

**On the categorical local Langlands  
conjectures for depth-zero regular  
supercuspidal representations**

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# Chapter 1

## Introduction

Let  $F$  be a non-archimedean local field. Let  $G$  be a reductive group over  $F$ . For simplicity, we assume  $G$  is split and simply connected (in particular, semisimple). Let  $\Lambda = \overline{\mathbb{Z}_\ell}$ . The categorical local Langlands conjecture predicts that there is a fully faithful embedding

$$\mathrm{Rep}_\Lambda(G(F)) \longrightarrow \mathrm{QCoh}(Z^1(W_F, \hat{G})_\Lambda / \hat{G})$$

from the category of smooth representations of the  $p$ -adic group  $G(F)$  to the category of quasi-coherent sheaves on the stack of Langlands parameters. In this paper, we compute the two sides explicitly for depth-zero regular supercuspidal part of a split simply connected group  $G$  and verify the categorical local Langlands conjecture for depth-zero supercuspidal part of  $GL_n$ .

Fix an irreducible depth-zero regular supercuspidal representation  $\pi \in \mathrm{Rep}_{\overline{\mathbb{F}_\ell}}(G(F))$ , the (classical) local Langlands conjecture predicts that it should correspond to a tame, regular semisimple, elliptic  $L$ -parameter (TRSELP for short)  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$  (See [DR09]). As mentioned above, this paper focus on the depth-zero regular supercuspidal part of the categorical local Langlands conjecture, which predicts that there is a fully faithful embedding

$$\mathrm{Rep}_\Lambda(G(F))_{[\pi]} \longrightarrow \mathrm{QCoh}([X_\varphi / \hat{G}])$$

from the block of  $\mathrm{Rep}_\Lambda(G(F))$  containing  $\pi$  to the category of quasi-coherent sheaves on the connected component  $[X_\varphi / \hat{G}]$  of the stack of  $L$ -parameters.

### 1.1 $L$ -parameter side

In this section, we explain Chapter 2 on how to compute  $\mathrm{QCoh}([X_\varphi / \hat{G}])$ .

This is done by describing  $[X_\varphi / \hat{G}]$  explicitly as a quotient stack over  $\Lambda = \overline{\mathbb{Z}_\ell}$ .

#### 1.1.1 Heuristics on the component $[X_\varphi / \hat{G}]$

In this subsection, we describe some heuristics on the component  $[X_\varphi / \hat{G}]$  which help us to guess what this component should look like.

First, let us recall what is known over  $\overline{\mathbb{Q}_\ell}$  instead of  $\Lambda = \overline{\mathbb{Z}_\ell}$ . Indeed, assuming  $G$  simply connected, the connected component of the stack of  $L$ -parameters over  $\overline{\mathbb{Q}_\ell}$  containing an elliptic  $L$ -parameter  $\varphi'$  is known to be one point. More precisely, it is isomorphic to the

quotient stack  $[*/S_{\varphi'}]$ , where  $S_{\varphi'} = C_{\hat{G}}(\varphi')$  is the centralizer of  $\varphi'$  (See [FS21, Section X.2]).

Second, let us explain what is the difference between the geometry of the connected components of the stack of  $L$ -parameters over  $\overline{\mathbb{Q}_\ell}$  and  $\overline{\mathbb{Z}_\ell}$ . This can be seen from the example  $G = GL_1$ . Indeed,

$$Z^1(W_F, \widehat{GL_1}) \cong \mu_{q-1} \times \mathbb{G}_m,$$

both over  $\overline{\mathbb{Q}_\ell}$  and  $\overline{\mathbb{Z}_\ell}$  (See Example 2.1.1). However,  $\mu_{q-1}$  is just  $q-1$  discrete points over  $\overline{\mathbb{Q}_\ell}$ , while the connected components of  $\mu_{q-1}$  are isomorphic to  $\mu_{\ell^k}$  (over  $\overline{\mathbb{F}_\ell}$ , hence also) over  $\overline{\mathbb{Z}_\ell}$ , where  $k$  is maximal such that  $\ell^k$  divides  $q-1$ . So when describing the connected components of the stack of  $L$ -parameters over  $\overline{\mathbb{Z}_\ell}$ , there will be possibly some non-reduced part  $\mu$  appearing.

These two features come together in the description of the connected component  $[X_\varphi/\hat{G}]$ . Under mild assumptions, we prove that

$$[X_\varphi/\hat{G}] \cong [*/S_\varphi] \times \mu,$$

where  $S_\varphi = C_{\hat{G}}(\varphi)$  (See Theorem 2.2.3).

### 1.1.2 Ingredients of the computation

The computation follows the theory of moduli space of Langlands parameters developed in [DHKM20, Section 2 and Section 4] (See also [Dat22, Section 3 and Section 4] for a more gentle introduction). It is very helpful to do the example of  $GL_2$  first (See Chapter 4).

To compute the component  $[X_\varphi/\hat{G}]$  over  $\overline{\mathbb{Z}_\ell}$ , let us fix a lift  $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$  of  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ .

Recall by [Dat22, Subsection 4.6],

$$X_\varphi = X_\psi \cong \left( \hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \right) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where  $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$  denotes the space of cocycles whose restriction to  $I_F^\ell$  equals  $\psi_\ell$  and whose image in  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$  is  $\overline{\psi}$ .

Here,  $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$  is essentially the space of cocycles of the torus

$$T := N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)$$

by our TRSELP assumption (See Definition 2.1.4) and that  $C_{\hat{G}}(\psi_\ell)$  is generalized reductive, hence split over  $\overline{\mathbb{Z}_\ell}$  (See Lemma 2.1.9). Since it's not hard to compute the space of tame cocycles of a commutative group scheme using the explicit presentation of the tame Weil group (See 2.1.3 and 2.1.4), we obtain that

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \cong T \times \mu,$$

where  $\mu$  is a product of  $\mu_{\ell^{k_i}}$ 's (See Theorem 2.1.6 for details). And it is not hard to see

$$C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell) = T.$$

Therefore, we get

$$X_\varphi \cong (\hat{G} \times T \times \mu) / T.$$

One need to be a bit careful about the  $T$  action on  $T$ , because here a twist by  $\psi(\text{Fr})$  is involved. Then one could compute that

$$[X_\varphi / \hat{G}] \cong [(T \times \mu) / T] \cong [T/T] \times \mu,$$

where  $T$  acts on  $T$  via twisted conjugacy. Then we could work in the category of diagonalizable group schemes (whose structure is clear, see [BCO14, p70, Section 5]) to identify  $[T/T]$  with  $[*/S_\varphi]$  under mild conditions.

## 1.2 Representation side

In this section, we explain Chapter 3 on how to compute  $\text{Rep}_\Lambda(G(F))_{[\pi]}$ .

### 1.2.1 Equivalence to the block of a finite group of Lie type

Recall that a depth-zero regular supercuspidal representation is of the form

$$\pi = \text{c-Ind}_{G_x}^{G(F)} \rho$$

for some representation  $\rho$  of the parahoric subgroup  $G_x$  corresponding to a vertex  $x$  in the Bruhat-Tits building of  $G$  over  $F$ . Moreover,  $\rho$  is the inflation of some regular supercuspidal representation  $\bar{\rho}$  of the finite group of Lie type  $\bar{G}_x := G_x/G_x^+$ .

Let  $\mathcal{A}_{x,1}$  denote the block  $\text{Rep}_\Lambda(\bar{G}_x)_{[\bar{\rho}]}$  of  $\text{Rep}_\Lambda(\bar{G}_x)$  containing  $\bar{\rho}$ . Similarly, let

$$\mathcal{B}_{x,1} := \text{Rep}_\Lambda(G_x)_{[\rho]}, \quad \mathcal{C}_{x,1} := \text{Rep}_\Lambda(G(F))_{[\pi]}.$$

Assume that the residue field of  $F$  is  $\mathbb{F}_q$ . For simplicity, we assume that  $q$  is greater than the Coxeter number of  $G$  (See Theorem 3.2.7 for reason). Then  $\mathcal{A}_{x,1}$  admits an explicit description as being equivalent to a block of a finite torus via Broué's equivalence 3.2.7. And it is not hard to show that the inflation induces an equivalence of categories  $\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}$ .

The main theorem we will prove for the representation side is Theorem 3.1.2: Assume  $q$  is greater than the Coxeter number of  $\bar{G}_x$ . Then the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}.$$

Once this is proven, then  $\mathcal{C}_{x,1} \cong \mathcal{A}_{x,1}$ , hence admits an explicit description. The proof of the Theorem 3.1.2 occupies the most of Chapter 3.

### 1.2.2 Proof of the main theorem for the representation side

In the rest of the section, let us briefly explain the idea of the proof of Theorem 3.1.2. The fully faithfulness of

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$$

is a usual computation by Frobenius reciprocity and Mackey's formula. Since a similar computation will be used later, we record it as a Theorem 3.1.4. The key point is that

$$\mathrm{Hom}_G(\mathrm{c}\text{-}\mathrm{Ind}_{G_x}^{G(F)} \rho_1, \mathrm{c}\text{-}\mathrm{Ind}_{G_y}^{G(F)} \rho_2)$$

could be computed explicitly assuming one of  $\rho_1, \rho_2$  has cuspidal reduction (i.e.  $\overline{\rho}_1$  or  $\overline{\rho}_2$  is cuspidal, see Definition 3.2.10).

There is a little subtlety that we want not only  $\rho$  to have cuspidal reduction, but also any representation  $\rho' \in \mathcal{B}_{x,1}$  to have cuspidal reduction. This is the content of Theorem 3.1.3. And this is why we need the **regular** supercuspidal assumption. To prove Theorem 3.1.3, we make a digression on regular cuspidal blocks of finite groups of Lie type in Section 3.2. As the **regular** assumption is impulsed here on purpose to make the result to hold, the readers are welcome to skip Section 3.2 and accept Theorem 3.1.3 in a first reading.

*Remark 1.2.1.* Although supercuspidal representations of  $G(F)$  come from **supercuspidal** representation  $\overline{\rho}$  of the finite group of Lie type  $\overline{G}_x$ , only the **cuspidality** of  $\overline{\rho}$  is relevant in our proof of Theorem 3.1.4. That's why we analyze cuspidality instead of supercuspidality in detail for finite group of Lie types in Chapter 3.

The difficulty for proving Theorem 3.1.2 lies in proving

$$\mathrm{c}\text{-}\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$$

is essentially surjective. For this, we prove that an explicit compact induction

$$\Pi_{x,1} := \mathrm{c}\text{-}\mathrm{Ind}_{G_x}^{G(F)} \sigma_{x,1}$$

is a projective generator of  $\mathcal{C}_{x,1}$ .

The first key is that  $\Pi_{x,1}$  is a summand of a projective generator of a larger category. Indeed,  $\Pi_{x,1}$  is a summand of

$$\Pi := \mathrm{c}\text{-}\mathrm{Ind}_{G_x^+}^{G(F)} \Lambda,$$

which is known to be a projective generator of the category  $\mathrm{Rep}_\Lambda(G(F))_0$  of depth-zero representations, i.e.,

$$\Pi = \Pi_{x,1} \oplus \Pi^{x,1}.$$

The second key is that the complement  $\Pi^{x,1}$  doesn't interfere with  $\Pi_{x,1}$ . More precisely, we could compute using Theorem 3.1.4 that

$$\mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = \mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = 0.$$

The above two keys allow us to conclude that  $\Pi_{x,1}$  is a projective generator of  $\mathcal{C}_{x,1}$ .

### 1.3 The example of $GL_n$

To illustrate the theory, we do the example of  $GL_n$  in Chapter 4 (Although  $GL_n$  is not simply connected, the theory still works without much change). It is quite concise once we have the theories developed in Chapter 2 and 3, so let us don't say anything more here. However, the readers are encouraged to do the example of  $GL_2$  throughout the paper, which will definitely help to understand the theories in Chapter 2 and 3. Actually, the author always do the example of  $GL_2$  first, and then figure out the theory for general groups.



## 1.4 The categorical local Langlands conjecture for $GL_n$

As an application, we could deduce the categorical local Langlands conjecture in Fargues-Scholze's form (See [FS21, Conjecture X.3.5]) for depth-zero supercuspidal blocks of  $GL_n$  (notice supercuspidal implies regular supercuspidal automatically in the  $GL_n$  case) in Chapter 5.

The idea is that we could unravel both sides of the categorical conjecture explicitly using our computation in Chapter 4. They both turn out to be

$$\bigoplus_{\mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu).$$

We have to show that the spectral action gives an equivalence. This reduces to the degree-zero part (of the  $\mathbb{Z}$ -grading) by compatibility of the spectral action with the  $\mathbb{Z}$ -grading (See 5.1.2). And the degree-zero part reduces to the theory of local Langlands in families (See [HM18]). Fortunately, several technical results we need to do the reductions are already available by [Zou22].

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## Chapter 2

# TRSELP components of the stack of $L$ -parameters

In this chapter, we compute the connected component  $[X_\varphi/\hat{G}]$  of the stack of  $L$ -parameters  $Z^1(W_F, \hat{G})/\hat{G}$  containing a tame regular semisimple elliptic  $L$ -parameter  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ . In Section 2.1, following the theory developed in [Dat22, Section 3 and Section 4], we first compute the connected component  $X_\varphi$  of the space of 1-cocycles  $Z^1(W_F, \hat{G})$  (without modulo out the  $\hat{G}$ -action). The result turns out to be very explicit:

$$X_\varphi \cong (\hat{G} \times T \times \mu)/T, \quad (2.0.1)$$

see Theorem 2.1.6 for details. In Section 2.2, we use 2.0.1 to obtain a particular simple description of  $[X_\varphi/\hat{G}]$  under mild conditions (See Theorem 2.2.3):

$$[X_\varphi/\hat{G}] \cong [*/S_\varphi] \times \mu. \quad (2.0.2)$$

### 2.1 The connected component $X_\varphi$ containing a TRSELP $\varphi$

The goal of this section is to compute the connected component  $X_\varphi$  containing a TRSELP  $\varphi$ . In 2.1.1, we recall the theory of moduli space of Langlands parameters. In 2.1.2, we define the class of  $L$ -parameters we are interested in – tame regular semisimple elliptic  $L$ -parameters (TRSELP for short). In 2.1.3, we compute  $X_\varphi$  explicitly as  $(\hat{G} \times T \times \mu)/T$  using the theory of moduli space of Langlands parameters. In 2.1.4, we spell out the  $T$ -action on  $(\hat{G} \times T \times \mu)$  to prepare for the next section.

#### 2.1.1 Recollections on the moduli space of Langlands parameters

Since our computation uses heavily the theory of moduli space of Langlands parameters, we recollect some basic facts here. For more sophisticated knowledge that will be used, we refer to [Dat22, Section 3 and Section 4], or [DHKM20, Section 2 and Section 4].

Let us first fix some notations.

- Let  $p \neq 2$  be a fixed prime number and  $\ell \neq 2$  be a prime number different from  $p$ .
- Let  $F$  be a non-archimedean local field with residue characteristic  $q = p^r$  for some  $r \in \mathbb{Z}_{\geq 1}$ .

- Let  $W_F$  be the Weil group of  $F$ ,  $I_F \subseteq W_F$  be the inertia subgroup,  $P_F$  be the wild inertia subgroup.
- Let  $W_t := W_F/P_F$  be the tame Weil group.
- Let  $I_t := I_F/P_F$  be the tame inertia subgroup.

Fix  $\text{Fr} \in W_F$  any lift of the arithmetic Frobenius. we will abuse the notation and also denote  $\text{Fr}$  the image of  $\text{Fr}$  in  $W_t$ . Then  $W_t \cong I_t \rtimes \langle \text{Fr} \rangle$ . Here  $I_t$  is non-canonically isomorphic to  $\prod_{p' \neq p} \mathbb{Z}_{p'}$ , which is procyclic. We fix such an isomorphism

$$I_t \cong \prod_{p' \neq p} \mathbb{Z}_{p'}. \quad (2.1.1)$$

This gives rise to a topological generator  $s_0$  of  $I_t$ , which corresponds to  $(1, 1, \dots)$  under the above isomorphism (2.1.1). Let us recall the following important relation in  $I_F/P_F$ :

$$\text{Fr} \cdot s_0 \cdot \text{Fr}^{-1} = s_0^q. \quad (2.1.2)$$

In fact, this is true for any  $s \in I_t$  instead of  $s_0$ .

Let

$$W_t^0 := \langle s_0, \text{Fr} \rangle = \mathbb{Z}[1/p]^{s_0} \rtimes \mathbb{Z}^{\text{Fr}}$$

be the subgroup of  $W_t$  generated by  $s_0$  and  $\text{Fr}$ . Denote  $W_F^0 \subseteq W_F$  the preimage of  $W_t^0$  under the natural projection  $W_F \rightarrow W_t$ .  $W_F^0$  is referred to as the discretization of the Weil group. To summarize,  $W_t^0$  is generated by two elements  $\text{Fr}$  and  $s_0$  with a single relation

$$W_t^0 = \langle \text{Fr}, s_0 \mid \text{Fr} \cdot s_0 \cdot \text{Fr}^{-1} = s_0^q \rangle. \quad (2.1.3)$$

Let  $G$  be a connected split reductive group over  $F$ . Let  $\hat{G}$  be its dual group over  $\mathbb{Z}$ . Then the space of cocycles from the discretization

$$Z^1(W_t^0, \hat{G}) = \underline{\text{Hom}}(W_t^0, \hat{G}) = \{(x, y) \in \hat{G} \times \hat{G} \mid yxy^{-1} = x^q\} \quad (2.1.4)$$

is an explicit closed subscheme of  $\hat{G} \times \hat{G}$  (See [Dat22, Section 3]). An important fact (See [Dat22, Proposition 3.9]) is that over a  $\mathbb{Z}_\ell$ -algebra  $R$  (the cases  $R = \overline{\mathbb{F}_\ell}, \overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}$  are most relevant for us), the restriction from  $W_t$  to  $W_t^0$  induces an isomorphism

$$Z^1(W_t, \hat{G}) \cong Z^1(W_t^0, \hat{G}).$$

Therefore, we can compute  $Z^1(W_t, \hat{G})$  using the explicit formula 2.1.4 above. This is fundamental for the study of the moduli space of Langlands parameters  $Z^1(W_t, \hat{G})$ . we refer the readers to [Dat22, Section 3 and Section 4] for the precise definition and properties of  $Z^1(W_t, \hat{G})$ .

