Group theory

Group theory study notes

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时间：2023 6 6

版本：0.0

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第一章 Basic Group Knowledge

1.1 Group

定义 1.1 (Equivalence Relation)

If X is a set, $\sim$ is **equivalence relation**. For $\forall a, b, c \in X$.

- $a \sim a$
- $a \sim b \quad \rightarrow b \sim a$
- $a \sim b \& b \sim c \quad \rightarrow a \sim c$

♣

定义 1.2 (equivalence class)

$[a] = \{x \in X, x \sim a\}$

♣

定义 1.3 (Group)

A group is a quartet, (G,m,I,e)

- G is a set
- m: $G \times G \rightarrow G$ is a map, called group multiplication map.
- I: $G \rightarrow G$ is a map, called inverse map.
- $e \in G$ is a distinguished element of G called identity element.

They have three significant property:

- associative: $\forall g_1 \ g_2 \ g_3 \in G \ m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$
- $\forall g \in G, \quad m(g, e) = m(e, g) = g$
- $\forall g \in G \quad m(I(g), g) = m(g, I(g)) = e$

♣

定义 1.4 (abelian group)

A group G is called abelian group if

$$\forall g_1 \quad g_2 \in G, \qquad g_1 g_2 = g_2 g_1 \tag{1.1}$$

♣

定义 1.5 (Subgroup)

(G,m,I,e) is a group, if $H \subset G$. m,I perserve H, which means $m : H \times H \rightarrow H$,$I : H \rightarrow H$ $(e \in H)$. Then we say that (H,m,I,e) is a subgroup of (G,m,I,e).

♣

定义 1.6 (Group representation)

A representation of Group G is a mapping D which maps elements of G onto a set of linear operators satisfying

- D(e)= $\mathcal{I}$ where $\mathcal{I}$ is the identity operator on the Linear space where the linear operators act.
- $D(g_1)D(g_2) = D(g_1 g_2)$

The dimension of linear operator acts is called the dimension of representation. (The dimension of linear operator equals to the dimension of the space of it acts.) ♣

定义 1.7 (General Linear Group)

$M_n(k)$ means a set of $n \times n$ matrixs with entries belongs to $K = \mathbb{R}$ or $k = \mathbb{C}$. However it is unital monoid, But it might not be a group cause some matrix don't have inverse matrix. So we can define a group called General Linear Group

$$GL(n,k) \equiv \{A|A = n \times n \text{ invertible matrix over k}\} \subset M_n(k) \tag{1.2}$$ ♣

There are many important groups:

$$SL(n,k) := \{A \in GL(n,k) \ : \ detA = 1\} \tag{1.3}$$

$$O(n,k) :== \{A \in GL(n,k) \ : \ AA^{tr} = 1\} \tag{1.4}$$

$$SO(n,k) :== \{A \in O(n,k) \ : \ detA = 1\} \tag{1.5}$$

$$U(n) :== \{A \in GL(n,C) \ : \ AA^{\dagger} = 1\} \tag{1.6}$$

$$SU(n) :== \{A \in U(n) \ : \ detA = 1\} \tag{1.7}$$

denote: Modular Group is called $SL(2,Z)$

定义 1.8 (Center of a group)

Z(G) is a center of a group, which all elements in this group commute with it.

$$Z(G) \equiv \{z \in G|zg = gz \ \forall g \in G\} \tag{1.8}$$ ♣

1.2 Homomorphism and Isomorphism

定义 1.9 (Homomorphism and Isomorphism)

Consider two groups, They are: (G,m,I,e) and (G',m',I',e')

- A homomorphism（同态）is a map from G to G', which preserve group law:

$$\begin{gathered}\varphi : G \to G' \ \ \forall g_1, g_2 \in G \\ \varphi(m(g_1, g_2)) = m'(\varphi(g_1), \varphi(g_2))\end{gathered} \tag{1.9}$$

- If φ is a 1-1 and onto map. Then it is called **Isomorphism**（同构）
- when G=G' and φ is an Isomorphism, φ is called the automorphism of G.

A common slogan is Isomorphic Group are the same. ♣

定义 1.10 (Vector Space)

Consider a vector space over field F, There is a non-empty set V. two binary operations. element in F are called scalars, element in V are called vectors.

There are 8 axioms.

- Associative of vector addition: $u + (v + w) = (u + v) + w$.
- Commutativity of vectors addition: $u + v = v + u$.
- Identity in vector addition: $\exists 0 \in V \ \ \forall u \in V \ u + 0 = u$
- Inverse in vector addition: $\forall u \in V, \ \exists -u \in V \ s.t. \ u + (-u) = 0$
- Compatibility of scalar multiplication with field multiplication: $a(bv) = (ab)v$
- Identity element of scalar multiplication: $\exists 1 \in F \ 1v = v$
- Distributivity of scalar multiplication with respect to vector addition $a(u+v) = au+av$
- Distributivity of scalar multiplication with respect to field addition $(a + b)u = au + bu$ ♣

定义 1.11 (Matrix Representation of group G)

A matrix representation of group G is a **homomorphism**（同态）:

$$T : \ G \to GL(n, k) \tag{1.10}$$

for some positive integral n and field k.

V is a vector space over a field k. $GL(V)$ represents all invertible linear transformations of V. Then

$$T : G \to GL(V) \tag{1.11}$$

is a Representation of group G. where V is a carrier space. ♣

1.3 Group actions on Sets

定义 1.12 (group action by group)

X is a set, G is a group,if X is a Group action by a Group G. We say that X is a G-set.
now, explain what does action actually means.
Left G-action on set X is a map $\varphi : G \times X \longrightarrow X$, which is compatible with group multiplication laws.

- $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ (compatible with group multiplication laws)
- $\forall x \in X \quad \varphi(1_G, x) = x$ (we want $1_G \circ x \mapsto x$ to be identity map)

I just wanna to say these two constrains are not the same, we cannot derive the second one from the first one.
From the first one, we know.

$$\varphi(1_G, \varphi(1_G, x)) = \varphi(1_G, x) \tag{1.12}$$

Does Not mean:

$$\forall x \in X \quad \varphi(1_G, x) = x \tag{1.13}$$

♣

定义 1.13 (Orbits)

If group G acts on set X. We can define a **equivalence relation** on X. We say that.

$$x_1, x_2 \in X \;\; if \;\; \exists g \in G \; \varphi(g, x_1) = x_2 \;\; then \;\; x_1 \sim x_2 \tag{1.14}$$

The equivalence class [x] is called orbit of G through a point x.

$$O_G(x) = \{y : \exists g \in G \;\; s.t. \;\; y = \varphi(g, x)\} \tag{1.15}$$

The set of orbits is denoted by X/G.

♣

proof why this definition of equivalence relation is valid.

First condition of equivalence relation would be $x \sim x$.
It is easy to know that:

$$\varphi(1_G, x) = x \tag{1.16}$$

Second condition of equivalence relation would be $x_1 \sim x_2 \longrightarrow x_2 \sim x_1$ we suppose that:

$$\varphi(g_1, x_1) = x_2 \tag{1.17}$$

Then, we consider this term:

$$\varphi(g_2, x_2) = \varphi(g_2, \varphi(g_1, x_1)) = \varphi(g_1 g_2, x_1) \tag{1.18}$$

We only need $g_2 = g_1^{-1}$, then $x_2 \sim x_1$

Third condition is $if x_1 \sim x_2$ & $x_2 \sim x_3$ $then$ $x_1 \sim x_3$
We suppose that:

$$\varphi(g_1, x_1) = x_2 \quad \varphi(g_2, x_2) = x_3 \tag{1.19}$$

It is obvious that:

$$\varphi(g_1 g_2, x_1) = x_3 \tag{1.20}$$

The above relation shows that $x_1 \sim x_3$

Now we consider **Group Actions On Sets Induce Group Actions On Associated Function Spaces.**
Consider there are two sets X and Y. $\mathcal{F}[X \to Y]$ is a set of functions from $X \to Y$
Now there is a Left G-action defined by φ

$$\varphi : G \times X \to X \tag{1.21}$$

G action on set $\mathcal{F}[X \to Y]$ can be induced by:

$$\tilde{\varphi} : \ G \times \mathcal{F} \to \mathcal{F} \tag{1.22}$$

which satisfies:

$$\tilde{\varphi}(g, F)(x) = F(\varphi(g^{-1}, x)) \tag{1.23}$$

Now I need to explain why this is true. (Why this kind of map is Group acting on set $\mathcal{F}$)
Consider

$$\tilde{\varphi}(g, F) \circ (x) = F(\varphi(g^{-1}, x)) \tag{1.24}$$

Then consider:

$$\tilde{\varphi}(g_1, \tilde{\varphi}(g_2, F)) \circ (x) \tag{1.25}$$

This would equal to:

$$\begin{aligned}
\tilde{\varphi}(g_2, F) \circ \big(\varphi(g_1^{-1}, x)\big) \\
&= F\big(\varphi(g_2^{-1}, \varphi(g_1^{-1}, x))\big) \\
&= F\big(\varphi(g_2^{-1} g_1^{-1}, x)\big) \\
&= F\big(\varphi((g_1 g_2)^{-1}, x)\big) \\
&= \tilde{\varphi}(g_1 g_2, F) \circ (x)
\end{aligned} \tag{1.26}$$

This means:

$$\tilde{\varphi}(g_1 g_2, F) = \tilde{\varphi}(g_1, \tilde{\varphi}(g_2, F)) \tag{1.27}$$

1.4 Symmetric Group

定理 1.1 (Cayley's theorem)
Any finite group is isomorphic to a subgroup of a permutation group S_N for some N. ♡

Proof
suppose there is a finite group G. $\{g_1, g_2, \cdots\}$. define a map called $L(h)$ $h \in G$.

$$L(h) \ : g \mapsto hg \ g \in G \ L(h) \in S_G \tag{1.28}$$

Why $\mathbf{L(h) \in S_G}$.
After impose $L(h)$ to the group G. we obtain:

$$\{hg_1, hg_2, \cdots\} \tag{1.29}$$

We say that $L(h)$ is a 1-1, onto map.(a permutation)
1-1: if

$$hg_i = hg_j \tag{1.30}$$

apply h^{-1} to the left, then: $g_i = g_j$
onto: We wanna to say that for any g_i, we can alwways find a g_x to let $hg_x = g_i$. This is true cause $g_x = h^{-1}g_i$.
So a 1-1, onto map from G to G is called **permutation**.
Also it is easy to say that element in the set $\{L(h)|h \in G\}$ conservers group multiplication of the group S_G.

$$L(h_1) \circ L(h_2) = L(h_1 h_2) \tag{1.31}$$

So $\{L(h)|h \in G\}$ a subgroup of group S_G, Also, the map $h \mapsto L(h)$ is a **1-1, onto** map.

$$L(h_1) = L(h_2) \ \text{ only when } \ h_1 = h_2 \tag{1.32}$$

So we say that G is isomorphic to a subgroup of S_G

Cyclic permutation and cylcle decomposition
cyclic permutation: let G is a set with **order** has n elements. Suppose that $a_1 \cdots a_l$ are l distinct number between 1 and n. here is the operation (called **cyclic permutation**):

$$g_{a_1} \longrightarrow g_{a_2} \longrightarrow g_{a_3} \cdots g_{a_l} \longrightarrow g_{a_1} \tag{1.33}$$

We call this: (obviously, there are l different ways to write this)

$$\phi : \quad (a_1, a_2 \cdots a_l) \tag{1.34}$$

Cycle decomposition means:Any permutation $\sigma \in S_n$ can be uniquely written as a product of disjoint cycles. For example, there is a cycle decomposition in S_{11}

$$\sigma = (12)(34)(10,11)(56789) \tag{1.35}$$

Any cycle can be written as a product of transportation

Consider a permutation φ

$$\begin{pmatrix} 1 & 2 & \cdots & k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \tag{1.36}$$

Now consider a relation:

$$\varphi \circ (1, 2, \cdots k) \circ \varphi^{-1} = (a_1, a_2 \cdots a_k) \tag{1.37}$$

The relation above can be proved like this, Consider the left hand side of equation:

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & \cdots & k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}\begin{pmatrix} 1 & 2 & \cdots & k \\ 2 & 3 & \cdots & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix} \\ &= \begin{pmatrix} 2 & \cdots & k & 1 \\ a_2 & \cdots & a_k & a_1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \cdots & k \\ 2 & 3 & \cdots & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix} = (a_1, a_2 \cdots a_k) \end{aligned} \tag{1.38}$$

However, we can alwawys write $(1, 2, \cdots k)$ as:

$$(1, k)(1, k-1) \cdots (1, 2) = (1, 2, \cdots k) \tag{1.39}$$

This equation can be proved by Mathematical induction...

In this case

$$\varphi \circ (1, 2, \cdots k) \circ \varphi^{-1} = \varphi \circ (1, k) \circ \varphi^{-1} \circ \varphi \circ (1, k-1) \circ \varphi^{-1} \cdots \varphi \circ (1, 2) \circ \varphi^{-1} \tag{1.40}$$

However:

$$\varphi \circ (1, k) \circ \varphi^{-1} = (a_1, a_k) \tag{1.41}$$

In this case:

$$(a_1, a_k)(a_1, a_{k-1}) \cdots (a_1, a_2) = (a_1, a_2 \cdots a_k) \tag{1.42}$$

This is the reason why we say that Any cycle can be written as a product of transportation. (I think transportation means exchange over two elements)

We should notice that every element in permutation group can be represented by a product of transportation. We say that transportation is the generator of the permutation group.

1.5 Generators and Relations

定义 1.14 (Generating set of a group)

A subset $S \subset G$ is a generating set of a Group, if every element $g \in G$ can be written as a product of elements of S.

$$g = s_{i_1} s_{i_2} \cdots s_{i_r} \tag{1.43}$$

♣

For **finitely generated group** (elements in generating group is finite). We write: (I think the left side represents the generators)

$$G = \langle g_2 \cdots g_n | R_1 \cdots R_r \rangle \tag{1.44}$$

In this relation, R_i means term represented by elements in S which will be set to 1.

However it is convenient to exclude 1 from the Generating Set S, There are two reasons. we can write $s^0 = 1$. And we can write s^n, for $n < 0$, this means: $s^{-|n|}$
A generating set that contains s^{-1} for every generator s is said to be **symmetric**. For this kind of set, we can construct 1 by: $ss^{-1} = 1_G$

Most General group with one generator and one relation:

$$\langle a | a^N = 1 \rangle \tag{1.45}$$

定义 1.15 (Free group)

If there is no relation on generating set S. We can define free group on set S, called $F(S)$. Which is generated by generating group S.

♣

定义 1.16 (Coxter Group)

Coxter group can be represented as: (let m be an $n \times n$ symmetric matrix)

$$\langle s_1 \cdots s_n | \forall i, j\, (s_i s_j)^{m_{ij}} = 1 \rangle \tag{1.46}$$

when $m_{ij} = +\infty$ it means there is no relation!
We have a restriction:

$$m_{ii} = 1 \tag{1.47}$$

Which means:

$$s_i s_i = 1 \tag{1.48}$$

This kind of element (group element that squares to 1) is called **involution**.

Then we consider another situation ($m_{ij} = 2$), In this situation: (we used $m_{ii} = 1$)

$$\begin{aligned} s_i s_j\ s_i s_j &= 1 \\ s_i s_j &= s_j s_i \end{aligned} \tag{1.49}$$

I think I should study detail into classification of coxter group.

Reflection Group

Actually, we will talk about The Reflection group Generated by Reflections in the plane orthogonal to vector v_i. ($v_i \in \mathbb{R}^N$).

