1 Description of the connected component X_φ containing a TRSELP φ

1.1 Recollections on the moduli space of Langlands parameters

Let $p \neq 2$ be a fixed prime number and $\ell \neq 2$ be a prime number different from p. Let F ba a non-archimedean local field with residue characteristic $q = p^r$ for some $r \in \mathbb{Z}_{\geq 1}$. Let W_F be the Weil group of F, $I_F \subset W_F$ be the inertia subgroup, P_F be the wild inertia subgroup. Fix $Fr \in W_F$ any lift of the arithmetic Frobenius. Let $W_t := W_F/P_F$ be the tame Weil group. Let $I_t := I_F/P_F$ be the tame inertia subgroup. I will abuse the notation and denote Fr the image of Fr in W_t. Then $W_t \simeq (I_t \rtimes < Fr >)$, where $< Fr > \simeq \mathbb{Z}$ is the subgroup generated by Fr. Here I_t is non-canonically isomorphic to $\prod_{p' \neq p} \mathbb{Z}_{p'}$, which is procyclic. We fix such an isomorphism. And we fix a topological generator s_0 of I_t. For example, we can choose s_0 which corresponds to $(1, 1, ...)$ under the chosen isomorphism $I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}$. Let us recall the following important relation in I_F/P_F:

$$Fr.s_0.Fr^{-1} = s_0^q.$$

In fact, this is true for any $s \in I_t$ instead of s_0.

Let $W_t^0 :=< s_0, Fr >= \mathbb{Z}[1/p]^{s_0} \rtimes \mathbb{Z}^{Fr}$ be the subgroup of W_t generated by s_0 and Fr. Denote $W_F^0 \subset W_F$ the preimage of W_t^0 under $W_F \to W_t$. This is known as the discretization of the Weil group. To summarize, W_t^0 is generated by two elements Fr and s_0 with a single relation $Fr.s_0.Fr^{-1} = s_0^q$.

Let G be a connected split reductive group over F. Let $\hat{G}$ be its dual group over $\mathbb{Z}$. Then the space of cocycles from the discretization

$$Z^1(W_t^0, \hat{G}) = \underline{Hom}(W_t^0, \hat{G}) = \{(x, y) \in \hat{G} \times \hat{G} | yxy^{-1} = x^q\} \tag{1}$$

is an explicit closed subscheme of $\hat{G} \times \hat{G}$ (See [3, Section 3]). An important fact (See [3, Proposition 3.9]) is that over a $\mathbb{Z}_\ell$-algebra R (the cases $R = \overline{\mathbb{F}_\ell}, \overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}$ are most relevant for us), the restriction from W_t to W_t^0 induces an isomorphism

$$Z^1(W_t, \hat{G}) \simeq Z^1(W_t^0, \hat{G}).$$

Therefore, we can compute $Z^1(W_t, \hat{G})$ using the explicit formula 1 above. This is fundamental for the analysis of the moduli space of Langlands parameters $Z^1(W_t, \hat{G})$. I refer the readers to [3, Section 3 and Section 4] for the precise definition and properties of $Z^1(W_t, \hat{G})$.

(maybe add an example here)

Let I_F^ℓ be the prime-to-ℓ inertia subgroup of W_F, i.e., $I_F^\ell := ker(t_\ell)$, where

$$t_\ell : I_F \to I_F/P_F \simeq \prod_{p' \neq p} \mathbb{Z}_{p'} \to \mathbb{Z}_\ell$$

is the composition. In other words, it is the maximal subgroup of I_F with pro-order prime to ℓ. This property makes I_F^ℓ important when determining the connected components of $Z^1(W_F, \hat{G})$ over $\overline{\mathbb{Z}_\ell}$ (See [3, Theorem 4.2 and Subsection 4.6]). I assume the readers to be familiar with the moduli space of Langlands parameters, see for example [3, Section 3 and Section 4], or [4, Section 2 and Section 4]. (I could also recollect the theory in the appendix.)

1.2 Tame regular semisimple elliptic L-parameters

I want to define a class of L-parameters, called TRSELP, which roughly corresponds to depth-zero regular supercuspidal representations. Before that, let me define the concept of schematic centralizer, which will be used throughout the article.

