Let's apply the theories in the previous chapters to the example of $GL_n(F)$. Throughout this chapter, $G := GL_n$.

That said, there is a little mismatch between the theories before and the example here, namely, we assumed for simplicity in the theories that G is simply connected (and in particular, semisimple), while this is not the case for $G = GL_n$. However, there is only some minor difference due to the center $\mathbb{G}_m$ of GL_n. I leave it as an exercise for the readers to figure out the details.

1 L-parameter side

Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be an irreducible tame L-parameter. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lift of φ. Let C_φ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ. By Need ref?, we compute that

$$C_\varphi \cong [T/T] \times \mu,$$

where $T = C_{\hat{G}}(\psi_\ell)$ is a maximal torus of GL_n, and $\mu = (T^{Fr=(-)^q})^0$, and the T-action on T is specified in Need ref?. To go further, let's choose a nice basis of the Weil group representations φ and ψ.

Indeed, every irreducible tame L-parameter with $\overline{\mathbb{F}_\ell}$-coefficients φ of GL_n are of the form $\varphi = Ind_{W_E}^{W_F}\eta$, where E is a degree n unramified extension of F, $W_E \cong I_F \rtimes \langle \mathrm{Fr}^n \rangle$ is the Weil group of E, and $\eta : W_E \to \overline{\mathbb{F}_\ell}^*$ is a tame (i.e., trivial on $P_E = P_F$) character of W_E such that $\{\eta, \eta^q, ..., \eta^{q^{n-1}}\}$ are distinct. To find a lift of it with $\overline{\mathbb{Z}_\ell}$-coefficients, we let $\tilde{\eta} : W_E \to \overline{\mathbb{Z}_\ell}^*$, and let $\psi := Ind_{W_E}^{W_F}\tilde{\eta}$. Then under a nice basis, we could specify the matrices corresponds to the topological generater s_0 and Fr:

$$\psi(s_0) = \begin{bmatrix} \tilde{\eta}(s_0) & 0 & 0 & \dots & 0 \\ 0 & \tilde{\eta}(s_0)^q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\eta}(s_0)^{q^{n-1}} \end{bmatrix}$$

and

$$\psi(\mathrm{Fr}) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \tilde{\eta}(\mathrm{Fr}^n) & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Under this basis, $T = C_{\hat{G}}(\psi_\ell)$ is the diagonal torus of GL_n, with Fr acting by conjugacy via ψ, i.e.,

$$\mathrm{Fr} \,.\, \mathrm{diag}(t_1, t_1, ..., t_{n-1}, t_n) = \mathrm{diag}(t_2, t_3, ..., t_n, t_1).$$

So one could compute that

$$T^{\mathrm{Fr}=(-)^q} \cong \mu_{q^n-1},$$

and that

$$(T^{\mathrm{Fr}=(-)^q})^0 \cong \mu_{\ell^k},$$

where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

To compute the quotient $[T/T]$, we note that T acts on T via twisted conjugacy

$$(t, t') \mapsto (tnt^{-1}n^{-1})t',$$

where n is same as $\psi(Fr)$ in effect. So in our case, this action is

$$(t_1, t_2, ..., t_n).(t_1', t_2', ..., t_n') = (t_n^{-1}t_1t_1', t_1^{-1}t_2t_2', ..., t_{n-1}^{-1}t_nt_n').$$

We see that the orbits of this action are determined by the determinants (hence are in bijection with $\mathbb{G}_m$), and the center $\mathbb{G}_m \cong Z \subset T$ acts trivially. Therefore,

$$[T/T] \cong [\mathbb{G}_m/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$.

In conclusion, we have that the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ is

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k},$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$, and $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

2 Representation side

By modular Deligne-Lusztig theory, the block $\mathcal{A}_{x,1}$ of $GL_n(\mathbb{F}_q)$ containing a cuspidal representation σ is equivalent to the block of an elliptic torus, which is isomorphic to $\mathbb{F}_{q^n}^*$. So this block is equivalent to $\overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)$, where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

$\mathcal{A}_{x,1}$ inflats to a block of $K := GL_n(\mathcal{O}_F)$ containing the inflation $\tilde{\sigma}$ of σ, and further corresponds to a block $\mathcal{B}_{x,1}$ of KZ containing ρ, a extension of $\tilde{\sigma}$ to KZ, where Z is the center of $GL_n(F)$. We have

$$\mathcal{B}_{x,1} \cong \mathcal{A}_{x,1} \otimes \mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\mathbb{Z}) \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod},$$

because

$$KZ \cong K \times \{\mathrm{diag}(\pi^m, ..., \pi^m | m \in \mathbb{Z})\} \cong K \times \mathbb{Z}.$$

Argue as before (See ?) we see that the compact induction c-Ind_{KZ}^G induces an equivalence of categories

$$\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1},$$

where $\mathcal{C}_{x,1}$ is the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$ containing $\pi := \text{c-Ind}_{KZ}^G \rho$.