We consider n vectors in the space $\mathbb{R}^N$.(I think it needs to satisfies $N \geq n$) There relation would be:

$$v_i \cdot v_j = -2\cos(\frac{\pi}{m_{i,j}}) \tag{1.50}$$

The reflection is a Map:

$$P_{v_i} : v \mapsto v - 2\frac{v \cdot v_i}{v_i \cdot v_i} v_i \tag{1.51}$$

For this , We can let these reflections be generators, and we construct a Reflection group:

$$\langle P_{v_i} | (P_{v_i} P_{v_j})^{m_{i,j}} = 1 \rangle \tag{1.52}$$

Okay, I wanna to say that what does $P_{v_i} P_{v_j}$ means. For simplicity, Consider 3-dimension situation:

$$v_i = \sqrt{2}\hat{i} \tag{1.53}$$

$$v_j = -\sqrt{2}\cos(\frac{\pi}{m_{i,j}})\hat{i} + \sqrt{2}\sin(\frac{\pi}{m_{i,j}})\hat{j} \tag{1.54}$$

$$v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{1.55}$$

It can be calculated that, After the double reflection, the vector would be: (Without the change of z direction)

$$\begin{pmatrix} \cos(2\frac{\pi}{m_{i,j}}) & \sin(2\frac{\pi}{m_{i,j}}) \\ -\sin(2\frac{\pi}{m_{i,j}}) & \cos(2\frac{\pi}{m_{i,j}}) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{1.56}$$

This states that $P_{v_i} P_{v_j}$ will make a clock-wise rotation in the plane determined by v_i and v_j. With the angle 2 times of the angle between v_i and v_j.

In this consideration, it is not hard to understand the relation:

$$(P_{v_i} P_{v_j})^{m_{i,j}} = 1 \tag{1.57}$$

With $m_{i,i} = 1$ and need m to be a symmetric matrix.

1.6 Cosets and Conjugacy

Lagrange Theorem

To begin with we will define **Left Coset of H**. Suppose H is a Subgroup of Group G. Then Left Coset of H is defined as:

定义 1.17 (Left Coset of H)

H is a sub group of G. Then the left coset of H is defined as: (it is not a Group! but a set)

$$gH := \{gh|h \in H\} \subset G \tag{1.58}$$

♣

There are three important properties of Left Coset of H.

1. Left Cosets are identical or disjoint.
2. Every elements $g \in G$ lies in some Left Coset.
3. We can define an equivalence principle $g_1 \sim g_2$ if $\exists h \in H$ s.t. $g_1 = g_2 h$

Then, I need to give a proof. To begin with, I need to review rearrangement lemma.
Rearrangement lemma states that:

$$\text{For a group H;} \quad hH = H \tag{1.59}$$

It is obvious that hH is close in group multiplication. and it has inverse element and Identity element. So it is a Group. But we wanna to state that this group is exactly H. This is because we can find all the element of H in group hH. This is obvious because:

$$\begin{aligned} hh_x &= h_1 \\ h_x &= h^{-1}h_1 \end{aligned} \tag{1.60}$$

Then we back to the three properties of the Left Coset of H. The second property is easy to prove. because H contains identity element.

Then, we would say that if $g_1 \sim g_2$, Then the left coset defined by them are the same, otherwise, they are disjoint.

证明 if $g_1 H$ and $g_2 H$ are not disjoint. Then, this means:

$$\exists g \in g_1 H \cap g_2 H \tag{1.61}$$

Then we would say:

$$\begin{aligned} g &= g_1 h_1 \\ g &= g_2 h_2 \end{aligned} \tag{1.62}$$

Which leads to:

$$g_1 = g_2 h_2 h_1^{-1} = g_2 h_3 \tag{1.63}$$

What we do means **if two Left Coset deduced by** g_1 **and** g_2 **are not disjoint, then** $\exists\ h\ s.t. g_1 = g_2 h$
Then we rearrangement lemma, the Left Coset deduced by g_1 and g_2 are the same!
End of proof

We can actually define a set of Cosets of Subgroup denoted by G/H. $g_1 H \cap g_2 H \neq \emptyset \longrightarrow g_1 \sim g_2 \longrightarrow g_1 H = g_2 H$

定理 1.2 (Lagrange Theorem)

if group H is a subgroup of a group G, Then the order of H divides the order of G.

证明 As the properties of Left Coset we have discussed, we know that:

$$G = \amalg_{i=1}^{m} g_i H \tag{1.64}$$

Now note that the order of H equals to the order of $g_i H$

$$|g_i H| = |H| \tag{1.65}$$

then the order of H divides the order of G:

$$|G|/|H| = m \tag{1.66}$$

♡

Conjugate

定义 1.18 (Conjugate)

Suppose G is a group, we say that group element h is conjugate to h' if:

$$\exists g \in G \ s.t. \ h' = ghg^{-1} \tag{1.67}$$

♣

Using this definition, we can define **Conjugate Class of h** or should I say equivalence class:

$$C(h) := \{ghg^{-1} \ : \ g \in G\} \tag{1.68}$$

Then if there is a subgroup of G: $H \subset G \quad K \subset G$. Then we say that H is conjugate to K if:

$$\exists g \in G s.t. K = gHg^{-1} := \{ghg^{-1} : h \in H\} \tag{1.69}$$

In abelian Group, every element form a conjugacy class.

Normal(invariant) subgroup and quotient group

定义 1.19 (normal subgroup / Invariant subgroup)

Subgroup H, $H \subseteq G$ is called normal subgroup or invariant subgroup if

$$gNg^{-1} = N \quad \forall g \in G \tag{1.70}$$

Sometimes denoted as: $H \lhd G$

♣

If $N \subset G$ is a normal subgroup, then the set of left cosets $G/N = \{gN | g \in G\}$ has a natural group structure with group multiplication defined by:

$$(g_1 N)(g_2 N) = (g_1 g_2) N \tag{1.71}$$

Group G/N is known as **quotient group**

∘ Why left Cosets can be quotient group

Now, need to explain why need to have normal subgroup to let it be a quotient group. Consider a element e, which is in the subgroup H. We have:

$$H = eH = hH \tag{1.72}$$

Which means: (let g can be any element in the group G)

$$gH = H \cdot gH = hH \cdot gH = hgH = gg^{-1}hgH \tag{1.73}$$

to let these two element be the same:

$$g^{-1}hg \in H \tag{1.74}$$

Which means that H has to be the normal subgroup.

Direct Product and Semi Direct Product

∘ The Direct product would be:

$$(h_1, h_2) \in H_1 \otimes H_2 \tag{1.75}$$

Consider a Group G, which has subgroup H_1 and H_2. If they satisfies:

1 $H_1 \bigcap H_2 = e$
2 All Group element of G can be represent by: $h_1 h_2 = g \quad h_1 \in H_1 \ h_2 \in H_2$
3 All h_1 commutes with h_2

In this case:

$$H_1 H_2 = H_1 \otimes H_2 \tag{1.76}$$

The requirement can be simplified:

1 $H_1 \bigcap H_2 = e$
2 All Group element of G can be represent by: $h_1 h_2 = g \quad h_1 \in H_1 \ h_2 \in H_2$
3 H_1 and H_2 are normal subgroup of Group G.

we prove that All h_1 commutes with h_2.

证明 Consider:

$$h_{2i} h_{1i} h_{2i}^{-1} = h_{1j} \in H_1 \tag{1.77}$$

Then:

$$\begin{aligned} h_{2i} &= h_{1j} h_{2i} h_{1i}^{-1} \\ &= h_{1j} h_{1i}^{-1} h_{1i} h_{2i} h_{1i}^{-1} \end{aligned} \tag{1.78}$$

Denote that:

$$h_{2j} = h_{1i} h_{2i} h_{1i}^{-1} \tag{1.79}$$

Then:

$$h_{2i} = h_{1j} h_{1i}^{-1} h_{2j} \tag{1.80}$$

Then, we need:

$$h_{1j} h_{1i}^{-1} = e \tag{1.81}$$

which means:

$$h_{2i}h_{1i}h_{2i}^{-1} = h_{1i} \tag{1.82}$$

Which means they commutes.

E.N.D

- ∘ Semi-Direct Product Group.

Consider:

1 N is normal subgroup of G
2 H is a subgroup of G
3 $N \bigcap H = e$
4 G=NH

We use the notation:

$$G = N \rtimes H \tag{1.83}$$

第二章 Representation Theory

2.1 Unitary Theory

A. finite groups have unitary representations

for finite groups and for compact Lie groups, all representations are equivalent to a unitary one.

proof ...

定理 2.1

every representation of a **finite group** is **equivalent** to a unitary representation.

证明 define

$$S = \Sigma_{g\in G} D(g)^\dagger D(g) \tag{2.1}$$

S is hermitian and is positive semidefinite. So we can decompose S matrix. (U is unitary matrix)

$$S = U^\dagger d U \tag{2.2}$$

$$d = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{bmatrix} \tag{2.3}$$

We want to say that$\forall j, d_j > 0$. if $d_j = 0$, then, $\exists \alpha \ s.t.\ S\alpha = 0$, then $\alpha^\dagger S\alpha = 0$, which means:

$$\Sigma_{g\in G} \alpha^\dagger D(g)^\dagger D(g)\alpha = \Sigma_g (D(g)\alpha)^\dagger (D(g)\alpha) = \Sigma ||D(g)\alpha||^2 = 0 \tag{2.4}$$

This is not possible because $D(e) = I$, Then we define Hermitian matrix X:

$$X = S^{1/2} = U^{-1} \begin{bmatrix} \sqrt{d_1} & & \\ & \sqrt{d_2} & \\ & & \ddots \end{bmatrix} U \tag{2.5}$$

Then we have a similarity transformation ($X^\dagger = X$, $(X^{-1})^\dagger = X^{-1}$)

$$D'(g) = XD(g)X^{-1} \tag{2.6}$$

Then (S=XX)

$$D'(g)^\dagger D'(g) = X^{-1} D(g)^\dagger X X D(g) X^{-1} \tag{2.7}$$

however: (gG=G)

$$\begin{aligned} D(g)^{\dagger} X X D(g) &= D(g)^{\dagger} S D(g) \\ &= D(g)^{\dagger}\left(\Sigma_{h\in G} D(h)^{\dagger} D(h)\right) D(g) \\ &= \Sigma_{h\in G} D(hg)^{\dagger} D(hg) \\ &= \Sigma_{h\in G} D(h)^{\dagger} D(h) = S = X^2 \end{aligned} \tag{2.8}$$

Then

$$D'(g)^{\dagger} D'(g) = I \tag{2.9}$$

Which means the representation D is equivalent to a unitary representation D'.

B. Unitary representations are always completely reducible.

定理 2.2 Every representation of a **finite group** is **completely reducible**.

Here have two proof,

证明 We need to prove that if V_1 is an invariant subspace, then its complement $V_1^{\perp}$ is an invariant subspace. if $v \in V_1$ and $w \in V_1^{\perp}$ then: $\langle v, w\rangle = 0$ Then:

$$0 = \langle D(g)v, w\rangle = \langle v, D(g)^{\dagger} w\rangle = \langle v, D(g^{-1})w\rangle \tag{2.10}$$

Which means:

$$D(g)w \in V_1^{\perp} \tag{2.11}$$

Then, its complement is an invariant subspace.

□

The below is a proof from the book (Lie groups and particle physics–I don't remember)

证明 If P is a projection operator, and the group is a reducible group, then:

$$PD(g)P = D(g)P \tag{2.12}$$

apply conjugate to the operator above:

$$PD^{\dagger}(g)P = PD(g)^{\dagger} \tag{2.13}$$

As the theory above, We can only consider unitary representation. Which means:

$$D(g)^{\dagger} = D(g)^{-1} = D(g^{-1}) \tag{2.14}$$

(I don't quite understand why $D(g)^{-1} = D(g^{-1})$–Because the definition of the group), We say that (don't know why–Rearrangement Lemma) when g travel through all the element in G, g^{-1} will travel through all the elements too. So,

$$PD(g)P = PD(g) \tag{2.15}$$

however, this equation is equal to:

$$(I - P)D(g)(I - P) = D(g)(I - P) \tag{2.16}$$

So we say that it is completely reducible.

□

2.2 Shur's Lemma

To begin with, we need to define a intertwiner.

定义 2.1 (Intertwiner)

And intertwiner between Representation D_1 and D_2 is a linear operator from space V_1 to space V_2.

$$F : V_1 \rightarrow V_2 \tag{2.17}$$

Which commutes with G:

$$FD_1(g) = D_2(g)F \tag{2.18}$$

♣

1. The kernel and the image of F are invariant subspaces of D_1 and D_2

Firstly, We talk about kernel:

$$v \in ker\ F \tag{2.19}$$

Then:

$$FD_1(g)v = D_2(g)Fv = D_2(g)0 = 0 \tag{2.20}$$

Which means:

$$D_1(g)v \in ker\ F \tag{2.21}$$

This proves that The kernel is an invariant subspace of D_1.

Then we talk about the Image space. Suppose:

$$w_2 = Fw_1 \tag{2.22}$$

Which means that w_2 is in the Image space, Then:

$$D_2(g)w_2 = D_2(g)Fw_1 = FD_1(g)w_1 \in Img\ F \tag{2.23}$$

Which states that the Image space is an invariant subspace of D_2.

2. If D_1 is irreducible, the only kernel is $\{0\}$ or V_1

However, this one is quite obvious, Because the kernel is the invariant subspace. To make D_1 irreducible, we need to let the invariant be the whole space or only the zero element.

So F is either **injective**(Kernel is 0) or zero(Kernel is V_1).

P.S Injective means one to one. We need to know why injective can be derived from $kerF = \{0\}$. let's suppose:

$$Fv_1 = v_2 \; \& \; Fv_1' = v_2 \tag{2.24}$$

This means that: (This is a special case in linear space)

$$F(v_1 - v_2) = 0 \tag{2.25}$$

Then, we know that:

$$v_1 - v_2 \in \; kerF \tag{2.26}$$

As we know that:

$$kerF = \{0\} \tag{2.27}$$

Then:

$$v_1 - v_2 = 0 \longrightarrow v_1 = v_2 \tag{2.28}$$

Which means F is injective.

3. If D_2 is irreducible, F is either surjective or zero

We need to know that **surjective** means onto. As the Image space of the map F is an invariant subspace. We need to let D_2 be an irreduciable representation. Which means the invariant subspace can only be $\{0\}$ or V_2 itself. So this map is surjective.

However **If D_1 and D_2 are both irreduciable and $V_1 \neq V_2$, then F is zero**. Because the number of elements in V_1 and V_2 are not the same. So F can not be surjective.

4. D_1 and D_2 are equivalent exactly if there exists an invertible intertwiner

Then we come to Shur's Lemma.

As we discussed, if the representation D_1 and D_2 are irreducible, then F is surjective and injective. which means the intertwiner F is invertible. Then we consider the relation:

$$FD_1(g) = D_2(g)F \tag{2.29}$$

When F is invertible, this means:

$$D_1(g) = F^{-1}D_2(g)F \tag{2.30}$$

However this is a group isomorphism.

定义 2.2 (Shur's Lemma)

Consider the situation of $D_1 = D_2$, Then F is an endomorphism($Hom(V_1, V_1)$) .

if D is an **irreducible** finite-dimensional representation on a complex vector space, and there is an endomorphism F of V which satisfies $FD(g) = D(g)F$. For all g, Then, F is the multiple of the identity $F = \lambda I$.