Definition 1 (schematic centralizer)**.** *Let H be an affine algebraic group over a ring R, Γ be a finite group. Let $u \in Z^1(\Gamma, H(R'))$ be a 1-cocycle for some R-algebra R'. Let $\alpha_u : H_{R'} \to Z^1(\Gamma, H)_{R'}, h \mapsto hu(-)h^{-1}$ be the orbit morphism. Then the schematic centralizer $C_H(u)$ is defined to be the fiber of α_u at u.*

$$\begin{tikzcd} C_H(u) \arrow[r] \arrow[d] & H_{R'} \arrow[d, "\alpha_u"] \\ R' \arrow[r, "u"] & Z^1(\Gamma, H)_{R'} \end{tikzcd}$$

One can show that $C_H(u)(R'') = C_{H(R'')}(u)$ is the set-theoretic centralizer for all R'-algebra R'', see for example [4, Appendix].

Remark. *Note this is enough for our applications where Γ is more generally taken as a profinite group, because $u : \Gamma \to H$ usually factors through a finite quotient Γ' of Γ.*

Let me now define a tame, regular semisimple, elliptic Langlands parameter (TRSELP for short) over $\overline{\mathbb{F}_\ell}$, roughly in the sense of [5, Section 3.4 and Section 4.1] in the case G is F-split, but with $\overline{\mathbb{F}_\ell}$-coefficients instead of $\mathbb{C}$-coefficients.

Definition 2. *A* ***tame regular semisimple elliptic L-parameter (TRSELP) over $\overline{\mathbb{F}_\ell}$*** *is a homomorphism $\varphi : W_F \to \hat{G}(\overline{\mathbb{F}_\ell})$ such that:*

1. *(smooth) $\varphi(I_F)$ is a finite subgroup of $\hat{G}(\overline{\mathbb{F}_\ell})$.*
2. *(Frobenius semisimple) $\varphi(Fr)$ is a semisimple element of $\hat{G}(\overline{\mathbb{F}_\ell})$.*
3. *(tame) The restriction of φ to P_F is trivial.*
4. *(elliptic) The identity component of the centralizer $C_{\hat{G}}(\varphi)^0$ is equal to the identity component of the center $Z(\hat{G})^0$.*
5. *(regular semisimple) The centralizer of the inertia $C_{\hat{G}}(\varphi|_{I_F})$ is a torus (in particular, connected).*

Concretely, a TRSELP consists of the following data:

1. The restriction to the inertia $\varphi|_{I_F}$, which is a direct sum of characters of a finite abelian group since $I_F/P_F \simeq \varprojlim \mathbb{F}_{q^n}^*$. In particular, it factors through (the $\overline{\mathbb{F}_\ell}$-points of) some maximal torus, say S. Then regular semisimple means that $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

2. The image of Frobenius $\varphi(Fr)$, which turns out to be an element of the normalizer $N_{\hat{G}(\overline{\mathbb{F}_\ell})}(S)$ (Since $Fr.s.Fr^{-1} = s^q \in I_t$ for any $s \in I_t$ implies that $\varphi(Fr)$ normalizes $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.). And "elliptic" means that the center $Z(\hat{G})$ has finite index in the centralizer $C_{\hat{G}}(\varphi)$. As we will see later, ellipticity implies that $\hat{G}(\overline{\mathbb{F}_\ell})$ acts transitively on the connected component $X_\varphi(\overline{\mathbb{F}_\ell})$ of the moduli space of L-parameters containing φ, which is essential for the description

$$[X_\varphi/\hat{G}] \simeq [*/\underline{S_\varphi}]$$

where $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is the centralizer of the whole L-parameter φ.

(Maybe add an example here)

Remark. *1. Let $\overline{\Lambda} \in \{\overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. It is important for my purpose to distinguish between the set-theoretic centralizer (for example, $C_{\hat{G}(\overline{\Lambda})}(\varphi(I_F))$) and the schematic centralizer (for example, $C_{\hat{G}}(\varphi)$). However, I might still use $\hat{G}$ to mean $\hat{G}(\overline{\Lambda})$ sometimes by abuse of notation, for which I hope the readers could recognize. One reason for that is that $\hat{G}$ is split over $\overline{\Lambda}$, hence $\hat{G}$ is completely determined by its $\overline{\Lambda}$-points. And many statements can either be phrased in terms of the $\overline{\Lambda}$-scheme or its $\overline{\Lambda}$-points (for example, 4 and 5).*

2. As we will see later, $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ turns out to be the $\overline{\mathbb{F}_\ell}$-points of the split torus $T = C_{\hat{G}}(\psi|_{I_F^\ell})$ for some lift ψ of φ over $\overline{\mathbb{Z}_\ell}$.