**Example 2.1.1.** If  $G = GL_1$ ,

$$\begin{aligned} Z^1(W_t, \hat{G}) &\cong Z^1(W_t^0, \hat{G}) \\ &= \{(x, y) \in GL_1 \times GL_1 \mid yxy^{-1} = x^q\} \\ &= \{(x, y) \in GL_1 \times GL_1 \mid x = x^q\} \cong \mu_{q-1} \times \mathbb{G}_m. \end{aligned}$$

Let  $I_F^\ell$  be the prime-to- $\ell$  inertia subgroup of  $W_F$ , i.e.,  $I_F^\ell := \ker(t_\ell)$ , where

$$t_\ell : I_F \rightarrow I_F/P_F \cong \prod_{p' \neq p} \mathbb{Z}_{p'} \rightarrow \mathbb{Z}_\ell$$

is the composition. In other words, it is the maximal subgroup of  $I_F$  with pro-order prime to  $\ell$ . This property makes  $I_F^\ell$  important when determining the connected components of  $Z^1(W_F, \hat{G})$  over  $\overline{\mathbb{Z}_\ell}$  (See [Dat22, Theorem 4.2 and Subsection 4.6]).

### 2.1.2 Tame regular semisimple elliptic $L$ -parameters

We want to define a class of  $L$ -parameters, called TRSELP, which roughly corresponds to depth-zero regular supercuspidal representations. Before that, let us define the concept of schematic centralizer, which will be used throughout the article.

**Definition 2.1.2** (Schematic centralizer). *Let  $H$  be an affine algebraic group over a ring  $R$ ,  $\Gamma$  be a finite group. Let  $u \in Z^1(\Gamma, H(R'))$  be a 1-cocycle for some  $R$ -algebra  $R'$ . Let*

$$\alpha_u : H_{R'} \longrightarrow Z^1(\Gamma, H)_{R'} \quad h \longmapsto hu(-)h^{-1}$$

*be the orbit morphism. Then the schematic centralizer  $C_H(u)$  is defined to be the fiber of  $\alpha_u$  at  $u$ .*

$$\begin{array}{ccc} C_H(u) & \longrightarrow & H_{R'} \\ \downarrow & & \downarrow \alpha_u \\ R' & \xrightarrow{u} & Z^1(\Gamma, H)_{R'} \end{array}$$

One can show that  $C_H(u)(R'') = C_{H(R'')}(u)$  is the set-theoretic centralizer for all  $R'$ -algebra  $R''$ , see for example [DHKM20, Appendix A].

*Remark 2.1.3.* Note this is enough for our applications where  $\Gamma$  is more generally taken as a profinite group, because  $u : \Gamma \rightarrow H$  will factor through a finite quotient  $\Gamma'$  of  $\Gamma$  in practice.

Let us now define a tame, regular semisimple, elliptic Langlands parameter (TRSELP for short) over  $\overline{\mathbb{F}_\ell}$ , roughly in the sense of [DR09, Section 3.4 and Section 4.1] in the case that  $G$  is  $F$ -split, but with  $\overline{\mathbb{F}_\ell}$ -coefficients instead of  $\mathbb{C}$ -coefficients.

**Definition 2.1.4.** *A tame regular semisimple elliptic  $L$ -parameter (TRSELP) over  $\overline{\mathbb{F}_\ell}$  is a homomorphism  $\varphi : W_F \rightarrow \hat{G}(\overline{\mathbb{F}_\ell})$  such that:*

1. (smooth)  $\varphi(I_F)$  is a finite subgroup of  $\hat{G}(\overline{\mathbb{F}_\ell})$ .
2. (Frobenius semisimple)  $\varphi(\text{Fr})$  is a semisimple element of  $\hat{G}(\overline{\mathbb{F}_\ell})$ .
3. (tame) The restriction of  $\varphi$  to  $P_F$  is trivial.
4. (elliptic) The identity component of the centralizer  $C_{\hat{G}}(\varphi)^0$  is equal to the identity component of the center  $Z(\hat{G})^0$ .
5. (regular semisimple) The centralizer of the inertia  $C_{\hat{G}}(\varphi|_{I_F})$  is a torus (in particular, connected).

Concretely, a TRSELF consists of the following data:

1. The restriction to the inertia  $\varphi|_{I_F}$ , which is essentially a direct sum of characters of a finite abelian group since  $I_F/P_F \cong \varprojlim \mathbb{F}_{q^n}^*$  is compact abelian and that

$$\mathrm{Hom}_{\mathrm{Cont}}(I_F/P_F, \overline{\mathbb{F}_\ell}^*) \cong \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*) \cong \varprojlim \mathrm{Hom}_{\mathrm{Cont}}(\mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*).$$

In particular, it factors through (the  $\overline{\mathbb{F}_\ell}$ -points of) some maximal torus, say  $S$ . Then regular semisimple means that  $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$ .

2. The image of the Frobenius  $\varphi(\mathrm{Fr})$ , which turns out to be an element of the normalizer  $N_{\hat{G}(\overline{\mathbb{F}_\ell})}(S)$  (Since  $\mathrm{Fr} \cdot s \cdot \mathrm{Fr}^{-1} = s^q \in I_t$  for any  $s \in I_t$  implies that  $\varphi(\mathrm{Fr})$  normalizes  $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$ ). And “elliptic” means that the center  $Z(\hat{G})$  has finite index in the centralizer  $C_{\hat{G}}(\varphi)$ . As we will see later, ellipticity implies that  $\hat{G}(\overline{\mathbb{F}_\ell})$  acts transitively on the connected component  $X_\varphi(\overline{\mathbb{F}_\ell})$  of the moduli space of  $L$ -parameters containing  $\varphi$  (See the proof of Lemma 2.2.4), which is essential for the description (roughly, see Theorem 2.2.3 for the precise statement)

$$[X_\varphi/\hat{G}] \cong [*/S_\varphi],$$

where  $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$  is the centralizer of the whole  $L$ -parameter  $\varphi$ .

See Section 4.1 for an example of TRSELF for  $GL_n$ , expressed in explicit matrices.

- Remark 2.1.5.* 1. Let  $\overline{\Lambda} \in \{\overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$ . It is important for our purpose to distinguish between the set-theoretic centralizer (for example,  $C_{\hat{G}(\overline{\Lambda})}(\varphi(I_F))$ ) and the schematic centralizer (for example,  $C_{\hat{G}}(\varphi)$ ). However, we might still use  $\hat{G}$  to mean  $\hat{G}(\overline{\Lambda})$  sometimes by abuse of notation, for which we hope the readers could recognize. One reason for doing this is that we assume throughout that  $\hat{G}$  is split over  $\overline{\Lambda}$ , hence  $\hat{G}$  is completely determined by its  $\overline{\Lambda}$ -points. And many statements can either be phrased in terms of the  $\overline{\Lambda}$ -scheme or its  $\overline{\Lambda}$ -points (for example, 4 and 5 in Definition 2.1.4).
2. As we will see later in Theorem 2.1.6,  $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$  turns out to be the  $\overline{\mathbb{F}_\ell}$ -points of the split torus  $T = C_{\hat{G}}(\psi|_{I_F^\ell})$  for some lift  $\psi$  of  $\varphi$  over  $\overline{\mathbb{Z}_\ell}$ .

### 2.1.3 Description of the component

Now given a TRSELF  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ . Pick any lift  $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$  of  $\varphi$ , whose existence is ensured by the flatness of  $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$  (See Lemma 2.1.8). Let  $\psi_\ell := \psi|_{I_F^\ell}$  denotes the restriction of  $\psi$  to the prime-to- $\ell$  inertia  $I_F^\ell$ . Note that  $\psi \in Z^1(W_F, \hat{G})$  factors through  $N_{\hat{G}}(\psi_\ell)$  (Since  $I_F^\ell$  is normal in  $W_F$ ). Let  $\overline{\psi}$  denotes the image of  $\psi$  in  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ . Let  $X_\varphi$  be the connected component of  $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$  containing  $\varphi$ . Note  $X_\varphi$  also contains  $\psi$  since  $\psi$  specializes to  $\varphi$ . So we sometimes also denote  $X_\varphi$  as  $X_\psi$ .

**Theorem 2.1.6.** *Let  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$  be a TRSELF over  $\overline{\mathbb{F}_\ell}$ . Let  $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$  be any lifting of  $\varphi$ . Then at least when the center  $Z(\hat{G})$  is smooth over  $\overline{\mathbb{Z}_\ell}$ , the connected component  $X_\varphi = X_\psi$  of  $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$  containing  $\varphi$  is isomorphic to*

$$\left( \hat{G} \times C_{\hat{G}}(\psi_\ell)^0 \times \mu \right) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where

1.  $C_{\hat{G}}(\psi_\ell)^0$  is the identity component of the schematic centralizer  $C_{\hat{G}}(\psi_\ell)$ , which turns out to be a split torus  $T$  over  $\overline{\mathbb{Z}_\ell}$  with  $\overline{\mathbb{F}_\ell}$ -points  $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ .
2.  $\mu$  is the identity component of  $T^{\text{Fr}=(-)^q}$  (the subscheme of  $T$  on which  $\text{Fr}$  acts by raising to  $q$ -th power) containing 1 (See [Dat22, Example 3.14]), which is a product of some  $\mu_{\ell^{k_i}}$  (the group scheme of  $\ell^{k_i}$ -th roots of unity over  $\overline{\mathbb{Z}_\ell}$ ),  $k_i \in \mathbb{Z}_{\geq 0}$ . Note that  $\mu$  could be trivial, depending on  $\hat{G}$  and some congruence relations between  $q, \ell$ .
3.  $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$  is the (schematic) stabilizer of  $\overline{\psi}$  in  $C_{\hat{G}}(\psi_\ell)$ .

In other words, we have the following isomorphism of schemes over  $\overline{\mathbb{Z}_\ell}$ :

$$X_\varphi \cong (\hat{G} \times T \times \mu) / T.$$

And we will specify in the next subsection what the  $T$ -action on  $(\hat{G} \times T \times \mu)$  is.

*Proof.* First, recall by [Dat22, Subsection 4.6],

$$X_\psi \cong (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where  $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$  denotes the space of cocycles whose restriction to  $I_F^\ell$  equals  $\psi_\ell$  and whose image in  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$  is  $\overline{\psi}$ . More precisely, recall (See [Dat22, Subsection 4.6]) first that the component  $X_\varphi = X_\psi$  morally consists of the  $L$ -parameters whose restriction to  $I_F^\ell$  and whose image in  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$  is  $\hat{G}$ -conjugate to  $(\psi_\ell, \overline{\psi})$ . Hence

$$X_\varphi \cong (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \quad g\eta(-)g^{-1} \leftarrow (g, \eta),$$

with  $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$  acting on  $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$  by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where  $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$  and  $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$ .

Second,  $\eta \cdot \psi \leftarrow \eta$  defines an isomorphism

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \cong Z_{\text{Ad}(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_{1_{I_F^\ell}} =: Z_{\text{Ad}(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$$

where  $Z_{\text{Ad}(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell))$  means the space of cocycles with  $W_F$  acting on  $N_{\hat{G}}(\psi_\ell)$  via conjugacy action through  $\psi$ , and the subscript  $1_{I_F^\ell}$  or 1 means the cocycles whose restriction to  $I_F^\ell$  is trivial. Indeed, this is clear by unraveling the definitions: two cocycles whose restriction to  $I_F^\ell$  are both  $\psi_\ell$  differ by something whose restriction to  $I_F^\ell$  is trivial; two cocycles whose pushforward to  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$  are both  $\overline{\psi}$  differ by something whose pushforward to  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$  is trivial, i.e., which factors through the identity component  $N_{\hat{G}}(\psi_\ell)^0$ .

Next, we show that  $C_{\hat{G}}(\psi_\ell)$  is a split torus over  $\overline{\mathbb{Z}_\ell}$ . By [Dat22, Subsection 3.1], the centralizer  $C_{\hat{G}}(\psi_\ell)$  is generalized reductive (See Lemma 2.1.9), hence split over  $\overline{\mathbb{Z}_\ell}$ , and  $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0$ . So we can determine  $C_{\hat{G}}(\psi_\ell)$  by computing its  $\overline{\mathbb{F}_\ell}$ -points. Indeed,

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)),$$

where the last equality follows since  $I_F/I_F^\ell$  doesn't contribute to the image of  $\varphi$  (See Lemma 2.1.11). Therefore,  $C_{\hat{G}}(\psi_\ell)$  is a split torus over  $\overline{\mathbb{Z}_\ell}$  with  $\overline{\mathbb{F}_\ell}$ -points  $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ . Denote  $T = C_{\hat{G}}(\psi_\ell)$ . In particular,  $C_{\hat{G}}(\psi_\ell)$  is connected, hence

$$N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell) = T.$$

Now we could compute

$$Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0) = Z_{Ad(\psi)}^1(W_F, T) \cong T \times T^{\text{Fr}=(-)^q},$$

where the last isomorphism is given by  $\eta \mapsto (\eta(\text{Fr}), \eta(s_0))$ , where  $s_0 \in W_t^0$  is the topological generator of  $I_t$  fixed before (See [Dat22, Example 3.14]).

Then we show that the identity component of  $T^{\text{Fr}=(-)^q}$  gives  $\mu$  in the statement of the theorem. Note  $T^{\text{Fr}=(-)^q}$  is a diagonalizable group scheme over  $\overline{\mathbb{Z}_\ell}$  of dimension zero (This can be seen either by  $\dim Z^1(W_F/P_F, T) = \dim T$ , or by noticing that  $\eta(s_0) \in T^{\text{Fr}=(-)^q}$  is semisimple with finitely many possible eigenvalues), hence of the form  $\prod_i \mu_{n_i}$  for some  $n_i \in \mathbb{Z}_{\geq 0}$ . Hence its connected component  $(T^{\text{Fr}=(-)^q})^0$  over  $\overline{\mathbb{Z}_\ell}$  is of the form  $\prod_i \mu_{\ell^{k_i}}$ , with  $k_i$  maximal such that  $\ell^{k_i}$  divides  $n_i$ . Therefore,

$$Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong (T \times T^{\text{Fr}=(-)^q})^0 \cong T \times (T^{\text{Fr}=(-)^q})^0 \cong T \times \mu,$$

(See Lemma 2.1.12 for the first isomorphism) where  $\mu$  is of the form  $\prod_i \mu_{\ell^{k_i}}$ .

Finally, we show that  $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$ . Recall  $C_{\hat{G}}(\psi_\ell)$  acts on  $Z^1(W_F, N_{\hat{G}}(\psi_\ell))$  by conjugation, inducing an action of  $C_{\hat{G}}(\psi_\ell)$  on  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ . And  $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$  is by definition the stabilizer of  $\overline{\psi} \in Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$  in  $C_{\hat{G}}(\psi_\ell)$ . Now  $C_{\hat{G}}(\psi_\ell) = T$  is connected, hence acts trivially on the component group  $\pi_0(N_{\hat{G}}(\psi_\ell))$ , hence also acts trivially on  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ . Therefore, the stabilizer  $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$ .

Above all, we have

$$X_\varphi \cong (\hat{G} \times Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \cong (\hat{G} \times T \times \mu) / T.$$

□

#### 2.1.4 The $T$ -action on $(\hat{G} \times T \times \mu)$

For later use, let us make it explicit the  $T$ -action on  $(\hat{G} \times T \times \mu)$ .

Recall (See [Dat22, Subsection 4.6]) first that the component  $X_\varphi = X_\psi$  morally consists of the  $L$ -parameters  $\psi'$  such that  $(\psi'_\ell, \overline{\psi'})$  is  $\hat{G}$ -conjugate to  $(\psi_\ell, \overline{\psi})$ . Hence  $X_\varphi$  is isomorphic to

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$$

via  $g\eta(-)g^{-1} \mapsto (g, \eta)$ , with  $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$  acting on  $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$  by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where  $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \cong T$  and  $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$ .

Next, recall that

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \cong Z_{Ad\psi}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong T \times \mu \quad \eta \cdot \psi \mapsto (\eta(\text{Fr}), \eta(s_0)).$$



Let's focus on the isomorphism  $\eta.\psi \leftarrow \eta$ :

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \bar{\psi}} \cong Z^1_{\text{Ad}\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1.$$

Recall that  $T \subseteq \hat{G}$  acts on  $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \bar{\psi}}$  via conjugation. Hence the above isomorphism induces an  $T$ -action on  $Z^1_{\text{Ad}\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$ , by

$$(t, \eta) \mapsto (t(\eta\psi(-))t^{-1})\psi^{-1}.$$

Hence in  $(\hat{G} \times T \times \mu)/T$ , we compute by tracking the above isomorphisms that

1.  $T$  acts on  $\hat{G}$  via  $(t, g) \mapsto gt^{-1}$ .
2.  $T = C_{\hat{G}}(\psi_\ell)_{\bar{\psi}}$  acts on  $T \subseteq (T \times \mu)$  (corresponds to  $\eta(\text{Fr})$ ) by twisted conjugacy (due to the isomorphisms  $\eta.\psi \leftarrow \eta \mapsto (\eta(\text{Fr}), \eta(s_0))$ ), i.e.,

$$(t, t') \mapsto (t(t'n)t^{-1})n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where  $n = \psi(\text{Fr})$ ; Note here  $n$ , a priori lies in  $\hat{G}$ , actually lies in  $N_{\hat{G}}(T)$  (Since  $\text{Fr} \cdot s. \text{Fr}^{-1} = s^q$  implies that  $\psi(\text{Fr})$  normalizes  $C_{\hat{G}}(\psi|_{I_F^\ell}) = T$ ). To summarize,  $t \in T$  acts on  $T$  via multiplication by  $tnt^{-1}n^{-1}$ .

3.  $T$  acts trivially on  $\mu$ . This is because  $\eta(s_0) \in T$  and  $\psi(s_0) \in T$ . So the conjugacy action is trivial.

On the other hand, recall we have the natural  $\hat{G}$ -action on  $Z^1(W_F, \hat{G})$  by conjugation, hence the  $\hat{G}$ -action on this component  $X_\varphi$ . Under the isomorphism  $X_\varphi \cong (\hat{G} \times T \times \mu)/T$ , the  $\hat{G}$ -action becomes

$$(g', (g, t, m)) \mapsto (g'g, t, m), \text{ for any } g' \in \hat{G} \text{ and } (g, t, m) \in (\hat{G} \times T \times \mu)/T.$$

Note that the  $T$ -action and the  $\hat{G}$ -action on  $(\hat{G} \times T \times \mu)$  commute with each other, we thus have the following:

**Proposition 2.1.7.**

$$[X_\varphi/\hat{G}] = \left[ \left( (\hat{G} \times T \times \mu)/T \right) / \hat{G} \right] \cong \left[ \left( (\hat{G} \times T \times \mu)/\hat{G} \right) / T \right] \cong [(T \times \mu)/T],$$

with  $t \in T$  acting on  $T$  via multiplication by  $tnt^{-1}n^{-1}$ , and  $t \in T$  acting trivially on  $\mu$ .