For a less formal representation: (we use the language of matrix to say this)
A matrix which commutes with all matrices of an irreducible representation is proportional to unit matrix. Need to know that we need V to be a complex vector space, a real vector space might not have real eigen vector. ♣

证明 If F is an matrix, then consider it has eigen value λ and eigen vector v. As we know the constrain for the intertwiner is that it commutes with Representation operator:

$$FD(g) = D(g)F \tag{2.31}$$

However, we notice that

$$F - \lambda I \tag{2.32}$$

is also an intertwiner. But this linear operator has a kernel space:

$$(F - \lambda I)v = 0 \tag{2.33}$$

But as we discussed before, if D is an irreducible representation, it means that the kernel space should be empty or be the vector space V itself. So we need to let this new intertwiner vanish all the vector in the space, which means that:

$$F = \lambda I \tag{2.34}$$

If D_1 are not equal to D_2, the intertwiner is unique upto a constant. This is because $F_2^{-1}F_1$ is a self-intertwiner in D_1 which is porpotional to identity operator, then:

$$F_1 = \lambda F_2 \tag{2.35}$$

2.3 Regular Representation

Group Algebra

Vector Space with an extra bilinear product operation is called Algebra. We have a vector space of linear combination of group elements: (We always let v_g to be complex number)

$$v = \sum_g v_g g \tag{2.36}$$

with an addition operation:

$$v + w = \sum_g (v_g + w_g)g \tag{2.37}$$

vector space with an extra bilinear product operation is called Algebra. The bilinear product operation would be:

$$v \cdot w = \sum_{gg'} v_g w_{g'} gg' = \sum_h (\sum_g v_g w_{g^{-1}h}) h \tag{2.38}$$

I will make slightly explain to this formula. Firstly, it would be easy to realize:

$$g' = g^{-1}h \tag{2.39}$$

while:

$$gg' = h \tag{2.40}$$

okay, then why

$$\sum_g \sum_{g'} = \sum_g \sum_h \tag{2.41}$$

Actually, we only need to know:

$$\sum_{g'} = \sum_h \tag{2.42}$$

this is because:

$$g' = g^{-1}h \tag{2.43}$$

Using Rearrangement lemma, we know $\{g^{-1}h |\ h \in G\}$ contains all the element in G.

Now define inner product:

$$\langle v, w \rangle = \sum_g v_g^* w_g \tag{2.44}$$

Regular representation

$$D_{reg}(g) :\ v \mapsto g \cdot v \tag{2.45}$$

Actually,

$$g \cdot v = \sum_h v_h gh = \sum_{h'} v_{g^{-1}h'} h' \tag{2.46}$$

Which means:

$$(g \cdot v)_h = v_{g^{-1}h} \tag{2.47}$$

in regular representation, there is only one element in each row and colume.

$$(D_{reg}(h))^i{}_j = 1 \text{ only when } hg_j = g_i \tag{2.48}$$

Regular representation is **unitary** because:

$$\begin{aligned} (D^\dagger_{reg}(h))^i{}_j &= 1 \text{ only when } hg_i = g_j \\ (D_{reg}(h^{-1}))^i{}_j &= 1 \text{ only when } h^{-1}g_j = g_i \end{aligned} \tag{2.49}$$

They are the same!

Regular representation is unitary, so it is completely reducible to irreducible components.

2.4 Great Orthogonality Theorem

A. Orthogonality theorem for representation

Defines:

$$(A^{(ja)}_{(\mu\nu)})^i{}_b = \sum_g (D_{(\mu)}(g))^i{}_j (D_{(\nu)}(g^{-1}))^a{}_b \tag{2.50}$$

It is a linear map:

$$A_{(\mu\nu)} : \ u^b \in V_\nu \mapsto (A^{(ja)}_{(\mu\nu)})^i{}_b u^b = v^i \in V_{(\mu)} \tag{2.51}$$

Then:

$$\begin{aligned} D_{(\mu)}(g)A_{\mu\nu} &= \sum_{g'} D_{(\mu)}(g)D_{(\mu)}(g')D_{(\nu)}(g'^{-1}) \\ &= \sum_{g'} D_{(\mu)}(gg')D_{(\nu)}(g'^{-1}) \\ &= \sum_{h} D_{(\mu)}(h)D_{(\nu)}(h^{-1}g) \\ &= \sum_{h} D_{(\mu)}(h)D_{(\nu)}(h^{-1})D_{(\nu)}(g) \\ &= A_{(\mu\nu)}D_{(\nu)}(g) \end{aligned} \tag{2.52}$$

According to 2.2(Shur's Lemma) , this is the intertwiner. If $\mu = \nu$, $A = \lambda\mathbb{I}$. If $\mu \neq \nu \& n_\mu \neq n_\nu$, $A = 0$. If $\mu \neq \nu \& n_\mu = n_\nu A = \lambda\mathbb{I}(A - \lambda\mathbb{I}$ is intertwiner) which means $D_{(\mu)} = D_{(\nu)}$ $(D_\mu A = D_{(\nu)}A)$

In all we say that:

$$(A^{(kl)}_{(\mu\nu)})^i{}_j = \delta_{\mu\nu}\delta^i_j \ \lambda^{(kl)}_\mu \tag{2.53}$$

$$\sum_g (D_{(\mu)}(g))^i{}_k (D_{(\nu)}(g^{-1}))^l{}_j = \delta_{\mu\nu}\delta^i_j \ \lambda^{(kl)}_\mu \tag{2.54}$$

Taking a trace can find the coefficient:

$$\begin{aligned} tr(A^{(kl)}_{(\mu\nu)}) = \delta_{\mu\nu}(A^{(kl)}_{\mu\nu})^i{}_i = \delta_{\mu\nu}\sum_g (D_{(\mu)}(g))^i{}_k (D_{(\nu)}(g^{-1}))^l{}_i = \delta_{\mu\mu}N(G)\delta^l_k \\ = \delta_{\mu\nu}\delta^i_i \ \lambda^{(kl)}_\mu = \delta_{\mu\nu}d_\mu \ \lambda^{(kl)}_\mu \end{aligned} \tag{2.55}$$

Used the property that: $D(g)D(g^{-1}) = \mathbb{I}$.

Then:

$$\lambda^{(kl)}_\mu = \frac{N(G)}{d_\mu}\delta^l_k \tag{2.56}$$

Orthogonality theorem for representation:

$$\sum_g (D_{(\mu)}(g))^i{}_k (D_{(\nu)}(g^{-1}))^l{}_j = \frac{N(G)}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k \tag{2.57}$$

If the representation is **unitary**:

$$\sum_g (D_{(\mu)}(g))^i{}_k (D^*_{(\nu)}(g))^j{}_l = \frac{N(G)}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k \tag{2.58}$$

Then define a Group Algebra Element

$$v^i_{\mu\ j} = \sqrt{\frac{d_\mu}{N(G)}}\sum_g (D_{(\mu)}(g))^i{}_j\ g \tag{2.59}$$

Since they are orthogonal with eachother:

$$\sum_\mu d^2_\mu \leq N(G) \tag{2.60}$$

For Regular Representation:

$$D_{reg}(g) = U(D_{(\mu1)} \oplus D_{(\mu2)} \cdots)U^\dagger \tag{2.61}$$

Also:

$$g = \sum_h (D_{reg}(h))^g{}_e h \tag{2.62}$$

Using decomposition of regular representation:

$$(D_{reg}(h))^g{}_e = \sum_{\mu,i,j} c^{geij}_\mu (D_{(\mu)}(h))^i{}_j \tag{2.63}$$

Then:

$$g = \sum_{\mu,i,j} c^{geij}_\mu \sum_h (D_{(\mu)}(h))^i{}_j h = \sum_{\mu,i,j} c^{geij}_\mu \sqrt{\frac{N(G)}{d_\mu}} v^i_{\mu\ j} \tag{2.64}$$

Then, for regular representation:

$$\sum_\mu d^2_\mu = N_G \tag{2.65}$$

B. Great Orthogonality Theorem for characters (column, 列)

Great Orthogonality Theorem is:

$$\sum_g (D_{(\mu)}(g))^i{}_k (D_{(\nu)}(g^{-1}))^l{}_j = \frac{N(G)}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k \tag{2.66}$$

Consider the character: (we take $k = i$ and $l = j$)

$$\sum_g \chi_{(\mu)}(g)\chi_{(\nu)}(g^{-1}) = N(G)\delta_{\mu\nu} \tag{2.67}$$

For Unitary Representation, (For Finite Group And Compact Lie Groups, All Representations are equivalent to a unitary one.)

$$\sum_g \chi_{(\mu)}(g)\chi^*_{(\nu)}(g) = N\delta_{\mu\nu} \tag{2.68}$$

let use n_r label the number of elements on class k_r, then attain **Great Orthogonality Theorem for**

characters

$$\sum_a n_a \chi^a_{(\mu)} \chi^{a*}_{(\nu)} = N(G)\delta_{\mu\nu} \tag{2.69}$$

We can construct a r different k-vector (k means the number of classes, r means number of representations)

$$\frac{1}{\sqrt{N(G)}}(\sqrt{n_1}\chi^1_{(\mu)}, \sqrt{n_2}\chi^2_{(\mu)}, \cdots, \sqrt{n_k}\chi^k_{(\mu)}) \tag{2.70}$$

So we say that: (r means number of representation)

$$r \leq k \tag{2.71}$$

C. Check if a representation is irreducible

Suppose a representation (or it is equivalent to this representation):

$$D = \oplus a^\mu D_{(\mu)} \tag{2.72}$$

Then

$$\chi = \sum a^\mu \chi_{(\mu)} \tag{2.73}$$

Consider:

$$\begin{aligned}\sum_a n_a \chi^a \chi^{a*} &= \sum_{\mu,\nu} a^\mu a^\nu \sum_a n_a \chi^a_{(\mu)} \chi^{a*}_{(\nu)} \\ &= \sum_{\mu,\nu} a^\mu a^\nu N(G)\delta_{\mu\nu} = N(G)\sum_\mu (a^\mu)^2\end{aligned} \tag{2.74}$$

By calculating

$$\frac{1}{N(G)}\sum_a n_a \chi^a \chi^{a*} = \sum_\mu (a^\mu)^2 \tag{2.75}$$

We can understand whether it is a irreducible representation

D. Find coefficient a^μ of each irreducible representations in a reducible representation

Consider:

$$\begin{aligned}\sum_a n_a \chi^{a*}_{(\nu)} \chi^a &= \sum_\mu a^\mu \sum_a n_a \chi^{a*}_{(\nu)} \chi^a_{(\mu)} \\ &= \sum_\mu a^\mu N(G)\delta_{\mu\nu} = N(G) a^\nu\end{aligned} \tag{2.76}$$

Which means:

$$a^\nu = \frac{1}{N(G)}\sum_a n_a \chi^{a*}_{(\nu)} \chi^a \tag{2.77}$$

E. r=k for regular representation

∘ coefficient of regular representation, and some properties

Consider Regular Representation:

$$\chi_{reg}(g) = \begin{cases} N \text{ for } g = e \\ 0 \text{ for } g \neq e \end{cases} \tag{2.78}$$

With this property, by using the conclusion in **D.**, Find a^{μ} of regular representation

$$a^{\mu} = \frac{1}{N(G)} \sum_{a} n_a \chi_{(\mu)}^{a*} \chi_{reg}^{a} = \frac{1}{N(G)} \chi_{(\mu)}^{1*} \chi_{reg}^{1} = d_{\mu} \tag{2.79}$$

After finding the coefficient of all irreducible representation, We have a useful property: (a=1 means class which contains identical element)

$$\left.\begin{matrix} N \text{ for } a = 1 \\ 0 \text{ for } a \neq 1 \end{matrix}\right\} = (\chi_{reg})_a = \sum_{\mu} a^{\mu} \chi_{(\mu)}^{a} = \sum_{\mu} d_{\mu} \chi_{(\mu)}^{a} = \sum_{\mu} \chi_{(\mu)}^{1} \chi_{(\mu)}^{a} \tag{2.80}$$

We focus on this property:

$$\sum_{\mu} \chi_{(\mu)}^{1} \chi_{(\mu)}^{a} = \begin{cases} N \text{ for } 1 = 1 \\ 0 \text{ for } a \neq 1 \end{cases} \tag{2.81}$$

- Class vector and class Algebra

Define class vector in Group Algebra.

$$\mathcal{K}_a = \sum_{g \in k_a} g \tag{2.82}$$

Class vector is invariant under conjugation:

$$g \mathcal{K}_a g^{-1} = \mathcal{K}_a \tag{2.83}$$

Product of Class vector is invariant under conjugation:

$$g \mathcal{K}_a \mathcal{K}_b g^{-1} = g \mathcal{K}_a g^{-1} g \mathcal{K}_b g^{-1} = \mathcal{K}_a \mathcal{K}_b \tag{2.84}$$

We state that: **If a vector is invariant under conjugation,(for all g) it must be a linear combination of class vectors.**

$$\mathcal{K}_a \mathcal{K}_b = \sum_{c} C_{abc} \mathcal{K}_c \tag{2.85}$$

Hence the $\mathcal{K}_a$ form an algebra themselves, Fixed by coefficients C_{abc}

For a given conjugacy class k_a, there is a class $k_{a'}$ whose elements are the inverse of those in k_a and $n_a = n_{a'}$ (n_a means number of elements in a clas), k_a might be equal to $k_{a'}$ Then $\mathcal{K}_a \mathcal{K}_{a'}$ contains n_a copies of identity.

$$C_{ab1} = \begin{cases} n_a \text{ for } b = a' \\ 0 \text{ for } b \neq a' \end{cases} \tag{2.86}$$

Consider a matrix:

$$D_{(\mu)}^{a} = \sum_{g \in K_a} D_{(\mu)}(g) \tag{2.87}$$

(Sorry that I used k_a to denote conjugacy class, and K_a here.)

This matrix commutes with all matrix in representation, Using Shur's Lemma.

$$D^a_{(\mu)} = \lambda^a_{(\mu)}\mathbb{I} \tag{2.88}$$

Taking trace we attain:

$$\begin{aligned} tr \sum_{g\in K_a} D_{(\mu)}(g) &= tr\ \lambda^a_{(\mu)}\mathbb{I} \\ n_a\chi^a_{(\mu)} &= \lambda^a_{(\mu)}d_\mu \\ n_a\chi^a_{(\mu)} &= \lambda^a_{(\mu)}\chi^1_{(\mu)} \\ \lambda^a_{(\mu)} &= \frac{n_a\chi^a_{(\mu)}}{\chi^1_{(\mu)}} \end{aligned} \tag{2.89}$$

The matrix we defined satisfies Class algebra 2.85.

$$\lambda^a_{(\mu)}\lambda^b_{(\mu)} = \sum_c C_{abc}\lambda^c_{(\mu)} \tag{2.90}$$

insert 2.89 we attain:

$$\begin{aligned} \frac{n_a\chi^a_{(\mu)}}{\chi^1_{(\mu)}}\frac{n_b\chi^b_{(\mu)}}{\chi^1_{(\mu)}} &= \sum_c C_{abc}\frac{n_c\chi^c_{(\mu)}}{\chi^1_{(\mu)}} \\ \sum_\mu \chi^a_{(\mu)}\chi^b_{(\mu)} &= \sum_c C_{abc}\frac{n_c}{n_a n_b}\sum_\mu \chi^1_{(\mu)}\chi^c_{(\mu)} \end{aligned} \tag{2.91}$$

By using the property of regular representation (2.81)

$$\sum_\mu \chi^a_{(\mu)}\chi^b_{(\mu)} = \sum_c C_{ab1}\frac{n_1}{n_a n_b}N(G) \tag{2.92}$$

Then using the property of the coefficient C (2.86)

$$\sum_\mu \chi^a_{(\mu)}\chi^b_{(\mu)} = \begin{cases} \dfrac{N(G)}{n_a} \text{ for } b = a' \\ 0 \text{ for } b \neq a' \end{cases} \tag{2.93}$$

Summarized as:

$$\sum_\mu \chi^a_{(\mu)}\chi^b_{(\mu)} = \frac{N(G)}{n_a}\delta_{a'b} \tag{2.94}$$

For regular representation (unitary):

$$\chi^{a'}_{(\mu)} = \chi^{a*}_{(\mu)} \tag{2.95}$$

Finally: (row orthogonality)

$$\sum_\mu \chi^{a*}_{(\mu)}\chi^b_{(\mu)} = \frac{N(G)}{n_a}\delta_{ab} \tag{2.96}$$

Then we can define k different r-dimensional orthogonal vectors:

$$\sqrt{\frac{n_a}{N}}(\chi^a_{(1)}, \chi^a_{(2)} \cdots \chi^a_{(r)}) \tag{2.97}$$

then:

$$k \leq r \tag{2.98}$$

So, for regular representation

$$r = k \tag{2.99}$$

2.5 Below is not the content

BELOW IS NOT THE CONTENT

定义 2.3 (similarity transformation)

$$D(g) \rightarrow D'(g) = S^{-1}D(g)S \tag{2.100}$$

we say that $D'(g)$ is equivalent representation of D. ♣

定义 2.4 (matrix element of representation)

First, for the element in a group (whose order is k)$\{e, a_1, a_2 \cdots\}$, construct k orthogonal vector $|e_i\rangle$ $i = 1 \cdots k$. let $|k_1\rangle$ represent e, $|k_2\rangle$ represent a_1 ...... Then the matrix element of representation would be:

$$D(g)_{ij} = \langle e_i|D(g)|e_j\rangle \tag{2.101}$$

♣

2.6 irreduciable representation

定义 2.5 (reducible representation)

A representation is reducible if: (P is a projection operator)

$$\exists P\ , \forall g \in G, PD(g)P = D(g)P \tag{2.102}$$

I think projection operator needs to satisfy:

$$P^n = P \tag{2.103}$$

I don't know why, but P seems needs to be Hermitian. $P^\dagger = P$ ♣

定义 2.6 (completely reducible)

if a representation is completely reducible if its representation is equivalent to a representation whose matrix element have the following form.

$$\begin{bmatrix} D_1(g) & & \\ & D_2(g) & \\ & & \ddots \end{bmatrix} = D_1 \oplus D_2 \cdots \tag{2.104}$$

a completely reducible reducible representation can be decomposed into a direct sum of irreducible representation.