1.3 Description of the component

Now given a TRSELP $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$. Pick any lift $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ of φ, whose existence is ensured by the flatness of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ (See Lemma 1). Let $\psi_\ell := \psi|_{I_F^\ell}$ denotes the restriction of ψ to the prime-to-ℓ inertia. Note that $\psi \in Z^1(W_F, \hat{G})$ factors through $N_{\hat{G}}(\psi_\ell)$ (Since I_F^ℓ is normal in W_F). Let $\overline{\psi}$ denotes the image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Let X_φ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ. Note X_φ also contains ψ since ψ specializes to φ. So we sometimes also denote X_φ as X_ψ.

Theorem 1. *Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a TRSELP over $\overline{\mathbb{F}_\ell}$. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of φ. Then at least when the center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$, the connected component $X_\varphi = X_\psi$ of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ is isomorphic to*

$$\left(\hat{G} \times C_{\hat{G}}(\psi_\ell)^0 \times \mu\right) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where

1. *$C_{\hat{G}}(\psi_\ell)^0$ is the identity component of the schematic centralizer $C_{\hat{G}}(\psi_\ell)$, which turns out to be a split torus T over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$.*

2. *μ is the connected component of $T^{Fr=(-)^q}$ (the subscheme of T on which Fr acts by raising to q-th power) containing 1 (See [3, Example 3.14]), which is a product of some $\mu_{\ell^{k_i}}$ (the group scheme of ℓ^{k_i}-th roots of unity over $\overline{\mathbb{Z}_\ell}$), $k_i \in \mathbb{Z}_{\geq 0}$. Note that μ could be trivial, depending on $\hat{G}$ and some congruence relations between q, ℓ.*

3. *$C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is the (schematic) stabilizer (definition see Appendix) of $\overline{\psi}$ in $C_{\hat{G}}(\psi_\ell)$.*

In other words, we have the following isomorphism of schemes over $\overline{\mathbb{Z}_\ell}$:

$$X_\varphi \simeq \left(\hat{G} \times T \times \mu\right)/T.$$

And we will specify in the next subsection what the T-action on $\left(\hat{G} \times T \times \mu\right)$ is.

Proof. First, recall by [3, Subsection 4.6],

$$X_\psi \simeq \left(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}}\right)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}}$ denotes the space of cocycles whose restriction to I_F^ℓ equals ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\overline{\psi}$. Explanation: Recall (See [3, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the L-parameters whose restriction to I_F^ℓ is $\hat{G}$-conjugate to ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $\overline{\psi}$. Hence X_φ is isomorphic to

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$$

via $g\eta(-)g^{-1} \hookleftarrow (g, \eta)$, with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$ by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ and $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$.

Second, $\eta.\psi \hookleftarrow \eta$ defines an isomorphism

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \simeq Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_{1_{I_F^\ell}} =: Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$$

where $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell))$ means the space of cocycles with W_F acting on $N_{\hat{G}}(\psi_\ell)$ via conjugacy action through ψ, and the subscript $1_{I_F^\ell}$ or 1 means the cocycles whose restriction to I_F^ℓ is trivial. Explanation: This is clear by unraveling the definitions: two cocycles whose restriction to I_F^ℓ are both ψ_ℓ differ by something whose restriction to I_F^ℓ is trivial; two cocycles whose pushforward to $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ are both $\overline{\psi}$ differ by something whose pushforward to $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is trivial, i.e., which factors through the identity component $N_{\hat{G}}(\psi_\ell)^0$.

Next, I show that $C_{\hat{G}}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$. By [3, Subsection 3.1], the centralizer $C_{\hat{G}}(\psi_\ell)$ is generalized reductive (See Lemma 2), hence split over $\overline{\mathbb{Z}_\ell}$, and $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0$. So we can determine $C_{\hat{G}}(\psi_\ell)$ by computing its $\overline{\mathbb{F}_\ell}$-points. Indeed,

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)),$$

where the last equality follows since I_F/I_F^ℓ doesn't contribute to the image of φ (See Lemma 3). Therefore, $C_{\hat{G}}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$. Denote $T = C_{\hat{G}}(\psi_\ell)$. In particular, $C_{\hat{G}}(\psi_\ell)$ is connected, hence

$$N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell) = T.$$

Now we could compute

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0) = Z^1_{Ad(\psi)}(W_F, T) \simeq T \times T^{Fr=(-)^q},$$

where the last isomorphism is given by $\eta \mapsto (\eta(Fr), \eta(s_0))$, where $s_0 \in W_t^0$ is the topological generator of I_t fixed before (See [3, Example 3.14]).

