(0) \int \frac{d^4p_1}{(2\pi)^4} \frac{1}{p_1^2 - m^2} e^{ip_1(x_2 - x)} \int \frac{d^4p_2}{(2\pi)^4} \frac{1}{p_2^2 - m^2} e^{ip_2(x - x_1)} d^4x \tag{13.2}$$

然后注意到对于 x 的积分会产生一个对于 p 的 delta 函数，也就是

$$\int d^4x e^{-ip_1x} e^{ip_2x} = (2\pi)^4 \delta^4(p_2 - p_1) \tag{13.3}$$

这个积分完了后，再对 p_2 进行积分，就会变成:

$$(i\hbar)^2\Delta(0) \int \frac{d^4p_1}{(2\pi)^4} \frac{1}{p_1^2 - m^2} \frac{1}{p_1^2 - m^2} e^{ip_1(x_2 - x_1)} \tag{13.4}$$

于是我们说，这一项只和 $x_2 - x_1$ 有关。

简单证明，坐标空间的 Feynman 图的连接等价于动量空间的 Feynman 图的连接

比如说两点的坐标空间的动量图的连接:(A 和 B 都是坐标空间的 Feynman 图，当然，他只和两个坐标的差值有关)

$$C(0, x) = \int A(0, y) B(y, x) d^4y \frac{1}{i} \tag{13.5}$$

然后对于 C 这个图，动量空间的 Feynman 图 $C(p)$ 是这么定义的。

$$C(p) = \int d^4x e^{-ipx} C(x) \tag{13.6}$$

具体来说

$$C(p) = \frac{1}{i} \int d^4x e^{-ipx} \int d^4y A(0, y) B(y, x) \tag{13.7}$$

可以把 e 指数拆开。

$$C(p) = \frac{1}{i} \int d^4y e^{-ipy} A(y) \int d^4x e^{-ip(x-y)} B(x - y) \tag{13.8}$$

然后做积分变换，变换是

$$(x, y) \to (x - y, y) \tag{13.9}$$

变换后的积分区间仍然是

$$x - y \in (-\infty, +\infty) y \in (-\infty, +\infty) \tag{13.10}$$

然后因为坐标变化而导致的 Jacobi 行列式是

$$det(\frac{\partial(x, y)}{\partial(x - y, y)}) = 1 \tag{13.11}$$

于是顺理成章的，就有动量图的连接:

$$C(p) = \frac{1}{i\hbar} A(p) B(p) \tag{13.12}$$

就是动量图的连接是有一个 $\frac{1}{i\hbar}$ 的

位置空间图片连接的例子

主要是想说如果有一个位置空间的图上面两个点跑到一起去了，那么应该有这么一项 $(i\hbar)\delta^4(x_1 - x_2)$。图13.2

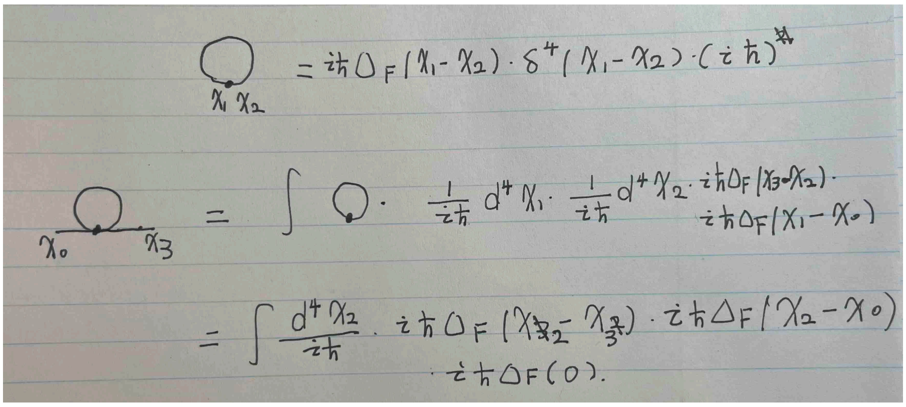


图 13.2: 2PointIrreducibleDiagram

用 1particle irreducible Diagram 构造 $G_c(x_1, x_2)$

首先, 通过方程:

$$\int d^4 y \frac{\delta^2 W[J]}{\delta J(y) \delta J(x_1)} \frac{\delta^2 \hat{\Gamma}[\phi - \bar{\phi}]}{\delta \phi(y) \delta \phi(x_2)} = \delta^4(x_1 - x_2) \tag{13.13}$$