2.7 useful theorem

定理 2.3 every representation of a finite group is equivalent to a unitary representation.

证明 define

$$S = \Sigma_{g\in G} D(g)^\dagger D(g) \tag{2.105}$$

S is hermitian and is positive semidefinite. So we can decompose S matrix. (for hermitian S, $U^\dagger = U^{-1}$)

$$S = U^\dagger d U \tag{2.106}$$

$$d = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{bmatrix} \tag{2.107}$$

We want to say that$\forall j, d_j > 0$. if $d_j = 0$, then, $\exists \alpha\ s.t.\ S\alpha = 0$, then $\alpha^\dagger S\alpha = 0$, which means:

$$\Sigma_{g\in G}\alpha^\dagger D(g)^\dagger D(g)\alpha = \Sigma_g (D(g)\alpha)^\dagger (D(g)\alpha) = \Sigma ||D(g)\alpha||^2 = 0 \tag{2.108}$$

This is not possible because $D(e) = I$, Then we define:

$$X = S^{1/2} = U^{-1} \begin{bmatrix} \sqrt{d_1} & & \\ & \sqrt{d_2} & \\ & & \ddots \end{bmatrix} U \tag{2.109}$$

Then we have a similarity transformation ($X^\dagger = X$, $(X^{-1})^\dagger = X^{-1}$)

$$D'(g) = XD(g)X^{-1} \tag{2.110}$$

Then (S=XX)

$$D'(g)^\dagger D'(g) = X^{-1}D(g)^\dagger XXD(g)X^{-1} \tag{2.111}$$

however: (gG=G)

$$\begin{aligned} D(g)^{\dagger}XXD(g) &= D(g)^{\dagger}SD(g) \\ &= D(g)^{\dagger}\left(\Sigma_{h\in G}D(h)^{\dagger}D(h)\right)D(g) \\ &= \Sigma_{h\in G}D(hg)^{\dagger}D(hg) \\ &= \Sigma_{h\in G}D(h)^{\dagger}D(h) = S = X^2 \end{aligned} \tag{2.112}$$

Then

$$D'(g)^{\dagger}D'(g) = I \tag{2.113}$$

Which means the representation D is equivalent to a unitary representation D'.

定理 2.4 Every representation of a finite group is completely reducible.

证明 If P is a projection operator, and the group is a reducible group, then:

$$PD(g)P = D(g)P \tag{2.114}$$

apply conjugate to the operator above:

$$PD^{\dagger}(g)P = PD(g)^{\dagger} \tag{2.115}$$

As the theory above, We can only consider unitary representation. Which means:

$$D(g)^{\dagger} = D(g)^{-1} = D(g^{-1}) \tag{2.116}$$

(I don't quite understand why $D(g)^{-1} = D(g^{-1})$), We say that (don't know why) when g travel through all the element in G, g^{-1} will travel through all the elements too. So,

$$PD(g)P = PD(g) \tag{2.117}$$

however, this equation is equal to:

$$(I-P)D(g)(I-P) = D(g)(I-P) \tag{2.118}$$

So we say that it is completely reducible.

第三章 Discrete Group

3.1 Young diagrams

For a symmetric group, the conjugacy will not change the circle structure. So we can find n-tuple ν_p to gives the number of p-circle. Which satisfies:

$$\sum_p p\nu_p = n \tag{3.1}$$

A conjugacy class is formed by all elements of given circle structure (ν)

Also, we can define a n-tuple λ to difine conjugacy class, this is done by:

$$\lambda_p = \sum_{q=p} \lambda_q \tag{3.2}$$

For these λ, they satisfies:

$$\sum_p \lambda_p = n \tag{3.3}$$

Also:

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \tag{3.4}$$

Conjugacy class are in one to ine correspondence with the partitions of n.

3.2 representation of Symmetric Group

We have shown that the number of irreducible representations is equal to the number of conjugacy classes, and we know that the conjugacy classes are in one-to-one correspondence with the Young tableaux.

第四章 Lie Group

4.1 Some Differential Geometry

A lie group is a group G. The group and manifold structure are required to be compatible in the sense that product and inverses are continuous maps. This can be combined in the requirement that the map

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2^{-1} \end{aligned} \tag{4.1}$$

are continous.

Manifold

The **manifold** is defined by the **atlas**. which means we cover the group G by **open sets** U_i. And define a set of **charts** from U_i to $\mathbb{R}^d$ i.e. invertible maps.

$$\varphi_i : G \supset U_i \to V_i \subset \mathbb{R}^d \tag{4.2}$$

we have to require that on the overlap of the open sets $U_{ij} = U_i \cap U_j$, the change of coordinates is differentiable. i.e. the maps:

$$\varphi_i \circ \varphi_j^{-1} : V_j \to V_i \tag{4.3}$$

are differentiable bijections

Then Consider about the functions. we want to talk about the differentiability of the functions:

$$\begin{aligned} f &: K \to G \\ g &: G \to K^m \end{aligned} \tag{4.4}$$

These functions are differentiable if:

$$\begin{aligned} \varphi_i \circ f &: K \to V_i \\ g \circ \varphi_i^{-1} &: V_i \to K^m \end{aligned} \tag{4.5}$$

are differentiable.

Vector Fields

To begin with

定义 4.1 ($C^\infty(\mathcal{M})$)

$$\text{for } f: \ \mathcal{M} \to \mathbb{R}, \text{ if } f \circ \varphi_i^{-1} \ \forall i \ V_i \to \mathbb{R} \text{ are } C^\infty \Rightarrow f \text{ are } C^\infty(\mathcal{M}) \tag{4.6}$$

♣

vector field is a linear map: $\xi : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$. that satisfies product rule:

$$\xi(fg) = f\xi(g) + \xi(f)g \tag{4.7}$$

We can define vector ξ_x:

$$\xi_x(f) = \xi(f)(x) \tag{4.8}$$

Tangent space

Tangent space at a point x of the manifold $\mathcal{M}$:

$$T_x = \{\xi_x | \xi \quad \text{is a (local) vector field}\} \tag{4.9}$$

Tangent Map

For a differentiable map $F : M_1 \longrightarrow M_2$, The tangent map $T_xF : T_xM_1 \longrightarrow T_{F(x)}M_2$ is defined by:

$$T_xF(v)(f) \equiv v(f \circ F) \tag{4.10}$$

where: $x \in M_a$, $v \in T_xM_1$, $f \in C^\infty(M_2)$. This is also denoted by F_*

$$(F_*\xi)_{F(x)} = T_xF(\xi_x) \tag{4.11}$$

Where ξ is a vector field on M_1, $F_*\xi$ is a vector field on M_2

Some properties of tangent map

(i) If there are two differential map $G : M_1 \longrightarrow M_2$ and $F : M_2 \longrightarrow M_3$. Then:

$$T_x(F \circ G) = T_{G(x)}(F)T_x(G) \tag{4.12}$$

证明 Consider : $f \in C^\infty(M_3)$ $v \in T_xM_1$ $w = T_xG(v)$ $w \in T_{G(x)}M_2$
Then:

$$\begin{aligned} T_x(F \circ G)(v)(f) &= v(f \circ F \circ G) = v\big((f \circ F) \circ G\big) \\ &= T_x(G)(v)(f \circ F) \\ &= w(f \circ F) \\ &= T_{G(x)}(F)(w)(f) \\ &= \big[T_{G(x)}(F) \circ T_x(G)\ (v)\big](f) \end{aligned} \tag{4.13}$$

End of Proof

(ii) Tangent map of identity map id_M is $T_x id_M = id_{T_xM}$. Called Identity map on the tangent space.

(iii) if F is invertible map: $F : M_1 \longrightarrow M_2$, Then:

$$(T_xF)^{-1} = T_{F(x)}(F^{-1}) \tag{4.14}$$

local identification of tangent vector

Consider chart (ϕ, U) of M,

$$T_x\phi: \ T_xM \longrightarrow T_{\phi(x)}V = T_{\phi(x)}\mathbb{R}^n \tag{4.15}$$

From my perspective, this is a some what definition

$$T_x\phi(v) = v^i(x)\frac{\partial}{\partial x^i} \tag{4.16}$$

ϕ is invertible $\Rightarrow T_x\phi$ is invertible

$$dim(T_xM) = dim(T_{\phi(x)}\mathbb{R}^n) = n \tag{4.17}$$

Tangent map in coordinates

Consider differentiable map $F: M \longrightarrow N$

$$\begin{aligned} M: \ (\phi, U) \quad \phi(x) = (x^1, x^2 \cdots x^n) \\ N: \ (\psi, W) \quad \phi(y) = (y^1, y^2 \cdots y^m) \end{aligned} \tag{4.18}$$

$$\begin{aligned} T_x\phi(v) = v^i(x)\frac{\partial}{\partial x^i} \in T_{\phi(x)}\mathbb{R}^n \\ T_y\phi(w) = w^i(x)\frac{\partial}{\partial y^i} \in T_{\psi(y)}\mathbb{R}^m \end{aligned} \tag{4.19}$$

Consider:

$$\begin{aligned} \mathcal{F} = \psi \circ F \circ \phi^{-1} \\ \mathcal{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ \mathcal{F}: x^i \mapsto \mathcal{F}^j \end{aligned} \tag{4.20}$$

And f is a function of $(y^1, y^2 \cdots y^m)$

$$\begin{aligned} & T_{\phi(x)}(\mathcal{F})(v^i\partial_{x^i})(f) \\ & \qquad = v^i\partial_{x^i}(f \circ \mathcal{F})(x) \\ & \qquad = v^i\frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}(x) \\ & \qquad = v^i(\partial_{x^i}\mathcal{F}^j)(x)\ (\partial_{y^j}f)(y) \\ & \qquad = \left[D\mathcal{F}(x)\right]^j_i v^i\partial_{y^j}f(y) \end{aligned} \tag{4.21}$$

Integral curve

Integral curve of vector field ξ on manifold $\mathcal{M}$ is a differentiable curve

$$\alpha_\xi: \ [a, b] \longrightarrow M \quad t \in [a, b] \mapsto \alpha_\xi(t) \in M \tag{4.22}$$

Such that:

$$\partial_t\alpha_\xi(t) \equiv T_t(\alpha_\xi)(\partial t) = \xi_{\alpha(t)} \tag{4.23}$$

Flow

Flow $\phi(t, x) = \phi_t(x)$ of the vector field ξ on M is given by unique integral curve with the initial condition $\phi(0, x) = x$

4.2 Lie Algebra

left translation

take a fixed element h, the left translation is defined by:

$$L_h : g \mapsto hg \tag{4.24}$$

In coordinates, the left translation would be:

$$L_h : \alpha^a \mapsto \beta^a(\alpha^a) \tag{4.25}$$

which satisfies:

$$\varphi(g) = \alpha^a \tag{4.26}$$

and

$$\varphi(hg) = \beta^a \tag{4.27}$$

The left translation also acts on functions on manifold. For a function f, the new function is the old function move along the manifold.

$$(L_h f)(hg) = f(g) \tag{4.28}$$

(这一段似乎有问题)It also induces a map on tangent vectors, the differential map (or push-forward), which maps tangent vector X at the point g to the vector $dL_h \circ X$ at the point hg, which satisfies:

$$(dL_h \circ X)[f(hg)] = X[f(g)] \tag{4.29}$$

The differential map allows us to single out a particular kind of vector fields, namely those that are invariant under the differential maps of all left translations

定义 4.2 (left-invariant)
A vector field is called left invariant if:

$$X|_{hg} = dL_h \circ X|_g \quad \text{for all } g, h \in G \tag{4.30}$$

♣

In the difinition, for left-invariant vector fields:

$$X|_{hg}[f(hg)] = X|_g[f(g)] \tag{4.31}$$

下面没问题了

Left-invariant vector field

$$T_x L_g(\xi_x) = \xi_{gx} \quad \forall x, g \in G, \tag{4.32}$$

Also denote as:

$$(g_*\xi)_{gx} = \xi_{gx} \tag{4.33}$$

Vector space isomorphism

The vector space of left invariant fields is isomorphic the tangentspace.

$$\mathcal{L}(G) \xrightarrow{\cong} T_eG \tag{4.34}$$

$$dim(\mathcal{L}(G)) = dim(T_eG) = dim(G) \tag{4.35}$$

Lie algebra is closed under commute operation

If ξ, η are left invariant vector fields on G, so is $[\xi, \eta]$

Firstly:

$$(g_*^{-1}\xi)_x(f) = \xi_{gx}(f \circ g^{-1}) = \xi(f \circ g^{-1})(gx) \tag{4.36}$$

$$g_*^{-1}\xi(f) = \xi(f \circ g^{-1}) \circ g \tag{4.37}$$

Then:

$$\begin{aligned}
[g_*^{-1}\xi, g_*^{-1}\eta](f) &= g_*^{-1}\xi\big(g_*^{-1}\eta(f)\big) - g_*^{-1}\eta\big(g_*^{-1}\xi(f)\big) \\
&= g_*^{-1}\xi\big(\eta(f \circ g^{-1}) \circ g\big) - g_*^{-1}\eta\big(\xi(f \circ g^{-1}) \circ g\big) \\
&= \xi\big(\eta(f \circ g^{-1})\big) \circ g - \eta\big(\xi(f \circ g^{-1})\big) \circ g \\
&= [\xi, \eta](f \circ g^{-1}) \circ g \\
&= g_*^{-1}[\xi, \eta](f)
\end{aligned} \tag{4.38}$$

In all:

$$g_*^{-1}[\xi, \eta] = [g_*^{-1}\xi, g_*^{-1}\eta] = [\xi, \eta] \tag{4.39}$$

Lie algebra g

定义 4.3

The Lie algebra $\boldsymbol{g}$ of a group G is the space of left-invariant vector fields with the Lie bracket as product. ♣

Some Properties of flow

Define integral curve of vector field ξ

$$\alpha_\xi(t) = \phi(t, e) \tag{4.40}$$

If ξ is a left invariant field,

$$\phi(t, x) = x\alpha_\xi(t) \tag{4.41}$$

证明 Obviously:

$$\phi(0, x) = x\alpha_\xi(0) = x \tag{4.42}$$

Consider:

$$\begin{aligned}\partial_t(x\alpha_\xi) &= T_t(L_x \circ \alpha_\xi)(\partial_t) \\ &= T_{\alpha_\xi(t)}L_x \circ T_t\alpha_\xi(\partial_t) = T_{\alpha_\xi(t)}L_x(\xi_{\alpha_\xi(t)}) \\ &= \xi_{L_x\alpha_\xi(t)}\end{aligned} \tag{4.43}$$