### 2.1.5 Some lemmas

**Lemma 2.1.8.** *Let  $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$ . Then there exists  $\psi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{Z}_\ell}))$  such that  $\psi'$  is a lift of  $\varphi'$ .*

*Proof.* In the statement,  $Z^1(W_t, \hat{G})$  is the abbreviation for  $Z^1(W_t, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ . Recall that  $Z^1(W_t, \hat{G}) \rightarrow \overline{\mathbb{Z}_\ell}$  is flat (See [Dat22, Proposition 3.3]), hence generalizing (See [SP, Stack, Tag 01U2]). Therefore, given  $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$ , there exists  $\xi \in Z^1(W_t, \hat{G}(\overline{\mathbb{Q}_\ell}))$  such that  $\xi$  specializes to  $\varphi'$ . In other words,  $\ker(\xi) \subseteq \ker(\varphi')$ . We are going to show that  $\xi : W_t \rightarrow \hat{G}(\overline{\mathbb{Q}_\ell})$  factors through  $\hat{G}(\overline{\mathbb{Z}_\ell})$ .

This is true by the following more general statement: Let  $Y = \operatorname{Spec}(R)$  be an affine scheme over  $\overline{\mathbb{Z}_\ell}$ , let  $y_\eta \in Y(\overline{\mathbb{Q}_\ell})$  specializing to  $y_s \in Y(\overline{\mathbb{F}_\ell})$ . Then  $y_\eta \in Y(\overline{\mathbb{Q}_\ell}) = \operatorname{Hom}(R, \overline{\mathbb{Q}_\ell})$  factors through  $\overline{\mathbb{Z}_\ell}$ .

To prove the above statement, let  $\mathfrak{p} := \ker(y_\eta)$  and  $\mathfrak{q} := \ker(y_s)$  be the corresponding prime ideals. Then “ $y_\eta$  specializes to  $y_s$ ” translates to “ $\mathfrak{p} \subseteq \mathfrak{q}$ ”. Recall that we are going to show that  $y_\eta : R \rightarrow \overline{\mathbb{Q}_\ell}$  factors through  $\overline{\mathbb{Z}_\ell}$ . We argue by contradiction. Otherwise there is some element  $f \in R$  mapping to  $\ell^{-m}u$  for some  $m \in \mathbb{Z}_{\geq 1}$  and  $u \in \overline{\mathbb{Z}_\ell}^*$ . Hence

$$\ell^m u^{-1} f - 1 \in \ker(y_\eta) \subseteq \ker(y_s). \quad (2.1.5)$$

However,  $\ell \in \ker(y_s)$  since  $y_s$  lives on the special fiber. This together with equation 2.1.5 implies that  $1 \in \ker(y_s)$ . Contradiction!  $\square$

**Lemma 2.1.9.** *The schematic centralizer  $C_{\hat{G}}(\psi_\ell)$  is a generalized reductive group scheme over  $\overline{\mathbb{Z}_\ell}$ .*

*Proof.* We are going to invoke [Dat22, Lemma 3.2]. We first show that

$$C_{\hat{G}}(\psi_\ell) = C_{\hat{G}}(\psi(I_F^\ell)),$$

where  $C_{\hat{G}}(\psi(I_F^\ell))$  is the schematic centralizer of the subgroup  $\psi(I_F^\ell) \subseteq \hat{G}(\overline{\mathbb{Z}_\ell})$  in  $\hat{G}$ . This can be checked by Yoneda Lemma on  $R$ -valued points for any  $\overline{\mathbb{Z}_\ell}$ -algebra  $R$ .

Then we could conclude by [Dat22, Lemma 3.2]. Indeed,  $\psi_\ell$  factors through some finite quotient  $Q$  of  $I_F^\ell$ , which has order invertible in the base  $\overline{\mathbb{Z}_\ell}$ . So the assumptions of [Dat22, Lemma 3.2] are satisfied (For details, see Remark 2.1.10 below).  $\square$

*Remark 2.1.10.* 1. While [Dat22, Lemma 3.2] is phrased in the setting that  $R$  is a normal subring of a number field, it still works for  $\overline{\mathbb{Z}_\ell} \subseteq \overline{\mathbb{Q}_\ell}$  instead of  $\mathbb{Z} \subseteq \mathbb{Q}$ . Indeed,  $\psi_\ell$  factors through some finite quotient  $Q$  of  $I_F^\ell$ , say of order  $|Q| = N$  (Note  $N$  is coprime to  $\ell$  since  $Q$  is a quotient of  $I_F^\ell$ ). Then we could use [Dat22, Lemma 3.2] to conclude that  $C_{\hat{G}}(\psi_\ell)$  is generalized reductive over  $\mathbb{Z}[1/pN]$ . Hence  $C_{\hat{G}}(\psi_\ell)$  is also generalized reductive over  $\overline{\mathbb{Z}_\ell}$  by base change.

2. There is also a small issue that  $\overline{\mathbb{Z}_\ell}$  is not finite over  $\mathbb{Z}_\ell$ , but this can be resolved since everything is already defined over some sufficiently large finite extension  $\mathcal{O}$  of  $\mathbb{Z}_\ell$ .

**Lemma 2.1.11.**

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)).$$

*Proof.* The first equation is by definition (and that  $C_{\hat{G}}(\psi_\ell)$  represents the set-theoretic centralizer).

For the second equation, note that  $\varphi|_{I_t} = \gamma_1 + \dots + \gamma_d$  is a direct sum of characters (Since  $I_t \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$ ), so it suffices to show that each  $\gamma_i$  is trivial on the summand  $\mathbb{Z}_\ell$  of  $I_t \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$ . Indeed,

$$\operatorname{Hom}_{\operatorname{Cont}}(\mathbb{Z}_\ell, \overline{\mathbb{F}_\ell}^*) = \operatorname{Hom}_{\operatorname{Cont}}(\varprojlim \mathbb{Z}/\ell^n \mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \varinjlim \operatorname{Hom}(\mathbb{Z}/\ell^n \mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \{1\}.$$

$\square$

**Lemma 2.1.12.**  $Z_{\operatorname{Ad}(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong (T \times T^{\operatorname{Fr}=(-)^q})^0$ .

*Proof.* We have omitted from the notations but here everything is over  $\overline{\mathbb{Z}_\ell}$ . Recall that  $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = T$  and that

$$Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0) \cong T \times T^{\text{Fr}=(-)^q}.$$

By [Dat22, Section 5.4, 5.5],  $Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$  is connected (over  $\overline{\mathbb{Z}_\ell}$ ). We check here that the assumptions of [Dat22, Section 5.4, 5.5] are satisfied. Indeed, since  $N_{\hat{G}}(\psi_\ell)^0 = T$  is a connected torus, the  $W_t^0$ -action on  $T$  automatically fixes a Borel pair of  $T$ . Moreover,  $s_0$  acts trivially on  $N_{\hat{G}}(\psi_\ell)^0 = T$  via  $\psi$ , so in particular the action of  $s_0$  (which is denoted by  $s$  in [Dat22, Section 5.5]) has order a power of  $\ell$  (which is  $1 = \ell^0$ ).

Therefore,

$$Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \subseteq Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \cong (T \times T^{\text{Fr}=(-)^q})^0.$$

By [Dat22, Section 4.6],

$$Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \hookrightarrow Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)$$

is open and closed. This is done by considering the restriction to the prime-to- $\ell$  inertia  $I_F^\ell$ , and then use [Dat22, Theorem 4.2].

Therefore,

$$Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 = Z_{Ad(\psi)}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \cong (T \times T^{\text{Fr}=(-)^q})^0.$$

□

## 2.2 Main Theorem: description of $[X_\varphi/\hat{G}]$

The goal of this section is to describe  $[X_\varphi/\hat{G}]$  explicitly (See 2.2.3 for the precise statement).

Let  $F$  be a non-archimedean local field,  $G$  be a connected split reductive group over  $F$ . Let  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$  be a tame, regular semisimple, elliptic  $L$ -parameter (TRSELP for short). Recall that this means that the centralizer

$$C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) =: S \subseteq \hat{G}(\overline{\mathbb{F}_\ell})$$

is a maximal torus, and  $\varphi(\text{Fr}) \in N_{\hat{G}}(S)$  gives rise to an element  $w = \overline{\varphi(\text{Fr})} \in N_{\hat{G}}(S)/S$  in the Weyl group (and that  $\varphi$  is tame and elliptic).

Assume that

1. The center  $Z(\hat{G})$  is smooth over  $\overline{\mathbb{Z}_\ell}$ .
2.  $Z(\hat{G})$  is finite.

Let  $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$  be any lifting of  $\varphi$ . Let  $\psi_\ell$  denotes the restriction  $\psi|_{I_F^\ell}$ , and  $\overline{\psi}$  denotes the image of  $\psi$  in  $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ . Recall that the schematic centralizer  $C_{\hat{G}}(\psi_\ell) = T$  is a split torus over  $\overline{\mathbb{Z}_\ell}$  with  $\overline{\mathbb{F}_\ell}$ -points  $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$ .

For later use, we record the following lemma –  $w$  can also be defined in terms of  $\psi$  instead of  $\varphi$ . This is helpful because we will reduce to a computation on the special fiber later. First, notice that since  $T$  is a split torus over  $\overline{\mathbb{Z}_\ell}$  with  $\ell \neq 2$ , we can identify

$$(N_{\hat{G}}(T)/T)(\overline{\mathbb{Z}_\ell}) \cong (N_{\hat{G}}(T)/T)(\overline{\mathbb{F}_\ell}),$$

and denote it by  $\Omega$ . (See Lemma 2.2.6 below)

*Remark 2.2.1.* Lemma 2.2.6 below shows that  $N_{\hat{G}}(T)/T$  is representable by a group scheme which is split over  $\overline{\mathbb{Z}_\ell}$ . Therefore, we will slightly abuse notations and use

$$\Omega, N_{\hat{G}}(T)/T, N_{\hat{G}}(S)/S$$

interchangeably.

**Lemma 2.2.2.** *The image of  $\varphi(\text{Fr})$  and  $\psi(\text{Fr})$  in the Weyl group  $\Omega$  agree, hence giving a well defined element  $w$  in the Weyl group  $\Omega$ .*

*Proof.* Let

$$\Omega = (N_{\hat{G}}(T)/T)(\overline{\mathbb{Z}_\ell}) = (N_{\hat{G}}(T)/T)(\overline{\mathbb{F}_\ell})$$

as above and  $\underline{\Omega}$  be the associated constant group scheme (See Lemma 2.2.6 below). Since  $\psi$  is a lift of  $\varphi$ ,  $\psi(\text{Fr})$  specializes to  $\varphi(\text{Fr})$  in  $N_{\hat{G}}(T)$ . Then the lemma follows since

$$N_{\hat{G}}(T) \rightarrow N_{\hat{G}}(T)/T = \underline{\Omega}$$

is a morphism of schemes, hence the following diagram commutes:

$$\begin{array}{ccc} N_{\hat{G}}(T)(\overline{\mathbb{Z}_\ell}) & \longrightarrow & N_{\hat{G}}(T)(\overline{\mathbb{F}_\ell}) \\ \downarrow & & \downarrow \\ \underline{\Omega}(\overline{\mathbb{Z}_\ell}) = \Omega & \longrightarrow & \underline{\Omega}(\overline{\mathbb{F}_\ell}) = \Omega \end{array}$$

□

Our main theorem is the following.

**Theorem 2.2.3.** *Assume that the center  $Z(\hat{G})$  is smooth over  $\overline{\mathbb{Z}_\ell}$ , and that  $Z(\hat{G})$  is finite. Let  $X_\varphi (= X_\psi)$  be the connected component of  $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$  containing  $\varphi$  (hence also containing  $\psi$ ). Then we have isomorphisms of quotient stacks*

$$[X_\varphi/\hat{G}] \cong [(T \times \mu)/T] \cong [* / C_T(n)] \times \mu,$$

where  $C_T(n)$  is the schematic centralizer of  $n = \psi(\text{Fr})$  in  $T = C_{\hat{G}}(\psi|_{I_F^\ell})$ , and  $\mu = \prod_{i=1}^m \mu_{\ell^{k_i}}$  for some  $k_i \in \mathbb{Z}_{\geq 1}$ ,  $m \in \mathbb{Z}_{\geq 0}$  is a product of group schemes of roots of unity.

If we moreover assume that  $\ell$  does not divide the order of  $w = \varphi(\text{Fr})$  in the Weyl group  $N_{\hat{G}}(S)/S$ , then

$$[X_\varphi/\hat{G}] \cong [(T \times \mu)/T] \cong [* / \underline{S}_\varphi] \times \mu,$$

where  $\underline{S}_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ , and  $\underline{S}_\varphi$  is the corresponding constant group scheme.

*Proof.* Recall that  $X_\varphi$  is isomorphic to the contracted product

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \bar{\psi}}) / C_{\hat{G}}(\psi_\ell)_{\bar{\psi}},$$

and that  $\eta.\psi \leftarrow \eta \mapsto (\eta(\text{Fr}), \eta(s_0))$  induces isomorphisms

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \bar{\psi}} \cong Z_{\text{Ad}\psi}^1(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong T \times \mu.$$

This implies that  $[X/\hat{G}] \cong [(T \times \mu)/T]$  with  $T$  acting on  $T$  by twisted conjugacy:

$$(t, t') \mapsto (t(t'n)t^{-1})n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where  $n = \psi(\text{Fr})$ . In other words,  $T$  acts on  $T$  via multiplication by  $tnt^{-1}n^{-1}$ . And  $T$  acts trivially on  $\mu$  (See Proposition 2.1.7).

So we are reduced to compute  $[T/T]$  with respect to a nice action of the split torus  $T$  on  $T$ , which should be and turns out to be very explicit.

For clarification, let us denote the source torus  $T$  by  $T^{(1)}$  and the target torus  $T$  by  $T^{(2)}$ . Consider the morphism

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \quad s \longmapsto sns^{-1}n^{-1}.$$

This is surjective on  $\overline{\mathbb{F}_\ell}$ -points by our assumption 2 that  $Z(\hat{G})$  is finite and  $\varphi$  is elliptic (See Lemma 2.2.4 below). Hence  $f$  is an epimorphism in the category of diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group schemes (See the same Lemma 2.2.4 below). Therefore,  $f$  induces an isomorphism

$$T^{(1)} / \ker(f) \cong T^{(2)} \tag{2.2.1}$$

as diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group schemes. Moreover, if we let  $t \in T$  act on  $T^{(1)}$  by left multiplication by  $t$ , and on  $T^{(2)}$  via multiplication by  $(tnt^{-1}n^{-1})$ , this isomorphism induced by  $f$  is  $T$ -equivariant.

Note  $T^{(1)} = T$  is commutative, so the  $T$ -action (via multiplication by  $tnt^{-1}n^{-1}$ ) and the  $\ker(f)$ -action (via left multiplication) on  $T$  commute with each other. Hence by the  $T$ -equivariant isomorphism (2.2.1), we have

$$[T/T] = [T^{(2)}/T] \cong \left[ \left( T^{(1)} / \ker(f) \right) / T \right] \cong \left[ \left( T^{(1)} / T \right) / \ker(f) \right] \cong [* / \ker(f)] = [* / C_T(n)].$$

For the last assertion, see Lemma 2.2.5 below. □

**Lemma 2.2.4.** *The morphism*

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \quad s \longmapsto sns^{-1}n^{-1}$$

*is epimorphic in the category of diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group schemes. And it induces an isomorphism  $T^{(1)} / \ker(f) \cong T^{(2)}$  as diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group schemes.*

*Proof.* Recall that  $T$  is a split torus over  $\overline{\mathbb{Z}_\ell}$ , hence a diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group scheme. Notice that  $f$  is a morphism of  $\overline{\mathbb{Z}_\ell}$ -group schemes, hence a morphism of diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group schemes. Recall that the category of diagonalizable  $\overline{\mathbb{Z}_\ell}$ -group schemes is equivalent to the category of abelian groups (See [BCO14, p70, Section 5] or [Con14]) via

$$D \mapsto \text{Hom}_{\overline{\mathbb{Z}_\ell}\text{-GrpSch}}(D, \mathbb{G}_m),$$

and the inverse is given by

$$\overline{\mathbb{Z}_\ell}[M] \hookleftarrow M,$$

where  $\overline{\mathbb{Z}_\ell}[M]$  is the group algebra of  $M$  with  $\overline{\mathbb{Z}_\ell}$ -coefficients.

Therefore, we could argue in the category of abelian groups via the above category equivalence:  $f$  is epimorphic if and only if the map  $f^*$  in the category of abelian groups is monomorphic. Note ellipticity and  $Z(\hat{G})$  finite imply that  $S_\varphi$  is finite, hence

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_T(n) = S_\varphi$$

is finite (See Equation (2.2.2) below), hence  $\operatorname{coker}(f^*)$  is finite. Therefore,

$$f^* : \operatorname{Hom}(T^{(2)}, \mathbb{G}_m) \rightarrow \operatorname{Hom}(T^{(1)}, \mathbb{G}_m)$$

is injective (i.e., monomorphism). Indeed, otherwise  $\ker(f^*)$  would be a nonzero sub- $\mathbb{Z}$ -module of the finite free  $\mathbb{Z}$ -module  $\operatorname{Hom}(T^{(2)}, \mathbb{G}_m)$ , hence a free  $\mathbb{Z}$ -module of positive rank, which contradicts with  $\operatorname{coker}(f^*)$  being finite.

The statement on the quotient follows from the corresponding result in the category of abelian groups:  $f^*$  induces an isomorphism

$$\operatorname{Hom}(T^{(1)}, \mathbb{G}_m) / \operatorname{Hom}(T^{(2)}, \mathbb{G}_m) \cong \operatorname{coker}(f^*)$$

(See [BCO14, p71, Subsection 5.3]). □

**Lemma 2.2.5.** *Assume that  $\ell$  doesn't divide the order of  $w$ . Then  $\ker(f) \cong \underline{S_\varphi}$  is the constant group scheme of the finite abelian group  $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ .*

*Proof.* We recall the following fact: Let  $H$  be a smooth affine group scheme over some ring  $R$ , let  $\Gamma$  be a finite group whose order is invertible in  $R$ . Then the fixed point functor  $H^\Gamma$  is representable and is smooth over  $R$ .

For a proof of the above fact, see [Edi92, Proposition 3.4] or [DHKM20, Lemma A.1, A.13].

In our case, let  $H = T$ ,  $\Gamma = \langle w \rangle$  the subgroup of the Weyl group  $W_{\hat{G}}(T)$  generated by  $w$ . Hence

$$\ker(f) = C_T(n) = H^\Gamma$$

is smooth over  $\overline{\mathbb{Z}_\ell}$ . Therefore,  $\ker(f)$  is finite étale over  $\overline{\mathbb{Z}_\ell}$  (Because it is smooth of relative dimension 0 over  $\overline{\mathbb{Z}_\ell}$ , by Equation (2.2.2) below). Hence  $\ker(f)$  is a constant group scheme over  $\overline{\mathbb{Z}_\ell}$ , since  $\overline{\mathbb{Z}_\ell}$  has no non-trivial finite étale cover.

