# 1 Description of the connected component $X_\varphi$ containing a TRSELP $\varphi$

## 1.1 Recollections on the moduli space of Langlands parameters

I assume the readers to be familiar with the theory of the moduli space of Langlands parameters, see for example [3, Section 3 and Section 4], or [4, Section 2 and Section 4]. (I could also recollect the theory in the appendix.)

Let us first fix some notations.

- Let $p \neq 2$ be a fixed prime number and $\ell \neq 2$ be a prime number different from $p$.
- Let $F$ be a non-archimedean local field with residue characteristic $q = p^r$ for some $r \in \mathbb{Z}_{\geq 1}$.
- Let $W_F$ be the Weil group of $F$, $I_F \subset W_F$ be the inertia subgroup, $P_F$ be the wild inertia subgroup.
- Let $W_t := W_F/P_F$ be the tame Weil group.
- Let $I_t := I_F/P_F$ be the tame inertia subgroup.

Fix $\mathrm{Fr} \in W_F$ any lift of the arithmetic Frobenius. I will abuse the notation and also denote Fr the image of Fr in $W_t$. Then $W_t \simeq I_t \rtimes \langle \mathrm{Fr} \rangle$. Here $I_t$ is non-canonically isomorphic to $\prod_{p' \neq p} \mathbb{Z}_{p'}$, which is procyclic. We fix such an isomorphism

$$I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}. \tag{1}$$

This gives rise to a topological generator $s_0$ of $I_t$, which correspond to $(1, 1, ...)$ under the above isomorphism (1). Let us recall the following important relation in $I_F/P_F$:

$$\mathrm{Fr}\,.s_0.\,\mathrm{Fr}^{-1} = s_0^q. \tag{2}$$

In fact, this is true for any $s \in I_t$ instead of $s_0$.

Let

$$W_t^0 := \langle s_0, \mathrm{Fr} \rangle = \mathbb{Z}[1/p]^{s_0} \rtimes \mathbb{Z}^{\mathrm{Fr}}$$

be the subgroup of $W_t$ generated by $s_0$ and Fr. Denote $W_F^0 \subset W_F$ the preimage of $W_t^0$ under the natural projection $W_F \to W_t$. This is known as the discretization of the Weil group. To summarize, $W_t^0$ is generated by two elements Fr and $s_0$ with a single relation $\mathrm{Fr}\,.s_0.\,\mathrm{Fr}^{-1} = s_0^q$.

Let $G$ be a connected split reductive group over $F$. Let $\hat{G}$ be its dual group over $\mathbb{Z}$. Then the space of cocycles from the discretization

$$Z^1(W_t^0, \hat{G}) = \underline{\mathrm{Hom}}(W_t^0, \hat{G}) = \{(x, y) \in \hat{G} \times \hat{G} \mid yxy^{-1} = x^q\} \tag{3}$$

is an explicit closed subscheme of $\hat{G} \times \hat{G}$ (See [3, Section 3]). An important fact (See [3, Proposition 3.9]) is that over a $\mathbb{Z}_\ell$-algebra $R$ (the cases $R = \overline{\mathbb{F}_\ell}, \overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}$ are most relevant for us), the restriction from $W_t$ to $W_t^0$ induces an isomorphism

$$Z^1(W_t, \hat{G}) \simeq Z^1(W_t^0, \hat{G}).$$

Therefore, we can compute $Z^1(W_t, \hat{G})$ using the explicit formula (3) above. This is fundamental for the study of the moduli space of Langlands parameters $Z^1(W_t, \hat{G})$. I refer the readers to [3, Section 3 and Section 4] for the precise definition and properties of $Z^1(W_t, \hat{G})$.

(maybe add an example here)

Let $I_F^\ell$ be the prime-to-$\ell$ inertia subgroup of $W_F$, i.e., $I_F^\ell := \ker(t_\ell)$, where

$$t_\ell : I_F \to I_F/P_F \simeq \prod_{p' \neq p} \mathbb{Z}_{p'} \to \mathbb{Z}_\ell$$

is the composition. In other words, it is the maximal subgroup of $I_F$ with pro-order prime to $\ell$. This property makes $I_F^\ell$ important when determining the connected components of $Z^1(W_F, \hat{G})$ over $\overline{\mathbb{Z}_\ell}$ (See [3, Theorem 4.2 and Subsection 4.6]).

## 1.2 Tame regular semisimple elliptic $L$-parameters

I want to define a class of $L$-parameters, called TRSELP, which roughly corresponds to depth-zero regular supercuspidal representations. Before that, let me define the concept of schematic centralizer, which will be used throughout the article.