Then we show that the identity component of $T^{Fr=(-)^q}$ gives μ in the statement of the theorem. Note $T^{Fr=(-)^q}$ is a diagonalizable group scheme over $\overline{\mathbb{Z}_\ell}$ of dimension zero (This can be seen either by $\dim Z^1(W_F/P_F, T) = \dim T$, or by noticing that $\eta(s_0) \in T^{Fr=(-)^q}$ is semisimple with finitely many possible eigenvalues), hence of the form $\prod_i \mu_{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$. Hence its connected component $(T^{Fr=(-)^q})^0$ over $\overline{\mathbb{Z}_\ell}$ is of the form $\prod_i \mu_{\ell^{k_i}}$, with k_i maximal such that ℓ^{k_i} divides n_i. Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq (T \times T^{Fr=(-)^q})^0 \simeq T \times (T^{Fr=(-)^q})^0 \simeq T \times \mu,$$

(See Lemma ? for the first isomorphism) where μ is of the form $\prod_i \mu_{\ell^{k_i}}$.

Finally, we show that $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$. Recall $C_{\hat{G}}(\psi_\ell)$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))$ by conjugation, inducing an action of $C_{\hat{G}}(\psi_\ell)$ on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. And $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is by definition the stabilizer of $\overline{\psi} \in Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ in $C_{\hat{G}}(\psi_\ell)$. Now $C_{\hat{G}}(\psi_\ell) = T$ is connected, hence acts trivially on the component group $\pi_0(N_{\hat{G}}(\psi_\ell))$ (See Lemma ?), hence also acts trivially on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Therefore, the stabilizer $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$.

Above all, we have

$$X_\varphi \simeq (\hat{G} \times Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \simeq (\hat{G} \times T \times \mu)/T.$$

□

1.4 The T-action on $(\hat{G} \times T \times \mu)$

For later use, let me make it explicit the T-action on $(\hat{G} \times T \times \mu)$.

Recall (See [3, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the L-parameters whose restriction to I_F^ℓ is $\hat{G}$-conjugate to ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $\overline{\psi}$. Hence X_φ is isomorphic to

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$$

via $g\eta(-)g^{-1} \mapsfrom (g, \eta)$, with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$ by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \simeq T$ and $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$.

Next, recall that $\eta.\psi \mapsfrom \eta \mapsto (\eta(Fr), \eta(s_0))$ defines isomorphisms

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq T \times \mu.$$

Let's focus on the isomorphism $\eta.\psi \mapsfrom \eta$:

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1.$$

Recall that $T \subset \hat{G}$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$ via conjugation. Hence the above isomorphism induces an T-action on $Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$, by

$$(t, \eta) \mapsto (t(\eta\psi(-))t^{-1})\psi^{-1}.$$

Hence in $(\hat{G} \times T \times \mu)/T$, we compute by tracking the above isomorphisms that

1. T acts on $\hat{G}$ via $(t, g) \mapsto gt^{-1}$.
2. $T = C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acts on $T \subset (T \times \mu)$ (corresponds to $\eta(Fr)$) by twisted conjugacy (due to the isomorphisms $\eta.\psi \mapsfrom \eta \mapsto (\eta(Fr), \eta(s_0))$), i.e.,
$$(t, t') \mapsto \left(t(t'n)t^{-1}\right)n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$
where $n = \psi(Fr)$; Note here n, a prior lies in $\hat{G}$, actually lies in $N_{\hat{G}}(T)$ (Since $Fr.s.Fr^{-1} = s^q$ implies that $\psi(Fr)$ normalizes $C_{\hat{G}}(\psi|_{I_F}) = C_{\hat{G}}(\psi|_{I_F^\ell}) = T$, See Lemma ?). To summarize, $t \in T$ acts on T via multiplication by $tnt^{-1}n^{-1}$.
3. T acts trivially on μ. This is because $\eta(s_0) \in T$ and $\psi(s_0) \in T$. (See Lemma ?)