End of Proof

推论 4.1

$$\alpha_\xi(s+t) = \alpha_\xi(s)\alpha_\xi(t) \tag{4.44}$$

♡

证明

$$\alpha_\xi(s+t) = \phi_{s+t}(e) = \phi(t, \phi_s(e)) = \phi_s(e)\alpha_\xi(t) = \alpha_\xi(s)\alpha_\xi(t) \tag{4.45}$$

End of Proof

4.3 Lie Group Representation

We want to represent Lie group on the vector space of Lie Algebra:

Adjoint is a Group Homomorphism

Consider conjugacy

$$C_g : \ G \to G \ \ C_g(x) = gxg^{-1} \tag{4.46}$$

Aut(G) is a Group of all invertible endomorphisms(Hom(G,G), Aut means automorphisms)

$$C_g \in Aut(G) \tag{4.47}$$

There is a Group homomorphism:

$$C : G \to Aut(G) \quad g \mapsto C_g \tag{4.48}$$

C is indeed a group homomorphism:

$$C_{g_1g_2}(x) = g_ag_2x(g_1g_2)^{-1} = g_1g_2xg_2^{-1}g_1^{-1} = C_{g_1} \circ C_{g_2}(x) \tag{4.49}$$

Representation of Lie Group

Consider:

$$C_g(e) = e \tag{4.50}$$

Tangent Map:

$$T_eC_g : T_eG \to T_eG \tag{4.51}$$

定义 4.4 (Adjoint Representatioin of Lie Group G)

$$Ad : G \to GL(\mathcal{L}(G)) \tag{4.52}$$

$$Ad(g) = T_eC_g : \ T_eG \to T_eG \tag{4.53}$$

♣

Ad is indeed a representation:

$$Ad(g_1g_2) = T_eC_{g_1g_2} = T_E(C_{g_1} \circ C_{g_2}) = T_eC_{g_1} \circ T_eC_{g_2} = Ad(g_1) \circ Ad(g_2) \tag{4.54}$$

4.4 Lie algebra representation

Adjoint Representation of the Lie algebra

定义 4.5 (Adjoint Representation of Lie algebra)

The adjoint representation of Lie algebra on itself is defined by:

$$ad = T_eAd \tag{4.55}$$

$$ad : \mathcal{L}(G) \to End\big(\mathcal{L}(G)\big) \tag{4.56}$$

♣

$$Ad(e) = id_{\mathcal{L}(G)} \tag{4.57}$$

∘ $End\big(\mathcal{L}(G)\big)$ Contains all $n \times n$ matrixes.
∘ The image of Ad, $Img(Ad) \subset GL(\mathcal{L}(G))$ is a n dimensional manifold,It's tangent space at $Ad(e) = id_{\mathcal{L}(G)}$ is a n dimensional vector space.
∘ T_eG is isomorphic with $\mathcal{L}(G)$

The vector at the space $Img(Ad) \subset GL(\mathcal{L}(G))$ is some what:

$$\partial_t\alpha(t) \tag{4.58}$$

In which:

$$\alpha(t) \in GL(\mathcal{L}(G)) \tag{4.59}$$

We consider:

$$\partial_t\left(\alpha(t)(\xi_e)\right) \quad \xi_e \in T_eG \to \partial_t\eta_e(t) \quad \eta_e \in T_eG \tag{4.60}$$

We will find $\partial_t\left(\alpha(t)(\xi)\right) \in T_eG$

推论 4.2

$$ad(\xi_e)(\eta_e) = [\xi, \eta]_e \tag{4.61}$$

♡

证明 Consider:

$$C_{\alpha_\xi(s)}\big(\alpha_\eta(t)\big) = \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(s)^{-1} = \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \tag{4.62}$$

Then:

$$\begin{aligned} Ad\big(\alpha_\xi(s)\big)(\eta_e) = T_e\big(C_{\alpha_\xi(s)}\big)(\eta_e) = T_e(C_{\alpha_\xi(s)})T_0(\alpha_\eta)(\partial t) = T_0(C_{\alpha_\xi(s)} \circ \alpha_\eta)(\partial t) \\ = \partial_t|_0 \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \end{aligned} \tag{4.63}$$

$$\begin{aligned} ad(\xi_e)(\eta_e) &= T_e Ad(\xi_e)(\eta_e) = T_e Ad\ T_0 \alpha_\xi(\partial_s)(\eta_e) \\ &= T_0(Ad \circ \alpha_\xi)(\partial_s)\eta_e \\ &= \partial_s|_0 Ad(\alpha_\xi(s))\eta_e \\ &= \partial_s \partial_t|_0 \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \\ ad(\xi_e)(\eta_e)(f) &= \partial_s\partial_t|_0 f(\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s)) \quad \text{这一步当作定义} \\ &= \partial_s\partial_t|_0 f(\alpha_\xi(s)\alpha_\eta(t)) - \partial_s\partial_t|_0 f(\alpha_\eta(t)\alpha_\xi(s)) \\ &= \xi \circ \eta(f)_e - \eta \circ \xi(f)_e \end{aligned} \tag{4.64}$$

End of Proof

推论 4.3

The adjoint Representation is a Lie algebra representation

证明

$$\begin{aligned} ad([\xi,\eta])(\lambda) &= [[\xi,\eta],\lambda] = [\xi,[\eta,\lambda]] - [\eta,[\xi,\eta]] \\ &= ad(\xi) \circ ad(\eta)(\lambda) - ad(\eta) \circ ad(\xi)(\eta) \\ &= [ad(\xi), ad(\eta)](\lambda) \end{aligned} \tag{4.65}$$

End of Proof

Suppose we choose a base, Then:

$$ad(\xi_i)(\xi_j) = [\xi_i, \xi_j] = f_{ij}{}^k \xi_k \tag{4.66}$$

denote:

$$[ad(\xi_i)]^k{}_j = f_{ij}{}^k \tag{4.67}$$

Commutation Relation

Will introduce some commutation relation

- Left Diagram:

$$F \circ C_g\ (x) = F\ (gxg^{-1}) = F(g)F(x)F(g)^{-1} = C_{F(g)} \circ F\ (x) \tag{4.68}$$

- Right Diagram

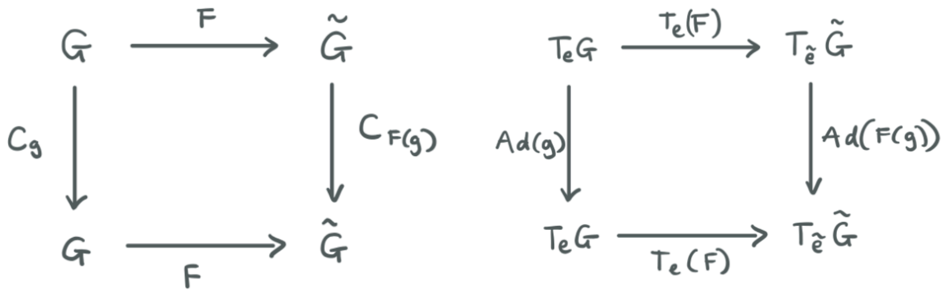

图 **4.1:** Commute Diagrams,Lie Group

$$T_e(F) \circ Ad(g) = T_e(F) \circ T_e(C_g) = T_e(F \circ C_g) = T_{\tilde{e}}(C_{F(g)}) \circ T_e(F) = Ad(F(g)) \circ T_e(F) \quad (4.69)$$

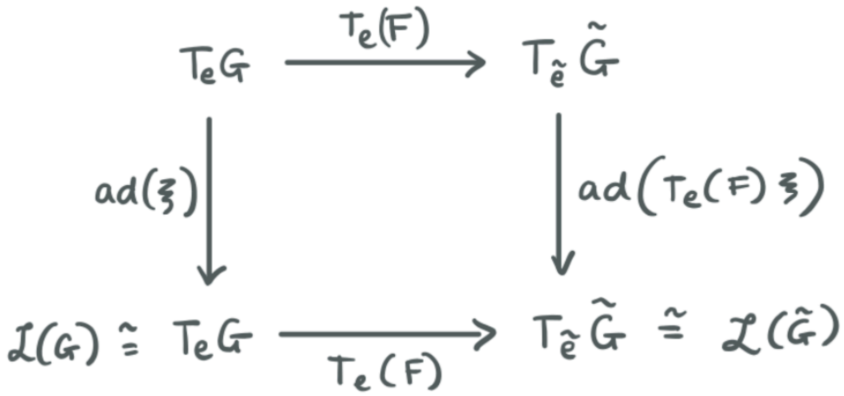

图 **4.2:** Commute Diagram,Lie Algebra

认为:

$$T_eF(\xi_e) = \tilde{\xi}_{\tilde{e}} \quad (4.70)$$

假设在 G 和 G' 中可以定义 Integral Curve:

$$\alpha_\xi(s) \quad \alpha_{\tilde{\xi}}(s) \quad (4.71)$$

$\tilde{G}$ 中的 Integral curve $\alpha_{\tilde{\xi}}(s)$ 需要满足的条件:

$$\tilde{\xi}_{\tilde{e}}(f) = T_0\alpha_{\tilde{\xi}}(\partial_s)(f) \quad f : \tilde{G} \to \mathbb{R} \quad (4.72)$$

考虑到:

$$\begin{cases} T_eF(\xi_e) = \tilde{\xi}_{\tilde{e}} \end{cases} \quad (4.73)$$

于是:

$$\tilde{\xi}_{\tilde{e}}(f) = T_eF(\xi_e)(f) = T_0\alpha_{\tilde{\xi}}(\partial_s)(f) \quad (4.74)$$

G 中的 integral curve $\alpha_\xi(s)$ 需要满足的条件:

$$\xi_e(f') = T_0\alpha_\xi(\partial_s)(f') \quad f' : G \to \mathbb{R} \quad (4.75)$$

下面证明:

$$\alpha_{\tilde{\xi}}(s) = F\alpha_{\xi}(s) \tag{4.76}$$

证明 直接将假设带入 $\tilde{G}$ 中 Integral curve $\alpha_{\tilde{\xi}}(s)$ 需要满足的条件:

$$\begin{cases} T_eF(\xi_e)(f) = T_0\alpha_{\tilde{\xi}}(\partial_s)(f) \end{cases} \tag{4.77}$$

注意到, 右式:

$$\begin{aligned} T_0\alpha_{\tilde{\xi}}(\partial_s)(f) &= T_0F\alpha_{\xi}(\partial_s)(f) \\ &= \partial_s|_0 f\left(F\alpha_{\xi}(s)\right) \end{aligned} \tag{4.78}$$

定义:

$$f' = f \circ F \quad G \to \mathbb{R} \tag{4.79}$$

于是:

$$\begin{aligned} \text{上式} &= \partial_s|_0 f'\left(\alpha_{\xi}(s)\right) \\ &= T_0\alpha_{\xi}(\partial_s)(f \circ F) \end{aligned} \tag{4.80}$$

由于 G 中 Integral Curve 满足条件:

$$\begin{cases} \xi_e(f') = T_0\alpha_{\xi}(\partial_s)(f') \quad f' : G \to \mathbb{R} \end{cases} \tag{4.81}$$

于是:

$$\begin{aligned} \text{上式} &= \xi_e(f \circ F) \\ &= T_eF(\xi_e)(f) \end{aligned} \tag{4.82}$$

□

接下来证明:

$$T_e(F) \circ ad(\xi_e)(\eta_e) = ad\left(T_eF(\xi_e)\right)\left(T_eF\eta_e\right) \tag{4.83}$$

证明 考虑到 adjoint representation 的性质:

$$\begin{cases} ad(\xi_e)(\eta_e) = \partial_s\partial_t|_0\alpha_{\xi}(s)\alpha_{\eta}(t)\alpha_{\xi}(-s) \end{cases} \tag{4.84}$$

于是，左式:

$$\begin{aligned} T_eF\left(ad(\xi_e)(\eta_e)\right)(f) &= T_eF\left(\partial_s\partial_t|_0\alpha_{\xi}(s)\alpha_{\eta}(t)\alpha_{\xi}(-s)\right)(f) \\ &= \partial_s\partial_t|_0 f\left(F\alpha_{\xi}(s)\alpha_{\eta}(t)\alpha_{\xi}(-s)\right) \end{aligned} \tag{4.85}$$

右式:

$$\begin{aligned} ad\left(T_eF(\xi_e)\right)\left(T_eF(\eta_e)\right)(f) &= ad(\tilde{\xi}_{\tilde{e}}, \tilde{\eta}_{\tilde{e}})(f) \\ &= \partial_s\partial_t|_0 f\left(\alpha_{\tilde{\xi}}(s)\alpha_{\tilde{\eta}}(t)\alpha_{\tilde{\xi}}(-s)\right) \end{aligned} \tag{4.86}$$

由于前面得到的对易关系:

$$\begin{cases} F \circ C_g\ (x) = F\ (gxg^{-1}) = F(g)F(x)F(g)^{-1} = C_{F(g)} \circ F\ (x) \end{cases} \tag{4.87}$$

于是:

$$\begin{aligned} F\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) &= F \circ C_{\alpha_\xi(s)}\left(\alpha_\eta(t)\right) \\ &= C_{F(\alpha_\xi(s))} \circ F\left(\alpha_\eta(t)\right) \\ &= F\left(\alpha_\xi(s)\right) F\left(\alpha_\eta(t)\right) F\left(\alpha_\xi(-s)\right) \\ &= \alpha_{\tilde\xi}(s)\alpha_{\tilde\eta}(t)\alpha_{\tilde\xi}(-s) \end{aligned} \tag{4.88}$$

也就是:

$$\partial_s\partial_t|_0 f\left(F\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s)\right) = \partial_s\partial_t|_0 f\left(\alpha_{\tilde\xi}(s)\alpha_{\tilde\eta}(t)\alpha_{\tilde\xi}(-s)\right) \tag{4.89}$$

$$T_e(F) \circ ad(\xi_e)(\eta_e) = ad\left(T_eF(\xi_e)\right)\left(T_eF\eta_e\right) \tag{4.90}$$

□

更近一步:

$$T_eF\left([\xi,\eta]_e\right) = [T_eF(\xi_e), T_eF(\eta_e)] \tag{4.91}$$

States $T_e(F)$ is a Lie Algebra morphism

定理 4.1

A linear map $f : \mathcal{L}(G) \to \mathcal{L}(\tilde G)$ is the tangent map of a group homomorphism F iff f is the tangent map of a group homomorphism F.

♡

4.5 Exponential map

Exponential map constructs the Lie Group From its Lie Algebra.

定义 4.6 (Exponential map)

For a left-invariant vector field ξ on group G, with $v = \xi_e$. we have an integral curve α_v. with condition $\alpha_v(0) = e$. Then, the exponential map: $Exp : T_eG \to G$ is defined by $Exp(v) = \alpha_v(1)$

♣

The exponential map is:

∘(1) differentiable at the origin and $T_0(Exp) = id_{\mathcal{L}(G)}$

∘(2) maps $\mathcal{L}(G) \cong T_e(G)$ diffeomorphically into a neighbourhood of $e \in G$.

∘(3) satisfies: $F \circ Exp = \tilde{Exp} \circ T_e(F)$ for a group homomorphism $F : G \to \tilde G$

For the (1) statement,

证明 denote: $m_v(t) = tv$.

$$Exp \circ m_v(t) = Exp(vt) = \alpha_{vt}(1) = \alpha_v(t) \tag{4.92}$$

Then:

$$\partial_t|_0 Exp \circ m_v(t) = T_0(Exp \circ m_v)(\partial_t) = T_0(Exp) \circ T_0(m_v)(\partial t) \tag{4.93}$$

State that:

$$T_0(m_v)(\partial t) = \partial_t|_0 m_v(t) = v \tag{4.94}$$

Then:

$$T_0(Exp)v = \partial_t|_0 \alpha_v(t) = v \tag{4.95}$$

End of Proof

For the (3) statement:

start with a lemma

引理 4.1 (Group homomorphism maps left-invariant field into left invariant field)

If $F : G \longrightarrow \tilde{G}$ is a group homomorphism and ξ is a left invariant vector field on G. then $\tilde{\xi} = F_*(\xi)$ is a Left-invariant vector field on $\tilde{G}$.