**Definition 1** (Schematic centralizer)**.** *Let $H$ be an affine algebraic group over a ring $R$, $\Gamma$ be a finite group. Let $u \in Z^1(\Gamma, H(R'))$ be a 1-cocycle for some $R$-algebra $R'$. Let*

$$\alpha_u : H_{R'} \longrightarrow Z^1(\Gamma, H)_{R'} \qquad h \longmapsto hu(-)h^{-1}$$

*be the orbit morphism. Then the schematic centralizer $C_H(u)$ is defined to be the fiber of $\alpha_u$ at $u$.*

$$\begin{array}{ccc} C_H(u) & \longrightarrow & H_{R'} \\ \downarrow & & \downarrow{\scriptstyle \alpha_u} \\ R' & \xrightarrow{u} & Z^1(\Gamma, H)_{R'} \end{array}$$

One can show that $C_H(u)(R'') = C_{H(R'')}(u)$ is the set-theoretic centralizer for all $R'$-algebra $R''$, see for example [4, Appendix A].

**Remark.** *Note this is enough for our applications where $\Gamma$ is more generally taken as a profinite group, because $u : \Gamma \to H$ usually factors through a finite quotient $\Gamma'$ of $\Gamma$.*

Let me now define a tame, regular semisimple, elliptic Langlands parameter (TRSELP for short) over $\overline{\mathbb{F}_\ell}$, roughly in the sense of [5, Section 3.4 and Section 4.1] in the case that $G$ is $F$-split, but with $\overline{\mathbb{F}_\ell}$-coefficients instead of $\mathbb{C}$-coefficients.

**Definition 2.** *A **tame regular semisimple elliptic** $L$**-parameter (TRSELP) over** $\overline{\mathbb{F}_\ell}$ is a homomorphism $\varphi : W_F \to \hat{G}(\overline{\mathbb{F}_\ell})$ such that:*

1. *(smooth) $\varphi(I_F)$ is a finite subgroup of $\hat{G}(\overline{\mathbb{F}_\ell})$.*
2. *(Frobenius semisimple) $\varphi(\mathrm{Fr})$ is a semisimple element of $\hat{G}(\overline{\mathbb{F}_\ell})$.*
3. *(tame) The restriction of $\varphi$ to $P_F$ is trivial.*
4. *(elliptic) The identity component of the centralizer $C_{\hat{G}}(\varphi)^0$ is equal to the identity component of the center $Z(\hat{G})^0$.*
5. *(regular semisimple) The centralizer of the inertia $C_{\hat{G}}(\varphi|_{I_F})$ is a torus (in particular, connected).*

Concretely, a TRSELP consists of the following data:

1. The restriction to the inertia $\varphi|_{I_F}$, which is a direct sum of characters of finite abelian groups since $I_F/P_F \simeq \varprojlim \mathbb{F}_{q^n}^*$ is compact abelian and that

   $$\mathrm{Hom}_{\mathrm{Cont}}(I_F/P_F, \overline{\mathbb{F}_\ell}^*) \simeq \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*) \simeq \varinjlim \mathrm{Hom}_{\mathrm{Cont}}(\mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*).$$

   In particular, it factors through (the $\overline{\mathbb{F}_\ell}$-points of) some maximal torus, say $S$. Then regular semisimple means that $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

2. The image of Frobenius $\varphi(\mathrm{Fr})$, which turns out to be an element of the normalizer $N_{\hat{G}(\overline{\mathbb{F}_\ell})}(S)$ (Since $\mathrm{Fr}.s.\mathrm{Fr}^{-1} = s^q \in I_t$ for any $s \in I_t$ implies that $\varphi(\mathrm{Fr})$ normalizes $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.). And "elliptic" means that the center $Z(\hat{G})$ has finite index in the centralizer $C_{\hat{G}}(\varphi)$. As we will see later, ellipticity implies that $\hat{G}(\overline{\mathbb{F}_\ell})$ acts transitively on the connected component $X_\varphi(\overline{\mathbb{F}_\ell})$ of the moduli space of $L$-parameters containing $\varphi$ (See the proof of Lemma 6), which is essential for the description (roughly, see Theorem 2 for the precise statement)

   $$[X_\varphi/\hat{G}] \simeq [*/\underline{S_\varphi}]$$

   where $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is the centralizer of the whole $L$-parameter $\varphi$.