可以推导出 $\hat{\Gamma}^{(n)}(x_1, x_2)$ 和 $G_c(x_1, x_2)$ 有一些关系，具体来说, 先把上面的变量 x_2 改一个名字叫做 y_2。然后再左右同时乘以 $W^{(2)}(y_2, x_2)$ 然后再对 y_2 积分。

$$\int \frac{d^4 y_1}{i\hbar} \frac{d^4 y_2}{i\hbar} (i\hbar)^2 (\frac{i}{\hbar})(\frac{\hbar}{i}) \frac{\delta^2 W[J]}{\delta J(x_1) \delta J(y_1)} \frac{1}{i\hbar} (i\hbar) \frac{\delta^2 \hat{\Gamma}[\phi - \bar{\phi}]}{\delta \phi(y_1) \delta \phi(y_2)} W^{(2)}(y_2, x_2) = W^{(2)}(x_1, x_2) \tag{13.14}$$

于是按照定义，这个式子实际上是:

$$(i\hbar)^2 \frac{i}{\hbar} \frac{1}{i\hbar} \int \frac{d^4 y_1}{i\hbar} \frac{d^4 y_2}{i\hbar} W^{(2)}(x_1, y_1) \hat{\Gamma}^{(2)}(y_1, y_2) W^{(2)}(y_2, x_2) = W^{(2)}(x_1, x_2) \tag{13.15}$$

也就是:

$$-\int \frac{d^4 y_1}{i\hbar} \frac{d^4 y_2}{i\hbar} W^{(2)}(x_1, y_1) \hat{\Gamma}^{(2)}(y_1, y_2) W^{(2)}(y_2, x_2) = W^{(2)}(x_1, x_2) \tag{13.16}$$

用图像表示式子13.16是图13.3 这个图里有一个负号需要注意一下。这一点比较特殊。

$\Gamma^2(x_1, x_2)$并不是不可约的。图13.4。中实心的才是不可约 irreducible 的。图13.3中实心的才是不可约 irreducible。

现在回到之前的图13.1。它实际上说明了这个式子13.4 假设动量空间的 1PI 图的表达式是 $i\Sigma(p^2)$ 对式子13.4取

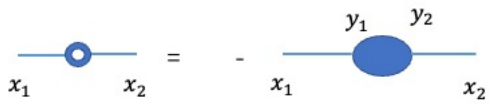


图 13.3: 1PIConstruct2Connected, 图里的实心是 Γ^2

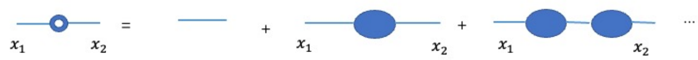


图 13.4: 1PIConstruct2Connected2(图里的实心是两点不可约表示)

Fourier 变换，得到动量空间的联系: (-1 是因为 $\frac{1}{i}\frac{1}{i}$)

$$i\Delta(p) = i\Delta_F(p) + i\Delta_F(p)\big(-i\Sigma(p^2)\big)i\Delta_F(p) + i\Delta_F(p)\big(-i\Sigma(p^2)\big)i\Delta_F(p)\big(-i\Sigma(p^2)\big)i\Delta_F(p)... \tag{13.17}$$

按照等比数列的求和

$$i\Delta(p) = i\Delta_F(p)\frac{1}{1+i\Sigma(p^2)i\Delta_F(p)} \tag{13.18}$$

把 Feynman 传播子的表达式带入得到:(应该有一小节专门讲一下 ϵ 的意义)

$$i\Delta(p) = \frac{i}{p^2-\mu_0^2-\Sigma(p^2)+i\epsilon} \tag{13.19}$$

然后再次观察这个等式:

$$\int d^4y \frac{\delta^2 W[J]}{\delta J(y)\delta J(x_1)}\frac{\delta^2\hat{\Gamma}[\phi-\bar{\phi}]}{\delta\phi(y)\delta\phi(x_2)} = \delta^4(x_1-x_2) \tag{13.20}$$