For this lemma, denote $\tilde{x} = F(x)$, $\tilde{g} = F(g)$, $\tilde{\xi} = F_*(\xi)$.

$$\begin{aligned} L_{\tilde{g}} \circ F(x) = F(g)F(x) = F(gx) &= F \circ L_g \\ L_{\tilde{g}} \circ F(x) &= F \circ L_g \end{aligned} \tag{4.96}$$

Then consider Lefi-invariant condition on $\tilde{x}$.

$$\begin{aligned} T_{\tilde{x}} L_{\tilde{g}}(\tilde{\xi}_{\tilde{x}}) = T_{\tilde{x}} L_{\tilde{g}} \circ T_x F(\xi_x) &= T_x(L_{\tilde{g}} \circ F)(\xi_x) \\ &= T_x(F \circ L_g)(\xi_x) \\ &= T_{gx} F \circ T_x L_g(\xi_x) \\ &= T_{gx} F(\xi_{gx}) = \tilde{\xi}_{F(gx)} = \tilde{\xi}_{\tilde{g}\tilde{x}} \end{aligned} \tag{4.97}$$

Then, back to (3)

证明 define $\beta_w = F \circ \alpha_v$ is an integral curve on $\tilde{G}$. associate to Left invariant vector field. Then,

$$w = \partial_t|_0 \beta_w(t) = \partial_t|_0 F \circ \alpha_v = T_0(F \circ \alpha_v)(\partial_t) = T_e F \circ (\dot{\alpha}_v(0)) = T_e F(v) \tag{4.98}$$

We prooved that:

$$w = T_e F(v) \tag{4.99}$$

Hence,

$$\begin{aligned} \tilde{Exp} \circ T_e F(v) = \tilde{Exp}\big(T_e F(v)\big) = \tilde{Exp}(w) &= \beta_w(1) \\ &= F \circ \alpha_v(1) = F \circ Exp(v) \end{aligned} \tag{4.100}$$

End of proof

4.6 Matrix Lie Group

Parametrices

Entries: Work in a chart around the group identity $\mathbb{I}_d$. parametrices:

$$g = g(t) \quad t = (t^1 \cdots t^n) \in \mathbb{R}^n \tag{4.101}$$

Where:

$$g(0) = \mathbb{I}_d \tag{4.102}$$

and g has entries $g_\mu^{\ \nu}$, where $\mu\nu = 1 \cdots d$.

Vector fields: $\xi_t = \xi^i(t)\frac{\partial}{\partial t^i} = \xi^i(t)\frac{\partial g_\mu^{\ \nu}}{\partial t^i}(t)\frac{\partial}{\partial g_\mu^{\ \nu}} = \xi^i(t)tr\left(\frac{\partial g}{\partial t^i}(t)\frac{\partial}{\partial g^T}\right)$

Generators: $T_i = \frac{\partial g}{\partial t^i}(0)$

Tangent space at identity

$$\begin{aligned} \xi_{t=0} &= \xi^i(0)tr(T_i\frac{\partial}{\partial g^T}) \\ T_{\mathbb{I}}G &= \left\{v^i tr(T_i\frac{\partial}{\partial g^T})|v \in \mathbb{R}^n\right\} \cong \left\{v^i T_i | v \in \mathbb{R}^n\right\} \cong \mathcal{L}(G) \end{aligned} \tag{4.103}$$

Left-invariant condition

Consider a tangent map that maps the vector field

$$T_x L_{g_0}(\xi_x)(f) = \xi_x(f \circ L_{g_0}) \tag{4.104}$$

The vectorfield ξ at the point x is:

$$\xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x)\frac{\partial}{\partial g_\rho^{\ \sigma}} \tag{4.105}$$

denote that:

$$f \circ L_{g_0}(g) = f(g_0 g) = f'(g) \tag{4.106}$$

Then:

$$\xi_x(f') = \xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x)\frac{\partial}{\partial g_\rho^{\ \sigma}} f(g_0 g)|_{g=x} \tag{4.107}$$

Consider the last part: $(g_1 = g_0 g)$

$$\begin{aligned}\frac{\partial}{\partial g_\rho^{\ \sigma}} &= \frac{\partial g_{1\mu}^{\ \ \nu}}{\partial g_\rho^{\ \sigma}} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\ &= \frac{\partial (g_0 g)_\mu^{\ \nu}}{\partial g_\rho^{\ \sigma}} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\ &= \frac{\partial (g_{0\mu}^{\ \ \tau} g_\tau^{\ \nu})_\mu^{\ \nu}}{\partial g_\rho^{\ \sigma}} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\ &= g_{0\mu}^{\ \ \rho} \delta^\nu_\sigma \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\ &= (T_x L_{g_0})^{\nu\rho}_{\mu\sigma} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}}\end{aligned} \tag{4.108}$$

Then:

$$\begin{aligned}T_x L_{g_0}(\xi_x)(f) &= \xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x) (T_x L_{g_0})^{\nu\rho}_{\mu\sigma} \frac{\partial}{\partial g_\mu^{\ \nu}}|_{g=g_0 x} f \\ &= \xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x) g_{0\mu}^{\ \ \rho} \delta^\nu_\sigma \frac{\partial}{\partial g_\mu^{\ \nu}}|_{g=g_0 x} f\end{aligned} \tag{4.109}$$

However:

$$\xi_{g_0 x} = \xi^i \frac{\partial g_\mu^{\ \nu}}{\partial t^i} \frac{\partial}{\partial g^\nu_\mu}(g_0 x) \tag{4.110}$$

The constrain would be:

$$\xi^i \frac{\partial g_\mu^{\ \nu}}{\partial t^i}(g_0 x) = \xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x) g_{0\mu}^{\ \ \rho} \delta^\nu_\sigma \tag{4.111}$$

Write as:

$$\xi^i(gx) \frac{\partial g}{\partial t^i}(gx) = \xi^i(x) g \ \frac{\partial g}{\partial t^i}(x) \tag{4.112}$$

Consider: $x = \mathbb{I}$ and $t = 0$

$$\xi^i(g) \frac{\partial g}{\partial t^i}(g) = \xi^i(0) g \ T_i \tag{4.113}$$

Consider a basis of Left invariant vector field:

$$L_i = \xi_i^j \partial_{t^j} \tag{4.114}$$

satisfies:

$$\xi_i^j(0) = \delta_i^j \tag{4.115}$$

Then:

$$\xi_i^j \frac{\partial g}{\partial t^j} = g T_i \tag{4.116}$$

The Left-invariant vector field:

$$L_i = \xi_i^j \frac{\partial}{\partial t^j} = \xi_i^j tr(\frac{\partial g}{\partial t^j}\frac{\partial}{\partial g^T})$$
$$= tr(gT_i \frac{\partial}{\partial g^T}) \tag{4.117}$$
$$L_{i,\mathbb{I}} = tr(T_i \frac{\partial}{\partial g^T})$$

We can regard Lie Algebra as a span of the generators.

Commutator

$$\begin{aligned}[L_i, L_j] &= tr(gT_i \frac{\partial}{\partial g^T}) \circ tr(gT_j \frac{\partial}{\partial g^T}) - \ (i \leftrightarrow j) \\ &= tr(g[T_i, T_j]\frac{\partial}{\partial g^T}) \in \mathcal{L}(G)\end{aligned} \tag{4.118}$$

In this case:

$$[T_i, T_j] = f_{ij}{}^k L_k \tag{4.119}$$

Exponential map

Left-invariant vector field can be written as $\xi^i = v^j \xi^i_j$. Associated with generator: $T = v^i T_i$. Integral curve: $t_i = t^i(s)$, $\alpha_v(s) = g(t(s))$ Satisfies:

$$\frac{dt^i}{ds}\partial_{t_i} = v^j \xi^i_j \partial_{t_i} \tag{4.120}$$

$$v^j \xi^i_j \partial_{t_i} = T_s(t(s))(\partial_s) \tag{4.121}$$

Then:

$$\frac{d\alpha_v}{ds} = \frac{\partial g}{\partial t^i}\frac{dt^i}{ds} = \frac{\partial g}{\partial t^i} v^j \xi^i_j = v^j g T_j = \alpha_v(s) T \tag{4.122}$$

Where used:

$$\xi_i^j \frac{\partial g}{\partial t^j} = gT_i \tag{4.123}$$

and

$$T = v^j T_j \tag{4.124}$$

Then:

$$Exp(T) = \alpha_v(1) = exp(T) \tag{4.125}$$

Ad representation

Consider the adjoint:

$$C_g(x) = gxg^{-1} \tag{4.126}$$

Then the adjoint representation would be:

$$\begin{aligned} Ad(g)(T) = T_e C_g(T) &= exp^{-1} \circ C_g \circ exp(T) \\ &= exp^{-1}(g e^T g^{-1}) \\ &= g exp^{-1}(e^T) g^{-1} \\ &= g T g^{-1} \end{aligned} \tag{4.127}$$

Which means Adjoint representation amounts to conjugation. Where we used:

$$F \circ Exp = \tilde{Exp} \circ T_e(F) \tag{4.128}$$

ad representation

$$ad(T)(S) = [T, S] \tag{4.129}$$

denote

$$\left[ad(T_i)\right]^k_{\ j} = f_{ij}^{\ \ k} \tag{4.130}$$

第五章 Lie algebras

The algebra is a **vector space**, we choose a **basis** T_i of Lie algebra $\boldsymbol{g}$.

Then, we introduce the **structure constants**

$$[T_i, T_j] = f_{ij}{}^k T_k \tag{5.1}$$

For which, we introduce the conventions:

$$[A, B]^\dagger = [B^\dagger, A^\dagger] \tag{5.2}$$

If $T_i^\dagger = \pm T_i$ (T_i is hermitean or anti-Hermitean), it can be deduced that:

$$(f_{ij}^k)^\dagger = -f_{ij}^k \tag{5.3}$$

which means the structure constant is purely imaginary, for this, we usually denote:

$$[T_i, T_j] = i f_{ij}^k T_k \tag{5.4}$$

定义 5.1 (Representation of Lie algebra)

A representation of Lie algebra is a **Lie algebra homomorphism** D from $\boldsymbol{g}$ to a **Lie algebra of matrices** with the matrix commutator as Lie bracket. The dimension of the representation is the dimension of the vector space. ♣

Notice:

- A representation is reducible if there is an invariant subspace.
- Representations which are related by a similarity transformation are called equivalent.

5.1 Structure of Lie Algebras

For a Lie Algebra $\mathcal{L}$, have the definition:

(i) A subset $\mathcal{A} \subset \mathcal{L}$ is called Lie sub-Algebra iff $\mathcal{A}$ $\mathcal{A}$ is a linear space and $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$.
(ii) A sub-algebra is called Abelian iff: $[\mathcal{A}, \mathcal{A}] = 0$
(iii) A sub-algebra is an ideal iff $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$. An ideal is called non-trival iff: $\mathcal{A} \neq \{0\}, \mathcal{L}$
(iv) The derived series $\{\mathcal{D}^k\mathcal{L}\}$ of $\mathcal{L}$ is defined by: $\mathcal{D}^1\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ and $\mathcal{D}^k\mathcal{L} = [\mathcal{D}^{k-1}\mathcal{L}, \mathcal{D}^{k-1}\mathcal{L}]$
(v) $\mathcal{L}$ is called solvable iff $\mathcal{D}^k\mathcal{L} = 0$ for some k.

A Lie Algebra is called:

(i) Simple (no non-trival ideals), if: $\nexists \mathcal{A}$ ($[\mathcal{L}, \mathcal{A}] \subset A$; $\mathcal{A} \neq \emptyset, \mathcal{L}$) s.t. $\mathcal{A} \subset \mathcal{L}$

(ii) Semi-simple (no non-zero solvable ideals), if: $\nexists \mathcal{A}$ ($\mathcal{A} \neq \emptyset$; $\mathcal{D}^k \mathcal{A} = 0$; $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$) s.t. $\mathcal{A} \subset \mathcal{L}$

引理 5.1 (semi-simple means has no non-zero Abelian ideals)

($\mathcal{L}$ is semi-simple) $\nexists \mathcal{A}$ ($\mathcal{A} \neq \emptyset$; $\mathcal{D}^k \mathcal{A} = 0$; $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$) s.t. $\mathcal{A} \subset \mathcal{L}$ $\Longleftrightarrow$ $\nexists \mathcal{A}$ ($\mathcal{A} \neq \emptyset$; $[\mathcal{A}, \mathcal{A}] = 0$; $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$) s.t. $\mathcal{A} \subset \mathcal{L}$ (not exist abelian non-zero ideal) ♡

证明 左逆命题 $\Leftarrow$ 右边逆命题: If $\mathcal{L}$ has non-zero Abelian ideals $\mathcal{A}$, then it is also non-zero solvable ideals.(k=1) which means it is not semi-simple.

左逆命题 $\Rightarrow$ 右边逆命题: 首先证明:

$$if : [\mathcal{L}, \mathcal{A}] \subset \mathcal{A} \Rightarrow [\mathcal{L}, \mathcal{D}^k \mathcal{A}] \subset \mathcal{D}^k \mathcal{A} \tag{5.5}$$

证明 用数学归纳法，假设

$$[\mathcal{L}, \mathcal{D}^{k-1} \mathcal{A}] \subset \mathcal{D}^{k-1} \mathcal{A} \tag{5.6}$$

于是:

$$\begin{aligned}
[\mathcal{L}, \mathcal{D}^k \mathcal{A}] &= \left[\mathcal{L}, [\mathcal{D}^{k-1} \mathcal{A}, \mathcal{D}^{k-1} \mathcal{A}]\right] \\
&= -\left[\mathcal{D}^{k-1} \mathcal{A}, [\mathcal{D}^{k-1} \mathcal{A}, \mathcal{L}]\right] - \left[\mathcal{D}^{k-1} \mathcal{A}, [\mathcal{L}, \mathcal{D}^{k-1} \mathcal{A}]\right] \\
&\subset \left[\mathcal{D}^{k-1} \mathcal{A}, \mathcal{D}^{k-1} \mathcal{A}\right] + \left[\mathcal{D}^{k-1} \mathcal{A}, \mathcal{D}^{k-1} \mathcal{A}\right] \\
&\subset \mathcal{D}^k \mathcal{A}
\end{aligned} \tag{5.7}$$

□

If $\mathcal{L}$ is not semi simple. Then it has solvable non-zero ideals $\mathcal{A}$. So $\mathcal{D}^{k-1} \mathcal{A}$ is ideal (proved above), $\mathcal{D}^{k-1} \mathcal{A}$ is abelian (Which means it has non-zero Abelian ideals $\mathcal{D}^{k-1} \mathcal{A}$)

End of Proof

5.2 Killing Form

定义 5.2 (Killing Form)

The symmetric bilinear form: $\Gamma : \quad \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ defined by:

$$\Gamma(T, S) = tr\big(ad(T) ad(S)\big) \tag{5.8}$$

is called the killing form of $\mathcal{L}$ need to proof this has no relation with basis ♣

Denote:

$$\begin{aligned}\gamma_{ij} &= \Gamma(T_i, T_j) = tr\big(ad(T_i)ad(T_j)\big) \\ &= f_{ik}{}^{l} f_{jl}{}^{k}\end{aligned} \tag{5.9}$$

In this case:

$$\Gamma(T, S) = \Gamma(t^i T_i, s^j T_j) = t^i s^j \gamma_{ij} \tag{5.10}$$

Killing Form and Lie-Algebra Structure

∘killing Form Γ is called non-degenerate iff $\forall T \in \mathcal{L}\ \Gamma(T, S) = 0$ implies $S = 0$

∘killing Form Γ is called non-degenerate iff matrix $\gamma_{ij}$is invertible.