(Maybe add an example here)

**Remark.** *1. Let $\overline{\Lambda} \in \{\overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. It is important for my purpose to distinguish between the set-theoretic centralizer (for example, $C_{\hat{G}(\overline{\Lambda})}(\varphi(I_F))$) and the schematic centralizer (for example, $C_{\hat{G}}(\varphi)$). However, I might still use $\hat{G}$ to mean $\hat{G}(\overline{\Lambda})$ sometimes by abuse of notation, for which I* *hope the readers could recognize. One reason for doing this is that $\hat{G}$ is split over $\overline{\Lambda}$, hence $\hat{G}$ is completely determined by its $\overline{\Lambda}$-points. And many statements can either be phrased in terms of the $\overline{\Lambda}$-scheme or its $\overline{\Lambda}$-points (for example, 4 and 5 in Definition 2).*

2. *As we will see later in Theorem 1, $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ turns out to be the $\overline{\mathbb{F}_\ell}$-points of the split torus $T = C_{\hat{G}}(\psi|_{I_F^\ell})$ for some lift $\psi$ of $\varphi$ over $\overline{\mathbb{Z}_\ell}$.*

### 1.3 Description of the component

Now given a TRSELP $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$. Pick any lift $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ of $\varphi$, whose existence is ensured by the flatness of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ (See Lemma 1). Let $\psi_\ell := \psi|_{I_F^\ell}$ denotes the restriction of $\psi$ to the prime-to-$\ell$ inertia. Note that $\psi \in Z^1(W_F, \hat{G})$ factors through $N_{\hat{G}}(\psi_\ell)$ (Since $I_F^\ell$ is normal in $W_F$). Let $\overline{\psi}$ denotes the image of $\psi$ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Let $X_\varphi$ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing $\varphi$. Note $X_\varphi$ also contains $\psi$ since $\psi$ specializes to $\varphi$. So we sometimes also denote $X_\varphi$ as $X_\psi$.

**Theorem 1.** *Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a TRSELP over $\overline{\mathbb{F}_\ell}$. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of $\varphi$. Then at least when the center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$, the connected component $X_\varphi = X_\psi$ of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing $\varphi$ is isomorphic to*

$$\left(\hat{G} \times C_{\hat{G}}(\psi_\ell)^0 \times \mu\right) / \, C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

*where*

1. *$C_{\hat{G}}(\psi_\ell)^0$ is the identity component of the schematic centralizer $C_{\hat{G}}(\psi_\ell)$, which turns out to be a split torus $T$ over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$.*

2. *$\mu$ is the connected component of $T^{\mathrm{Fr}=(-)^q}$ (the subscheme of $T$ on which $\mathrm{Fr}$ acts by raising to q-th power) containing 1 (See [3, Example 3.14]), which is a product of some $\mu_{\ell^{k_i}}$ (the group scheme of $\ell^{k_i}$-th roots of unity over $\overline{\mathbb{Z}_\ell}$), $k_i \in \mathbb{Z}_{\geq 0}$. Note that $\mu$ could be trivial, depending on $\hat{G}$ and some congruence relations between $q, \ell$.*

3. *$C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is the (schematic) stabilizer (definition see Appendix. Could be defined by Yoneda) of $\overline{\psi}$ in $C_{\hat{G}}(\psi_\ell)$.*

*In other words, we have the following isomorphism of schemes over $\overline{\mathbb{Z}_\ell}$:*

$$X_\varphi \simeq \left(\hat{G} \times T \times \mu\right) / T.$$

*And we will specify in the next subsection what the $T$-action on $\left(\hat{G} \times T \times \mu\right)$ is.*

*Proof.* First, recall by [3, Subsection 4.6],

$$X_\psi \simeq \left(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}}\right)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}}$ denotes the space of cocycles whose restriction to $I_F^\ell$ equals $\psi_\ell$ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\overline{\psi}$. Explanation: Recall (See [3, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the $L$-parameters whose restriction to $I_F^\ell$ is $\hat{G}$-conjugate to $\psi_\ell$ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $\overline{\psi}$. Hence

$$X_\varphi \cong (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \qquad g\eta(-)g^{-1} \leftarrow\!\shortmid (g,\eta),$$

with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$ by

$$(t,(g,\psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ and $(g,\psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$.

Second, $\eta.\psi \leftarrow\!\shortmid \eta$ defines an isomorphism

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \simeq Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_{1_{I_F^\ell}} =: Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$$

where $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell))$ means the space of cocycles with $W_F$ acting on $N_{\hat{G}}(\psi_\ell)$ via conjugacy action through $\psi$, and the subscript $1_{I_F^\ell}$ or 1 means the cocycles whose restriction to $I_F^\ell$ is trivial. Explanation: This is clear by unraveling the definitions: two cocycles whose restriction to $I_F^\ell$ are both $\psi_\ell$ differ by something whose restriction to $I_F^\ell$ is trivial; two cocycles whose pushforward to $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ are both $\overline{\psi}$ differ by something whose pushforward to $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is trivial, i.e., which factors through the identity component $N_{\hat{G}}(\psi_\ell)^0$.