第一项变换的 Fourier 变换的结果是 $\frac{i}{\hbar}\big(i\Delta(p)\big) = \frac{1}{\hbar}(-\Delta(p))$。第二项 Fourier 变换的结果是 $\frac{1}{(i\hbar)}\big(i\Gamma^2(p)\big) = \frac{1}{\hbar}\Gamma^2(p^2)$ (这个 $\Gamma^{(2)}(p)$ 的定义是:$i\Gamma^{(2)}(p)$ 和 $\Gamma^{(2)}(x_1,x_2)$ 有 Fourier 变换关系)。[1] 于是就有这样的性质

$$\frac{1}{\hbar}\Gamma(p^2)\frac{1}{\hbar}(-\Delta(p)) = 1 \tag{13.21}$$

这个式子就说明了

$$-\Gamma(p^2) = \frac{1}{\Delta(p)} \tag{13.22}$$

也就是说:

$$-\Gamma(p^2) = p^2-\mu_0^2-\Sigma(p^2)+i\epsilon \tag{13.23}$$

13.1.2 简单的判断可不可以使用重整化

判断的依据是根据函数的发散项数目是否可数。

考虑函数

$$\Gamma^n(p_1...p_n) \tag{13.24}$$

他有 n 个外腿。假设里面有 V 个节点。P 个传播子。如果是 phi4 理论，点和线守恒:

$$2P+n = 4V \tag{13.25}$$

需要积分的量是每一个传播子都要积分，但是每一个格点可以有一个动量守恒, 也就是节省一个积分。同时由于外动量守恒，需要加一。所以总共的积分数量是。

$$I = P-V+1 \tag{13.26}$$

利用式子13.25。

$$I = V-\frac{n}{2}+1 \tag{13.27}$$

如果积分的维度是 d。最后的积分大概会有这样的形式:

$$\int\frac{d^dp_1...d^dp_I}{(p_1^2-m^2)...(p_P^2-m^2)} \tag{13.28}$$

他收敛的条件是:

$$dI-2P = (d-4)V+(1-\frac{d}{2})n+d<0 \tag{13.29}$$

Now let's consider general situation, For the action: The original one has $\hbar$ and this is after wick rotation

$$\mathcal{A} = \int d^dx\Big(\frac{1}{2}(\partial\phi_0)^2+\frac{m_0^2}{2}\phi_0^2+\sum_{k=3}^{N}\frac{\lambda_{0k}}{k!}\phi_0^k\Big) \tag{13.30}$$

[1] 这样，$\Delta(p)\Gamma^{(2)}(p)\Delta(p) = -\Delta(p)$, 其中的负号和公式 13.3里的负号是一个意思

Let's consider the dimension analysis. For $c = \hbar = 1$.

$$\begin{aligned} \hbar &= [E][T] = [M][L]^2[T]^{-2}[T] = [M][L]^2[T]^{-1} \\ c &= [L][T]^{-1} = 1 \end{aligned} \tag{13.31}$$

We can use mass unit to represent all units. From the above equation, It can be easily know that:

$$[T] = [M] \quad [L] = [M]^{-1} \tag{13.32}$$

Consider the dimension of ϕ. The dimension of $\mathcal{A}$ is 1. ($[\mathcal{A}] = 1$) Consider the second term in action.

$$\begin{aligned} [\phi]^2[M]^2[L]^d &= [M]^0 \\ [\phi]^2[M]^2[M]^{-d} &= [M]^0 \\ [\phi] &= [M]^{\frac{d-2}{2}} \end{aligned} \tag{13.33}$$

Consider the dimension of λ_{0k}, Consider the last term in the action.

$$\begin{aligned} [\lambda_{0k}][\phi]^k[L]^d &= 1 \\ [\lambda_{0k}][M]^{k(\frac{d-2}{2})-d} &= 1 \\ [\lambda_{0k}] &= [M]^{d-\frac{d-2}{2}k} \end{aligned} \tag{13.34}$$

Consider the unit of $\tilde{\Gamma}^{(n)}$, Consider the difination of proper diagram: This is not the original difination, the original one has $\hbar$, etc

$$\tilde{\Gamma}^{(n)} = \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots} \tag{13.35}$$

The dimension of $J(x_1)$ is:

$$[J(x_1)] = [\phi]^{-1} = [M]^{-\frac{d-2}{2}} \tag{13.36}$$

The dimension of $\delta(x - x_1)$ is:

$$[\delta(x - x_1)] = [L]^{-d} = [M]^d \tag{13.37}$$

The derivation of $\frac{\delta}{\delta J}$ will have the ability to change J into $\delta(x - x_1)$. So we say that the dimension of proper propagator is:

$$[\Gamma^{(n)}] = (\frac{[\delta]}{[J]})^n = [M]^{n(d+\frac{d-2}{2})} \tag{13.38}$$

Consider the unit of $\tilde{\Gamma}^{(n)}(p)$, Consider the propagator in momentum diagram.