On the other hand, recall we have the natural $\hat{G}$-action on $Z^1(W_F, \hat{G})$ by conjugation, hence the $\hat{G}$-action on this component X_φ. Under the isomorphism $X_\varphi \simeq (\hat{G} \times T \times \mu)/T$, the $\hat{G}$-action becomes

$$(g', (g, t, m)) \mapsto (g'g, t, m), \text{ for any } g' \in \hat{G} \text{ and } (g, t, m) \in (\hat{G} \times T \times \mu)/T.$$

Note that the T-action and the $\hat{G}$-action on $(\hat{G} \times T \times \mu)$ commute with each other, we thus have the following:

Proposition 1.

$$[X_\varphi/\hat{G}] = \left[\left((\hat{G}\times T\times\mu)/T\right)/\hat{G}\right] \simeq \left[\left((\hat{G}\times T\times\mu)/\hat{G}\right)/T\right] \simeq [(T\times\mu)/T],$$

with $t \in T$ acting on T via multiplication by $tnt^{-1}n^{-1}$, and $t \in T$ acting trivially on μ.

1.5 Some lemmas

Lemma 1. *Let $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$. Then there exists $\psi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{Z}_\ell}))$ such that ψ' is a lift of φ'.*

Proof. In the statement, $Z^1(W_t, \hat{G})$ is the abbreviation for $Z^1(W_t, \hat{G})_{\overline{\mathbb{Z}_\ell}}$. Recall that $Z^1(W_t, \hat{G}) \to \overline{\mathbb{Z}_\ell}$ is flat (See [3, Proposition 3.3]), hence generalizing (See Stack Project, 01U2). Therefore, given $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$, there exists $\xi \in Z^1(W_t, \hat{G}(\overline{\mathbb{Q}_\ell}))$ such that ξ specializes to φ'. In other words, $ker(\xi) \subset ker(\varphi')$. I'm going to show that $\xi : W_t \to \hat{G}(\overline{\mathbb{Q}_\ell})$ factors through $\hat{G}(\overline{\mathbb{Z}_\ell})$.

This is true by the following more general statement: Let $Y = \mathrm{Spec}(R)$ be an affine scheme over $\overline{\mathbb{Z}_\ell}$, let $y_\eta \in Y(\overline{\mathbb{Q}_\ell})$ specializing to $y_s \in Y(\overline{\mathbb{F}_\ell})$. Then $y_\eta \in Y(\overline{\mathbb{Q}_\ell}) = \mathrm{Hom}(R, \overline{\mathbb{Q}_\ell})$ factors through $\overline{\mathbb{Z}_\ell}$.

To prove the above statement, let $\mathfrak{p} := \ker(y_\eta)$ and $\mathfrak{q} := \ker(y_s)$ be the corresponding prime ideas. Then "y_η specializes to y_s" translates to "$\mathfrak{p} \subset \mathfrak{q}$". Recall that we are going to show that $y_\eta : R \to \overline{\mathbb{Q}_\ell}$ factors through $\overline{\mathbb{Z}_\ell}$. We argue by contradiction. Otherwise there is some element $f \in R$ mapping to $\ell^{-m}u$ for some $m \in \mathbb{Z}_{\geq 1}$ and $u \in \overline{\mathbb{Z}_\ell}^*$. Hence

$$\ell^m u^{-1} f - 1 \in ker(y_\eta) \subset ker(y_s). \tag{2}$$

However, $\ell \in ker(y_s)$ since y_s lives on the special fiber. This together with equation 2 implies that $1 \in ker(y_s)$. Contradiction! □

Lemma 2. *The schematic centralizer $C_{\hat{G}}(\psi_\ell)$ is a generalized reductive group scheme over $\overline{\mathbb{Z}_\ell}$.*

Proof. To invoke [3, Lemma 3.2], I first show that

$$C_{\hat{G}}(\psi_\ell) = C_{\hat{G}}(\psi(I_F^\ell)),$$

where $C_{\hat{G}}(\psi(I_F^\ell))$ is the schematic centralizer of the subgroup $\psi(I_F^\ell) \subset \hat{G}(\overline{\mathbb{Z}_\ell})$ in $\hat{G}$. This can be checked by Yoneda Lemma on R-valued points for any $\overline{\mathbb{Z}_\ell}$-algebra R.