Next, I show that $C_{\hat{G}}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$. By [3, Subsection 3.1], the centralizer $C_{\hat{G}}(\psi_\ell)$ is generalized reductive (See Lemma 2), hence split over $\overline{\mathbb{Z}_\ell}$, and $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0$. So we can determine $C_{\hat{G}}(\psi_\ell)$ by computing its $\overline{\mathbb{F}}_\ell$-points. Indeed,

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)),$$

where the last equality follows since $I_F/I_F^\ell$ doesn't contribute to the image of $\varphi$ (See Lemma 3). Therefore, $C_{\hat{G}}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}}_\ell$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$. Denote $T = C_{\hat{G}}(\psi_\ell)$. In particular, $C_{\hat{G}}(\psi_\ell)$ is connected, hence

$$N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell) = T.$$

Now we could compute

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0) = Z^1_{Ad(\psi)}(W_F, T) \simeq T \times T^{\mathrm{Fr}=(-)^q},$$

where the last isomorphism is given by $\eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$, where $s_0 \in W_t^0$ is the topological generator of $I_t$ fixed before (See [3, Example 3.14]).

Then we show that the identity component of $T^{\mathrm{Fr}=(-)^q}$ gives $\mu$ in the statement of the theorem. Note $T^{\mathrm{Fr}=(-)^q}$ is a diagonalizable group scheme over $\overline{\mathbb{Z}_\ell}$ of dimension zero (This can be seen either by $\dim Z^1(W_F/P_F, T) = \dim T$, or by noticing that $\eta(s_0) \in T^{\mathrm{Fr}=(-)^q}$ is semisimple with finitely many possible eigenvalues), hence of the form $\prod_i \mu_{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$. Hence its connected component $(T^{\mathrm{Fr}=(-)^q})^0$ over $\overline{\mathbb{Z}_\ell}$ is of the form $\prod_i \mu_{\ell^{k_i}}$, with $k_i$ maximal such that $\ell^{k_i}$ divides $n_i$. Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0 \simeq T \times (T^{\mathrm{Fr}=(-)^q})^0 \simeq T \times \mu,$$

(See Lemma 4 for the first isomorphism[1]) where $\mu$ is of the form $\prod_i \mu_{\ell^{k_i}}$.

Finally, we show that $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$. Recall $C_{\hat{G}}(\psi_\ell)$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))$ by conjugation, inducing an action of $C_{\hat{G}}(\psi_\ell)$ on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. And $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is by definition the stabilizer of $\overline{\psi} \in Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ in $C_{\hat{G}}(\psi_\ell)$. Now $C_{\hat{G}}(\psi_\ell) = T$ is connected, hence acts trivially on the component group $\pi_0(N_{\hat{G}}(\psi_\ell))$ [2], hence also acts trivially on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Therefore, the stabilizer $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$ (Check).

Above all, we have

$$X_\varphi \simeq (\hat{G} \times Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \simeq (\hat{G} \times T \times \mu)/T.$$

□

## 1.4 The $T$-action on $(\hat{G} \times T \times \mu)$

For later use, let me make it explicit the $T$-action on $(\hat{G} \times T \times \mu)$.

Recall (See [3, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the $L$-parameters whose restriction to $I_F^\ell$ is $\hat{G}$-conjugate to $\psi_\ell$ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $\overline{\psi}$. Hence $X_\varphi$ is isomorphic to

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$$

via $g\eta(-)g^{-1} \leftarrow\!\shortmid (g, \eta)$, with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$ by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \simeq T$ and $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$.

Next, recall that

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq T \times \mu \quad \eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0)).$$

[1] This can be rewritten more elegantly if I have time, with as less dependence on [3] as possible.

[2] Explain this if have time

Let's focus on the isomorphism $\eta.\psi \leftarrow\!\shortmid \eta$:

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1.$$

Recall that $T \subset \hat{G}$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}}$ via conjugation. Hence the above isomorphism induces an $T$-action on $Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$, by

$$(t, \eta) \mapsto (t(\eta\psi(-))t^{-1})\psi^{-1}.$$

Hence in $(\hat{G} \times T \times \mu)/T$, we compute by tracking the above isomorphisms that

1. $T$ acts on $\hat{G}$ via $(t, g) \mapsto gt^{-1}$.
2. $T = C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acts on $T \subset (T \times \mu)$ (corresponds to $\eta(\mathrm{Fr})$) by twisted conjugacy (due to the isomorphisms $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$), i.e.,

   $$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

   where $n = \psi(\mathrm{Fr})$; Note here $n$, a prior lies in $\hat{G}$, actually lies in $N_{\hat{G}}(T)$ (Since $\mathrm{Fr}\,.s.\,\mathrm{Fr}^{-1} = s^q$ implies that $\psi(\mathrm{Fr})$ normalizes $C_{\hat{G}}(\psi|_{I_F^\ell}) = T$. Check!). To summarize, $t \in T$ acts on $T$ via multiplication by $tnt^{-1}n^{-1}$.
3. $T$ acts trivially on $\mu$. This is because $\eta(s_0) \in T$ and $\psi(s_0) \in T$ (Check!).

