1 Equivalent Multiplicative Submonoids of $\mathbb{Z}^+$

Consider the set of positive integers $\mathbb{Z}^+$ as a monoid equipped with multiplication. Let Y be a submonoid of $\mathbb{Z}^+$. Define $F(Y) := \{\frac{m}{n} \mid m, n \in Y\}$ and $\overline{Y} := F(Y) \cap \mathbb{Z}^+$. We call $\overline{Y}$ the completion of Y. Note that $F(Y)$ is a multiplicative subgroup of $\mathbb{Q}^+$, so it is free. Now, for two submonoids Y, Z of $\mathbb{Z}^+$, we define $Y \sim Z$ if $\overline{Y} = \overline{Z}$. It is easy to check that $\sim$ is an equivalent relation. We then raise the following question.

Question 1. *Is it true that any submonoid Y of $\mathbb{Z}^+$ is equivalent to some freely generated submonoid Z of $\mathbb{Z}^+$? In particular, if Y is finitely generated, then Z should have the same rank as $F(Y)$.*

One should be wary that the completion of a monoid is not necessarily free, even if the original one is free. For example, let Y be the free monoid generated by $\{4, 6\}$. We then have $9 \notin Y$ but $9 = 4^{-1} \cdot 6^2 \in \overline{Y}$. One can check that if $\overline{Y}$ is free, then its generating set must contain 2^2 and 3^2, so there will be nontrivial relations when we write $2 \cdot 3$ in terms of the remaining generators. Hence, $\overline{Y}$ is not free as a monoid.