Then we could conclude by [3, Lemma 3.2]. Indeed, ψ_ℓ factors through some finite quotient Q of I_F^ℓ, which has order invertible in the base $\overline{\mathbb{Z}_\ell}$. So the conditions of [3, Lemma 3.2] are satisfied.

Some explanations to use [3, Lemma 3.2]: While [3, Lemma 3.2] is phrased in the setting that R is a normal subring of a number field, it still works for $\overline{\mathbb{Z}_\ell} \subset \overline{\mathbb{Q}_\ell}$ instead of $\mathbb{Z} \subset \mathbb{Q}$ (Why?). There is also a small issue that $\overline{\mathbb{Z}_\ell}$ is not finite over $\mathbb{Z}_\ell$, but this can be resolved since everything is already defined over some sufficiently large finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$. □

Lemma 3.
$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)).$$

Proof. The first equation is by definition (and that $C_{\hat{G}}(\psi_\ell)$ represents the set-theoretic centralizer).

For the second equation, note that $\varphi|_{I_t} = \gamma_1 + ... + \gamma_d$ is a direct sum of characters (Since $I_t \simeq \prod_{p'\neq p} \mathbb{Z}_{p'}$), so it suffices to show that each γ_i is trivial on the summand $\mathbb{Z}_\ell$ of $I_t \simeq \prod_{p'\neq p} \mathbb{Z}_{p'}$. Indeed,

$$\mathrm{Hom}_{Cont}(\mathbb{Z}_\ell, \overline{\mathbb{F}_\ell}^*) = \mathrm{Hom}_{Cont}(\varprojlim \mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \varinjlim \mathrm{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \{1\}.$$

□

2 Main Theorem: description of $[X_\varphi/\hat{G}]$

Let F be a non-archimedean local field, G be a connected split reductive group over F. Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a tame, regular semisimple, elliptic L-parameter (TRSELP for short). Recall that this means that the centralizer

$$C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) =: S \subset \hat{G}(\overline{\mathbb{F}_\ell})$$

is a maximal torus, and $\varphi(Fr) \in N_{\hat{G}}(S)$ gives rise to an element $w = \overline{\varphi(Fr)} \in N_{\hat{G}}(S)/S$ in the Weyl group (and that φ is tame and elliptic). (Maybe add a remark on the relation between $\varphi(Fr), n = \psi(Fr)$, and w)

Assume that

(assumption 1) The center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$.

(assumption 2) $Z(\hat{G})$ is finite.

Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of φ. Let ψ_ℓ denotes the restriction $\psi|_{I_F^\ell}$, and $\overline{\psi}$ denotes the image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Recall that the schematic centralizer $C_{\hat{G}}(\psi_\ell) = T$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

For later use, I record the following lemma – w can also be defined in terms of ψ instead of φ. This is helpful because we will reduce to a computation on the special fiber later. First, notice that since T is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\ell \neq 2$, we can identify

$$\left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) \simeq \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell}),$$

and denote it by Ω. (See Lemma 7 below)

Remark. *Lemma 7 below shows that $N_{\hat{G}}(T)/T$ is representable by a group scheme which is split over $\overline{\mathbb{Z}_\ell}$. Therefore, we will slightly abuse notations and use $\Omega, N_{\hat{G}}(T)/T, N_{\hat{G}}(S)/S$ interchangeably.*

Lemma 4. *The image of $\varphi(Fr)$ and $\psi(Fr)$ in the Weyl group Ω agree, hence giving a well defined element w in the Weyl group Ω. (Check carefully!)*

Proof. Let
$$\Omega = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell})$$
as above and $\underline{\Omega}$ be the associated constant group scheme. Since ψ is a lift of φ, $\psi(Fr)$ specializes to $\varphi(Fr)$ in $N_{\hat{G}}(T)$. Then the lemma follows since
$$N_{\hat{G}}(T) \to N_{\hat{G}}(T)/T = \underline{\Omega}$$
is a morphism of schemes, hence the following diagram commutes:
$$\begin{tikzcd}
N_{\hat{G}}(T)(\overline{\mathbb{Z}_\ell}) \arrow[r] \arrow[d] & N_{\hat{G}}(T)(\overline{\mathbb{F}_\ell}) \arrow[d] \\
\underline{\Omega}(\overline{\mathbb{Z}_\ell}) = \Omega \arrow[r] & \underline{\Omega}(\overline{\mathbb{F}_\ell}) = \Omega
\end{tikzcd}$$
□

Our main theorem is the following.