On the other hand, recall we have the natural $\hat{G}$-action on $Z^1(W_F, \hat{G})$ by conjugation, hence the $\hat{G}$-action on this component $X_\varphi$. Under the isomorphism $X_\varphi \simeq (\hat{G} \times T \times \mu)/T$, the $\hat{G}$-action becomes

$$(g', (g, t, m)) \mapsto (g'g, t, m), \text{ for any } g' \in \hat{G} \text{ and } (g, t, m) \in (\hat{G} \times T \times \mu)/T.$$

Note that the $T$-action and the $\hat{G}$-action on $(\hat{G} \times T \times \mu)$ commute with each other, we thus have the following:

**Proposition 1.**

$$[X_\varphi/\hat{G}] = \left[\left((\hat{G} \times T \times \mu)/T\right)/\hat{G}\right] \simeq \left[\left((\hat{G} \times T \times \mu)/\hat{G}\right)/T\right] \simeq [(T \times \mu)/T],$$

*with $t \in T$ acting on $T$ via multiplication by $tnt^{-1}n^{-1}$, and $t \in T$ acting trivially on $\mu$.*

## 1.5 Some lemmas

**Lemma 1.** *Let $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$. Then there exists $\psi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{Z}_\ell}))$ such that $\psi'$ is a lift of $\varphi'$.*

*Proof.* In the statement, $Z^1(W_t, \hat{G})$ is the abbreviation for $Z^1(W_t, \hat{G})_{\overline{\mathbb{Z}_\ell}}$. Recall that $Z^1(W_t, \hat{G}) \to \overline{\mathbb{Z}_\ell}$ is flat (See [3, Proposition 3.3]), hence generalizing (See Stack Project, 01U2). Therefore, given $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$, there exists $\xi \in Z^1(W_t, \hat{G}(\overline{\mathbb{Q}_\ell}))$ such that $\xi$ specializes to $\varphi'$. In other words, $\ker(\xi) \subset \ker(\varphi')$. I'm going to show that $\xi : W_t \to \hat{G}(\overline{\mathbb{Q}_\ell})$ factors through $\hat{G}(\overline{\mathbb{Z}_\ell})$.

This is true by the following more general statement: Let $Y = \operatorname{Spec}(R)$ be an affine scheme over $\overline{\mathbb{Z}_\ell}$, let $y_\eta \in Y(\overline{\mathbb{Q}_\ell})$ specializing to $y_s \in Y(\overline{\mathbb{F}_\ell})$. Then $y_\eta \in Y(\overline{\mathbb{Q}_\ell}) = \operatorname{Hom}(R, \overline{\mathbb{Q}_\ell})$ factors through $\overline{\mathbb{Z}_\ell}$.

To prove the above statement, let $\mathfrak{p} := \ker(y_\eta)$ and $\mathfrak{q} := \ker(y_s)$ be the corresponding prime ideas. Then "$y_\eta$ specializes to $y_s$" translates to "$\mathfrak{p} \subset \mathfrak{q}$". Recall that we are going to show that $y_\eta : R \to \overline{\mathbb{Q}_\ell}$ factors through $\overline{\mathbb{Z}_\ell}$. We argue by contradiction. Otherwise there is some element $f \in R$ mapping to $\ell^{-m}u$ for some $m \in \mathbb{Z}_{\geq 1}$ and $u \in \overline{\mathbb{Z}_\ell}^*$. Hence

$$\ell^m u^{-1} f - 1 \in (y_\eta) \subset \ker(y_s). \tag{4}$$

However, $\ell \in \ker(y_s)$ since $y_s$ lives on the special fiber. This together with equation 4 implies that $1 \in \ker(y_s)$. Contradiction! □

**Lemma 2.** *The schematic centralizer $C_{\hat{G}}(\psi_\ell)$ is a generalized reductive group scheme over $\overline{\mathbb{Z}_\ell}$.*

*Proof.* To invoke [3, Lemma 3.2], I first show that

$$C_{\hat{G}}(\psi_\ell) = C_{\hat{G}}(\psi(I_F^\ell)),$$

where $C_{\hat{G}}(\psi(I_F^\ell))$ is the schematic centralizer of the subgroup $\psi(I_F^\ell) \subset \hat{G}(\overline{\mathbb{Z}_\ell})$ in $\hat{G}$. This can be checked by Yoneda Lemma on $R$-valued points for any $\overline{\mathbb{Z}_\ell}$-algebra $R$.

Then we could conclude by [3, Lemma 3.2]. Indeed, $\psi_\ell$ factors through some finite quotient $Q$ of $I_F^\ell$, which has order invertible in the base $\overline{\mathbb{Z}_\ell}$. So the assumptions of [3, Lemma 3.2] are satisfied.