Since every depth-zero supercuspidal representation π arises as above, we have that the block containing π satisfies

$$\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]} \cong \mathcal{C}_{x,1} \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod}.$$

In this chapter, I prove the categorical local Langlands conjecture for depth-zero supercuspidal part of $G = GL_n$ with coefficients $\Lambda = \overline{\mathbb{Z}_\ell}$.

Let $\varphi \in Z^1(W_E, \hat{G}(\overline{\mathbb{F}_\ell}))$ be an irreducible tame L-parameter. Let C_φ be the connected component of $Z^1(W_E, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ.

The goal is to show that there is an equivalence

$$D_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong D_{Coh,Nilp}^{b,qc}(C_\varphi)$$

of derived (?) categories.

As a first step, let's unravel the definition of both sides and describe them explicitly.

3 Unraveling definitions

3.1 L-parameter side

Let's first state a lemma that makes the decorations in $D_{Coh,Nilp}^{b,qc}(C_\varphi)$ go away. We postpone its proof to a later subsection.

Lemma 1. $D_{Coh,Nilp}^{b,qc}(C_\varphi) \cong D_{Coh,\{0\}}^{b}(C_\varphi) \cong \mathrm{Perf}(C_\varphi)$.

Let's assume the lemma for the moment and continue. By our computation before,

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k} \cong \mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k},$$

where $k \in \mathbb{Z}_{\geq 0}$ is maximal such that ℓ^k divides $q^n - 1$. So

$$\mathrm{Perf}(C_\varphi) \cong \mathrm{Perf}(\mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k}) \simeq \mathrm{Perf}(\mathbb{G}_m) \otimes \mathrm{Perf}([*/\mathbb{G}_m]) \otimes \mathrm{Perf}(\mu_{\ell^k}).$$

Here,

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell})\chi \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell}),$$

where χ runs over characters of $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m | m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

In conclusion, we have

$$\mathrm{Perf}(C_\varphi) \cong \bigoplus_\chi \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where χ runs over characters of $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m | m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

3.2 Bun_G side

Since φ is irreducible,

$$D_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong D_{lis}^{C_\varphi}(Bun_G^{ss}, \overline{\mathbb{Z}_\ell})^\omega.$$

Since

$$Bun_G^{ss} = \sqcup_{b \in B(G)_{basic}}[*/G_b(F)],$$

we have

$$D_{lis}^{C_\varphi}(Bun_G^{ss}, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{basic}} D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega.$$

Let's look closer into each direct summand. In our case $G = GL_n$,

$$B(G)_{basic} \cong \pi_1(G)_\Gamma \cong \mathbb{Z}$$

.

Let's first look at the summand for $b = 1$ (corresponding to $0 \in \mathbb{Z} \cong B(G)_{basic}$). For $b = 1$, $G_b \cong GL_n$, and

$$D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega \cong D^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong D(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]})^\omega,$$

where $\pi \in \mathrm{Rep}_{\overline{\mathbb{F}_\ell}}(GL_n(F))$ is the representation with L-parameter φ, and $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]}$ is the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))$ containing π. And we've computed that

$$\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]} \cong \overline{\mathbb{Z}_\ell}[t, t^{-1}] \otimes \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)\text{-Mod} \cong \mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where $k \in \mathbb{Z}_{\geq 0}$ is again maximal such that ℓ^k divides $p^n - 1$. So we have

$$D^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong D(\mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}))^\omega \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

We could get a similar description of $D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})$ for free by the spectral action and the compatibility of Fargues-Scholze with $\pi_1(G)_\Gamma$-grading. For this, we consider the composition

$$q : C_\varphi \cong \mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k} \to [*/\mathbb{G}_m].$$

Recall that

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell})\chi,$$

we denote by $\mathcal{M}_\chi$ the corresponding simple object in $\mathrm{Perf}([*/\mathbb{G}_m])$. Moreover, $\mathcal{M}_\chi$ pullbacks to a line bundle

$$\mathcal{L}_\chi := q^*\mathcal{M}_\chi.$$

We could now state the key proposition that allows us to get to arbitrary $b \in B(G)_{basic}$ from the $b = 1$ case, using the spectral action.

Proposition 1. *1. The restriction of the spectral action by $\mathcal{L}_\chi$ to $D(G_b(F), \overline{\mathbb{Z}_\ell})$ factors through $D(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell})$.*

$$\begin{tikzcd}
\mathcal{L}_\chi * - : & D_{lis}(Bun_G, \overline{\mathbb{Z}_\ell}) \arrow[r] & D_{lis}(Bun_G, \overline{\mathbb{Z}_\ell}) \\
& D(G_b(F), \overline{\mathbb{Z}_\ell}) \arrow[r] \arrow[u, "\subset"] & D(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell}) \arrow[u, "\subset"]
\end{tikzcd}$$

2. $\mathcal{L}_\chi * - : D(G_b(F), \overline{\mathbb{Z}_\ell}) \to D(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell})$ is an equivalence of categories, with inverse $\mathcal{L}_{\chi^{-1}} * -$.