13.1.3 维数正规化的 phi4 重整化

这个先看纸质笔记吧

13.2 重整化群

本来对 Lagrangian 是:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{13.39}$$

如果加入 Counter-Term 之后:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta Z\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \tag{13.40}$$

对于 Bare Field 是:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \tag{13.41}$$

满足的关系是:

$$Z = 1 + \delta Z \quad \phi_0 = Z^{1/2}\phi \quad m_0^2 = (m^2 + \delta m^2)Z^{-1} \quad \lambda_0 = (\lambda + \delta\lambda)Z^{-1} \tag{13.42}$$

微扰重整化就是用 Counter-Term 来消除原本的发散项。有一些重整化的规范，在选定后，可以按照级数解出各个抵消项。这些抵消想现在就可以表示为 λ 的级数。而且各级的系数是 ϵ(维度正规化) 和 μ 的函数。然后现在说 $\lambda = \lambda(\mu)$ 于是所有东西都是 μ 的函数。

Callan-Symanzik equation 在 $m = 0$ 没有质量时。(critical submanifold)。(λ,μ) 是一个 physics，$(\lambda(\mu+\delta\mu),\mu+\delta\mu)$ 又是一种 physics。定义

$$\lambda(\mu+\delta\mu) = g(\lambda|\frac{\tilde{\mu}}{\mu}) = 1 + \frac{\delta\mu}{\mu}\beta(\lambda) \tag{13.43}$$

其中 $\tilde{\mu} = \mu + \delta\mu, \beta(\lambda) = \frac{\partial}{\partial\xi}g(\lambda|\xi)|_{\xi=1}$。
考虑 ϕ 的大小。变化是:

$$\phi_{(\mu)} = Z_{(\mu)}^{-1/2}\phi_0 \tag{13.44}$$

于是有

$$\phi_{(\tilde{\mu})} = Z_{(\mu)}^{1/2}Z_{(\tilde{\mu})}^{-1/2}\phi_{(\mu)} \tag{13.45}$$

讲义里面定义他们的变换是:

$$\phi_{(\tilde{\mu})} = Z_{finite}^{-1/2}(\lambda|\frac{\tilde{\mu}}{\mu})\phi_{(\mu)} = (1 - \frac{\delta\mu}{\mu}\gamma(\lambda))\phi_{(\mu)} \tag{13.46}$$

当然, 其中的 $\gamma(\lambda)$ 是 $-\frac{\partial}{\partial\xi}Z_{finite}^{-1/2}(\lambda|\frac{\tilde{\mu}}{\mu})|_{\xi=1}$。现在考虑一个物理量:

$$\begin{aligned} &< \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >= (Z_{\mu}^{-1/2})^n < \phi_0(x_1)\cdots\phi_0(x_n) > \\ &< \phi_{(\tilde{\mu})}(x_1)\cdots\phi_{(\tilde{\mu})}(x_n) >= (Z_{\tilde{\mu}}^{-1/2})^n < \phi_0(x_1)\cdots\phi_0(x_n) > \end{aligned} \tag{13.47}$$

其中，不应该随着 μ 的不同而不同的量应是:

$$(Z_{\mu}^{1/2})^n < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > \tag{13.48}$$

也就是说应该有一个等式:

$$(Z_{\mu}^{1/2})^n < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >= (Z_{\tilde{\mu}}^{1/2})^n < \phi_{(\tilde{\mu})}(x_1)\cdots\phi_{(\tilde{\mu})}(x_n) > \tag{13.49}$$

结合13.46. 得到:

$$(1 - n\frac{\delta\mu}{\mu}\gamma(\lambda)) < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >=< \phi_{(\tilde{\mu})}(x_1)\cdots\phi_{(\tilde{\mu})}(x_n) > \tag{13.50}$$

