Explicit mod-ℓ categorical local Langlands correspondence

for depth-zero supercuspidal part of GL_2.

§1. Introduction

F : NA local. $\Lambda = \bar{\mathbb{Z}}_\ell$. $G = GL_2$.

* Categorical LLC: There exists a fully faithful embed

$$\mathrm{Rep}_\Lambda(G(F)) \hookrightarrow \mathrm{QCoh}(Z^1(W_F, \hat{G})_\Lambda / \hat{G})$$

This talk: verify CLLC for depth-0 supercuspidal part.

Setting: Let $\pi \in \mathrm{Rep}_{\bar{\mathbb{F}}_\ell}(G(F))$ irred depth-zero & supercuspidal,

$\varphi := \varphi_\pi \in Z^1(W_F, \hat{G}(\bar{\mathbb{F}}_\ell))$ corresponding L-param, tame & irred.

$$\text{Goal: } \underset{\text{"block containing } \pi\text{"}}{\mathrm{Rep}_\Lambda(G(F))_{[\pi]}} \dashrightarrow \mathrm{QCoh}(\underbrace{C_\varphi}_{\text{"connected comp containing } \varphi\text{"}})$$

$$\mathrm{Rep}_\Lambda(G(F))_{[\pi]} \cong \mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k})$$

$$\mathrm{QCoh}(C_\varphi) \cong \bigoplus_{2} \mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k})$$

(10 min)

Dat's IHES notes.

§2. The component C_φ. || reference: dat.pdf.

$$Z^1(W_F, \hat{G})_\Lambda = \underline{\mathrm{Hom}}(W_F, \hat{G})_\Lambda : \quad \Lambda\text{-alg} \longrightarrow \mathrm{Sets}$$

$$\begin{array}{l} \cup\!\!\mid \\ Z^1(W_t, \hat{G})_\Lambda \quad (W_t := W_F/P_F) \end{array} \qquad R \longmapsto \mathrm{Hom}(W_F, \hat{G}(R))$$

$$\| \ \text{Fact}$$

$$Z^1(W_t^\circ, \hat{G})_\Lambda \qquad W_t^\circ = \langle \underset{\leftrightarrow}{s_0}, \mathrm{Fr} \mid \mathrm{Fr}\cdot s_0 \cdot \mathrm{Fr}^{-1} = s_0^q \rangle$$

topological generator of $I_F/P_F \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$

$\underline{\text{Eg.1}}$ $G = GL_1$

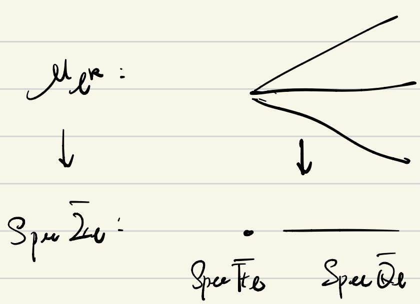
$$Z^1(W_t^\circ, GL_1)_{\overline{\mathbb{Z}}_\ell} = \{(x, y) \in GL_1 \times GL_1 \mid \underbrace{yxy^{-1}}_{x} = x^q\}$$

$$\cong \mu_{q-1} \times GL_1$$

$\Rightarrow$ the connected components are of form $\mu_{\ell^k} \times GL_1$

where $k = \max\{n \in \mathbb{Z}_{\geq 0} \mid \ell^n \text{ divides } q-1\}$.

(To see this, first do the $\Lambda = \overline{\mathbb{F}}_\ell$ case)



E.g. 2 $\quad G = GL_2 \qquad E/F$ deg 2, unramified. $W_E = I_F \rtimes \langle Fr^2 \rangle$.