Theorem 2. *Let X_φ $(= X_\psi)$ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ (hence also containing ψ). The we have isomorphisms of quotient stacks*
$$[X_\varphi/\hat{G}] \simeq [(T \times \mu)/T] \simeq [*/C_T(n)] \times \mu,$$
where $C_T(n)$ is the schematic centralizer of $n = \psi$ in $T = C_{\hat{G}}(\psi|_{I_F^\ell})$, and $\mu = \prod_{i=1}^m \mu_{\ell^{k_i}}$ for some $k_i \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}_{\geq 0}$ is a product of group schemes of roots of unity.

If moreover assume

(assumption 3) ℓ doesn't divide the order of $w = \overline{\varphi(Fr)}$ in the Weyl group $N_{\hat{G}}(S)/S$;

then
$$[X_\varphi/\hat{G}] \simeq [(T \times \mu)/T] \simeq [*/\underline{S_\varphi}] \times \mu,$$
where $S_\varphi = C_{\hat{G}}(\overline{\mathbb{F}_\ell})(\varphi(W_F))$, and $\underline{S_\varphi}$ is the corresponding constant group scheme.

Proof. Recall that X_φ is isomorphic to the contracted product
$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$
and that $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(Fr), \eta(s_0))$ induces isomorphisms
$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq T \times \mu.$$

This implies that $[X/\hat{G}] \simeq [(T \times \mu)/T]$ with T acting on T by twisted conjugacy:
$$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where $n = \psi(Fr)$. In other words, T acts on T via multiplication by $tnt^{-1}n^{-1}$. And T acts trivially on μ (See Proposition 1).

So we are reduced to compute $[T/T]$ with respect to a nice action of the split torus T on T, which should be and turns out to be very explicit.

For clarification, let me denote the source torus T by $T^{(1)}$ and the target torus T by $T^{(2)}$. Consider the morphism

$$f : T^{(1)} = T \to T = T^{(2)}, s \mapsto sns^{-1}n^{-1}.$$

This is surjective on $\overline{\mathbb{F}_\ell}$-points by our assumption 2 that $Z(\hat{G})$ is finite and φ is elliptic (See Lemma below). Hence f is an epimorphism in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes (See the same Lemma below)(maybe add an appendix on diagonalizable group schemes?). Therefore, f induces an isomorphism

$$T^{(1)}/ker(f) \simeq T^{(2)},$$

as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Moreover, if we let $t \in T$ act on $T^{(1)}$ by left multiplication by t, and on $T^{(2)}$ via multiplication by $(tnt^{-1}n^{-1})$, this isomorphism induced by f is T-equivariant.

Note $T^{(1)} = T$ is commutative, so the T-action (via multiplication by $tnt^{-1}n^{-1}$) and the $ker(f)$-action (via left multiplication) on T commutes with each other. Hence by the T-equivariant isomorphism $T^{(1)}/ker(f) \simeq T^{(2)}$ above, we have

$$[T/T] = [T^{(2)}/T] \simeq \left[\left(T^{(1)}/ker(f)\right)/T\right] \simeq \left[\left(T^{(1)}/T\right)/ker(f)\right] \simeq [*/ker(f)] = [*/C_T(n)].$$

For the last assertion, see Lemma 6 below.

$\square$

Does $C_T(n) \simeq C_{\hat{G}}(\psi)$ holds?

2.0.1 Some lemmas

Lemma 5. *The morphism*

$$f : T^{(1)} = T \to T = T^{(2)}, s \mapsto sns^{-1}n^{-1}$$

is epimorphic in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. And it induces and isomorphism $T^{(1)}/ker(f) \simeq T^{(2)}$ as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes.

Proof. Recall that T is a split torus over $\overline{\mathbb{Z}_\ell}$, hence a diagonalizable $\overline{\mathbb{Z}_\ell}$-group scheme. Notice that f is a morphism of $\overline{\mathbb{Z}_\ell}$-group schemes, hence a morphism of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Recall that the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes is equivalent to the category of abelian groups (See [1, p70, Section 5] or [2]) via

$$D \mapsto \mathrm{Hom}_{\overline{\mathbb{Z}_\ell}-GrpSch}(D, \mathbb{G}_m),$$

and the inverse is given by

$$\overline{\mathbb{Z}_\ell}[M] \leftarrow M,$$

where $\overline{\mathbb{Z}_\ell}[M]$ is the group algebra of M with $\overline{\mathbb{Z}_\ell}$-coefficients.