对 $< \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >$ 去微分得到 $< \phi_{(\tilde{\mu})}(x_1)\cdots\phi_{(\tilde{\mu})}(x_n) >$

$$\begin{aligned} &< \phi_{(\tilde{\mu})}(x_1)\cdots\phi_{(\tilde{\mu})}(x_n) > \\ &\qquad = (1 + \delta\lambda\frac{\partial}{\partial\lambda} + \delta\mu\frac{\partial}{\partial\mu}) < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > \end{aligned} \tag{13.51}$$

考虑到13.43:

$$\begin{aligned} &< \phi_{(\tilde{\mu})}(x_1)\cdots\phi_{(\tilde{\mu})}(x_n) > \\ &\qquad = (1 + \delta\lambda\frac{\partial}{\partial\lambda} + \delta\mu\frac{\partial}{\partial\mu}) < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > \\ &\qquad = (1 + \frac{\delta\mu}{\mu}\beta(\lambda)\frac{\partial}{\partial\lambda} + \delta\mu\frac{\partial}{\partial\mu}) < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > \end{aligned} \tag{13.52}$$

于是:

$$(\beta(\lambda)\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu} + n\gamma(\lambda)) < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >= 0 \tag{13.53}$$

如果假设:

$$< \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >= \mu^n C^{(n)}(\mu x_1, \mu x_2\cdots\mu x_n, \lambda) \tag{13.54}$$

就有 (∂_i 是指对第 i 个指标求导数)

$$\begin{aligned}\frac{\partial}{\partial x_i} < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >= \mu^{(n+1)}\partial_i C^{(n)}(\mu x_1\cdots\mu x_n,\lambda)\\ \frac{\partial}{\partial\mu}C^{(n)}(\mu x_1,\mu x_2\cdots\mu x_n,\lambda) = x_1\partial_i C^{(n)}(\mu x_1,\mu x_2\cdots\mu x_n,\lambda)\\ \frac{\partial}{\partial\mu}\mu^n C^{(n)}(\mu x_1,\mu x_2\cdots\mu x_n,\lambda) = \frac{n}{\mu}\mu^n C^{(n)}(\mu x_1,\mu x_2\cdots\mu x_n,\lambda)+\mu^n x_i\partial_i C^{(n)}(\mu x_1,\mu x_2\cdots\mu x_n,\lambda)\\ = \frac{n}{\mu} < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > +\frac{1}{\mu}x_i\frac{\partial}{\partial x_i} < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >\end{aligned} \tag{13.55}$$

于是13.53。就变成了:

$$(\Sigma_i x_i\frac{\partial}{\partial x_i}+n(1+\gamma(\lambda))+\beta(\lambda)\frac{\partial}{\partial\lambda}) < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >= 0 \tag{13.56}$$

讲义里说这个叫做坐标变化被另外两个变化抵消掉了。这几个变化分别是:

$$x^\mu \to (1+\delta L)x^\mu \quad \lambda\to\lambda-\beta(\lambda)\delta L \quad \phi\to\phi-(1+\gamma(\lambda))\phi\delta L \tag{13.57}$$

假如说 μ 和 L 有这个关系:

$$\frac{\delta L}{L} = -\frac{\delta\mu}{\mu} = -\frac{\delta\lambda}{\beta(\lambda)} \quad \lambda(L=1)=\lambda \tag{13.58}$$

于是, 我们想验证有这个等式:

$$Z^{n/2}(L) < \phi(\frac{x_1}{L}...\phi(\frac{x_n}{L})) >_{\lambda(L)}= Z^{n/2}(\tilde{L}) < \phi(\frac{x_1}{\tilde{L}})...\phi(\frac{x_n}{\tilde{L}}) >_{\lambda(\tilde{L})} \tag{13.59}$$