Since  $\ker(f)$  is constant, we can determine it by computing its  $\overline{\mathbb{F}_\ell}$ -points:

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_{T(\overline{\mathbb{F}_\ell})}(n) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F)), \quad (2.2.2)$$

where the middle equality follows by noticing  $T(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$  and  $n = \varphi(\operatorname{Fr})$ .

Finally, note by our TRSELF assumption,  $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$  is (the  $\overline{\mathbb{F}_\ell}$ -points of) a torus. Hence  $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$  is abelian, hence finite abelian as we've noticed in the proof of the previous lemma that  $S_\varphi$  is finite (by ellipticity and  $Z(\hat{G})$  finite). □

**Lemma 2.2.6.** *Let  $\hat{G}$  be a connected reductive group over  $\overline{\mathbb{Z}_\ell}$ , and  $T$  a maximal torus of  $\hat{G}$ . Then the Weyl group  $N_{\hat{G}}(T)/T$  is split over  $\overline{\mathbb{Z}_\ell}$ .*

*Proof.* By [Con14, Proposition 3.2.8], the Weyl group  $N_{\hat{G}}(T)/C_{\hat{G}}(T)$  is finite étale over  $\overline{\mathbb{Z}_\ell}$  and hence split over  $\overline{\mathbb{Z}_\ell}$ . In our case,  $C_{\hat{G}}(T) = T$  since  $\hat{G}$  is connected (For example, use the proof of [Con14, Theorem 3.1.12]).  $\square$





## Chapter 3

# Depth-zero regular supercuspidal blocks

The goal of this chapter is to describe the block  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$  (denoted  $\mathcal{C}_{x,1}$  later) of  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$  containing a depth-zero regular supercuspidal representation  $\pi$ .

Recall that a depth-zero regular supercuspidal representation  $\pi$  is of the form

$$\pi = \text{c-Ind}_{G_x}^{G(F)} \rho,$$

where  $\rho$  is a representation of  $G_x$  whose reduction  $\bar{\rho}$  to the finite reductive group  $\overline{G_x} = G_x/G_x^+$  is supercuspidal.

In the end, assuming that  $G$  is simply connected, the block  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$  would be equivalent to the block  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(\overline{G_x})_{[\bar{\rho}]}$  (denoted  $\mathcal{A}_{x,1}$  later) of  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(\overline{G_x})$  containing  $\bar{\rho}$ . And  $\mathcal{A}_{x,1}$  has an explicit description via the Broué equivalence 3.2.7.

Indeed, let  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G_x)_{[\rho]}$  (denoted  $\mathcal{B}_{x,1}$  later) be the block of  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G_x)$  containing  $\rho$ . It is not hard to see that the inflation along  $G_x \rightarrow \overline{G_x}$  induces an equivalence of categories  $\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}$ . The main theorem we prove in this chapter is that the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}.$$

The proof of this main theorem 3.1.2 would occupy most of this chapter, from Section 3.1 to 3.4. The proof relies on three theorems. In Section 3.1, we prove the main theorem modulo the three theorems. And the proofs of the three theorems are given in Sections 3.2, 3.3, 3.4, respectively.

### 3.1 The compact induction induces an equivalence

In this section, we prove the Main Theorem 3.1.2 modulo Theorem 3.1.3 3.1.4 3.1.5.

Let  $G$  be a split reductive group scheme over  $\mathbb{Z}$ , which is simply connected. Let  $F$  be a non-archimedean local field, with ring of integers  $\mathcal{O}_F$  and residue field  $k_F \cong \mathbb{F}_q$  of characteristic  $p$ . For simplicity, we assume that  $q$  is greater than the Coxeter number of  $G$  (See Theorem 3.2.7 for reason).

Let  $x$  be a vertex of the Bruhat-Tits building  $\mathcal{B}(G, F)$ . Let  $G_x$  be the parahoric subgroup associated to  $x$ , and  $G_x^+$  be its pro-unipotent radical. Recall that  $\overline{G_x} := G_x/G_x^+$  is a generalized Levi subgroup of  $G(k_F)$  with root system  $\Phi_x$ , see [Rab03, Theorem 3.17].

Let  $\Lambda = \overline{\mathbb{Z}_\ell}$ , with  $\ell \neq p$ . Let  $\rho \in \text{Rep}_\Lambda(G_x)$  be an irreducible representation of  $G_x$ , which is trivial on  $G_x^+$  and whose reduction to the finite group of Lie type  $\overline{G_x} = G_x/G_x^+$  is regular cuspidal. Here **regular cuspidal** (See Definition 3.2.8 for precise definition.) means  $\rho$  is cuspidal and lies in a **regular block** of  $\text{Rep}_\Lambda(\overline{G_x})$ , in the sense of [Bro90]. The reason we want the regularity assumption is that we want to work with a block of  $\text{Rep}_\Lambda(\overline{G_x})$  which consists purely of cuspidal representations. See Section 3.2 for details. We make this a definition for later use.

**Definition 3.1.1.** *Let  $\rho \in \text{Rep}_\Lambda(G_x)$ . We say  $\rho$  has cuspidal reduction (resp. has regular cuspidal reduction), if  $\rho$  is trivial on  $G_x^+$  and whose reduction to the finite group of Lie type  $\overline{G_x} = G_x/G_x^+$  is cuspidal (resp. regular cuspidal). Let's denote the reduction of  $\rho$  modulo  $G_x^+$  by  $\bar{\rho} \in \text{Rep}_\Lambda(\overline{G_x})$ .*

Let  $\mathcal{B}_{x,1}$  be the block of  $\text{Rep}_\Lambda(G_x)$  containing  $\rho$ . Let  $\mathcal{C}_{x,1}$  be the block of  $\text{Rep}_\Lambda(G(F))$  containing  $\pi := \text{c-Ind}_{G_x}^{G(F)} \rho$ . Now we can state the Main Theorem of this chapter.

**Theorem 3.1.2** (Main Theorem). *Let  $x$  be a vertex of the Bruhat-Tits building  $\mathcal{B}(G, F)$ . Let  $\rho \in \text{Rep}_\Lambda(G_x)$  which has regular cuspidal reduction. Let  $\mathcal{B}_{x,1}$  be the block of  $\text{Rep}_\Lambda(G_x)$  containing  $\rho$ . Let  $\mathcal{C}_{x,1}$  be the block of  $\text{Rep}_\Lambda(G(F))$  containing  $\pi := \text{c-Ind}_{G_x}^{G(F)} \rho$ . Then the compact induction  $\text{c-Ind}_{G_x}^{G(F)}$  induces an equivalence of categories  $\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}$ .*

As mentioned before, the reason we want the regular cuspidal assumption is the following Theorem.

**Theorem 3.1.3.** *Let  $\rho \in \text{Rep}_\Lambda(G_x)$  be an irreducible representation of  $G_x$ , which has regular cuspidal reduction. Let  $\mathcal{B}_{x,1}$  be the block of  $\text{Rep}_\Lambda(G_x)$  containing  $\rho$ . Then any  $\rho' \in \mathcal{B}_{x,1}$  has cuspidal reduction.*

The proof of the Main Theorem 3.1.2 basically splits into two parts – fully faithfulness and essentially surjectivity. It is convenient to have the following theorem available at an early stage, which implies fully faithfulness immediately and is also used in the proof of essentially surjectivity.

**Theorem 3.1.4.** *Let  $x, y$  be two vertices of the Bruhat-Tits building  $\mathcal{B}(G, F)$ . Let  $\rho_1$  be a representation of the parahoric  $G_x$  which is trivial on the pro-unipotent radical  $G_x^+$ . Let  $\rho_2$  be a representation of  $G_y$  which is trivial on  $G_y^+$ . Assume one of them has cuspidal reduction. Then exactly one of the following happens:*

1. *If there exists an element  $g \in G(F)$  such that  $g.x = y$ , then*

$$\text{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_y}^{G(F)} \rho_2) = \text{Hom}_{G_x}(\rho_1, {}^g \rho_2).$$

2. *If there is no elements  $g \in G(F)$  such that  $g.x = y$ , then*

$$\text{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_y}^{G(F)} \rho_2) = 0.$$

The proof of the above Theorem is basically a computation using Mackey's formula. See Section 3.3.

*Proof of Theorem 3.1.2.* Now we proceed by steps towards our goal: The compact induction  $\text{c-Ind}_{G_x}^{G(F)}$  induces an equivalence of categories  $\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}$ .

First, we show that  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is well-defined. We need to show that the image of  $\mathcal{B}_{x,1}$  under  $\text{c-Ind}_{G_x}^{G(F)}$  lies in  $\mathcal{C}_{x,1}$ . By Theorem 3.1.3 and Theorem 3.1.4 above,

$$\text{c-Ind}_{G_x}^{G(F)}|_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \rightarrow \text{Rep}_\Lambda(G(F))$$

is fully faithful (See Lemma 3.1.6, note here we used Theorem 3.1.3 that any representation in  $\mathcal{B}_{x,1}$  has cuspidal reduction, so that we can apply Theorem 3.1.4), hence an equivalence onto the essential image. Since  $\mathcal{B}_{x,1}$  is indecomposable as an abelian category, so is its essential image (See Lemma 3.1.7), hence its essential image is contained in a single block of  $\text{Rep}_\Lambda(G(F))$ . But such a block must be  $\mathcal{C}_{x,1}$  since  $\text{c-Ind}_{G_x}^{G(F)}$  maps  $\rho$  to  $\pi \in \mathcal{C}_{x,1}$ . Therefore,  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is well-defined.

Second, we show that  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is fully faithful. This is already noticed in the proof of “well-defined” in the last paragraph. Indeed,

$$\text{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_x}^{G(F)} \rho_2) = \text{Hom}_{G_x}(\rho_1, \rho_2)$$

by Theorem 3.1.3 and Theorem 3.1.4 (See Lemma 3.1.6). Therefore,  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is fully faithful.

Finally, we show that  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is essentially surjective. This will occupy the rest of this section.

The idea is to find a projective generator of  $\mathcal{C}_{x,1}$  and show that it is in the essential image. Fix a vertex  $x$  of the Bruhat-Tits building  $\mathcal{B}(G, F)$  as before. Let  $V$  be the set of equivalence classes of vertices of the Bruhat-Tits building  $\mathcal{B}(G, F)$  up to  $G(F)$ -action. For  $y \in V$ , let  $\sigma_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$ . Let  $\Pi := \bigoplus_{y \in V} \Pi_y$  where  $\Pi_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$ . Then  $\Pi$  is a projective generator of the category of depth-zero representations  $\text{Rep}_\Lambda(G(F))_0$ , see [Dat09, Appendix]. Let  $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xrightarrow{\text{summand}} \text{Rep}_\Lambda(G_x)$  be the  $\mathcal{B}_{x,1}$ -summand of  $\sigma_x$ . And let  $\Pi_{x,1} := \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ . Note  $\Pi_{x,1}$  is a summand of  $\Pi_x = \text{c-Ind}_{G_x}^{G(F)} \sigma_x$ , hence a summand of  $\Pi$ . Using Theorem 3.1.4, one can show that the rest of the summands of  $\Pi$  don't interfere with  $\Pi_{x,1}$  (See Lemma 3.4.2 and Lemma 3.4.3 for precise meaning), hence  $\Pi_{x,1}$  is a projective generator of  $\mathcal{C}_{x,1}$ . Let us state it as a Theorem, see Section 2 for details.

**Theorem 3.1.5.**  $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$  is a projective generator of  $\mathcal{C}_{x,1}$ .

Now we've found a projective generator  $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$  of  $\mathcal{C}_{x,1}$ , and it is clear that  $\Pi_{x,1}$  is in the essential image of  $\text{c-Ind}_{G_x}^{G(F)}$ . We now deduce from this that  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is essentially surjective. Indeed, for any  $\pi' \in \mathcal{C}_{x,1}$ , we can resolve  $\pi'$  by some copies of  $\Pi_{x,1}$ :

$$\Pi_{x,1}^{\oplus I} \xrightarrow{f} \Pi_{x,1}^{\oplus J} \rightarrow \pi' \rightarrow 0.$$

Using Theorem 3.1.4 and  $\text{c-Ind}_{G_x}^{G(F)}$  commutes with arbitrary direct sums (See Lemma 3.1.8) we see that  $f \in \text{Hom}_G(\Pi_{x,1}^{\oplus I}, \Pi_{x,1}^{\oplus J})$  comes from a morphism  $g \in \text{Hom}_{G_x}(\sigma_{x,1}^{\oplus I}, \sigma_{x,1}^{\oplus J})$ . Using  $\text{c-Ind}_{G_x}^{G(F)}$  is exact we see that  $\pi'$  is the image of  $\text{coker}(g) \in \mathcal{B}_{x,1}$  under  $\text{c-Ind}_{G_x}^{G(F)}$ . Therefore,  $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \rightarrow \mathcal{C}_{x,1}$  is essentially surjective.  $\square$

**Lemma 3.1.6.**  $\text{c-Ind}_{G_x}^{G(F)}|_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \rightarrow \text{Rep}_\Lambda(G(F))$  is fully faithful.

*Proof.* Let  $\rho_1, \rho_2 \in \mathcal{B}_{x,1}$ . By the regular cuspidal assumption and Theorem 3.1.3,  $\rho_1, \rho_2$  has cuspidal reduction. Hence the assumption of Theorem 3.1.4 is satisfied and we compute using the first case of Theorem 3.1.4 that

$$\text{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_x}^{G(F)} \rho_2) \cong \text{Hom}_{G_x}(\rho_1, \rho_2).$$

In other words,  $\text{c-Ind}_{G_x}^{G(F)}|_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \rightarrow \text{Rep}_\Lambda(G(F))$  is fully faithful.  $\square$

**Lemma 3.1.7.** The image of  $\mathcal{B}_{x,1}$  under  $\text{c-Ind}_{G_x}^{G(F)}$  is indecomposable as an abelian category.

*Proof.* The point is that  $\text{c-Ind}_{G_x}^{G(F)}|_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \rightarrow \text{Rep}_\Lambda(G(F))$  is not only fully faithful, i.e., an equivalence of categories onto the essential image, but also an equivalence of **abelian** categories onto the essential image. Indeed, it suffices to show that  $\text{c-Ind}_{G_x}^{G(F)}|_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \rightarrow \text{Rep}_\Lambda(G(F))$  preserves kernels, cokernels, and finite (bi-)products. But this follows from the next Lemma 3.1.8.

Assume otherwise that the essential image of  $\mathcal{B}_{x,1}$  under  $\text{c-Ind}_{G_x}^{G(F)}$  is decomposable, then so is  $\mathcal{B}_{x,1}$ . But  $\mathcal{B}_{x,1}$  is a block, hence indecomposable, contradiction!  $\square$

**Lemma 3.1.8.**  $\text{c-Ind}_{G_x}^{G(F)}$  is exact and commutes with arbitrary direct sums.

*Proof.* For  $\text{c-Ind}_{G_x}^{G(F)}$  is exact, we refer to [Vig96, I.5.10].

We show that  $\text{c-Ind}_{G_x}^{G(F)}$  commutes with arbitrary direct sums. Indeed,  $\text{c-Ind}_{G_x}^{G(F)}$  is a left adjoint (See [Vig96, I.5.7]), hence commutes with arbitrary colimits. In particular, it commutes with arbitrary direct sums.  $\square$

## 3.2 Regular cuspidal blocks for finite groups of Lie type

In this section, we prove Theorem 3.1.3. As mentioned before, we made the **regular** assumption in order that the conclusion of Theorem 3.1.3 – all representations in such a block have cuspidal reduction – is true. So the readers are welcome to skip this section for a first reading and pretend that we begin with a block in which all representations have cuspidal reduction.

The main body of this section is to define regular cuspidal blocks with  $\Lambda = \overline{\mathbb{Z}_\ell}$ -coefficients of a finite group of Lie type, and to show that a regular cuspidal block consists purely of cuspidal representations.

Let  $\Lambda := \overline{\mathbb{Z}_\ell}$  be the coefficients of representations. Fix a prime number  $p$ . Let  $\ell$  be a prime number different from  $p$ . Let  $q$  be a power of  $p$ .

**Definition 3.2.1** ([Vig96, I.4.1]). *Let  $\Lambda'$  be any ring.*

1. *Let  $H$  be a profinite group, a **representation of  $H$  with  $\Lambda'$ -coefficients**  $(\pi, V)$  is a  $\Lambda'$ -module  $V$ , together with a  $H$ -action  $\pi : H \rightarrow GL_{\Lambda'}(V)$ .*
2. *A representation of  $H$  with  $\Lambda'$ -coefficients is called **smooth** if for any  $v \in V$ , the stabilizer  $Stab_H(v) \subseteq H$  is open.*

From now on, all representations are assumed to be smooth. The category of smooth representations of  $H$  with  $\Lambda'$ -coefficients is denoted by  $\text{Rep}_{\Lambda'}(H)$ .

### 3.2.1 The definition of regular cuspidal blocks

The following notations are used in this subsection only. Let  $\mathcal{G}$  be a split reductive group scheme over  $\mathbb{Z}$ . Let  $\mathbb{G} := \mathcal{G}(\overline{\mathbb{F}}_q)$ ,  $G := \mathbb{G}^F = \mathcal{G}(\mathbb{F}_q)$ , where  $F$  is the Frobenius. By abuse of notation, we sometimes identify the group scheme  $\mathcal{G}_{\overline{\mathbb{F}}_q}$  with its  $\overline{\mathbb{F}}_q$ -points  $\mathbb{G}$ . Let  $\mathbb{G}^*$  be the dual group (over  $\overline{\mathbb{F}}_q$ ) of  $\mathbb{G}$ , and  $F^*$  the dual Frobenius (See [Car85, Section 4.2]). Fix an isomorphism  $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ .

The definition of regular cuspidal blocks and regular cuspidal representations of a finite group of Lie type  $\Gamma$  involves modular Deligne-Lusztig theory and block theory. We refer to [DL76], [Car85], and [DM20] for Deligne-Lusztig theory, [BM89] and [Bro90] for modular Deligne-Lusztig theory, and [Bon11, Appendix B] for generalities on blocks.

First, let's recall a result in Deligne-Lusztig theory (See [DM20, Proposition 11.1.5]).

**Proposition 3.2.2.** *The set of  $\mathbb{G}^F$ -conjugacy classes of pairs  $(\mathbb{T}, \theta)$ , where  $\mathbb{T}$  is a  $F$ -stable maximal torus of  $\mathbb{G}$  and  $\theta \in \widehat{\mathbb{T}^F}$ , is in non-canonical bijection to the set of  $\mathbb{G}^{*F^*}$ -conjugacy classes of pairs  $(\mathbb{T}^*, s)$ , where  $s$  is a semisimple element of  $\mathbb{G}^*$  and  $\mathbb{T}^*$  is a  $F^*$ -stable maximal torus of  $\mathbb{G}^*$  such that  $s \in \mathbb{T}^{*F^*}$ . Moreover, we could and will fix a compatible system of isomorphisms  $\mathbb{F}_{q^n}^* \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$  to pin down this bijection.*

Now let  $s$  be a **strongly regular semisimple** element of  $G^* = \mathbb{G}^{*F^*}$  (note we require  $s$  to be fixed by  $F^*$  here), i.e., the centralizer  $C_{G^*}(s)$  is a  $F^*$ -stable maximal torus, denoted  $\mathbb{T}^*$ . Let  $\mathbb{T}$  be the dual torus of  $\mathbb{T}^*$ . Let  $T = \mathbb{T}^F$  and  $T^* = \mathbb{T}^{*F^*}$ . Let  $T_\ell$  denote the  $\ell$ -part of  $T$ .