Proof. For the first assertion, see [2, Lemma 5.3.2]. For the second assertion, note that $\mathcal{L}_\chi$ and $\mathcal{L}_{\chi^{-1}}$ are clearly inverse to each other once they are well-defined, since q^* preserves tensor product. $\square$

So we have

$$D^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{basic}} D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell}) \cong \bigoplus_{b \in B(G)_{basic}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

3.3 Proof of Lemma 1

Now we prove Lemma 1.

The first isomorphism is because C_φ is connected, hence the quasicompact support condition qc is automatic.

The second isomorphism needs some computation. For the definition and properties of the nilpotent singular support condition $Nilp$, I refer to [1, Section VIII.2]. At the end of the day, it boils to the fact that

$$H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda(1)) \cap Nilp(\hat{\mathfrak{g}}^*) = \{0\}.$$

(Maybe elaborate more.)

4 The spectral action induces an equivalence of categories

To summarize, we have (abstract) equivalence of categories

$$D^{b,qc}_{Coh,Nilp}(C_\varphi) \cong \bigoplus_{\chi \in \mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong \bigoplus_{b \in \mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega,$$

where I identified both $X^*(\mathbb{G}_m) \cong X^*(Z(\hat{G}))$ and $B(G)_{basic} \cong \pi_1(G)_\Gamma$ with $\mathbb{Z}$. The next goal is to show that the spectral action induces an equivalence of categories

$$D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong D^{b,qc}_{Coh,Nilp}(C_\varphi). \tag{1}$$

4.1 Definition of the functor

Let's first define the functor. For this, let's choose a Whittaker datum consisting of a Borel $B \subset G$ and a generic character $\vartheta : U(F) \to \overline{\mathbb{Z}_\ell}^*$. Let $\mathcal{W}_\vartheta$ be the sheaf concentrated on Bun_G^1 corresponding to the representation $W_\vartheta := \text{c-Ind}_{U(F)}^{G(F)} \vartheta$. Let $W_{\vartheta,[\pi]}$ be the restriction of W_ϑ to the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$, and $\mathcal{W}_{\vartheta,[\pi]}$ the corresponding sheaf.

We define our desired functor by spectral acting on $\mathcal{W}_{\vartheta,[\pi]}$:

$$\Theta : D^{b,qc}_{Coh,Nilp}(C_\varphi) \cong \mathrm{Perf}(C_\varphi) \longrightarrow D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega, \qquad A \mapsto A * \mathcal{W}_{\vartheta,[\pi]}.$$

4.2 Equivalence on degree zero part

We now show that Θ induces an equivalence on degree zero part. At the end of the day, this is similar to the following fact: If I have a functor $F : R\text{-Mod} \to R\text{-Mod}$, which is ($R$-Mod)-linear and sends R to R, then F is an equivalence of category.

By compatibility with $\pi_1(G)_\Gamma$-grading, Θ restricts to a map

$$\Theta_0 := \Theta|_{\mathrm{Perf}(C_\varphi)_{\chi=0}} : \mathrm{Perf}(C_\varphi)_{\chi=0} \longrightarrow D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0},$$

where $\mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ and

$$D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0} \cong D(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega \cong D(\mathrm{End}(W_{\vartheta,[\pi]})\text{-Mod})^\omega.$$

By tracking the definition, the structure sheaf $\mathcal{O} \in \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ goes to the Whittaker representation $W_{\vartheta,[\pi]} \in D(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega$, and further goes to $\mathrm{End}(W_{\vartheta,[\pi]}) \in D(\mathrm{End}(W_{\vartheta,[\pi]})\text{-Mod})$. Moreover, by local Langlands in family (See ?),

$$\mathrm{End}(W_{\vartheta,[\pi]}) \cong \mathcal{Z}(G)_{[\pi]} \cong \mathcal{O}(C_\varphi) \cong \mathcal{O}(\mathbb{G}_m \times \mu_{\ell^k}).$$

Therefore, we have a functor $\Theta_0 : \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \to \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ which is $\mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$-linear and sends the structure sheaf to the structure sheaf, hence an equivalence of categories.

4.3 The full equivalence

Finally, we use the spectral action to get the full equivalence. Indeed, on the L-parameter side, for any character $\chi' \in X^*(\mathbb{G}_m)$, tensoring with $\mathcal{L}_{\chi'}$ induces an equivalence

$$\mathcal{L}_{\chi'} \otimes - : \mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(C_\varphi)_{\chi=\chi'}.$$

Similarly, on the Bun_G side, by Proposition 1, spectral acting by $\mathcal{L}_{\chi'}$ induces an equivalence

$$\mathcal{L}_{\chi'} * - : D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0} \cong D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=-\chi'}.$$

Therefore, we get the full equivalence via the spectral action.

References

[1] Laurent Fargues and Peter Scholze. Geometrization of the local langlands correspondence. *arXiv preprint arXiv:2102.13459*, 2021.

[2] Konrad Zou. The categorical form of fargues' conjecture for tori. *arXiv preprint arXiv:2202.13238*, 2022.