证明 结合式子13.58和式子13.46, 式子13.55。得到:

$$\delta\mu\frac{\partial}{\partial\mu} < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > \ = \delta\mu(\frac{n}{\mu} < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) > +\frac{1}{\mu}x_i\frac{\partial}{\partial x_i} < \phi_{(\mu)}(x_1)\cdots\phi_{(\mu)}(x_n) >) \tag{13.60}$$

$$\begin{aligned}&(\frac{Z^{1/2}(\tilde{L})}{Z^{1/2}(L)})^n < \phi(\frac{x_1}{\tilde{L}})...\phi(\frac{x_n}{\tilde{L}}) >_{\lambda(\tilde{L})}\\ &= (Z^{1/2}_{finite}(\lambda,\frac{\tilde{\mu}(\tilde{L})}{\mu(L)}))^{-n} < \phi(\frac{x_1}{\tilde{L}})...\phi(\frac{x_n}{\tilde{L}}) >_{\lambda(\tilde{L})}\\ &= (1-n\frac{\delta\mu}{\mu}\gamma(\lambda))(1-\frac{\delta L}{L^2}x_i\partial_i-\frac{\delta L}{L}n+\delta\lambda\frac{\partial}{\partial\lambda}) < \phi(\frac{x_1}{L})...\phi(\frac{x_n}{L}) >_{\lambda(L)}\\ &=< \phi(\frac{x_1}{L})...\phi(\frac{x_n}{L}) >_{\lambda(L)} -\frac{\delta L}{L}(n(\gamma(\lambda)+1)+\frac{x_i}{L}\partial_i+\beta(\lambda)\frac{\partial}{\partial\lambda}) < \phi(\frac{x_1}{L})...\phi(\frac{x_n}{L}) >_{\lambda(L)}\end{aligned} \tag{13.61}$$

注意, 上面的 ∂_i 是指 $\frac{\partial}{\partial(\frac{x_i}{L})} = L\frac{\partial}{\partial x_i}$。于是:

$$\begin{aligned}&(\frac{Z^{1/2}(\tilde{L})}{Z^{1/2}(L)})^n < \phi(\frac{x_1}{\tilde{L}})...\phi(\frac{x_n}{\tilde{L}}) >_{\lambda(\tilde{L})}\\ &=< \phi(\frac{x_1}{L})...\phi(\frac{x_n}{L}) >_{\lambda(L)} -\frac{\delta L}{L}(n(\gamma(\lambda)+1)+x_i\frac{\partial}{\partial x_i}+\beta(\lambda)\frac{\partial}{\partial\lambda}) < \phi(\frac{x_1}{L})...\phi(\frac{x_n}{L}) >_{\lambda(L)}\end{aligned} \tag{13.62}$$

由于式子13.56。可以证明:

$$Z^{n/2}(L) < \phi(\frac{x_1}{L}...\phi(\frac{x_n}{L})) >_{\lambda(L)}= Z^{n/2}(\tilde{L}) < \phi(\frac{x_1}{\tilde{L}})...\phi(\frac{x_n}{\tilde{L}}) >_{\lambda(\tilde{L})} \tag{13.63}$$

第十四章 数学基础

14.1 泛函

14.1.1 泛函微分

泛函就是有无限个自变量的传统函数。。。。暂且这么理解吧，或者可以理解为把一个函数当作自变量的一个函数，(把一个函数映射到一个值上面去)。首先考虑有 N 个自变量的多元函数，他在 $y = 0$ 地方的泰勒展开自然而然可以写成这样。

$$F(y_1, y_2...y_N) = \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{k_1=1}^{N} ... \sum_{k_n=1}^{N} \frac{\delta F(y_1...y_N)}{\delta y_{k_1}...\delta y_{k_n}} \delta y_{k_1}...\delta y_{k_n} \tag{14.1}$$

接下来考虑无穷自变量的函数, 无穷个自变量的函数写作 $J(x)$, 其中 x 的取值范围是 $[-\infty, +\infty]$.
我们认为函数 F[J] 可以表示成 $\int dx J(x) f(x)$ 的形式, 来方便理解吧.. 然后我们说在 x_1 这个地方函数值该变量是 $dJ(x_1)$(以前是 $J(x_1)$) 那么这个函数因为在这个地方 J 的改变而发生的改变就是 $f(x_1)\epsilon J(x_1)$. 这个 ϵ 表示的是积分的时候分段的长度。
实际上我们把他写成这个样子

$$\frac{\delta \int dx J(x) f(x)}{\delta J(x_1)} = \int dx \delta(x - x_1) f(x) \ \ \epsilon \tag{14.2}$$