Recall for  $s$  strongly regular semisimple, the (rational) Lusztig series  $\mathcal{E}(G, (s))$  consists of only one element, namely,  $\pm R_T^G(\hat{s})$ , where  $\hat{s} = \theta$  is such that  $(\mathbb{T}, \theta)$  corresponds to  $(\mathbb{T}^*, s)$  via the previous bijection in Proposition 3.2.2. Here and after the sign  $\pm$  is taken such that it is an honest representation (See [Car85, Section 7.5]).

**From now on, assume moreover that  $s \in \mathbb{G}^{*F^*}$  has order prime to  $\ell$ .** In other words, assume  $s \in G^* = \mathbb{G}^{*F^*}$  is a **strongly regular semisimple  $\ell'$ -element**. We are going to define regular blocks, we refer to [Bon11, Appendix B] for generalities on blocks.

Define the  **$\ell$ -Lusztig series**

$$\mathcal{E}_\ell(G, (s)) := \{\pm R_T^G(\hat{s}\eta) \mid \eta \in \widehat{T}_\ell\}.$$

Note the notation  $\mathcal{E}_\ell(T, (s))$  also makes sense by putting  $G = T$ .

By [BM89],  $\mathcal{E}_\ell(G, (s))$  is a union of  $\ell$ -blocks of  $\text{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ . Such a block (or more precisely, a union of blocks) is called a **( $\ell$ -)regular block**. Let  $e_s^G \in \overline{\mathbb{Z}}_\ell G$  denote the

corresponding central idempotent. Note  $e_s^T$  also makes sense by putting  $G = T$ . We shall see later that a regular block is indeed a block, i.e., indecomposable. (This follows from, for example, Broué's equivalence. See Theorem 3.2.7 below.)

**Definition 3.2.3** (Regular blocks). *Let  $s \in G^* = \mathbb{G}^{*F^*}$  be a strongly regular semisimple  $\ell'$ -element. We call the block  $\overline{\mathbb{Z}_\ell}Ge_s^G$  of the group algebra  $\overline{\mathbb{Z}_\ell}G$  corresponding to the central idempotent  $e_s^G$  the **regular  $\overline{\mathbb{Z}_\ell}$ -block** associated to  $s$ . Let  $\mathcal{A}_s := \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$  be the corresponding category of modules, this is also referred to as a regular block, by abuse of notation.*

*Similarly, the block  $\overline{\mathbb{F}_\ell}Ge_s^G$  is called a  $\overline{\mathbb{F}_\ell}$ -block. (However, this notion won't be used later.)*

*Remark 3.2.4.* Above all, “a block” could have three different meanings:  $\ell$ -block,  $\overline{\mathbb{Z}_\ell}$ -block, and  $\overline{\mathbb{F}_\ell}$ -block. But they are in one-one correspondence to each other, so we often abuse the notation and simply call it “a block”.

**Definition 3.2.5.** *By a **regular cuspidal block**, we mean a regular block which contains a cuspidal representation.*

*Remark 3.2.6.* We will see later in Theorem 3.2.12 that a regular cuspidal block consists only of cuspidal representations. In the end, the above definition is equivalent to requiring the torus  $\mathbb{T}^F$  to be elliptic, i.e., not contained in any proper parabolic subgroup of  $\mathbb{G}^F$  (See Lemma 3.2.14). This is because  $\overline{\mathbb{Z}_\ell}$ -cuspidality could be checked over  $\overline{\mathbb{Q}_\ell}$  (See the proof of Theorem 3.2.12).

Thanks to [Bro90], we understand the category  $\mathcal{A}_s = \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$  quite well. Roughly speaking, it is equivalent to the category of representations of a torus, via Deligne-Lusztig induction. This is what we are going to explain now.

Let  $\mathbb{B} \subseteq \mathbb{G}$  be a Borel subgroup containing our torus  $\mathbb{T}$ , let  $\mathbb{U}$  be the unipotent radical of  $\mathbb{B}$ . Let  $X_{\mathbb{U}}$  be the Deligne-Lusztig variety defined by

$$X_{\mathbb{U}} := \{g \in \mathbb{G} \mid g^{-1}F(g) \in \mathbb{U}\}.$$

The main result of [Bro90] is the following: The Deligne-Lusztig induction

$$\pm R_T^G : \overline{\mathbb{Z}_\ell}T\text{-Mod} \rightarrow \overline{\mathbb{Z}_\ell}G\text{-Mod}$$

induces an equivalence of categories between  $\overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod}$  and  $\overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$ . In particular, one deduce that the irreducible objects in  $\overline{\mathbb{F}_\ell}Ge_s^G\text{-Mod}$  lifts to  $\overline{\mathbb{Z}_\ell}$ . More precisely, let us state it as the following theorem.

**Theorem 3.2.7** (Broué's equivalence, [Bro90, Theorem 3.3]). *With the previous assumptions and notations, assume  $X_{\mathbb{U}}$  is affine of dimension  $d$  (which is the case if  $q$  is greater than the Coxeter number of  $\mathbb{G}$ ). The cohomology complex  $R\Gamma_c(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell}) = R\Gamma_c(X_{\mathbb{U}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}_\ell}$  is concentrated in degree  $d = \dim X_{\mathbb{U}}$ . And the  $(\overline{\mathbb{Z}_\ell}Ge_s^G, \overline{\mathbb{Z}_\ell}Te_s^T)$ -bimodule  $e_s^G H_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell}) e_s^T$  induces an equivalence of categories*

$$e_s^G H_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell}) e_s^T \otimes_{\overline{\mathbb{Z}_\ell}Te_s^T} - : \overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod} \rightarrow \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}.$$

From now on, we assume the above Theorem holds for all finite groups of Lie type we encountered in this paper. we hope this is not a severe restriction. This is the case at least when  $q$  is greater than the Coxeter number of  $\mathbb{G}$ .

We now define regular cuspidal representations as those representations that occur in some regular cuspidal block. The term “cuspidal” in the name “regular cuspidal” shall be justified later by Theorem 3.2.12.

**Definition 3.2.8.** Let  $G$  be a finite group of Lie type. Let  $\Lambda = \overline{\mathbb{Z}_\ell}$ . Let  $\rho \in \text{Rep}_\Lambda(G)$ . Then  $\rho$  is called **regular cuspidal** if each of its irreducible subquotient  $\rho_i$  is cuspidal (See Definition 3.2.10) and lies in a regular cuspidal  $\overline{\mathbb{Z}_\ell}$ -block  $\mathcal{A}_{s_i}$  of  $G$ .

### 3.2.2 Pure Cuspidality

#### A digression on cuspidality

Before stating the theorem of pure cuspidality, let's define cuspidality for representations with arbitrary coefficients. Let  $\Lambda'$  be any ring. For example,  $\Lambda'$  can be  $\overline{\mathbb{Q}_\ell}$ ,  $\overline{\mathbb{Z}_\ell}$ , or  $\overline{\mathbb{F}_\ell}$ .

First, we define two functors.

**Definition 3.2.9** (Parabolic induction and restriction). Let  $G$  be a finite group of Lie type. Let  $P$  be a parabolic subgroup and  $M$  the corresponding Levi subgroup.

1. The **parabolic induction functor** is defined to be the composition

$$i_M^G := \text{Ind}_P^G \circ f^*,$$

where

$$f^* : \text{Rep}_{\Lambda'}(M) \rightarrow \text{Rep}_{\Lambda'}(P)$$

is the inflation along the natural projection  $f : P \rightarrow M$ .

2. The **parabolic restriction functor** is defined to be the composition

$$r_M^G := (-)_U \circ \text{Res}_P^G,$$

where

$$(-)_U : \text{Rep}_{\Lambda'}(P) \rightarrow \text{Rep}_{\Lambda'}(M), V \mapsto V / \langle \{u.v - v \mid u \in U, v \in V\} \rangle_{\Lambda'U\text{-Mod}}$$

is the functor of taking coinvariance.

We recall that  $r_M^G$  is left adjoint to  $i_M^G$  and they are both exact under our assumption  $\ell \neq p$  (See [Vig96, II.2.1]).

**Definition 3.2.10** (Cuspidal). Let  $G$  be a finite group of Lie type. Let  $\rho \in \text{Rep}_{\Lambda'}(G)$  be a representation of  $G$ . Then  $\rho$  is called **( $\Lambda'$ -)cuspidal** if  $\rho$  is not a subrepresentation of any proper parabolic induction, i.e.,

$$\text{Hom}_G(\rho, i_P^G(\sigma)) = 0$$

for any proper parabolic subgroup  $P$  of  $G$  and any representation  $\sigma \in \text{Rep}_{\Lambda'}(M)$ , where  $M$  is the Levi subgroup corresponding to  $P$ .



For example, let  $s \in G^*$  strongly regular semisimple and  $T$  is not contained in any proper parabolic subgroup, then

$$\pm R_T^G(\hat{s})$$

is cuspidal in  $\text{Rep}_{\overline{\mathbb{Q}_\ell}}(G)$  (See [DL76, Theorem 8.3]).

we record the following equivalent definition of cuspidality for later use.

**Lemma 3.2.11.** [Vig96, II.2.3]  $\rho \in \text{Rep}_{\Lambda'}(G)$  is cuspidal if and only if  $r_M^G \rho = 0$ , for any proper Levi subgroup  $M$  of  $G$ .

### The theorem of pure cuspidality

We can now state the theorem of pure cuspidality.

As in Broué's paper [Bro90], we fix a finite integral extension  $\mathcal{O}$  of  $\mathbb{Z}_\ell$ , which is big enough. One good thing to work with  $\mathcal{O}$  instead of  $\overline{\mathbb{Z}_\ell}$  is that  $\mathcal{O}$  is a discrete valuation ring, while  $\overline{\mathbb{Z}_\ell}$  is not (even not Noetherian). We assume  $\mathcal{O}$  to be big enough (for example,  $\mathcal{O}$  contains all roots of unity we encounter) so that all things we need to do representation theory are available without change.

**Theorem 3.2.12** (Pure Cuspidality). *Let  $G$  be a finite group of Lie type. Let  $s \in G^* = \mathbb{G}^{*F^*}$  be a strongly regular semisimple  $\ell'$ -element, with corresponding torus  $T = \mathbb{T}^F$  and character  $\hat{s} \in \hat{T}$  as in Proposition 3.2.2. Assume that  $\pm R_T^G(\hat{s})$  is  $\overline{\mathbb{Q}_\ell}$ -cuspidal. Then the  $\overline{\mathbb{Z}_\ell}$ -block  $\mathcal{A}_s = \overline{\mathbb{Z}_\ell} G e_s^G\text{-Mod}$  consists purely of cuspidal representations.*

*Proof.* Recall Broué's equivalence 3.2.7: For  $\mathcal{O}$  a finite integral extension of  $\mathbb{Z}_\ell$ , big enough, we have

$$F := e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T \otimes_{\mathcal{O} T e_s^T} - : \mathcal{O} T e_s^T\text{-Mod} \rightarrow \mathcal{O} G e_s^G\text{-Mod}$$

is an equivalence of categories. This is moreover an equivalence of abelian categories (See Lemma 3.2.13). Let  $V := F(\mathcal{O} T e_s^T) = e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T$ . Then  $V$  is a projective generator of  $\mathcal{A}_s$ , since  $\mathcal{O} T e_s^T$  is a projective generator of  $\mathcal{O} T e_s^T\text{-Mod}$ . We first show that  $V$  is  $\mathcal{O}$ -cuspidal.

By classical Deligne-Lusztig theory,  $\overline{\mathbb{Q}_\ell} V \cong \bigoplus_{\eta \in \hat{T}_\ell} \pm R_T^G(\hat{s}\eta)$  is  $\overline{\mathbb{Q}_\ell}$ -cuspidal (For details, see Lemma 3.2.14 below.). In other words,

$$r_{M, \overline{\mathbb{Q}_\ell}}^G(\overline{\mathbb{Q}_\ell} V) := \overline{\mathbb{Q}_\ell} V / \langle \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} \rangle_{\overline{\mathbb{Q}_\ell} U\text{-Mod}} = 0.$$

However, note

$$\langle \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} \rangle_{\overline{\mathbb{Q}_\ell} U\text{-Mod}} = \langle \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} \rangle_{\mathcal{O} U\text{-Mod}}.$$

So we have

$$r_{M, \mathcal{O}}^G(\overline{\mathbb{Q}_\ell} V) := \overline{\mathbb{Q}_\ell} V / \langle \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} \rangle_{\mathcal{O} U\text{-Mod}} = 0.$$

Note  $V$  is finitely presented and projective over  $\mathcal{O} T e_s^T$  (See [Bro90, Proof of Theorem 3.3]), hence projective over  $\mathcal{O}$  (because the restriction functor  $\mathcal{O} T\text{-Mod} \rightarrow \mathcal{O}\text{-Mod}$  preserves projectivity, since it's left adjoint to an exact functor, the induction functor), which is a local ring, hence  $V$  is free over  $\mathcal{O}$  (See [Vak17, Theorem 24.4.5]). We thus have an inclusion

$$V \hookrightarrow \overline{\mathbb{Q}_\ell} V := \overline{\mathbb{Q}_\ell} \otimes_{\mathcal{O}} V$$



as  $\mathcal{O}G$ -modules. Recall that the parabolic restriction  $r_{M,\mathcal{O}}^G$  is exact (See [Vig96, II.2.1]), hence

$$r_{M,\mathcal{O}}^G(\overline{\mathbb{Q}_\ell}V) = 0$$

implies that

$$r_{M,\mathcal{O}}^G(V) = 0,$$

i.e.,  $V$  is  $\mathcal{O}$ -cuspidal.

Moreover, base change to  $\overline{\mathbb{Z}_\ell}$  we see that  $\overline{\mathbb{Z}_\ell}V$  is  $\overline{\mathbb{Z}_\ell}$ -cuspidal. Indeed,

$$r_{M,\overline{\mathbb{Z}_\ell}}^G(\overline{\mathbb{Z}_\ell}V) = \overline{\mathbb{Z}_\ell}V/\overline{\mathbb{Z}_\ell}V(U) = \overline{\mathbb{Z}_\ell} \otimes_{\mathcal{O}} (V/V(U)) = \overline{\mathbb{Z}_\ell} \otimes_{\mathcal{O}} r_{M,\mathcal{O}}^G(V) = 0.$$

For general  $V' \in \mathcal{A}_s$ , we can resolve it by some direct sum of  $V$ 's, and we see that

$$r_{M,\overline{\mathbb{Z}_\ell}}^G(V') = 0,$$

(using  $r_{M,\overline{\mathbb{Z}_\ell}}^G$  is exact and commutes with arbitrary direct sum) i.e.,  $V'$  is  $\overline{\mathbb{Z}_\ell}$ -cuspidal.  $\square$

**Lemma 3.2.13.**

$$F := e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T \otimes_{\mathcal{O}Te_s^T} - : \mathcal{O}Te_s^T\text{-Mod} \rightarrow \mathcal{O}Ge_s^G\text{-Mod}$$

is an equivalence of abelian categories.

*Proof.* We already know that  $F$  is an equivalence of categories. It remains to show that  $F$  is exact and commutes with product.

Now  $e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T$  is projective over  $\mathcal{O}Te_s^T$  (See [Bro90, Proof of Theorem 3.3]), hence flat over  $\mathcal{O}Te_s^T$ . Hence  $F := e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T \otimes_{\mathcal{O}Te_s^T} -$  is exact.

It is clear that  $F := e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T \otimes_{\mathcal{O}Te_s^T} -$  commutes with product.  $\square$

**Lemma 3.2.14.** *Let  $G$  be a finite group of Lie type. Let  $s \in G^* = \mathbb{G}^{*F^*}$  be a strongly regular semisimple  $\ell'$ -element, with corresponding torus  $T = \mathbb{T}^F$  and character  $\hat{s} \in \hat{T}$  as before. Assume that  $\pm R_T^G(\hat{s})$  is  $\overline{\mathbb{Q}_\ell}$ -cuspidal. Then  $\pm R_T^G(\hat{s}\eta)$  is  $\overline{\mathbb{Q}_\ell}$ -cuspidal for any  $\eta \in \hat{T}_\ell$ .*

*Proof.* Broué's equivalence 3.2.7 makes sure that  $\pm R_T^G(\hat{s}\eta)$  is irreducible, hence  $\hat{s}\eta$  is in general position. And  $\pm R_T^G(\hat{s})$  being cuspidal implies that  $T$  is not contained in any proper parabolic subgroup, hence  $\pm R_T^G(\hat{s}\eta)$  is  $\overline{\mathbb{Q}_\ell}$ -cuspidal for any  $\eta \in \hat{T}_\ell$  (See [Car85, Theorem 9.3.2]).  $\square$

### 3.2.3 Proof of Theorem 3.1.3 on cuspidal reduction

We now apply the previous results on finite groups of Lie type to representations of the parahoric subgroups of a  $p$ -adic group. So let's get back to the notation at the beginning of this chapter.

Let  $G$  be a split reductive group scheme over  $\mathbb{Z}$ , which is simply connected. Let  $F$  be a non-archimedean local field, with ring of integers  $\mathcal{O}_F$  and residue field  $k_F \cong \mathbb{F}_q$  of residue characteristic  $p$ . Let  $x$  be a vertex of the Bruhat-Tits building  $\mathcal{B}(G, F)$ ,  $G_x$  the parahoric subgroup associated to  $x$ ,  $G_x^+$  its pro-unipotent radical. Recall that  $\overline{G_x} := G_x/G_x^+$  is a generalized Levi subgroup of  $G(k_F)$  with root system  $\Phi_x$ , see [Rab03, Theorem 3.17].

Let  $\Lambda = \overline{\mathbb{Z}_\ell}$ , with  $\ell \neq p$ . Let  $\rho \in \text{Rep}_\Lambda(G_x)$  be an irreducible representation of  $G_x$ , which is trivial on  $G_x^+$  and whose reduction to the finite group of Lie type  $\overline{G_x} = G_x/G_x^+$  is regular cuspidal.

In other words, we start with an irreducible representation  $\rho \in \text{Rep}_\Lambda(G_x)$  which has regular cuspidal reduction. Let  $\mathcal{B}_{x,1}$  be the  $(\overline{\mathbb{Z}_\ell})$ -block of  $\text{Rep}_\Lambda(G_x)$  containing  $\rho$ . We can now prove Theorem 3.1.3, which we restate as follows.