$\varphi := \mathrm{Ind}_{W_E}^{W_F} \varsigma$ irred tame. $\quad \varsigma : W_E \to \overline{\mathbb{F}}_\ell^\times \overset{\text{lift}}{\rightsquigarrow} \tilde{\varsigma} : W_E \to \overline{\mathbb{Z}}_\ell^\times$

$\psi := \mathrm{Ind}_{W_E}^{W_F} \tilde{\varsigma} : \; s_0 \mapsto \begin{pmatrix} \tilde{\varsigma}(s_0) & \\ & \tilde{\varsigma}(s_0)^q \end{pmatrix}, \quad Fr \mapsto \begin{pmatrix} & \tilde{\varsigma}(Fr^2) \\ 1 & \end{pmatrix}$

$\in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}}_\ell))$

$\psi_\ell := \psi|_{I_F^\ell}$, $\quad I_F^\ell \subseteq I_F$ max subgp of (pro-)order prime to ℓ

$\bar{\psi} :=$ image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$

dat.pdf section 4 $\qquad := \{\psi' \mid \psi'_\ell = \psi_\ell,\ \bar{\psi}' = \bar{\psi}\}$.

$X_\varphi \xleftarrow{\cong} \left(\hat{G} \times \boxed{Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \bar{\psi}}}\right) \Big/ C_{\hat{G}}(\psi_\ell)_{\bar{\psi}} \qquad g\psi'(-)g^{-1} \leftarrow\!\shortmid (g, \psi')$

"The conn comp (of $Z^1(\cdot)$) containing φ consists of L-param ψ' s.t. $(\psi'_\ell, \bar{\psi}') \sim (\psi_\ell, \bar{\psi})$"

$\cong \left(\hat{G} \times \underbrace{(T^{Fr=(-)^q})^\circ}_{\cong} \times T\right) \Big/ T \qquad (g,\ \psi'(s_0)\psi(s_0)^{-1},\ \psi'(Fr)\psi(Fr)^{-1})$

$\qquad \begin{pmatrix} * & \\ & * \end{pmatrix} \begin{pmatrix} * & \\ & * \end{pmatrix} \quad \begin{pmatrix} & * \\ * & \end{pmatrix} \begin{pmatrix} & * \\ * & \end{pmatrix}$

$\mu_{\ell^k} \qquad k$ max s.t. $\ell^k \mid q^2 - 1$

where $T = N_{\hat{G}}(\psi_\ell)^\circ = C_{\hat{G}}(\psi_\ell)$, $\qquad \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}, \begin{pmatrix} & a \\ b & \end{pmatrix} \mapsto \begin{pmatrix} & t_1 t_2^{-1} a \\ t_1^{-1} t_2 b & \end{pmatrix}$

$\Rightarrow \quad C_\varphi = [X_\varphi / \hat{G}] \cong [(\mu_{\ell^k} \times T)/T] \cong \mu_{\ell^k} \times [T/T] \cong \mu_{\ell^k} \times [\mathbb{G}_m / \mathbb{G}_m]$

$\left(\Rightarrow \quad \mathrm{QCoh}(C_\varphi) \cong \bigoplus_{\mathbb{Z}} \mathrm{QCoh}(\mu_{\ell^k} \times \mathbb{G}_m).\right)$

(30 min)

§3. The block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$.

<u>E.g 1 / Exer</u> (depending on time) $G = GL_1$

$$G(F) = F^\times = \mathcal{O}_F^\times \times \mathbb{Z}$$

$$\begin{aligned}
\mathrm{Rep}_\Lambda(G(F))_0 &\cong \mathrm{Rep}_\Lambda(\underbrace{\mathcal{O}_F^\times / \varpi \mathcal{O}_F^\times}_{\mathbb{Z}/(q-1)\mathbb{Z}}) \otimes \mathrm{Rep}_\Lambda(\mathbb{Z}) \\
&\cong \mathcal{H}_\Lambda(\mathbb{Z}/(q-1)\mathbb{Z})\text{-mod} \otimes \mathcal{H}_\Lambda(\mathbb{Z})\text{-mod} \\
&\cong \mathbb{Z}[s]/(s^{q-1}-1)\text{-mod} \otimes \mathbb{Z}[t, t^{-1}]\text{-mod} \\
&\cong \mathrm{QCoh}(\mu_{q-1} \times \mathbb{G}_m).
\end{aligned}$$

$$\Rightarrow \mathrm{Rep}_\Lambda(G(F))_{[\pi]} \cong \mathrm{QCoh}(\mu_{\ell^k} \times \mathbb{G}_m), \quad \ell^k \mid q-1.$$

§3. The block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$. $\qquad G = GL_2$.