Some explanations to use [3, Lemma 3.2]:

1. While [3, Lemma 3.2] is phrased in the setting that $R$ is a normal subring of a number field, it still works for $\overline{\mathbb{Z}_\ell} \subset \overline{\mathbb{Q}_\ell}$ instead of $\mathbb{Z} \subset \mathbb{Q}$. Indeed, $\psi_\ell$ factors through some finite quotient $Q$ of $I_F^\ell$, say of order $|Q| = N$ (Note $N$ is coprime to $\ell$ since $Q$ is a quotient of $I_F^\ell$). Then we could use [3, Lemma 3.2] to conclude that $C_{\hat{G}}(\psi_\ell)$ is generalized reductive over $\mathbb{Z}[1/pN]$ (Check!). Hence $C_{\hat{G}}(\psi_\ell)$ is also generalized reductive over $\overline{\mathbb{Z}_\ell}$ by base change.

2. There is also a small issue that $\overline{\mathbb{Z}_\ell}$ is not finite over $\mathbb{Z}_\ell$, but this can be resolved since everything is already defined over some sufficiently large finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$.

□

**Lemma 3.**
$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)).$$

*Proof.* The first equation is by definition (and that $C_{\hat{G}}(\psi_\ell)$ represents the set-theoretic centralizer).

For the second equation, note that $\varphi|_{I_t} = \gamma_1 + ... + \gamma_d$ is a direct sum of characters (Since $I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}$), so it suffices to show that each $\gamma_i$ is trivial on the summand $\mathbb{Z}_\ell$ of $I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}$. Indeed,

$$\mathrm{Hom}_{\mathrm{Cont}}(\mathbb{Z}_\ell, \overline{\mathbb{F}_\ell}^*) = \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \varinjlim \mathrm{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \{1\}.$$

□

**Lemma 4.** $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0$.

*Proof.* I have omitted from the notations but here everything is over $\overline{\mathbb{Z}_\ell}$. Recall that $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = T$ and that

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0) \simeq T \times T^{\mathrm{Fr}=(-)^q}.$$

By [3, Section 5.4, 5.5], $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$ is connected (over $\overline{\mathbb{Z}_\ell}$). I check here that the assumptions of [3, Section 5.4, 5.5] are satisfied. Indeed, since $N_{\hat{G}}(\psi_\ell)^0 = T$ is a connected torus, the $W_t^0$-action on $T$ automatically fixes a Borel pair of $T$. Moreover, $s_0$ acts trivially on $N_{\hat{G}}(\psi_\ell)^0 = T$ via $\psi$, so in particular $s_0$ (which is denoted by $s$ in [3, Section 5.5]) has order a power of $\ell$ (which is $1 = \ell^0$).

Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \subset Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0.$$

By [3, Section 4.6],

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \hookrightarrow Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)$$

is open and closed. This is done by considering the restriction to the prime-to-$\ell$ inertia $I_F^\ell$, and then use [3, Theorem 4.2].

Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 = Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0.$$

□

# 2 Main Theorem: description of $[X_\varphi/\hat{G}]$

Let $F$ be a non-archimedean local field, $G$ be a connected split reductive group over $F$. Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a tame, regular semisimple, elliptic $L$-parameter (TRSELP for short). Recall that this means that the centralizer

$$C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) =: S \subset \hat{G}(\overline{\mathbb{F}_\ell})$$

is a maximal torus, and $\varphi(\mathrm{Fr}) \in N_{\hat{G}}(S)$ gives rise to an element $w = \overline{\varphi(\mathrm{Fr})} \in N_{\hat{G}}(S)/S$ in the Weyl group (and that $\varphi$ is tame and elliptic).

Assume that

1. The center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$.

2. $Z(\hat{G})$ is finite.

Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of $\varphi$. Let $\psi_\ell$ denotes the restriction $\psi|_{I_F^\ell}$, and $\overline{\psi}$ denotes the image of $\psi$ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Recall that the schematic centralizer $C_{\hat{G}}(\psi_\ell) = T$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

For later use, I record the following lemma – $w$ can also be defined in terms of $\psi$ instead of $\varphi$. This is helpful because we will reduce to a computation on the special fiber later. First, notice that since $T$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\ell \neq 2$, we can identify

$$\left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) \simeq \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell}),$$

and denote it by $\Omega$. (See Lemma 8 below)

**Remark.** *Lemma 8 below shows that $N_{\hat{G}}(T)/T$ is representable by a group scheme which is split over $\overline{\mathbb{Z}_\ell}$. Therefore, we will slightly abuse notations and use $\Omega, N_{\hat{G}}(T)/T, N_{\hat{G}}(S)/S$ interchangeably.*

**Lemma 5.** *The image of $\varphi(\mathrm{Fr})$ and $\psi(\mathrm{Fr})$ in the Weyl group $\Omega$ agree, hence giving a well defined element $w$ in the Weyl group $\Omega$. (Check carefully!)*