**Theorem 3.2.15.** *Let  $\rho \in \text{Rep}_\Lambda(G_x)$  be an irreducible representation of  $G_x$ , which has regular cuspidal reduction. Let  $\mathcal{B}_{x,1}$  be the  $\overline{\mathbb{Z}_\ell}$ -block of  $\text{Rep}_\Lambda(G_x)$  containing  $\rho$ . Then any  $\rho' \in \mathcal{B}_{x,1}$  has cuspidal reduction.*

*Proof.* Let  $\bar{\rho} \in \text{Rep}_\Lambda(\overline{G_x})$  be the reduction of  $\rho$  modulo  $G_x^+$ .  $\bar{\rho}$  is irreducible (since  $\rho$  is) and regular cuspidal by assumption, so it is of the form  $r_\ell(\pm R_T^G(\hat{s}))$  (i.e., the  $\ell$ -reduction of  $\pm R_T^G(\hat{s})$ ; Here we used any regular cuspidal representation with  $\overline{\mathbb{F}_\ell}$ -coefficients lifts to  $\overline{\mathbb{Z}_\ell}$ , by Broué-equivalence 3.2.7), for some strongly regular semisimple  $\ell'$ -element  $s$  of  $\overline{G_x}^*$  (See Definition 3.2.8.).

Let  $\text{Rep}_\Lambda(G_x)_0$  be the full subcategory of  $\text{Rep}_\Lambda(G_x)$  consists of representations of  $G_x$  that are trivial on  $G_x^+$ . The key observation is that  $\text{Rep}_\Lambda(G_x)_0$  is a summand (as abelian category) of  $\text{Rep}_\Lambda(G_x)$  (See Lemma 3.2.16).

Then since  $\rho \in \text{Rep}_\Lambda(G_x)_0$ , its block  $\mathcal{B}_{x,1}$  is a summand of  $\text{Rep}_\Lambda(G_x)_0$ .

On the other hand, notice the inflation induces an equivalence of categories between  $\text{Rep}_\Lambda(\overline{G_x})$  and  $\text{Rep}_\Lambda(G_x)_0$ , with inverse the reduction modulo  $G_x^+$ .

So the blocks of  $\text{Rep}_\Lambda(\overline{G_x})$  and  $\text{Rep}_\Lambda(G_x)_0$  are in one-one correspondence. Let  $\mathcal{A}_{x,1}$  be the corresponding block of  $\text{Rep}_\Lambda(\overline{G_x})$  to  $\mathcal{B}_{x,1}$ . Then  $\mathcal{A}_{x,1}$  is contained in the regular cuspidal block  $\mathcal{A}_s$  corresponding to  $s$  (recall  $\bar{\rho} = r_\ell(\pm R_T^G(\hat{s}))$ ). By Theorem 3.2.12,  $\mathcal{A}_s$  consists purely of cuspidal representation. Therefore,  $\mathcal{B}_{x,1}$  consists purely of representations that have cuspidal reductions.  $\square$

**Lemma 3.2.16.** *Let  $\text{Rep}_\Lambda(G_x)_0$  be the full subcategory of  $\text{Rep}_\Lambda(G_x)$  consists of representations of  $G_x$  that are trivial on  $G_x^+$ . Then  $\text{Rep}_\Lambda(G_x)_0$  is a summand as abelian category of  $\text{Rep}_\Lambda(G_x)$ .*

*Remark 3.2.17.* A similar proof as [Dat09, Appendix] should work. Nevertheless, I include an alternative computational proof below.

*Proof.* Note  $G_x^+$  is pro- $p$  (See [Vig96, II.5.2.(b)]), in particular, it has pro-order invertible in  $\Lambda$ . So we have a normalized Haar measure  $\mu$  on  $G_x$  such that  $\mu(G_x^+) = 1$  (See [Vig96, I.2.4]). The characteristic function  $e := 1_{G_x^+}$  is an idempotent of the Hecke algebra  $\mathcal{H}_\Lambda(G_x)$  under convolution with respect to the Haar measure  $\mu$ . We shall show that  $e = 1_{G_x^+}$  cuts out  $\text{Rep}_\Lambda(G_x)_0$  as a summand of  $\text{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)\text{-Mod}$ .

Let's first check that  $e = 1_{G_x^+}$  is central. This can be done by an explicit computation. Recall that we have a descending filtration  $\{G_{x,r} | r \in \mathbb{R}_{>0}\}$  of  $G_x$  such that

1.  $\forall r \in \mathbb{R}_{>0}, G_{x,r}$  is an open compact pro- $p$  subgroup of  $G_x$ .
2.  $\forall r \in \mathbb{R}_{>0}, G_{x,r}$  is a normal subgroup of  $G_x$ .
3.  $G_{x,r}$  form a neighborhood basis of 1 inside  $G_x$ .

(See [Vig96, II.5.1].) Therefore, to check  $e*f = f*e$ , for all  $f \in \mathcal{H}_\Lambda(G_x)$ , it suffices to check for all  $f$  of the form  $1_{gG_{x,r}}$ , the characteristic function of the (both left and right) coset  $gG_{x,r}(= G_{x,r}g$ , by normality) for some  $g \in G(F)$  and  $r \in \mathbb{R}_{>0}$ . Indeed, one can compute that  $(e*1_{gG_{x,r}})(y) = \mu(G_x^+ \cap G_{x,r}yg^{-1})$  and that  $(1_{gG_{x,r}}*e)(y) = \mu(gG_{x,r} \cap yG_x^+)$ , for any  $y \in G_x$ . Note that  $G_{x,r} \subseteq G_x^+$ , we get that  $\mu(G_x^+ \cap G_{x,r}yg^{-1}) = \mu(G_{x,r})$  if  $yg^{-1} \in G_x^+$  and 0 otherwise. Same for  $\mu(gG_{x,r} \cap yG_x^+)$ . Therefore,  $e$  is central.

Finally, under the isomorphism  $\text{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)\text{-Mod}$ ,  $\text{Rep}_\Lambda(G_x)_0$  corresponds to the summand  $\mathcal{H}_\Lambda(G_x, G_x^+)\text{-Mod} = e\mathcal{H}_\Lambda(G_x)e\text{-Mod}$  corresponding to the central idempotent  $e := 1_{G_x^+} \in \mathcal{H}_\Lambda(G_x)$  of  $\mathcal{H}_\Lambda(G_x)\text{-Mod}$ , hence  $\text{Rep}_\Lambda(G_x)_0$  is a summand of  $\text{Rep}_\Lambda(G_x)$ .  $\square$

**Lemma 3.2.18.** *The inflation induces an equivalence of categories between  $\text{Rep}_\Lambda(\overline{G_x})$  and  $\text{Rep}_\Lambda(G_x)_0$ . In particular, let  $\rho$  as in Theorem 3.2.15 and let  $\mathcal{A}_{x,1}$  be the block of  $\text{Rep}_\Lambda(\overline{G_x})$  containing  $\overline{\rho}$ , then the inflation induces an equivalence of categories*

$$\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}.$$

*Proof.* The inverse functor is given by the reduction modulo  $G_x^+$ . One could check by hand that they are equivalences of categories.  $\square$

### 3.3 Hom between compact inductions

Let's now prove Theorem 3.1.4 which computes the Hom between compact inductions of  $\rho_1$  and  $\rho_2$ , assuming one of them has cuspidal reduction.

*Proof of Theorem 3.1.4.*

$$\begin{aligned} & \text{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_y}^{G(F)} \rho_2) \\ &= \text{Hom}_{G_x} \left( \rho_1, (\text{c-Ind}_{G_y}^{G(F)} \rho_2)|_{G_x} \right) \\ &= \text{Hom}_{G_x} \left( \rho_1, \bigoplus_{g \in G_y \setminus G(F)/G_x} \text{c-Ind}_{G_x \cap g^{-1}G_yg}^{G_x} \rho_2(g - g^{-1}) \right) \end{aligned}$$

Recall that  $g^{-1}G_yg = G_{g^{-1}.y}$ . So it suffices to show that for  $g \in G(F)$  with  $G_x \cap g^{-1}G_yg \neq G_x$ , or equivalently, for  $g \in G(F)$  with  $g.x \neq y$  (since  $x$  and  $y$  are vertices), it holds that

$$\text{Hom}_{G_x} \left( \rho_1, \text{c-Ind}_{G_x \cap g^{-1}G_yg}^{G_x} \rho_2(g - g^{-1}) \right) = 0.$$

Note  $G_x/(G_x \cap g^{-1}G_yg)$  is compact, hence  $\text{c-Ind}_{G_x \cap g^{-1}G_yg}^{G_x} = \text{Ind}_{G_x \cap g^{-1}G_yg}^{G_x}$ , and we have Frobenius reciprocity in the other direction

$$\text{Hom}_{G_x} \left( \rho_1, \text{c-Ind}_{G_x \cap g^{-1}G_yg}^{G_x} \rho_2(g - g^{-1}) \right) \cong \text{Hom}_{G_x \cap g^{-1}G_yg} \left( \rho_1, \rho_2(g - g^{-1}) \right).$$

So it suffices to show that for  $g \in G(F)$  with  $g.x \neq y$ ,

$$\text{Hom}_{G_x \cap g^{-1}G_yg} \left( \rho_1, \rho_2(g - g^{-1}) \right) = 0.$$

Note now this expression is symmetric with respect to  $\rho_1$  and  $\rho_2$ , so is the following argument.

First, if  $\rho_2$  has cuspidal reduction (denoted  $\overline{\rho_2}$ ),

$$\begin{aligned}
& \text{Hom}_{G_x \cap g^{-1}G_y g}(\rho_1, \rho_2(g - g^{-1})) \\
&= \text{Hom}_{G_x \cap G_{g^{-1}.y}}(\rho_1, \rho_2(g - g^{-1})) \\
&\subseteq \text{Hom}_{G_x^+ \cap G_{g^{-1}.y}}(\rho_1, \rho_2(g - g^{-1})) \\
&= \text{Hom}_{G_x^+ \cap G_{g^{-1}.y}}(1^{\oplus d_1}, \rho_2(g - g^{-1})) && \rho_1 \text{ is trivial on } G_x^+ \\
&= \text{Hom}_{G_{g.x}^+ \cap G_y}(1^{\oplus d_1}, \rho_2) && \text{Conjugate by } g^{-1} \\
&= \text{Hom}_{U_y(g.x)}(1^{\oplus d_1}, \overline{\rho_2}) && \text{Reduction modulo } G_y^+. \text{ See below.} \\
&= 0 && \overline{\rho_2} \text{ is cuspidal. See below.}
\end{aligned}$$

The last two equations need some explanation.

The former one uses the following consequence from Bruhat-Tits theory: If  $x_1$  and  $x_2$  are two different vertices of the Bruhat-Tits building, then  $\overline{G_{x_i}} := G_{x_i}/G_{x_i}^+$  is a generalized Levi subgroup of  $\overline{G} = G(\mathbb{F}_q)$ , for  $i = 1, 2$ . Moreover,  $G_{x_1} \cap G_{x_2}$  projects onto a proper parabolic subgroup  $P_{x_1}(x_2)$  of  $\overline{G_{x_1}}$  under the reduction map  $G_{x_1} \rightarrow \overline{G_{x_1}}$ . And  $G_{x_1} \cap G_{x_2}^+$  projects onto  $U_{x_1}(x_2)$ , the unipotent radical of  $P_{x_1}(x_2)$ , under the reduction map  $G_{x_1} \rightarrow \overline{G_{x_1}}$ . For details, see Lemma 3.3.1 below. Note that the assumption of Lemma 3.3.1 is satisfied since without loss of generality we may assume  $x_1 = x$  and  $x_2 = y$  lies in the closure of a common alcove (since  $G$  acts simply transitively on the set of alcoves).

The latter one uses that for a cuspidal representation  $\rho$  of a finite group of Lie type  $\Gamma$ ,

$$\text{Hom}_U(1, \rho|_U) = \text{Hom}_U(\rho|_U, 1) = 0,$$

for the unipotent radical  $U$  of  $P$ , where  $P$  is any proper parabolic subgroup of  $\Gamma$ . For details, see Lemma 3.3.2 below.

Symmetrically, a similar argument works if  $\rho_1$  has cuspidal reduction. Indeed, if  $\rho_1$  has cuspidal reduction (denoted  $\overline{\rho_1}$ ),

$$\begin{aligned}
& \text{Hom}_{G_x \cap g^{-1}G_y g}(\rho_1, \rho_2(g - g^{-1})) \\
&= \text{Hom}_{gG_x g^{-1} \cap G_y}(\rho_1(g^{-1} - g), \rho_2) && \text{Conjugate by } g^{-1} \\
&\subseteq \text{Hom}_{gG_x g^{-1} \cap G_y^+}(\rho_1(g^{-1} - g), \rho_2) \\
&= \text{Hom}_{gG_x g^{-1} \cap G_y^+}(\rho_1(g^{-1} - g), 1^{\oplus d_2}) && \rho_2 \text{ is trivial on } G_y^+ \\
&= \text{Hom}_{G_x \cap g^{-1}G_y^+ g}(\rho_1, 1^{\oplus d_2}) && \text{Conjugate by } g \\
&= \text{Hom}_{G_x \cap G_{g^{-1}.y}^+}(\rho_1, 1^{\oplus d_2}) \\
&= \text{Hom}_{U_x(g^{-1}.y)}(\overline{\rho_1}, 1^{\oplus d_2}) && \text{Reduction modulo } G_x^+ \\
&= 0 && \overline{\rho_1} \text{ is cuspidal.}
\end{aligned}$$

□

**Lemma 3.3.1.** *Let  $x_1$  and  $x_2$  be two points of the Bruhat-Tits building  $\mathcal{B}(G, F)$ . Assume they lie in the closure of a same alcove.*

(i) The image of  $G_{x_1} \cap G_{x_2}$  in  $\overline{G_{x_1}}$  is a parabolic subgroup of  $\overline{G_{x_1}}$ . Let's denote it by  $P_{x_1}(x_2)$ . Moreover, the image of  $G_{x_1} \cap G_{x_2}^+$  in  $\overline{G_{x_1}}$  is the unipotent radical of  $P_{x_1}(x_2)$ . Let's denote it by  $U_{x_1}(x_2)$ .

(ii) Assume moreover that  $x_1$  and  $x_2$  are two different vertices of the building. Then  $P_{x_1}(x_2)$  is a proper parabolic subgroup of  $\overline{G_{x_1}}$ .

*Proof.* (i) is [Vig96, II.5.1.(k)].

Let's prove (ii). It suffices to show that  $G_{x_1} \neq G_{x_2}$ . Assume otherwise that  $G_{x_1} = G_{x_2}$ , then  $x_1$  and  $x_2$  lie in the same facet, which contradicts with the assumption that  $x_1$  and  $x_2$  are two different vertices.  $\square$

**Lemma 3.3.2.** *Let  $\bar{\rho}$  be a cuspidal representation of a finite group of Lie type  $\Gamma$ . Let  $P$  be a proper parabolic subgroup of  $\Gamma$ , with unipotent radical  $U$ . Then*

$$\mathrm{Hom}_U(1_U, \bar{\rho}) = \mathrm{Hom}_U(\bar{\rho}, 1_U) = 0.$$

*Proof.*  $\mathrm{Hom}_U(\bar{\rho}|_U, 1_U) = \mathrm{Hom}_\Gamma(\bar{\rho}, \mathrm{Ind}_P^\Gamma(\sigma)) = 0$ , where  $\sigma = \mathrm{Ind}_U^P(1_U)$ . The last equality holds because  $\bar{\rho}$  is assumed to be cuspidal (Recall Definition 3.2.10).

Moreover, since  $U$  is a successive extension of additive groups,  $U$  is of order a power of  $p$ . In particular,  $\ell$  does not divide the order of  $U$ . Hence the category of representations of  $U$  with  $\Lambda = \mathbb{Z}_\ell$ -coefficients is semisimple **No! Even  $\mathbb{Z}_\ell$  is not semisimple as a  $\mathbb{Z}_\ell$  module.**, and

$$\mathrm{Hom}_U(1_U, \bar{\rho}) = \mathrm{Hom}_U(\bar{\rho}, 1_U) = 0.$$

**No!  $\mathrm{Hom}_1(\mathbb{Z}_\ell, \mathbb{F}_\ell) \neq \mathrm{Hom}_1(\mathbb{F}_\ell, \mathbb{Z}_\ell)$ , where 1: trivial group.**  $\square$

### 3.4 $\Pi_{x,1}$ is a projective generator

In this subsection, we prove Theorem 3.1.5:  $\Pi_{x,1}$  is a projective generator of  $\mathcal{C}_{x,1}$ . Before doing this, let's recall the setting. Fix a vertex  $x$  of the building of  $G$ . Let  $\rho \in \mathrm{Rep}_\Lambda(G_x)$  which is trivial on  $G_x^+$  and whose reduction to  $\overline{G_x} = G_x/G_x^+$  is regular cuspidal (even supercuspidal, but we will only use it is cuspidal),  $\pi = \mathrm{c}\text{-Ind}_{G_x}^{G(F)} \rho$  as before. Let  $\mathcal{B}_{x,1}$  be the block of  $\mathrm{Rep}_\Lambda(G_x)$  containing  $\rho$ , and  $\mathcal{C}_{x,1}$  the block of  $\mathrm{Rep}_\Lambda(G(F))$  containing  $\pi$ .

Let  $V$  be the set of equivalence classes of vertices of the Bruhat-Tits building  $\mathcal{B}(G, F)$  up to  $G(F)$ -action. For  $y \in V$ , let  $\sigma_y := \mathrm{c}\text{-Ind}_{G_y^+}^{G_y} \Lambda$ . Let  $\Pi := \bigoplus_{y \in V} \Pi_y$  where  $\Pi_y := \mathrm{c}\text{-Ind}_{G_y^+}^{G(F)} \Lambda$ . Then  $\Pi$  is a projective generator of the category of depth-zero representations  $\mathrm{Rep}_\Lambda(G(F))_0$ , see [Dat09, Appendix]. Let  $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xrightarrow{\text{summand}} \mathrm{Rep}_\Lambda(G_x)$  be the  $\mathcal{B}_{x,1}$ -summand of  $\sigma_x$ . And let  $\Pi_{x,1} := \mathrm{c}\text{-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ .

Let's summarize the setting in the following diagram.

$$\begin{array}{ccc}
\text{Rep}_\Lambda(G_x) & \xrightarrow{\text{c-Ind}_{G_x}^{G(F)}} & \text{Rep}_\Lambda(G(F)) \\
\cup & & \cup \\
\text{Rep}_\Lambda(G_x)_0 & \longrightarrow & \text{Rep}_\Lambda(G(F))_0 \\
\cup & & \cup \\
\mathcal{B}_{x,1} & \longrightarrow & \mathcal{C}_{x,1} \\
\parallel & & \parallel \\
\text{block of } \rho & & \text{block of } \pi
\end{array}$$

**Theorem 3.4.1.**  $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$  is a projective generator of  $\mathcal{C}_{x,1}$ .