Recall: all irreducible depth-zero supercuspidal representations of $G(F)$ are of form $\pi = \text{c-Ind}_{\underbrace{G(\mathcal{O}_F)\cdot Z}_{K}}^{G(F)} \rho$, where ρ is a extension of $\tilde{\lambda} \in \mathrm{Rep}_\Lambda(G(\mathcal{O}_F))$, $\tilde{\lambda}$: inflation of cuspidal $\lambda \in \mathrm{Rep}_\Lambda(G(\mathbb{F}_q))$

Thm (Broué) $\underbrace{T(\mathbb{F}_q)}_{\mathbb{F}_{q^2}^\times} \subseteq G(\mathbb{F}_q)$ elliptic torus, Deligne-Lusztig induction

$$\ell^k \mid q^2-1 \qquad R_T^G : \underbrace{\mathrm{Rep}_\Lambda(T(\mathbb{F}_q))_{[\theta]}}_{\text{block containing } \theta} \xrightarrow{\cong} \underbrace{\mathrm{Rep}_\Lambda(G(\mathbb{F}_q))_{[\lambda]}}_{\text{block containing } \lambda = R_T^G\theta}$$

$\mathrm{Rep}_\Lambda(\mathbb{Z}/\ell^k\mathbb{Z}) \cong \mathrm{Rep}_\Lambda(T(\mathbb{F}_q)_\ell) \cong \mathrm{Rep}_\Lambda(T(\mathbb{F}_q))_{[\theta]}$

$$\rightsquigarrow \Big[\mathrm{Rep}_\Lambda(\underbrace{G(\mathcal{O}_F)\cdot Z}_{G(\mathcal{O}_F)\times\left\{\begin{pmatrix}\varpi^m & \\ & \varpi^m\end{pmatrix}\right\} \cong G(\mathcal{O}_F)\times\mathbb{Z}})_{[\rho]} \cong \underbrace{\mathrm{Rep}_\Lambda(\mathbb{Z}/\ell^k\mathbb{Z}) \otimes \mathrm{Rep}_\Lambda(\mathbb{Z})}_{\mathcal{O}(\mu_{\ell^k}\times\mathbb{G}_m)\text{-mod} \,\cong\, \mathrm{QCoh}(\mu_{\ell^k}\times\mathbb{G}_m)}\Big] \qquad \ell^k \mid q^2-1.$$

Thm $\text{c-Ind}_{G(\mathcal{O}_F)\cdot Z}^{G(F)}$ induces an equivalence of categories

$$\mathrm{Rep}_\Lambda(K\cdot Z)_{[\rho]} \cong \mathcal{B}_{[\rho]} \cong \mathcal{C}_{[\pi]} \cong \mathrm{Rep}_\Lambda(G(F))_{[\pi]}.$$

Sketch: Fully faithful: $\mathrm{Hom}_G(\text{c-Ind}_{KZ}^G \rho_1, \text{c-Ind}_{KZ}^G \rho_2) \overset{\text{FR+Mackey}}{\cong} \mathrm{Hom}_{KZ}(\rho_1, \rho_2)$

Essential surj: To show: $\overset{\Pi_{[\rho]}\,\cong}{c\text{-}\mathrm{Ind}_{KZ}^{G}(\sigma|_{\mathcal{B}_{[\rho]}})}$ is a projective generator of $\mathcal{C}_{[\pi]}$, where $\sigma := c\text{-}\mathrm{Ind}_{K_1}^{KZ}\Lambda$, $K_1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$.

* $\Pi_{[\rho]}$ is a generator, i.e. $\mathrm{Hom}_G(\Pi_{[\rho]}, X) \neq 0$, $\forall X \in \mathcal{C}_{[\pi]}$.

Key 1

<u>Fact</u>: [$\Pi := c\text{-}\mathrm{Ind}_{K_1}^{G}\Lambda$ is a projective generator of $\mathrm{Rep}_\Lambda(G(F))_0$

$\Rightarrow \Pi_{[\rho]}$ projective ⊕ All depth 0 $\mathrm{Rep}_\Lambda(G(F))$.