Therefore, we could argue in the category of abelian groups via the above category equivalence: f is epimorphic if and only if the map f^* in the category of abelian groups is monomorphic. Note ellipticity and $Z(\hat{G})$ finite imply that S_φ is finite, hence

$$ker(f)(\overline{\mathbb{F}_\ell}) = C_T(n) = S_\varphi$$

is finite (See Lemma ? for the last equality Should be true over $\overline{\mathbb{F}_\ell}$. But maybe not true over $\overline{\mathbb{Z}_\ell}$?), hence $coker(f^*)$ is finite. Therefore,

$$f^* : \operatorname{Hom}(T^{(2)}, \mathbb{G}_m) \to \operatorname{Hom}(T^{(1)}, \mathbb{G}_m)$$

is injective (i.e., monomorphism). Indeed, otherwise $ker(f^*)$ would be a nonzero sub-$\mathbb{Z}$-module of the finite free $\mathbb{Z}$-module $\operatorname{Hom}(T^{(2)}, \mathbb{G}_m)$, hence a free $\mathbb{Z}$-module of positive rank, which contradicts with $coker(f^*)$ being finite.

The statement on the quotient follows from the corresponding result in the category of abelian groups: f^* induces an isomorphism

$$\operatorname{Hom}(T^{(1)}, \mathbb{G}_m)/\operatorname{Hom}(T^{(2)}, \mathbb{G}_m) \simeq coker(f^*)$$

(See [1, p71, Subsection 5.3].) □

Lemma 6. *(Assume 2 (Need adjust): ℓ doesn't divide the order of w.) $ker(f) \simeq \underline{S_\varphi}$ is the constant group scheme of the finite abelian group $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$.*

Proof. We recall the following fact: Let H be a smooth affine group scheme over some ring R, let Γ be a finite group whose order is invertible in R. Then the fixed point functor H^Γ is representable and is smooth over R.

For a proof of the above fact, see [6, Proposition 3.4] or [4, Lemma A.1, A.13].

In our case, let $H = T$, $\Gamma =< w >$ the subgroup of the Weyl group $W_{\hat{G}}(T)$ generated by w. Hence

$$ker(f) = C_T(n) = H^\Gamma$$

(See Lemma ? for the last equality) is smooth over $\overline{\mathbb{Z}_\ell}$. Therefore, $ker(f)$ is finite etale over $\overline{\mathbb{Z}_\ell}$ (See Lemma ?). Hence $ker(f)$ is a constant group scheme over $\overline{\mathbb{Z}_\ell}$, since $\overline{\mathbb{Z}_\ell}$ has no non-trivial finite etale cover.

Since $ker(f)$ is constant, we can determine it by computing its $\overline{\mathbb{F}_\ell}$-points:

$$ker(f)(\overline{\mathbb{F}_\ell}) = C_{T(\overline{\mathbb{F}_\ell})}(n) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F)),$$

where the middle equality follows by noticing $T(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ and $n = \varphi(Fr)$.

Finally, note by our TRSELP assumption, $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ is (the $\overline{\mathbb{F}_\ell}$-points of) a torus. Hence $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is abelian, hence finite abelian as we've noticed in the proof of the previous lemma that S_φ is finite (by ellipticity and $Z(\hat{G})$ finite). □

Lemma 7. *Let $\hat{G}$ be a connected reductive group over $\overline{\mathbb{Z}_\ell}$, and T a maximal torus of $\hat{G}$. Then the Weyl group $N_{\hat{G}}(T)/T$ is split over $\overline{\mathbb{Z}_\ell}$.*

Proof. By [2, Proposition 3.2.8], the Weyl group $N_{\hat{G}}(T)/C_{\hat{G}}(T)$ is finite etale over $\overline{\mathbb{Z}_\ell}$ and hence split over $\overline{\mathbb{Z}_\ell}$. In our case, $C_{\hat{G}}(T) = T$ since $\hat{G}$ is connected (For example, use the proof of [2, Theorem 3.1.12]). $\square$

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