*Proof.* First, let  $\text{Rep}_\Lambda(G_x)_0$  be the full subcategory of  $\text{Rep}_\Lambda(G_x)$  consisting of representations that are trivial on  $G_x^+$  (Don't confuse with  $\text{Rep}_\Lambda(G(F))_0$ , the depth-zero category of  $G$ ). Note  $\text{Rep}_\Lambda(G_x)_0$  is a summand of  $\text{Rep}_\Lambda(G_x)$  (see Lemma 3.2.16).

Second, note that  $\text{Rep}_\Lambda(G_x)_0 \cong \text{Rep}_\Lambda(\overline{G}_x)$ . We may assume

$$\text{Rep}_\Lambda(G_x)_0 = \mathcal{B}_{x,1} \oplus \dots \oplus \mathcal{B}_{x,m}$$

is its block decomposition. So that  $\sigma_x = \sigma_{x,1} \oplus \dots \oplus \sigma_{x,m}$  accordingly. Write  $\sigma_x^1 := \sigma_{x,1} \oplus \dots \oplus \sigma_{x,m}$ . Then  $\sigma_x = \sigma_{x,1} \oplus \sigma_x^1$ , and  $\Pi_x = \Pi_{x,1} \oplus \Pi_x^1$  accordingly, where  $\Pi_x^1 := \text{c-Ind}_{G_x}^{G(F)} \sigma_x^1$ . And

$$\Pi = \Pi_{x,1} \oplus \Pi_x^1 \oplus \Pi^x,$$

where  $\Pi^x := \bigoplus_{y \in V, y \neq x} \Pi_y$ . Let  $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$ , then we have

$$\Pi = \Pi_{x,1} \oplus \Pi^{x,1}.$$

Recall that  $\Pi$  is a projective generator of the category of depth-zero representations  $\text{Rep}_\Lambda(G(F))_0$ . This implies that

$$\text{Hom}_G(\Pi, -) : \text{Rep}_\Lambda(G(F))_0 \rightarrow \text{Mod-End}_G(\Pi)$$

is an equivalence of categories. See [Ber92, Lemma 22].

Next, it is not hard to see that Theorem 3.1.4 implies that

$$\text{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \text{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0,$$

see Lemma 3.4.2. This implies that

$$\text{Mod-End}_G(\Pi) \cong \text{Mod-End}_G(\Pi_{x,1}) \oplus \text{Mod-End}_G(\Pi^{x,1})$$

is an equivalence of categories.

Now we can combine the above to show that  $\Pi^{x,1}$  does not interfere with  $\Pi_{x,1}$ , i.e.,

$$\text{Hom}_G(\Pi^{x,1}, X) = 0,$$

for any object  $X \in \mathcal{C}_{x,1}$  (see Important Lemma 3.4.3).

However, since  $\Pi$  is a projective generator of  $\text{Rep}_\Lambda(G(F))_0$ , we have

$$\text{Hom}_G(\Pi, X) \neq 0,$$

for any  $X \in \mathcal{C}_{x,1}$ . This together with the last paragraph implies that

$$\text{Hom}_G(\Pi_{x,1}, X) \neq 0,$$

for any  $X \in \mathcal{C}_{x,1}$ , i.e.  $\Pi_{x,1}$  is a generator of  $\mathcal{C}_{x,1}$ .

Finally, note  $\Pi_{x,1}$  is projective in  $\text{Rep}_\Lambda(G(F))_0$  since it is a summand of the projective object  $\Pi$ . Hence  $\Pi_{x,1}$  is projective in  $\mathcal{C}_{x,1}$ . This together with the last paragraph implies that  $\Pi_{x,1}$  is a projective generator of  $\mathcal{C}_{x,1}$ .  $\square$

**Lemma 3.4.2.**

$$\text{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \text{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0.$$

*Proof.* Recall that  $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$ .

First, we compute

$$\text{Hom}_G(\Pi_{x,1}, \Pi_x^1) = \text{Hom}_{G_x}(\sigma_{x,1}, \sigma_x^1) = 0,$$

where the first equality is the first case of Theorem 3.1.4 (note  $\sigma_{x,1} \in \mathcal{B}_{x,1}$ , hence has cuspidal reduction by Theorem 3.1.3, and hence the condition of Theorem 3.1.4 is satisfied), and the second equality is because  $\sigma_{x,1}$  and  $\sigma_x^1$  lie in different blocks of  $\text{Rep}_\Lambda(G_x)$  by definition.

Second, recall that  $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$  with  $\sigma_{x,1}$  having cuspidal reduction, and  $\Pi_y = \text{c-Ind}_{G_y}^{G(F)} \sigma_y$ . We compute

$$\text{Hom}_G(\Pi_{x,1}, \Pi^x) = \bigoplus_{y \in V, y \neq x} \text{Hom}_G(\Pi_{x,1}, \Pi_y) = 0,$$

by the second case of Theorem 3.1.4.

Combining the above three paragraphs, we get  $\text{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = 0$ .

A same argument shows that  $\text{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0$ .  $\square$

**Lemma 3.4.3** (Important Lemma).  $\text{Hom}_G(\Pi^{x,1}, X) = 0$ , for any object  $X \in \mathcal{C}_{x,1}$ .

*Proof.* Recall that

$$\text{Hom}_G(\Pi, -) : \text{Rep}_\Lambda(G(F))_0 \rightarrow \text{Mod-End}_G(\Pi) \cong \text{Mod-End}_G(\Pi_{x,1}) \oplus \text{Mod-End}_G(\Pi^{x,1})$$

is an equivalence of categories. It is even an equivalence of abelian categories since  $\text{Hom}_G(\Pi, -)$  is exact and commutes with direct product. Hence the image of  $\mathcal{C}_{x,1}$  must be indecomposable as  $\mathcal{C}_{x,1}$  is indecomposable, i.e.,

$$\text{Hom}_G(\Pi, -) = \text{Hom}_G(\Pi_{x,1}, -) \oplus \text{Hom}_G(\Pi^{x,1}, -)$$

can map  $\mathcal{C}_{x,1}$  nonzeroly to only one of  $\text{Mod-End}_G(\Pi_{x,1})$  and  $\text{Mod-End}_G(\Pi^{x,1})$  (See the diagram below).

$$\begin{array}{ccc}
\mathrm{Rep}_\Lambda(G(F))_0 & \xrightarrow{\mathrm{Hom}_G(\Pi, -)} & \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi) \\
\uparrow \cup & & \uparrow \cong \\
\mathcal{C}_{x,1} & \xrightarrow{\mathrm{Hom}_G(\Pi_{x,1}, -) \oplus \mathrm{Hom}_G(\Pi^{x,1}, -)} & \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{x,1}) \oplus \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi^{x,1})
\end{array}$$

Then it must be  $\mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{x,1})$  (that  $\mathrm{Hom}_G(\Pi, -)$  maps  $\mathcal{C}_{x,1}$  nonzeroly to) since

$$\mathrm{Hom}_G(\Pi_{x,1}, \pi) = \mathrm{Hom}_{G_x}(\sigma_{x,1}, \rho) = \mathrm{Hom}_{G_x}(\sigma_x, \rho) \neq 0.$$

In other words,  $\mathrm{Hom}_G(\Pi^{x,1}, -)$  is zero on  $\mathcal{C}_{x,1}$ . □

### 3.5 Application: description of the block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$

Recall we denote  $\mathcal{A}_{x,1} = \mathrm{Rep}_\Lambda(\overline{G_x})_{[\bar{\rho}]}$ ,  $\mathcal{B}_{x,1} = \mathrm{Rep}_\Lambda(G_x)_{[\rho]}$ , and  $\mathcal{C}_{x,1} = \mathrm{Rep}_\Lambda(G(F))_{[\pi]}$ .

We have proven that the inflation along  $G_x \rightarrow \overline{G_x}$  induces an equivalence of categories

$$\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1},$$

see Lemma 3.2.18. And we have also proven that the compact induction induces an equivalence of categories

$$\mathrm{c}\text{-}\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}.$$

Hence  $\mathcal{C}_{x,1} \cong \mathcal{A}_{x,1}$ , where the latter is isomorphic to the block of a finite torus via Broué's equivalence 3.2.7.

We will see in the example (See Chapter 4) of  $GL_n$  that (up to central characters) such a block of a finite torus corresponds to  $\mathrm{QCoh}(\mu)$ , where  $\mu$  is the group scheme of roots of unity appearing in the computation of the  $L$ -parameter side (See Theorem 2.2.3).



## Chapter 4

### Example: $GL_n(F)$

Let's apply the theories in the previous chapters to the example of  $GL_n(F)$ . Throughout this chapter,  $G := GL_n$ .

That said, there is a little mismatch between the theories before and the example here. Namely, we assumed for simplicity in the theories that  $G$  is simply connected (and in particular, semisimple), while this is not the case for  $G = GL_n$ . However, there is only some minor difference due to the center  $\mathbb{G}_m$  of  $GL_n$ . Let us leave it as an exercise for the readers to figure out the details.

#### 4.1 $L$ -parameter side

Let  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$  be an irreducible tame  $L$ -parameter. Let  $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$  be any lift of  $\varphi$ . Let  $C_\varphi$  be the connected component of  $[Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}} / \hat{G}]$  containing  $\varphi$ . By Proposition 2.1.7, we compute that

$$C_\varphi \cong [T/T] \times \mu,$$

where  $T = C_{\hat{G}}(\psi_\ell)$  is a maximal torus of  $GL_n$ , and  $\mu = (T^{Fr=(-)^q})^0$ , and the  $T$ -action on  $T$  is specified in Proposition 2.1.7. To go further, let's choose a nice basis of the Weil group representations  $\varphi$  and  $\psi$ .

Indeed, every irreducible tame  $L$ -parameter  $\varphi$  with  $\overline{\mathbb{F}_\ell}$ -coefficients of  $GL_n$  are of the form  $\varphi = \text{Ind}_{W_E}^{W_F} \eta$ , where  $E$  is a degree  $n$  unramified extension of  $F$ ,  $W_E \cong I_F \rtimes \langle \text{Fr}^n \rangle$  is the Weil group of  $E$ , and  $\eta : W_E \rightarrow \overline{\mathbb{F}_\ell}^*$  is a tame (i.e., trivial on  $P_E = P_F$ ) character of  $W_E$  such that  $\{\eta, \eta^q, \dots, \eta^{q^{n-1}}\}$  are distinct. To find a lift of it with  $\overline{\mathbb{Z}_\ell}$ -coefficients, we let  $\tilde{\eta} : W_E \rightarrow \overline{\mathbb{Z}_\ell}^*$  be any lift of  $\eta$ , and let  $\psi := \text{Ind}_{W_E}^{W_F} \tilde{\eta}$ . Then under a nice basis, we could specify the matrices correspond to the topological generator  $s_0$  and  $\text{Fr}$ :

$$\psi(s_0) = \begin{bmatrix} \tilde{\eta}(s_0) & 0 & 0 & \dots & 0 \\ 0 & \tilde{\eta}(s_0)^q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\eta}(s_0)^{q^{n-1}} \end{bmatrix}$$

and

$$\psi(\text{Fr}) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \tilde{\eta}(\text{Fr}^n) & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Under this basis,  $T = C_{\hat{G}}(\psi_\ell)$  is the diagonal torus of  $GL_n$ , with  $\text{Fr}$  acting by conjugacy via  $\psi$ , i.e.,

$$\text{Fr} \cdot \text{diag}(t_1, t_2, \dots, t_{n-1}, t_n) = \text{diag}(t_2, t_3, \dots, t_n, t_1).$$

So one could compute that

$$T^{\text{Fr}=(-)^q} \cong \mu_{q^n-1},$$

and that

$$(T^{\text{Fr}=(-)^q})^0 \cong \mu_{\ell^k},$$

where  $k \in \mathbb{Z}$  is maximal such that  $\ell^k$  divides  $q^n - 1$ .

To compute the quotient  $[T/T]$ , we note that  $T$  acts on  $T$  via twisted conjugacy

$$(t, t') \mapsto (tnt^{-1}n^{-1})t',$$

where  $n$  is same as  $\psi(\text{Fr})$  in effect. So in our case, this action is

$$(t_1, t_2, \dots, t_n) \cdot (t'_1, t'_2, \dots, t'_n) = (t_n^{-1}t_1t'_1, t_1^{-1}t_2t'_2, \dots, t_{n-1}^{-1}t_nt'_n).$$

We see that the orbits of this action are determined by the determinants (hence are in bijection with  $\mathbb{G}_m$ ), and the center  $\mathbb{G}_m \cong Z \subseteq T$  acts trivially. Therefore,

$$[T/T] \cong [\mathbb{G}_m/\mathbb{G}_m],$$

where  $\mathbb{G}_m$  acts trivially on  $\mathbb{G}_m$ .

In conclusion, we have that the connected component of  $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$  containing  $\varphi$  is

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k},$$

where  $\mathbb{G}_m$  acts trivially on  $\mathbb{G}_m$ , and  $k \in \mathbb{Z}$  is maximal such that  $\ell^k$  divides  $q^n - 1$ .

## 4.2 Representation side

By modular Deligne-Lusztig theory, the block  $\mathcal{A}_{x,1}$  of  $GL_n(\mathbb{F}_q)$  containing a cuspidal representation  $\sigma$  is equivalent to the block of an elliptic torus. Such an elliptic torus is isomorphic to  $\mathbb{F}_{q^n}^*$ . So this block is equivalent to  $\overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)\text{-Mod}$ , where  $k \in \mathbb{Z}$  is maximal such that  $\ell^k$  divides  $q^n - 1$ .

$\mathcal{A}_{x,1}$  inflats to a block of  $K := GL_n(\mathcal{O}_F)$  containing the inflation  $\tilde{\sigma}$  of  $\sigma$ , and further corresponds to a block  $\mathcal{B}_{x,1}$  of  $KZ$  containing  $\rho$ , an extension of  $\tilde{\sigma}$  to  $KZ$ , where  $Z$  is the center of  $GL_n(F)$ . We have

$$\mathcal{B}_{x,1} \cong \mathcal{A}_{x,1} \otimes \text{Rep}_{\overline{\mathbb{Z}_\ell}}(\mathbb{Z}) \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod},$$

because

$$KZ \cong K \times \{\text{diag}(\pi^m, \dots, \pi^m) \mid m \in \mathbb{Z}\} \cong K \times \mathbb{Z}.$$

Argue as in the proof of Theorem 3.1.2 we see that the compact induction  $\text{c-Ind}_{KZ}^G$  induces an equivalence of categories

$$\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1},$$

where  $\mathcal{C}_{x,1}$  is the block of  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$  containing  $\pi := \text{c-Ind}_{KZ}^G \rho$ .

Since every irreducible depth-zero supercuspidal representation  $\pi$  arises as above, we have that the block containing an irreducible depth-zero supercuspidal representation  $\pi$  satisfies

$$\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]} \cong \mathcal{C}_{x,1} \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod},$$

where  $k \in \mathbb{Z}$  is maximal such that  $\ell^k$  divides  $q^n - 1$ .



## Chapter 5

# The categorical local Langlands conjecture

In this chapter, we prove the categorical local Langlands conjecture for depth-zero supercuspidal part of  $G = GL_n$  with coefficients  $\Lambda = \overline{\mathbb{Z}_\ell}$  in Fargues-Scholze's form (See [FS21, Conjecture X.3.5]).

Let  $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$  be an irreducible tame  $L$ -parameter. Let  $C_\varphi$  be the connected component of  $[Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}} / \hat{G}]$  containing  $\varphi$ .

The goal is to show that there is an equivalence

$$\mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}_{Coh, Nilp}^{b, qc}(C_\varphi)$$

of derived categories.

As a first step, let's unravel the definitions of both sides and describe them explicitly.

### 5.1 Unraveling definitions

#### 5.1.1 $L$ -parameter side

Let's first state a lemma that makes the decorations in  $\mathcal{D}_{Coh, Nilp}^{b, qc}(C_\varphi)$  go away. We postpone its proof to a later subsection.

**Lemma 5.1.1.**  $\mathcal{D}_{Coh, Nilp}^{b, qc}(C_\varphi) \cong \mathcal{D}_{Coh, Nilp}^b(C_\varphi) \cong \mathcal{D}_{Coh, \{0\}}^b(C_\varphi) \cong \text{Perf}(C_\varphi)$ .

Let's assume the lemma for the moment and continue. By our computation before,

$$C_\varphi \cong [\mathbb{G}_m / \mathbb{G}_m] \times \mu_{\ell^k} \cong \mathbb{G}_m \times [* / \mathbb{G}_m] \times \mu_{\ell^k},$$

where  $k \in \mathbb{Z}_{\geq 0}$  is maximal such that  $\ell^k$  divides  $q^n - 1$ . So

$$\text{Perf}(C_\varphi) \cong \text{Perf}(\mathbb{G}_m \times [* / \mathbb{G}_m] \times \mu_{\ell^k}) \cong \text{Perf}(\mathbb{G}_m) \otimes \text{Perf}([* / \mathbb{G}_m]) \otimes \text{Perf}(\mu_{\ell^k}).$$

Here,

$$\text{Perf}([* / \mathbb{G}_m]) \cong \bigoplus_{\chi} \text{Perf}(\overline{\mathbb{Z}_\ell})_\chi \cong \bigoplus_{\chi} \text{Perf}(\overline{\mathbb{Z}_\ell}),$$

where  $\chi$  runs over characters of  $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

In conclusion, we have

$$\text{Perf}(C_\varphi) \cong \bigoplus_{\chi} \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where  $\chi$  runs over characters of  $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

### 5.1.2 $Bun_G$ side

Since  $\varphi$  is irreducible,

$$\mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}_{lis}^{C_\varphi}(Bun_G^{ss}, \overline{\mathbb{Z}_\ell})^\omega.$$

See [FS21, Section X.2].