当然, 如果整个函数都有发生变化, 那么函数对应的微分就可以用积分来表达

$$\int dx_1 dJ(x_1) \int dx \delta(x - x_1) f(x) \tag{14.3}$$

这样我们定义泛函微分这样定义

$$\frac{\delta J(x)}{\delta J(x_1)} = \delta(x - x_1) \tag{14.4}$$

这样, 我们说泛函展开:

$$W[J] = \Sigma_{n=0}^{+\infty} \int dx_1...dx_n \frac{1}{n!} \frac{\delta W[J]}{\delta J(x_1)...\delta J(x_n)} J(x_1)...J(x_n) \tag{14.5}$$

是这样的，然后在场论里面，因为我们一般说 $W[J]$ 是 Generating Function. 生成的是 n-Point Function

$$W[J] = \Sigma_{n=0}^{+\infty} \int dx_1...dx_n \frac{1}{n!} (\frac{i}{\hbar})^n \ (\frac{\hbar}{i})^n \frac{\delta W[J]}{\delta J(x_1)...\delta J(x_n)} J(x_1)...J(x_n) \tag{14.6}$$

$$G^{(n)}(x_1...x_n) = (\frac{\hbar}{i})^n \frac{\delta W[J]}{\delta J(x_1)...\delta J(x_n)} \tag{14.7}$$

14.2 群的知识

14.2.1 Group

群是定义了乘法的集合

- 群对群乘法有封闭性 $\forall g_1, g_2 \in G$ 则 $g_1 g_2 \in G$
- 群乘法满足结合律 $g_1(g_2 g_3) = (g_1 g_2) g_3$ for $\forall g_1, g_2, g_3 \in G$
- 群中一定有恒元 e: s.t eg=ge=g for $\forall g \in G$
- 任意群元 g 一定有逆元 g^{-1} for $\forall g \in G, \exists g^{-1} \in G$, s.t $g^{-1} g = g g^{-1} = e$

一些特殊的群

- Abel 群: $g_1 g_2 = g_2 g_1$
- 子群: 群 G 的子集 H。H 满足群的定义则称 H 是 G 的子群，记为 $H < G$

S 代表行列式是 1。O 是 orthogo 正交。U 是 unitary 幺正。SO(1,3) 是固有保时向 Lorentz 群。

14.2.2 群的线性表示

线性表示: 一些 m 阶方正 D(G) 和 G 中元素的乘法关系完全相同。这样就可以用这些方阵来表示群 G. 矩阵们构成了 G 的线性表示。并且 D(G) 中的一个元素至少对应于 G 中的一个群元。
同态: 群 G 的线性表示 D(G) 和群 G 之间有同态关系。

$$D(g_1 g_2) = D(g_1)D(g_2) \tag{14.8}$$

- 恒等表示: 1 维线性表示, 矩阵取 1.
- 同构/忠实表示: G 中的群元素和 D(g) 中的群元素一一对应
- 幺正表示: D(G) 中的矩阵是幺正的
- 自身表示/基础表示: 群元素本来就是矩阵，然后要表示自己
- 等价表示: 两个群线性表示之间存在变换:$D_2(g) = S^{-1}D_1(g)S$
- 可约表示: 如果存在一个矩阵 S。则说 D(g) 是可约表示，否则就是不可约表示。

$$S^{-1}D_1(g)S = \begin{bmatrix} D_1(g) & M(g) \\ 0 & D_2(g) \end{bmatrix} \tag{14.9}$$

14.2.3 $SO(1,3)^\dagger$ 固有保时 Lorentz 群

emm, 我其实不知道写的对不对..... 其实我只是想说狭义相对论罢了。。
我只是想说几个喜闻乐见的性质。
保度归条件:

$$g_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g_{\mu\nu} \tag{14.10}$$

证明他只需要保证 lorentz 变化前后的两个矢量的度量是一样的 $A'^{\mu} = \Lambda^{\mu}{}_{\nu}A^{\nu}$

洛伦兹变换的逆矩阵:

$$(\Lambda^{-1})^{\mu}{}_{\nu} = g_{\nu\alpha}g^{\mu\beta}\Lambda^{\alpha}{}_{\beta} \tag{14.11}$$

证明这个式子只用证明他和 Λ 乘之后是一个科洛内科 delta 符号就好了。

逆变矢量的 lorentz 逆变换:

$$x_{\nu} = x'_{a}\Lambda^{a}{}_{\nu} \tag{14.12}$$

证明他也不难，反正就是把逆变矢量的定义和 Lorentz 逆变换结合起来就好了。