定理 5.1 (semi-simple and non-degenerate)

A Lie Algebra $\mathcal{L}$ is semi-simple iff Γ is non-degenerate

证明 ⇐If Γ is non-degenerate, Consider Lie Algebra $\mathcal{L}$ has an abelian ideal $\mathcal{A}$.(含有 abelian ideal 等价于不是 semi-simple) Consider basis:(T_a, T_α), T_a is basis of $\mathcal{A}$, T_α the basis of the remainder. Consider $\forall T \in \mathcal{L}, \forall S \in \mathcal{A}$.

$$\begin{aligned} ad(T) \circ ad(S)(T_a) &= [T, [S, T_a]] = [T, 0] = 0 \\ ad(T) \circ ad(S)(T_\alpha) &= [T, [S, T_\alpha]] \in \mathcal{A} \end{aligned} \tag{5.11}$$

For a matrix of $ad(T) \circ ad(S)$

$$ad(T) \circ ad(S) = \begin{bmatrix} 0_1 & * \\ 0_2 & 0_3 \end{bmatrix} \tag{5.12}$$

For this, $0_1 = 0_{a,a}$ $0_3 = 0_{\alpha,\alpha}$

Killing Form:

$$\Gamma(T, S) = tr(ad(T)ad(S)) = 0 \tag{5.13}$$

Then, $S = 0$, $\mathcal{A} = 0$

⇒ ...

End of Proof

定理 5.2 (Negative semi-definite and Compact)

If G is compact, then the Killing form Γ on $\mathcal{L}(G)$ is negative semi-definite.

...

Jacobi identity for structure constants

Consider the relation:

$$[[T_i, T_j], T_k] + [[T_j, T_k], T_i] + [[T_k, T_i], T_j] = 0 \tag{5.14}$$

As for:

$$[T_i, T_j] = f_{ij}{}^{l} T_l \tag{5.15}$$

By insertion:

$$f_{ij}{}^{l} f_{lk}{}^{n} + f_{jk}{}^{l} f_{li}{}^{n} + f_{ki}{}^{l} f_{lj}{}^{n} = 0 \tag{5.16}$$

Totally anti-symmetric structure constant

introduce the structure constants

$$f_{ijk} = \gamma_{kl} f_{ij}{}^{l} \tag{5.17}$$

Insert the difinition of γ, and use the Jacobi indentity for structure constants talked before, we find that

$$f_{ijk} = tr(f_k f_j f_i) + tr(\tilde{f}_k \tilde{f}_i \tilde{f}_j) \tag{5.18}$$

for the tilde term:

$$(\tilde{f}_i)_j{}^{k} = f_{ji}{}^{k} \tag{5.19}$$

Cyclicity of the trace shows that the structure constants f_{ijk} are unchanged under cycle permutation, Consider the anti-symmetric for the first two term. We Conclude that f_{ijk} is totally anti-symmetric.

Quadratic Casimir If the algebra $\mathcal{L}$ is semi-simple, we define the quadratic Casimir operator:

$$C = \gamma^{ij} T_i T_j \tag{5.20}$$

Its relevance is that it commutes with the entire Lie algebra.

定理 5.3 (commutation of Casimir operator)

The Casimir operator satisfies $\forall T \in \mathcal{L} \quad [C, T] = 0$

By using Shur's Lemma, if the Lie Algebra is irreduciable, Then

$$C = \lambda \mathbb{I} \tag{5.21}$$

5.3 Cartan-Weyl basis

定义 5.3 (Cartan Algebra)

A maximal, diagonalisable **Abelian** subalgebra $H \subset \mathcal{L}$ is called a Cartan subalgebra. The dimension of H is called the rank of $\mathcal{L}$ denoted by $rk(\mathcal{L}) = dim(H)$

定理 5.4 (Commute Matrix and their mutual eigen values)

Consider two commute matrices A and B.

(a) We can always decompose the eigen space with eigen value α of operator A into direct sum of eigen space of operator B.

(b) We can find the common eigen space of operators A and B.

证明

$$AB|\alpha\rangle = BA|\alpha\rangle = \alpha(B|\alpha\rangle) \tag{5.22}$$

Which means $B|\alpha\rangle \in$ subspace of eigen value α
(i) If subspace of eigen value α is non degenerate:

$$B|\alpha\rangle = \beta|\alpha\rangle \tag{5.23}$$

(ii) If subspace of eigen value α is degenerate:

$$\begin{aligned} B|\alpha_1\rangle &= C_{11}|\alpha_1\rangle + C_{12}|\alpha_2\rangle \cdots \\ B|\alpha_2\rangle &= C_{21}|\alpha_1\rangle + C_{22}|\alpha_2\rangle \cdots \\ &\vdots \end{aligned} \tag{5.24}$$

we can decompose the subspace spaned by $|\alpha_1\rangle, |\alpha_2\rangle \cdots$ into eigenspace of operator B. by studying the eigen value of matrix C^T.
This means, (a) We can always decompose the eigen space with eigen value α of operator A into direct sum of eigen space of operator B. (b) We can find the common eigen space of operators A and B.

End of proof

Cartan Decomposition We can study the simultaneous eigenvectors $T \in \mathcal{L}(G)$ Satisfies the equation:

$$ad(H)(T) = \alpha(H)T \tag{5.25}$$

For which $\alpha(H)$ depends on elements in H linearly. $\alpha \in \mathcal{H}'$, where $\mathcal{H}'$ is the dual space of $\mathcal{H}'$
Denote mutual eigen-space $\mathcal{L}_\alpha \subset \mathcal{L}$ the eigen space of root α. The Lie Algebra can be written as:

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_\alpha \mathcal{L}_\alpha \tag{5.26}$$

This is called the **Cartan Decomposition**. Sometimes, we write: $\mathcal{H} = \mathcal{L}_0$
The dimension of $\mathcal{H}$ is called the rank of $\mathcal{L}$, $rk(\mathcal{L}) = dim(\mathcal{H})$.

Root A non-zero linear functional $\alpha \in \mathcal{H}'$ is called a root of the Lie Algebra $\mathcal{L}$ if there is a non-zero $T \in \mathcal{L}$ such that $ad(H)(T) = \alpha(H)T$. The set $\Delta = \{\alpha \in \mathcal{H}'|\alpha$ is a root$\}$ is called **roots**, the lattice generated by roots are called root lattice Λ_R

Structure of Cartan decomposition

1-The Cartan decomposition is consistent with the commutator if $T \in \mathcal{L}_\alpha$ & $S \in \mathcal{L}_\beta$. $\Rightarrow$ $[T, S] \in \mathcal{L}_{\alpha+\beta}$

证明

$$[H,T]=\alpha(H)T \quad [H,S]=\beta(H)S \tag{5.27}$$

Then:

$$\begin{aligned}\big[H,[T,S]\big] &= \big[T,[H,S]\big]-\big[S,[H,T]\big] \\ &= \beta(H)[T,S]-\alpha(H)[S,T]=(\alpha(H)+\beta(H))[T,S]\end{aligned} \tag{5.28}$$

Sometimes, written as:

$$[\mathcal{L}_\alpha,\mathcal{L}_\beta]\subset\mathcal{L}_{\alpha+\beta} \tag{5.29}$$

End of Proof

2-Relation between cartan decomposition and killing form

$$T\in\mathcal{L}_\alpha\ ;\ S\in\mathcal{L}_\beta\ ;\ \alpha+\beta\neq 0\Longrightarrow\Gamma(T,S)=tr\big(ad(T)\circ ad(S)\big)=0 \quad \big(\mathcal{L}_\alpha\perp\mathcal{L}_\beta\big) \tag{5.30}$$

证明 首先寻找:

$$T\in\mathcal{L}_\alpha \quad S\in\mathcal{L}_\beta \quad U\in\mathcal{L}_\gamma \tag{5.31}$$

将其作用于 U:

$$ad(T)\circ ad(S)(U)=\big[T,[S,U]\big]\in\mathcal{L}_{\alpha+\beta+\gamma}\ \big([\mathcal{L}_\alpha,\mathcal{L}_\beta]\subset\mathcal{L}_{\alpha+\beta}\big) \tag{5.32}$$

This means the $ad(T)\circ ad(S)$ has vanishing diagonal elements.

$$\Gamma(T,S)=tr\big(ad(T)\circ ad(S)\big)=0 \quad \mathcal{L}_\alpha\perp\mathcal{L}_\beta \quad \text{for } \alpha+\beta\neq 0 \tag{5.33}$$

□

3-$\Gamma_{\mathcal{H}\times\mathcal{H}}$ is non-degenerate.

$$\nexists H\in\mathcal{H}\ ;\ H\neq 0\ s.t.\ \forall\tilde{H}\in\mathcal{H}\ \Gamma(H,\tilde{H})=0 \tag{5.34}$$

And

$$\forall\alpha\in\mathcal{H}'\ ,\ \exists H_\alpha\in\mathcal{H}\ s.t.\ \Gamma(H,H_\alpha)=\alpha(H) \tag{5.35}$$

Define:

$$(\alpha,\beta)=\Gamma(H_\alpha,H_\beta) \tag{5.36}$$

证明 $\mathcal{L}$ is semi-simple $\Rightarrow$ Killing form is non degenerate. on $\mathcal{L}\times\mathcal{L}$.
$\exists H\in\mathcal{H}$, s.t. $\forall\tilde{H}\in\mathcal{H}$, $\Gamma(H,\tilde{H})=0$. We need to show that $H=0$.
$\forall S\in\mathcal{L}$

$$S=\tilde{H}+\sum_{\alpha\in\Delta}S_\alpha \quad S_\alpha\in\mathcal{L}_\alpha \quad \tilde{H}\in\mathcal{H} \tag{5.37}$$

if $\exists H\in\mathcal{H}$, s.t. $\forall\tilde{H}\in\mathcal{H}$, $\Gamma(H,\tilde{H})=0$

Noticed that:

$$\forall S_\alpha \quad \Gamma(H, S_\alpha) = 0 \tag{5.38}$$

Then:

$$\forall S_\alpha \in \mathcal{L}_\alpha \;\; \forall \tilde{H} \in \mathcal{H} \quad \sum_\alpha \Gamma(H, S_\alpha) + \Gamma(H, \tilde{H}) = 0 \tag{5.39}$$

Which means:

$$\forall S \in \mathcal{L} \quad \Gamma(H, S) = 0 \tag{5.40}$$

From theorem about relation of semi-simple and non-degenerate 5.1.

$$H = 0 \tag{5.41}$$

Which means $\Gamma_{\mathcal{H}\times\mathcal{H}}$ is non-degenerate.
To find an $H_\alpha \in \mathcal{H}$,

$$\Gamma(H, H_\alpha) = \alpha(H) \Rightarrow \Gamma(H_i, H_\alpha) = \alpha(H_i) \tag{5.42}$$

Consider:

$$H_\alpha = H_\alpha^j H_j \tag{5.43}$$

This relation means:

$$\gamma_{ij} H_\alpha^j = \alpha(H_i) \tag{5.44}$$

if γ matrix is invertible, the exact H_α can be recovered from this equation.
□

4- If Δ Contains α, it contains $-\alpha$.

证明 Assume that $-\alpha \notin \Delta$, Then $\mathcal{L}_\alpha \perp \mathcal{L}_\beta \forall \beta \in \Delta$. As well as $\mathcal{L}_\alpha \perp \mathcal{H}$ So $\mathcal{L}_\alpha \perp \mathcal{L}$. however, this is contradiction since Γ is non-degenerate.
□

5- For $T \in \mathcal{L}_\alpha$, $S \in \mathcal{L}_{-\alpha}$. we have $[T, S] = \Gamma(T, S) H_\alpha$. One can normalise as: $\Gamma(T, S) = 1$.

证明 Let $H \in \mathcal{H}, T \in \mathcal{L}_\alpha, S \in \mathcal{L}_{-\alpha}$

$$\begin{aligned} \Gamma(H, [T, S]) &= \Gamma([H, T], S) = \alpha(H)\Gamma(T, S) \\ &= \Gamma(H, H_\alpha)\Gamma(T, S) = \Gamma(H, \Gamma(T, S) H_\alpha) \end{aligned} \tag{5.45}$$

with non-degeneracy of Γ, $[T, S] = \Gamma(T, S) H_\alpha$. There must be a $T \in \mathcal{L}_\alpha$ and $S \in \mathcal{L}_{-\alpha}$. with $\Gamma(T, S) \neq 0$. (Γ is non-degenerate in $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}$). by suitable normalising T, S . $\Gamma(T, S) = 1$

6- $dim(\mathcal{L}_\alpha) = 1$ **for all** $\alpha \in \Delta$

7- Let $\alpha \in \Delta$**, from** $\{k\alpha | k \in \mathbb{Z}\}$**, only** α **and** $-\alpha$ **are roots** .

证明 We proof these two statement together.
Choose: $T \in \mathcal{L}_\alpha$, $S \in \mathcal{L}_{-\alpha}$, $H_\alpha \in \mathcal{H}$,such that $[T, S] = H_\alpha$.
Define space:

$$V = \mathbb{C}S + \mathbb{C}H_\alpha + \sum_{k\geq 1} \mathcal{L}_{k\alpha} \tag{5.46}$$

Which is invariant under $ad(H)$. We proof this two statement by computin trace of $ad(H_\alpha)|_V$ in two different ways.
(1)

$$tr\big(ad(H_\alpha)|_V\big) = tr\big(ad([T, S])|_V\big) = tr\big([ad(T), ad(S)]|_V\big) = 0 \tag{5.47}$$

(2) evaluating it on a basis of V.

$$\begin{aligned} ad(H_\alpha)(S) &= [H_\alpha, S] = -\alpha(H_\alpha)S = -\Gamma(H_\alpha, H_\alpha)S = -(\alpha, \alpha)S \\ ad(H_\alpha)(H_\alpha) &= [H_\alpha, H_\alpha] = 0 \\ \underbrace{ad(H_\alpha)(U)}_{\text{U is basis of } \mathcal{L}_{k\alpha}} &= [H_\alpha, U] = k\alpha(H_\alpha)U = k(\alpha, \alpha)U \end{aligned} \tag{5.48}$$

Which implies:

$$tr\big(ad(H_\alpha)|_V\big) = (\alpha, \alpha)\Big(-1 + \sum_{k\geq 1} k dim(\mathcal{L}_{k\alpha})\Big) \tag{5.49}$$

Which means $dim(\mathcal{L}_{k\alpha}) = 1$ for $k = 1$, others: $dim(\mathcal{L}_{k\alpha}) = 0$
□

8- For $H, \tilde{H} \in \mathcal{H}$**, we have** $\Gamma(H, \tilde{H}) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(\tilde{H})$.

证明 Start with: $T \in \mathcal{L}_\alpha$

$$ad(H) \circ ad(\tilde{H})(T) = \big[H, [\tilde{H}, T]\big] = \alpha(H)\alpha(\tilde{H})(T) \tag{5.50}$$

In this case:

$$\Gamma(H, \tilde{H}) = tr\big(ad(H) \circ ad(\tilde{H})\big) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(\tilde{H}) \tag{5.51}$$

□

9- Δ **contains a basis of** $\mathcal{H}'$ **(roots span root space)**.

证明 Assuming $Span(\Delta) \neq \mathcal{H}'$. choose basis of $Span(\Delta)$ as $(\alpha_1, \cdots \alpha_n)$. Complete it to $(\alpha_1 \cdots \alpha_N)$. Introducing its dual basis: $(H_1, \cdots H_N)$ of $\mathcal{H}$. This means:

$$\alpha_i(H_j) = \delta_{ij} \Rightarrow \alpha(H_N) = 0 \text{ for all } \alpha \in \Delta \tag{5.52}$$

This means:

$$\Gamma(H_N, H) = 0 \text{ for all } H \in \mathcal{H} \tag{5.53}$$

Which is contradict with non-degeneracy of Γ.