*Proof.* Let

$$\Omega = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell})$$

as above and $\underline{\Omega}$ be the associated constant group scheme. Since $\psi$ is a lift of $\varphi$, $\psi(\mathrm{Fr})$ specializes to $\varphi(\mathrm{Fr})$ in $N_{\hat{G}}(T)$. Then the lemma follows since

$$N_{\hat{G}}(T) \to N_{\hat{G}}(T)/T = \underline{\Omega}$$

is a morphism of schemes, hence the following diagram commutes:

$$\begin{tikzcd}
N_{\hat{G}}(T)(\overline{\mathbb{Z}_\ell}) \arrow[r] \arrow[d] & N_{\hat{G}}(T)(\overline{\mathbb{F}_\ell}) \arrow[d] \\
\underline{\Omega}(\overline{\mathbb{Z}_\ell}) = \Omega \arrow[r] & \underline{\Omega}(\overline{\mathbb{F}_\ell}) = \Omega
\end{tikzcd}$$

□

Our main theorem is the following.

**Theorem 2.** *Let $X_\varphi (= X_\psi)$ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing $\varphi$ (hence also containing $\psi$). The we have isomorphisms of quotient stacks*

$$[X_\varphi/\hat{G}] \simeq [(T \times \mu)/T] \simeq [*/C_T(n)] \times \mu,$$

*where $C_T(n)$ is the schematic centralizer of $n = \psi$ in $T = C_{\hat{G}}(\psi|_{I_F^\ell})$, and $\mu = \prod_{i=1}^m \mu_{\ell^{k_i}}$ for some $k_i \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}_{\geq 0}$ is a product of group schemes of roots of unity.*

*If we moreover assume that $\ell$ doesn't divide the order of $w = \overline{\varphi(\mathrm{Fr})}$ in the Weyl group $N_{\hat{G}}(S)/S$, then*

$$[X_\varphi/\hat{G}] \simeq [(T \times \mu)/T] \simeq [*/\underline{S_\varphi}] \times \mu,$$

*where $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$, and $\underline{S_\varphi}$ is the corresponding constant group scheme.*

*Proof.* Recall that $X_\varphi$ is isomorphic to the contracted product

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

and that $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$ induces isomorphisms

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq T \times \mu.$$

This implies that $[X/\hat{G}] \simeq [(T \times \mu)/T]$ with $T$ acting on $T$ by twisted conjugacy:

$$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where $n = \psi(\mathrm{Fr})$. In other words, $T$ acts on $T$ via multiplication by $tnt^{-1}n^{-1}$. And $T$ acts trivially on $\mu$ (See Proposition 1).

So we are reduced to compute $[T/T]$ with respect to a nice action of the split torus $T$ on $T$, which should be and turns out to be very explicit.

For clarification, let me denote the source torus $T$ by $T^{(1)}$ and the target torus $T$ by $T^{(2)}$. Consider the morphism

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \qquad s \longmapsto sns^{-1}n^{-1}.$$

This is surjective on $\overline{\mathbb{F}_\ell}$-points by our assumption 2 that $Z(\hat{G})$ is finite and $\varphi$ is elliptic (See Lemma 6 below). Hence $f$ is an epimorphism in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes (See the same Lemma 6 below)(maybe add an appendix on diagonalizable group schemes?). Therefore, $f$ induces an isomorphism

$$T^{(1)}/\ker(f) \simeq T^{(2)} \tag{5}$$

as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Moreover, if we let $t \in T$ act on $T^{(1)}$ by left multiplication by $t$, and on $T^{(2)}$ via multiplication by $(tnt^{-1}n^{-1})$, this isomorphism induced by $f$ is $T$-equivariant.

Note $T^{(1)} = T$ is commutative, so the $T$-action (via multiplication by $tnt^{-1}n^{-1}$) and the $\ker(f)$-action (via left multiplication) on $T$ commutes with each other. Hence by the $T$-equivariant isomorphism (5), we have

$$[T/T] = [T^{(2)}/T] \simeq \left[\left(T^{(1)}/\ker(f)\right)/T\right] \simeq \left[\left(T^{(1)}/T\right)/\ker(f)\right] \simeq [*/\ker(f)] = [*/C_T(n)].$$

For the last assertion, see Lemma 7 below.

□

Does $C_T(n) \simeq C_{\hat{G}}(\psi)$ holds?