Since

$$Bun_G^{ss} = \bigsqcup_{b \in B(G)_{basic}} [* / G_b(F)],$$

we have

$$\mathcal{D}_{lis}^{C_\varphi}(Bun_G^{ss}, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{basic}} \mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega.$$

Let's look closer into each direct summand. In our case  $G = GL_n$ ,

$$B(G)_{basic} \cong \pi_1(G)_\Gamma \cong \mathbb{Z}.$$

Let's first look at the summand for  $b = 1$  (corresponding to  $0 \in \mathbb{Z} \cong B(G)_{basic}$ ). For  $b = 1$ ,  $G_b \cong GL_n$ , and

$$\mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}(\text{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]})^\omega,$$

where  $\pi \in \text{Rep}_{\overline{\mathbb{F}_\ell}}(GL_n(F))$  is the representation with  $L$ -parameter  $\varphi$ , and  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]}$  is the block of  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))$  containing  $\pi$ . And we've computed in Chapter 4 that

$$\text{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]} \cong \overline{\mathbb{Z}_\ell}[t, t^{-1}] \otimes \overline{\mathbb{Z}_\ell}[s] / (s^{\ell^k} - 1) \text{-Mod} \cong \text{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where  $k \in \mathbb{Z}_{\geq 0}$  is again maximal such that  $\ell^k$  divides  $p^n - 1$ . So we have

$$\mathcal{D}^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}(\text{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}))^\omega \cong \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

We could get a similar description of  $\mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})$  (with arbitrary  $b$ ) for free by the spectral action and its compatibility with  $\pi_1(G)_\Gamma$ -grading. For this, we consider the composition

$$q : C_\varphi \cong \mathbb{G}_m \times [* / \mathbb{G}_m] \times \mu_{\ell^k} \rightarrow [* / \mathbb{G}_m].$$

Recall that

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_{\chi} \mathrm{Perf}(\overline{\mathbb{Z}}_{\ell})\chi.$$

For any  $\chi$ , we denote by  $\mathcal{M}_{\chi}$  the corresponding simple object in  $\mathrm{Perf}([*/\mathbb{G}_m])$ . Moreover,  $\mathcal{M}_{\chi}$  pullbacks to a line bundle on  $C_{\varphi}$

$$\mathcal{L}_{\chi} := q^* \mathcal{M}_{\chi}.$$

We could now state the key proposition that allows us to get to arbitrary  $b \in B(G)_{\mathrm{basic}}$  from the  $b = 1$  case, using the spectral action.

**Proposition 5.1.2.** *1. The restriction of the spectral action by  $\mathcal{L}_{\chi}$  to  $\mathcal{D}(G_b(F), \overline{\mathbb{Z}}_{\ell})$  factors through  $\mathcal{D}(G_{b-\chi}(F), \overline{\mathbb{Z}}_{\ell})$ .*

$$\begin{array}{ccc} \mathcal{L}_{\chi} * - : & \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_{\ell}) & \longrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_{\ell}) \\ & \uparrow \cup & \uparrow \cup \\ & \mathcal{D}(G_b(F), \overline{\mathbb{Z}}_{\ell}) & \cdots \cdots \cdots \mathcal{D}(G_{b-\chi}(F), \overline{\mathbb{Z}}_{\ell}) \end{array}$$

2.  $\mathcal{L}_{\chi} * - : \mathcal{D}(G_b(F), \overline{\mathbb{Z}}_{\ell}) \rightarrow \mathcal{D}(G_{b-\chi}(F), \overline{\mathbb{Z}}_{\ell})$  is an equivalence of categories, with inverse  $\mathcal{L}_{\chi^{-1}} * -$ .

*Proof.* For the first assertion, see [Zou22, Lemma 5.3.2]. For the second assertion, note that  $\mathcal{L}_{\chi}$  and  $\mathcal{L}_{\chi^{-1}}$  are clearly inverse to each other once they are well-defined, since  $q^*$  preserves tensor product.  $\square$

So we have

$$\mathcal{D}^{C_{\varphi}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_{\ell})^{\omega} \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \mathcal{D}^{C_{\varphi}}(G_b(F), \overline{\mathbb{Z}}_{\ell})^{\omega} \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

### 5.1.3 The nilpotent singular support condition

Now we prove Lemma 5.1.1.

The first isomorphism is because  $C_{\varphi}$  is connected, hence the quasicompact support condition  $qc$  is automatic.

The second isomorphism needs some computation. For the definition and properties of the nilpotent singular support condition  $Nilp$ , we refer to [FS21, Section VIII.2]. At the end of the day, it boils down to the fact that for any point  $\varphi'$  in  $C_{\varphi}$  valued in an algebraically closed  $\Lambda$ -field  $k$ ,

$$\left( x_{\varphi'}^*, \mathrm{Sing}_{[Z^1(W_F, \hat{G})/\hat{G}]/\Lambda} \right) \cap \left( \mathcal{N}_{\hat{G}}^* \otimes_{\mathbb{Z}_{\ell}} k \right) \cong H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_{\ell}} k(1)) \cap \left( \mathcal{N}_{\hat{G}}^* \otimes_{\mathbb{Z}_{\ell}} k \right) = \{0\},$$

where  $\hat{\mathfrak{g}}^*$  is the dual of the adjoint representation of  $\hat{G}$ ,  $W_F$  acts by conjugacy on  $\hat{\mathfrak{g}}$  through  $\varphi'$  (and then taking dual and Tate twist to get the action on  $\hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_{\ell}} k(1)$ ), and  $\mathcal{N}_{\hat{G}}^* \subseteq \hat{\mathfrak{g}}^*$  is the nilpotent cone.

In our case,  $\hat{G} = GL_n$ ,  $\hat{\mathfrak{g}} = M_{n \times n}$  is the set of  $n \times n$  matrices. Take  $\varphi' = \varphi$  for example (the similar argument works for any  $\varphi'$  in  $C_\varphi$ ).  $W_F$  acts by conjugacy on  $\hat{\mathfrak{g}} = M_{n \times n}$  through  $\varphi$ , hence induces an action of  $W_F$  on the dual space with Tate twist  $\hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1)$ . One could use the explicit matrices 4.1 of  $s_0$  to compute that the fixed points  $H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1))$  is contained in the (dual of) diagonal torus of  $M_{n \times n}^*$ , the dual Lie algebra  $\hat{\mathfrak{g}}^*$ . On the other hand, the nilpotent cone  $\mathcal{N}_{\hat{G}}^*$  is nothing else than the (dual of) nilpotent matrices in  $M_{n \times n}^*$ . So we conclude that

$$H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1)) \cap (\mathcal{N}_{\hat{G}}^* \otimes_{\mathbb{Z}_\ell} k) = \{0\}.$$

The last isomorphism of Lemma 5.1.1 is [FS21, Theorem VIII.2.9].

## 5.2 The spectral action induces an equivalence of categories

To summarize, we have (abstract) equivalences of categories

$$\mathcal{D}_{Coh, Nilp}^{b, qc}(C_\varphi) \cong \bigoplus_{\chi \in \mathbb{Z}} \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong \bigoplus_{b \in \mathbb{Z}} \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong \mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega,$$

where we identified both  $X^*(\mathbb{G}_m) \cong X^*(Z(\hat{G}))$  and  $B(G)_{basic} \cong \pi_1(G)_\Gamma$  with  $\mathbb{Z}$ . The next goal is to show that the spectral action induces an equivalence of categories

$$\mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}_{Coh, Nilp}^{b, qc}(C_\varphi). \quad (5.2.1)$$

### 5.2.1 Definition of the functor

Let's first define the functor. For this, let's choose a Whittaker datum consisting of a Borel  $B \subseteq G$  and a generic character  $\vartheta : U(F) \rightarrow \overline{\mathbb{Z}_\ell}^*$ , where  $U$  is the unipotent radical of  $B$ . Let  $\mathcal{W}_\vartheta$  be the sheaf concentrated on  $Bun_G^1$  corresponding to the representation  $W_\vartheta := \text{c-Ind}_{U(F)}^{G(F)} \vartheta$ . Let  $W_{\vartheta, [\pi]}$  be the restriction of  $W_\vartheta$  to the block  $\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$ , and  $\mathcal{W}_{\vartheta, [\pi]}$  the corresponding sheaf.

We define our desired functor by spectral acting on  $\mathcal{W}_{\vartheta, [\pi]}$ :

$$\Theta : \mathcal{D}_{Coh, Nilp}^{b, qc}(C_\varphi) \cong \text{Perf}(C_\varphi) \longrightarrow \mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega, \quad A \mapsto A * \mathcal{W}_{\vartheta, [\pi]}.$$

### 5.2.2 Equivalence on degree zero part

We now show that  $\Theta$  induces a derived equivalence on degree zero part. Before that, we do some preparations.

The main input is local Langlands in families (See [HM18]): For  $G = GL_n$ , there are natural isomorphisms

$$\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda / \hat{G}) \cong \mathcal{Z}_\Lambda(G(F)) \cong \text{End}_G(W_\vartheta),$$

where  $\mathcal{Z}_\Lambda(G(F))$  is the Bernstein center of  $\text{Rep}_\Lambda(G(F))$ ; the first map is the unique map between  $\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda / \hat{G})$  and  $\mathcal{Z}_\Lambda(G(F))$  that is compatible with the classical local Langlands correspondence for  $GL_n$ , hence also same as the map defined in [FS21, Section VIII.4]; the second map is given by the action of the Bernstein center on the representation  $W_\vartheta$ .

We shall also use the following two Lemmas:



**Lemma 5.2.1.** *The restriction of the Whittaker representation  $W_{\vartheta, [\pi]}$  is a finitely generated projective generator of  $\text{Rep}_\Lambda(G(F))_{[\pi]}$ .*

*Proof.* For projectivity, see [Hel16, Section 3].

For being a generator, in the  $GL_2$  case one could argue similarly as the  $\overline{\mathbb{Q}_\ell}$ -case in [BH06, Section 39]. (Note their definition of Whittaker representation is dual to our definition, as an induction instead of compact induction. But it still go through by taking dual everywhere. See also, [BH03, Section 2.1 and others].) See [BH03] for the  $GL_n$  case.

For finitely generation, it's enough to observe that  $W_{\vartheta, [\pi]}$  has finitely many irreducible subquotients (by our explicit description of the block  $\text{Rep}_\Lambda(G(F))_{[\pi]}$  with multiplicity one (again, argue similarly as in [BH06, Section 39] for the multiplicity one property).  $\square$

**Lemma 5.2.2.** *The spectral action is compatible with the map*

$$\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda / \hat{G}) \cong \mathcal{Z}_\Lambda(G(F)).$$

*Proof.* See [Zou22, Section 5].  $\square$

Now we state the main result of this subsection.

By compatibility with  $\pi_1(G)_\Gamma$ -grading 5.1.2 (problem with the hyperlink),  $\Theta$  restricts to a map between degree-0 parts of both sides

$$\Theta_0 := \Theta|_{\text{Perf}(C_\varphi)_{\chi=0}} : \text{Perf}(C_\varphi)_{\chi=0} \longrightarrow \mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})_{b=0}^\omega,$$

where  $\text{Perf}(C_\varphi)_{\chi=0} \cong \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$  and

$$\mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})_{b=0}^\omega \cong \mathcal{D}(\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega.$$

**Proposition 5.2.3.** *Under the above identifications, the functor*

$$\Theta_0 : \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \longrightarrow \mathcal{D}(\text{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega \quad A \mapsto A * W_{\vartheta, [\pi]}$$

*is an equivalence of derived categories.*

*Proof.* Let's first prove  $\Theta_0$  is fully faithful. The key observation is that fully faithfulness could be checked on generators of the triangulated category  $\text{Perf}(C_\varphi)_{\chi=0} \cong \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$  (See Lemma 5.2.5). In our case, the structure sheaf  $\mathcal{O}$  is a generator of  $\text{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ , hence it suffices to check fully faithfulness on the structure sheaf. Recall this map sends the structure sheaf  $\mathcal{O} \in \text{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$  to the restriction of the Whittaker representation  $W_{\vartheta, [\pi]}$ . So it suffices to show that the map between Hom-sets in derived category

$$\text{Hom}(\mathcal{O}, \mathcal{O}[n]) \rightarrow \text{Hom}(W_{\vartheta, [\pi]}, W_{\vartheta, [\pi]}[n])$$

is a bijection for all  $n \in \mathbb{Z}$ . The case  $n \neq 0$  follows from the vanishing of higher Ext for projective objects ( $\mathcal{O}$  and  $W_{\vartheta, [\pi]}$ ). For  $n = 0$ ,  $\text{Hom}(\mathcal{O}, \mathcal{O}) \cong \mathcal{O}(C_\varphi)$ , and the above map fits into the following commutative diagram by Lemma 5.2.2, hence a bijection.

$$\begin{array}{ccc} \mathcal{O}(Z^1(W_F, \hat{G})_\Lambda / \hat{G}) & \xrightarrow{\cong} & \text{End}_G(W_\vartheta) \\ \uparrow \cup & & \uparrow \cup \\ \mathcal{O}(C_\varphi) & \longrightarrow & \text{End}_G(W_{\vartheta, [\pi]}) \end{array}$$

The essentially surjectivity follows from Lemma 5.2.1 that  $W_{\vartheta, [\pi]}$  is a finitely generated projective generator of  $\text{Rep}_\Lambda(G(F))_{[\pi]}$ .  $\square$

*Remark 5.2.4.* We remark that to use Lemma 5.2.5 in the above proof, we need the fact that the spectral action commutes with direct sums. Indeed, it commutes with colimits. This boils down to the fact that the Hecke operators commutes with colimits, as they are defined using pullback, tensor product, and shriek pushforward, all of which are left adjoints, hence commutes with colimits.

**Lemma 5.2.5.** *Let  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  be a triangulated functor between triangulated categories that commutes with direct sums, and let  $E$  be a generator of  $\mathcal{D}_1$ . Assume  $F$  induces isomorphisms*

$$\text{Hom}(E, E[n]) \cong \text{Hom}(F(E), F(E[n]))$$

*for all  $n \in \mathbb{Z}$ , then  $F$  is fully faithful.*

*Proof.* We use the general lemma [SP, Stack, Tag 0ATH] twice.

To check that condition (1) and (3) in the general lemma holds, we use  $F$  commutes with direct sums.

To check that condition (2) in the general lemma holds, we use the five lemma.

We first apply it with the property  $T = T_1$ : an object  $M \in \mathcal{D}_1$  has the property  $T_1$  (written  $T_1(M)$ ) if  $F$  induces isomorphisms

$$\text{Hom}(M, E[n]) = \text{Hom}(F(M), F(E[n]))$$

for all  $n \in \mathbb{Z}$ . The assumption implies that condition (4) in the general lemma holds:  $T_1(E[n])$  for all  $n \in \mathbb{Z}$ . Therefore,  $T_1(M)$  for all  $M \in \mathcal{D}_1$ .

We then apply it with the property  $T = T_2$ : an object  $N \in \mathcal{D}_1$  has the property  $T_2$  (written  $T_2(N)$ ) if  $F$  induces isomorphisms

$$\text{Hom}(M, N) = \text{Hom}(F(M), F(N))$$

for all  $M \in \mathcal{D}_1$ . By the last paragraph,  $T_1(M)$  for all  $M \in \mathcal{D}_1$ , i.e.,  $T_2(E[n])$  for all  $n \in \mathbb{Z}$ . Therefore,  $T_2(N)$  for all  $N \in \mathcal{D}_1$ .  $\square$

### 5.2.3 The full equivalence

Finally, we use the spectral action to get the full equivalence. Indeed, on the  $L$ -parameter side, for any character  $\chi' \in X^*(\mathbb{G}_m)$ , tensoring with  $\mathcal{L}_{\chi'}$  induces an equivalence

$$\mathcal{L}_{\chi'} \otimes - : \text{Perf}(C_\varphi)_{\chi=0} \cong \text{Perf}(C_\varphi)_{\chi=\chi'}.$$

Similarly, on the  $Bun_G$  side, by Proposition 5.1.2, spectral acting by  $\mathcal{L}_{\chi'}$  induces an equivalence

$$\mathcal{L}_{\chi'} * - : \mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})_{b=0}^\omega \cong \mathcal{D}_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})_{b=-\chi'}^\omega.$$

Therefore, we get the full equivalence via the spectral action.

# Chapter 6

## Conclusions and questions

In this last chapter, let us make some concluding remarks and raise some further questions.

### 6.1 How do Chapters 2, 3 help?

First, let us reflect on how do Chapters 2, 3 help to prove the categorical conjecture in Chapter 5. It helps to write the  $L$ -parameter side as a  $\mathbb{Z}$ -grading of derived categories of modules over some ring (so that we could reduce to the degree-zero case and that we could check fully faithfulness on the generator). It does not help much thereafter if one accept the local Langlands in families (LLIF)

$$\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda / \hat{G}) \cong \mathcal{Z}_\Lambda(G(F)) \cong \text{End}_G(W_\vartheta).$$

Indeed, the description of the representations side could be reproved using LLIF and that  $W_{\vartheta, [\pi]}$  is a projective generator of  $\text{Rep}_\Lambda(G(F))_{[\pi]}$ .

However, we note that (assuming compatibility of Fargues-Scholze with the usual local Langlands correspondence for  $GL_n$ ) it's possible to use Chapters 2, 3 to reprove the first isomorphism in LLIF. It boils down to the fact that if you have a morphism

$$f : \mathbb{G}_m \times \mu \longrightarrow \mathbb{G}_m \times \mu$$

over  $\overline{\mathbb{Z}_\ell}$ , which becomes an isomorphism after base change to  $\overline{\mathbb{Q}_\ell}$ , then  $f$  is an isomorphism over  $\overline{\mathbb{Z}_\ell}$ .

Moreover, assuming the result in Chapter 3, the second isomorphism in LLIF (when restricted to the block  $\text{Rep}_\Lambda(G(F))_{[\pi]}$ ) is almost equivalent to the statement that  $W_{\vartheta, [\pi]}$  is a projective generator of  $\text{Rep}_\Lambda(G(F))_{[\pi]}$ . The latter could be proven almost by hand as in Lemma 5.2.1.

### 6.2 Relation to Bernstein's projective generator

In [Ber92, p46, Section 3.3], Bernstein constructed certain projective generator

$$\text{c-Ind}_{G^0}^{G(F)}(\rho|_{G^0})$$

of a cuspidal block of  $G(F)$  by inducing from  $G^0$ , the subgroup generated by compact subgroups (for representations with  $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$  coefficients). It is interesting to understand the relation between the projective generators constructed in Chapter 3 and Bernstein's projective generators.

### 6.3 The categorical conjecture for general groups

Since our results in Chapter 2, 3 also work for general reductive groups (other than  $GL_n$ ), it is expected that they could be used to prove the categorical local Langlands conjectures for the depth-zero supercuspidal block of general reductive groups. In particular, the  $\mu$  occurring in the result of the  $L$ -parameter side (See Theorem 2.2.3) should match with the block  $\mathcal{A}_{x,1}$  (See Section 3.5) occurring on the representation side: we should have

$$\mathrm{QCoh}(\mu) \cong \mathcal{A}_{x,1}.$$

Indeed,  $\mu = (T^{\mathrm{Fr}=(-)^q})^0$  is certain fixed point of a torus (See Theorem 2.1.6), and  $\mathcal{A}_{x,1}$  is also a block of some finite torus via Broué's equivalence 3.2.7. And these two finite torus should match.

One possible way to do this is via the (so far unknown in general) compatibility of Fargues-Scholze with classical local Langlands correspondences for depth-zero supercuspidal representations, say the work of DeBacker-Reeder [DR09]. Then these two finite torus should be related by local Langlands for tori (See [DR09, Section 4.3]).

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