$\Pi = \Pi_{[\rho]} \oplus \Pi^{[\rho]}$] $\left(\Pi^{[\rho]} := c\mathrm{Ind}_{KZ}^{G}(\sigma|_{\mathcal{B}_{[\rho]}^{\perp}})\right)$

FR + Mackey $\Rightarrow$ Key 2: the complement doesn't interfere.

[$\mathrm{Hom}_G(\Pi_{[\rho]}, \Pi^{[\rho]}) = \mathrm{Hom}_G(\Pi^{[\rho]}, \Pi_{[\rho]}) = 0$]

$$\begin{tikzcd}
\rightsquigarrow \mathrm{Rep}_\Lambda(G(F))_0 \arrow[rr, "\mathrm{Hom}_G(\Pi, -)", "\cong"'] & & \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi) \arrow[d, phantom, "\cong" rotate=90] \\
\mathcal{C}_{[\pi]} \arrow[u, hook] \arrow[rr, "\mathrm{Hom}_G(\Pi_{[\rho]}, -)\,\oplus\,\mathrm{Hom}_G(\Pi^{[\rho]}, -)"] & & \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{[\rho]}) \oplus \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi^{[\rho]})
\end{tikzcd}$$

$\mathcal{C}_{[\pi]}$ indecomposable $\Rightarrow$ $\mathrm{Hom}_G(\Pi^{[\rho]}, -) = 0$ on $\mathcal{C}_{[\pi]}$.

$\Rightarrow \mathrm{Hom}_G(\Pi_{[\rho]}, X) \neq 0$, $\forall X \in \mathcal{C}_{[\pi]}$ $\square$

(30 min).

Summary: k max s.t. $\ell^k \mid q^2-1$

$$\begin{array}{ccc}
\mathrm{Rep}_\Lambda(G(F))_{[\pi]} & \overset{\text{abstractly}}{\longleftrightarrow} & \mathrm{QCoh}([X_\varphi/\hat{G}]) \\
\wr\Vert & & \wr\Vert \\
\mathrm{Rep}_\Lambda(G(\mathcal{O}_F)\cdot Z)_{[\rho]} & & \mathrm{QCoh}(\mu_{\ell^k} \times \mathbb{G}_m \times [*/\mathbb{G}_m]) \\
\wr\Vert & & \wr\Vert \\
\mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}) & & \bigoplus\limits_{\chi \in X^*(\mathbb{G}_m) \cong \mathbb{Z}} \mathrm{QCoh}(\mu_{\ell^k} \times \mathbb{G}_m).
\end{array}$$

§4 Categorical LLC (in Fargues-Scholze form).

Ind the end:

$$\begin{array}{ccccc} F: & \mathcal{C} \circlearrowleft A\text{-mod} & \longrightarrow & \mathcal{D} \circlearrowleft A\text{-mod} \\ & \wr\!\| & & \wr\!\| \\ \sim\ G: & A\text{-mod} & \dashrightarrow & A\text{-mod} \end{array}$$

s.t. (1) F is A-mod linear

(2) G sends A to A

Then F is an equivalence of categories.

In our case: $A\text{-mod} = \mathrm{Perf}(C_g)$

$$\mathcal{C} = D^{b,qc}_{Coh,Nilp}(C_g) \overset{\text{Fact}}{\cong} \mathrm{Perf}(C_g), \quad \mathcal{D} = D^{C_g}(\mathrm{Bun}_G, \bar{\mathbb{Z}}_\ell)^\omega$$

$$F: \mathcal{C} \longrightarrow \mathcal{D}, \quad A \longmapsto A * \mathcal{H}(G(F))_{[\pi]}$$

Guess: Same as Whittaker representation restricted to $\mathrm{Rep}(G(F))_{[\pi]}$

Can check:

* $\mathcal{D} \cong \mathrm{Perf}(C_g)$ using $\mathrm{Rep}_\Lambda(G(F))_{[\pi]} \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$.
* F is $\mathrm{Perf}(C_g)$-linear, and sends $\mathcal{O}(C_g)$ to $\mathcal{O}(C_g)$.

☺ F is an equiv of categories.

[the end ☺]