### 2.0.1 Some lemmas

**Lemma 6.** *The morphism*

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \qquad s \longmapsto sns^{-1}n^{-1}$$

*is epimorphic in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. And it induces an isomorphism $T^{(1)}/\ker(f) \simeq T^{(2)}$ as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes.*

*Proof.* Recall that $T$ is a split torus over $\overline{\mathbb{Z}_\ell}$, hence a diagonalizable $\overline{\mathbb{Z}_\ell}$-group scheme. Notice that $f$ is a morphism of $\overline{\mathbb{Z}_\ell}$-group schemes, hence a morphism of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Recall that the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes is equivalent to the category of abelian groups (See [1, p70, Section 5] or [2]) via

$$D \mapsto \mathrm{Hom}_{\overline{\mathbb{Z}_\ell}-GrpSch}(D, \mathbb{G}_m),$$

and the inverse is given by

$$\overline{\mathbb{Z}_\ell}[M] \leftarrow\!\shortmid M,$$

where $\overline{\mathbb{Z}_\ell}[M]$ is the group algebra of $M$ with $\overline{\mathbb{Z}_\ell}$-coefficients.

Therefore, we could argue in the category of abelian groups via the above category equivalence: $f$ is epimorphic if and only if the map $f^*$ in the category of abelian groups is monomorphic. Note ellipticity and $Z(\hat{G})$ finite imply that $S_\varphi$ is finite, hence

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_T(n) = S_\varphi$$

is finite (See Equation (6) below), hence $\mathrm{coker}(f^*)$ is finite. Therefore,

$$f^* : \mathrm{Hom}(T^{(2)}, \mathbb{G}_m) \to \mathrm{Hom}(T^{(1)}, \mathbb{G}_m)$$

is injective (i.e., monomorphism). Indeed, otherwise $\ker(f^*)$ would be a nonzero sub-$\mathbb{Z}$-module of the finite free $\mathbb{Z}$-module $\mathrm{Hom}(T^{(2)}, \mathbb{G}_m)$, hence a free $\mathbb{Z}$-module of positive rank, which contradicts with $\mathrm{coker}(f^*)$ being finite.

The statement on the quotient follows from the corresponding result in the category of abelian groups: $f^*$ induces an isomorphism

$$\mathrm{Hom}(T^{(1)}, \mathbb{G}_m)/\mathrm{Hom}(T^{(2)}, \mathbb{G}_m) \simeq \mathrm{coker}(f^*)$$

(See [1, p71, Subsection 5.3].)

□

**Lemma 7.** *Assume that $\ell$ doesn't divide the order of $w$. $\ker(f) \simeq \underline{S_\varphi}$ is the constant group scheme of the finite abelian group $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$.*

*Proof.* We recall the following fact: Let $H$ be a smooth affine group scheme over some ring $R$, let $\Gamma$ be a finite group whose order is invertible in $R$. Then the fixed point functor $H^\Gamma$ is representable and is smooth over $R$.

For a proof of the above fact, see [6, Proposition 3.4] or [4, Lemma A.1, A.13].

In our case, let $H = T$, $\Gamma =< w >$ the subgroup of the Weyl group $W_{\hat{G}}(T)$ generated by $w$. Hence

$$\ker(f) = C_T(n) = H^\Gamma$$

(See Lemma ? for the last equalityShould be able to check by Yoneda.) is smooth over $\overline{\mathbb{Z}_\ell}$. Therefore, $\ker(f)$ is finite etale over $\overline{\mathbb{Z}_\ell}$ (Because it is smooth of relative dimension 0 over $\overline{\mathbb{Z}_\ell}$, by Equation (6) below). Hence $\ker(f)$ is a constant group scheme over $\overline{\mathbb{Z}_\ell}$, since $\overline{\mathbb{Z}_\ell}$ has no non-trivial finite etale cover.

Since $\ker(f)$ is constant, we can determine it by computing its $\overline{\mathbb{F}_\ell}$-points:

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_{T(\overline{\mathbb{F}_\ell})}(n) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F)), \tag{6}$$

where the middle equality follows by noticing $T(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ and $n = \varphi(\mathrm{Fr})$.

Finally, note by our TRSELP assumption, $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ is (the $\overline{\mathbb{F}_\ell}$-points of) a torus. Hence $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is abelian, hence finite abelian as we've noticed in the proof of the previous lemma that $S_\varphi$ is finite (by ellipticity and $Z(\hat{G})$ finite). ☐

**Lemma 8.** *Let $\hat{G}$ be a connected reductive group over $\overline{\mathbb{Z}_\ell}$, and $T$ a maximal torus of $\hat{G}$. Then the Weyl group $N_{\hat{G}}(T)/T$ is split over $\overline{\mathbb{Z}_\ell}$.*

*Proof.* By [2, Proposition 3.2.8], the Weyl group $N_{\hat{G}}(T)/C_{\hat{G}}(T)$ is finite etale over $\overline{\mathbb{Z}_\ell}$ and hence split over $\overline{\mathbb{Z}_\ell}$. In our case, $C_{\hat{G}}(T) = T$ since $\hat{G}$ is connected (For example, use the proof of [2, Theorem 3.1.12]). ☐

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