□

Cartan Weyl basis The basis would be: (H_i, E_α) where $i = 1 \cdots r = rk(\mathcal{L})$, $\mathcal{L}_\alpha = Span(E_\alpha)$, and $\Gamma(E_\alpha, E_{-\alpha}) = 1$

Relative to basis of Cartan, elements $\alpha \in \mathcal{H}'$ can be described by $(\alpha_1 \cdots \alpha_r)$ where $\alpha_i = \alpha(H_i)$

– Killing form can be used to upper or lower the index Consider $\alpha \in \mathcal{H}'$, $H_\alpha = \lambda^i H_i$

$$\alpha_j \equiv \alpha(H_j) = \Gamma(H_j, H_\alpha) = \lambda^i \Gamma(H_i, H_j) = \lambda^i \gamma_{ij} \tag{5.54}$$

take inverse and denote λ as upper index α

$$\alpha^i = \gamma^{ij} \alpha_j \tag{5.55}$$

– Commutation relations for the Cartan-Weyl basis

$$\begin{aligned} &[H_i, H_j] = 0 \quad [H_i, E_\alpha] = \alpha_i E_\alpha \\ &[E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^i H_i \quad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & 0 \neq \alpha + \beta \in \Delta \\ 0 & 0 \neq \alpha + \beta \notin \Delta \end{cases} \end{aligned} \tag{5.56}$$

– Subalgebra of Cartan-Wel basis For any root $\alpha \in \Delta$, the three generators $(H_\alpha, E_\alpha, E_{-\alpha})$ form a subalgebra, with commutation relation:

$$\begin{aligned} &[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha) E_{\pm\alpha} = \pm(\alpha, \alpha) E_{\pm\alpha} \\ &[E_\alpha, E_{-\alpha}] = H_\alpha \end{aligned} \tag{5.57}$$

Weights

– Basic of weights For a representation $r : \mathcal{L} \to End(V)$, we call $w \in \mathcal{H}'$ a weight of r if there is a non-zero vector $v \in V$ s.t.

$$r(H)(v) = w(H)v \ \forall H \in \mathcal{H} \tag{5.58}$$

The eigen space of weight w denoted by V_w consists of all $v \in V$ satisfies above equation. Representation vector space:

$$V = \oplus_w V_w \tag{5.59}$$

A weight $w \in \mathcal{H}'$ is represented by:

$$(w(H_1) \cdots w(H_r)) \qquad r = rk(\mathcal{L}) \tag{5.60}$$

– Raising and Lowering operator

$$\begin{aligned} r(H)\big(r(E_\alpha)v\big) &= \Big(r(E_\alpha)r(H) + \big[r(H), r(E_\alpha)\big]\Big)v \\ &= \Big(r(E_\alpha)w(H) + r\big([H, E_\alpha]\big)\Big)v \\ &= \Big(r(E_\alpha)w(H) + r\big(\alpha(H)E_\alpha\big)\Big)v \\ &= \big(w(H) + \alpha(H)\big)r(E_\alpha)v \end{aligned} \tag{5.61}$$

Shows that:

$$r(E_\alpha)v \begin{cases} \in V_{w+\alpha} & w+\alpha \text{ is a weight} \\ = 0 & w+\alpha \text{ is not a weight} \end{cases} \tag{5.62}$$

命题 5.1 (Weights in representation differ by roots.)

If representation $r : \mathcal{L} \to End(V)$ is irreducible, Then any two weights w_1, w_2 of r satisfy $w_1 - w_2 \in \Lambda_R$.

证明 If $w_1 - w_2 \notin \Lambda_R$, $\oplus_{\alpha \in \Lambda_R} V_{w_1+\alpha} \subset V$ is invariant under r, but it dose not contains V_{w_2}. which is contradict with irreducible condition of r.

□

–Weight Lattice Consider the commutation relation of $(H_\alpha, E_\alpha, E_{-\alpha})$

$$\begin{aligned} [H_\alpha, E_{\pm\alpha}] &= \pm\alpha(H_\alpha)E_{\pm\alpha} = \pm(\alpha,\alpha)E_{\pm\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha \end{aligned} \tag{5.63}$$

Eigen values of H_α,$w(H_\alpha)$ has to be $\frac{(\alpha,\alpha)}{2}\mathbb{Z}$

$$(w,\alpha) = w(H_\alpha) \in \frac{(\alpha,\alpha)}{2}\mathbb{Z} \tag{5.64}$$

remember: $\Gamma(H, H_\alpha) = \alpha(H)$, $(\alpha,\beta) = \Gamma(H_\alpha, H_\beta)$ The weight lattice:

$$\Lambda_w = \{w \in \mathcal{H}' | \frac{2(w,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \ \forall \alpha \in \Delta\} \tag{5.65}$$

It follows that all weights of a representation of $\mathcal{L}$ lie in Λ_w

–Positive and negative roots Choose a direction $l \in \mathcal{H}'$ in root space. Define two subset of roots:

$$\Delta_+ = \{\alpha \in \Delta | l(\alpha) > 0\}$$
$$\Delta_- = \{\alpha \in \Delta | l(\alpha) < 0\} \tag{5.66}$$
$$l(\alpha) = \alpha(H_l) = \Gamma(H_l, H_\alpha)$$

Need to be careful that $l(\alpha) \neq 0$ for all $\alpha \in \Delta$ Since Δ is a **finite site** of roots, this is always possible. In this case $\Delta = \Delta_+ \cup \Delta_-$. E_α with $\alpha \in \Delta_+$ are raising operator, E_α with ... are lowering operator.

–Highest weight vector Let $r : \mathcal{L} \to End(V)$ be a representation. A non-zero vector $v \in V$ is called the highest weight vector if $E_\alpha(v) = 0$ $\forall \alpha \in \Delta_+$. (Definition of Highest weight vector). The weight λ of highest weight vector is called highest weight.

Properties of Cartan-Weyl decomposition For a semi-simple complex Lie Algebra $\mathcal{L}$ with representation $r : \mathcal{L} \to End(V)$.

1- r has a highest weight vector .

证明 Choose a weight λ as highest weight which means $l(\lambda) = \lambda(H_l)$ is a highest over all weights. Consider

$$\forall \alpha \in \Delta_+ \quad (\lambda + \alpha)(H_l) = \lambda(H_l) + \alpha(H_l) > \lambda(H_l) \tag{5.67}$$

This means $\lambda + \alpha$ can not be weight.

$$E_\alpha(v) = 0 \tag{5.68}$$

□

2- Construct irrep successive application of E_α, where $\alpha \in \Delta_-$ on v gives a sub-representation of r. If r is irreducible representation, it is obtained in this way.

证明 Define

$$W_k = Span\{E_{\alpha_1} \cdots E_{\alpha_k} v | \alpha_i \in \Delta_-\} \tag{5.69}$$

$$W = \oplus_k W_k \in V \tag{5.70}$$

We can show that W is invariant under all generators.$\cdots$

□

3- If r is an irrep, the highest weight vector is unique up to re-scaling .

证明 Suppose there are two linearly indenpendent highest weight vector, $v_1, v_2 \in V_\lambda$, Then v_1 generates irrep which does not contain v_2, So V is not irreducible, so $dim(V_\lambda) = 1$

□

Simple roots A positive (or negative) roots is called simple if it cannot be written as a sum of other two positive (negative) roots.

推论 5.1 (Statement about simple roots)

(i) An irrep r can be obtained by successively applying lowering operators of simple roots E_α

(ii) The simple positive roots form a basis of $\mathcal{H}'$

♡

– Dynkin labels and Cartan Matrix Choose a basis formed by positive simple roots $(\alpha_1 \cdots \alpha_r)$. For a weight w, construct:

$$a_i = \frac{2(w, \alpha_i)}{(\alpha_i, \alpha_i)} \tag{5.71}$$

The vector $(a_1 \cdots a_r)$ Carateries weight w. (Dynkin label)

Dynkin labels for positive simple roots:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)} \tag{5.72}$$

This is called Cartan Matrix.

第六章 Lorentz 群

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6.1}$$

6.1 Lorentz 变换

定义 6.1 (Lorentz 变换)

Lorentz 变换定义为:

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \tag{6.2}$$

$$x'_{\mu} = \Lambda_{\mu}^{\ \nu} x_{\nu} \tag{6.3}$$

满足

$$x'^{\mu} x'_{\mu} = x^{\mu} x_{\mu} \tag{6.4}$$

也就是

$$\Lambda^{\mu}_{\ \nu} \Lambda_{\mu}^{\ \rho} = \delta^{\rho}_{\nu} \tag{6.5}$$

考虑到度规算符。

$$\Lambda^{\mu}_{\ \nu}\ g_{\mu\lambda}\ \Lambda^{\lambda}_{\ \sigma}\ g^{\sigma\rho} = \delta^{\rho}_{\nu} \tag{6.6}$$

总的来说，Lorentz 变换:

$$\left\{ \Lambda^{\mu}_{\ \nu} \middle|\ \Lambda^{\mu}_{\ \nu}\ g_{\mu\lambda}\ \Lambda^{\lambda}_{\ \sigma}\ g^{\sigma\rho} = \delta^{\rho}_{\nu} \right\} \tag{6.7}$$

♣

引理 6.1 (Lorentz 变换矩阵的行列式)

$$\det \Lambda = \pm 1 \tag{6.8}$$

♡

证明 由6.1:

$$\left\{ \Lambda^{\mu}_{\ \nu}\ g_{\mu\lambda}\ \Lambda^{\lambda}_{\ \sigma}\ g^{\sigma\rho} = \delta^{\rho}_{\nu} \right. \tag{6.9}$$

对上式取行列式:

$$\begin{aligned} \det(\Lambda^{T}) \det(g_{\mu\lambda}) \det(\Lambda) \det(g^{\sigma\rho}) = \det(\mathbb{I}) = 1 \\ \det(\Lambda)^2 = 1 \det \Lambda \quad = \pm 1 \end{aligned} \tag{6.10}$$

□

Lorentz 群的参数 当 Lorentz 变换在恒等变换附近时，可以假设:

$$\Lambda^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \Delta w^{\nu}{}_{\mu} \tag{6.11}$$

考虑 Lorentz 变换的约束条件:

$$\Big\{ \Lambda^{\mu}{}_{\nu} \Lambda^{\rho}_{\mu} = \delta^{\rho}_{\nu} \tag{6.12}$$

于是:

$$\left(\delta^{\mu}_{\nu} + \Delta w^{\mu}{}_{\nu}\right)\left(\delta^{\rho}_{\mu} + \Delta w_{\mu}{}^{\rho}\right) = \delta^{\rho}_{\nu} \tag{6.13}$$

忽略二阶小量:

$$\Delta w^{\rho}{}_{\nu} + \Delta w_{\nu}{}^{\rho} = 0 \tag{6.14}$$

升指标后得到关系:

$$\Delta w^{\rho\nu} + \Delta w^{\nu\rho} = 0 \tag{6.15}$$

也就是说 Δw 矩阵中只有 $(4 \times 4 - 4)/2 = 6$ 个独立变量

x 方向运动 Lorentz 变换 此时

$$\Delta w^{10} = -\Delta w^{01} = -\Delta\beta \tag{6.16}$$

考虑 Covariant vector x 的变换。

$$\Delta w^{1}{}_{0} = -\Delta\beta \quad \Delta w^{0}{}_{1} = -\Delta\beta \tag{6.17}$$

也就是:

$$\begin{aligned} (x')^0 &= x^0 - \Delta\beta x_1 \\ (x')^1 &= -\Delta\beta x^0 + x^1 \\ (x')^2 &= x^2 \\ (x')^3 &= x^3 \end{aligned} \tag{6.18}$$

之后要用 infinitesimal transformation 构造 finite transformation。考虑矩阵:

$$I_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad I_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (I_1)^3 = I_1 \tag{6.19}$$

于是这种微小的 Lorentz 变换写为:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \Delta\beta (I_1)^{\mu}{}_{\nu} \tag{6.20}$$

把 $\Delta\beta$ 写为 $\frac{w}{N}$, N 是一个比较大的数，所以它表示着微元 Lorentz 变换之后, 考虑变换:

$$\lim_{N\to+\infty}\left(\mathbb{I}+\frac{w}{N}I_1\right)^N = e^{wI_1} \tag{6.21}$$

下面是一些化简

$$\begin{aligned} e^{wI_1} &= \cosh(wI_1)+\sinh(wI_1) \\ &= \left[\left(\mathbb{I}+\frac{(wI_1)^2}{2!}+\frac{(wI_1)^4}{4!}+\cdots\right)+\left(wI_1+\frac{(wI_1)^3}{3!}\cdots\right)\right] \\ &= \left[\left(\mathbb{I}+\frac{(wI_1)^2}{2!}+\frac{(wI_1)^2}{4!}+\cdots\right)+\left(wI_1+\frac{(wI_1)}{3!}\cdots\right)\right] \\ &= \mathbb{I}-(I_1)^2+\cosh(w)(I_1)^2+\sinh(w)I_1 \end{aligned} \tag{6.22}$$

具体来说, 对于 Covariant vector，变换可以写为:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{6.23}$$

为了把它和传统的侠义相对论知识联系在一起。首先考虑 $x'^0=0$ 时, 原始坐标系中的坐标

$$\cosh w(x^1-x^0\tanh w)=0 \quad \frac{x^1}{x^0}=\frac{x^1}{ct^1}=\frac{v_x}{c}=\tanh w=\beta \tag{6.24}$$

考虑到关系 $\cosh^2 w-\sinh^2 w=1$。$\cosh w=\frac{\cosh w}{\sqrt{\cosh^2 w-\sinh^2 w}}=\frac{1}{\sqrt{1-\tanh^2 w}}=\frac{1}{\sqrt{1-\beta^2}}$ 那么:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{6.25}$$

绕 z 轴旋转 (坐标系绕 z 轴旋转) 此时:

$$\Delta w^{21}=-\Delta w^{12}=\Delta\varphi \quad others=0 \tag{6.26}$$

考虑 Covariant vector x 的变换:

$$\Delta w^2{}_1=-\Delta\varphi \quad \Delta w^1{}_2=\Delta\varphi \tag{6.27}$$

也就是:

$$\begin{aligned} (x')^0 &= x^0 \\ (x')^1 &= x^1+\Delta\varphi x^2 \\ (x')^2 &= -\Delta\varphi x^1+x^2 \\ (x')^3 &= x^3 \end{aligned} \tag{6.28}$$

接下来，想用 infinitesmial Lorentz 变换构造 finite Lorentz 变换, 先定义矩阵 I_6

$$I_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (I_6)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (I_6)^3 = -I_6 \tag{6.29}$$

这样，这种微小 Lorentz 变换可以写为:

$$\Lambda = \mathbb{I} + \Delta\varphi I_6 \tag{6.30}$$

考虑:

$$\lim_{N\to+\infty} (\mathbb{I} + \frac{w}{N} I_6)^N = e^{wI_6} \tag{6.31}$$

$$\begin{aligned} e^{wI_6} &= \cosh(wI_6) + \sinh(wI_6) \\ &= \left(\mathbb{I} + \frac{(wI_6)^2}{2!} + \frac{(wI_6)^4}{4!} + \cdots\right) + \left(wI_6 + \frac{(wI_6)^3}{3!} + \cdots\right) \\ &= \mathbb{I} + (I_6)^2 + \left(-\mathbb{I} + \frac{w^2}{2!} - \frac{w^4}{4!} + \cdots\right)(I_6)^2 + \left(w - \frac{w^3}{3!} + \cdots\right)(I_6) \\ &= \mathbb{I} + (I_6)^2 - \cos(w)(I_6)^2 + \sin(w) I_6 \end{aligned} \tag{6.32}$$

具体来说，对于 Covariant vector, 它的变换是:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w & \sin w & 0 \\ 0 & -\sin w & \cos w & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{6.33}$$

相当于坐标系绕着 z 轴转动