On the categorical local Langlands conjectures for depth-zero regular supercuspidal representations

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Chapter 1

Introduction

Let F be a non-archimedean local field. Let G be a reductive group over F. For simplicity, we assume that G is split and simply connected (in particular, semisimple). Let $\Lambda = \overline{\mathbb{Z}_\ell}$. The categorical local Langlands conjecture predicts that there is a fully faithful embedding

$$\mathrm{Rep}_\Lambda(G(F)) \longrightarrow \mathrm{QCoh}(Z^1(W_F, \hat{G})_\Lambda/\hat{G})$$

from the category of smooth representations of the p-adic group $G(F)$ to the category of quasi-coherent sheaves on the stack of Langlands parameters. In this paper, we compute the two sides explicitly for depth-zero regular supercuspidal part of a split simply connected group G and verify the categorical local Langlands conjecture for depth-zero supercuspidal part of GL_n.

Fixing an irreducible depth-zero regular supercuspidal representation $\pi \in \mathrm{Rep}_{\overline{\mathbb{F}_\ell}}(G(F))$, the (classical) local Langlands conjecture predicts that it should correspond to a tame, regular semisimple, elliptic L-parameter (TRSELP for short) $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ (See [DR09]). As mentioned above, this paper focuses on the depth-zero regular supercuspidal part of the categorical local Langlands conjecture, which predicts a fully faithful embedding

$$\mathrm{Rep}_\Lambda(G(F))_{[\pi]} \longrightarrow \mathrm{QCoh}([X_\varphi/\hat{G}])$$

from the block of $\mathrm{Rep}_\Lambda(G(F))$ containing π to the category of quasi-coherent sheaves on the connected component $[X_\varphi/\hat{G}]$ of the stack of L-parameters.

1.1 L-parameter side

In this section, we explain Chapter 2 on how to compute $\mathrm{QCoh}([X_\varphi/\hat{G}])$.

This is done by describing $[X_\varphi/\hat{G}]$ explicitly as a quotient stack over $\Lambda = \overline{\mathbb{Z}_\ell}$.

1.1.1 Heuristics on the component $[X_\varphi/\hat{G}]$

In this subsection, we describe some heuristics on the component $[X_\varphi/\hat{G}]$ which help us to guess what this component should look like.

First, let us recall what is known over $\overline{\mathbb{Q}_\ell}$ instead of $\Lambda = \overline{\mathbb{Z}_\ell}$. Indeed, assuming that G is simply connected, the connected component of the stack of L-parameters over $\overline{\mathbb{Q}_\ell}$ containing an elliptic L-parameter φ' is known to be one point. More precisely, it is isomorphic to the quotient stack $[*/S_{\varphi'}]$, where $S_{\varphi'} = C_{\hat{G}}(\varphi')$ is the centralizer of φ' (See [FS21, Section X.2]).

Second, let us explain the difference between the geometry of the connected components of the stack of L-parameters over $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{Z}_\ell}$. This can be seen from the example $G = GL_1$. Indeed,

$$Z^1(W_F, \widehat{GL_1}) \cong \mu_{q-1} \times \mathbb{G}_m,$$

both over $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{Z}_\ell}$ (See Example 2.1.1). However, μ_{q-1} is just $q-1$ discrete points over $\overline{\mathbb{Q}_\ell}$, while the connected components of μ_{q-1} are isomorphic to μ_{ℓ^k} (over $\overline{\mathbb{F}_\ell}$, hence also) over $\overline{\mathbb{Z}_\ell}$, where k is the maximal integer such that ℓ^k divides $q-1$. So when describing the connected components of the stack of L-parameters over $\overline{\mathbb{Z}_\ell}$, there will be possibly some non-reduced part μ appearing.

These two features come together in the description of $[X_\varphi/\hat{G}]$, the connected component of $Z^1(W_F, \hat{G})/\hat{G}$ containing φ. Under mild assumptions, we prove that

$$[X_\varphi/\hat{G}] \cong [*/S_\varphi] \times \mu,$$

where $S_\varphi = C_{\hat{G}}(\varphi)$ (See Theorem 2.2.3).

1.1.2 Ingredients of the computation

The computation follows the theory of moduli space of Langlands parameters developed in [DHKM20, Section 2, 4] (See also [Dat22, Section 3, 4] for a more gentle introduction). It is very helpful to do the example of GL_2 first (See Chapter 4).

To compute the component $[X_\varphi/\hat{G}]$ over $\overline{\mathbb{Z}_\ell}$, let us fix a lift $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ of $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$.

Recall by [Dat22, Subsection 4.6],

$$X_\varphi = X_\psi \cong \left(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}\right) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$ denotes the space of cocycles whose restriction to I_F^ℓ equals ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\overline{\psi}$.

Here, $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$ is essentially the space of cocycles of the torus

$$T := N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)$$

by our TRSELP assumption (See Definition 2.1.4) and that $C_{\hat{G}}(\psi_\ell)$ is generalized reductive, hence split over $\overline{\mathbb{Z}_\ell}$ (See Lemma 2.1.12). Since it's not hard to compute the space of tame cocycles of a commutative group scheme using the explicit presentation of the tame Weil group (See 2.1.3 and 2.1.4), we obtain that

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \cong T \times \mu,$$

where μ is a product of $\mu_{\ell^{k_i}}$'s (See Theorem 2.1.7 for details). And it is not hard to see that

$$C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell) = T.$$

Therefore, we get

$$X_\varphi \cong \left(\hat{G} \times T \times \mu\right) / T.$$

One needs to be a bit careful about the T action on T, because here a twist by $\psi(\mathrm{Fr})$ is involved. Then one could compute that

$$[X_\varphi/\hat{G}] \cong [(T \times \mu)/T] \cong [T/T] \times \mu,$$

where T acts on T via twisted conjugacy. After that, we could work in the category of diagonalizable group schemes (whose structure is clear, see [BCO14, p70, Section 5]) to identify $[T/T]$ with $[*/S_\varphi]$ under mild conditions.

1.2 Representation side

In this section, we explain Chapter 3 on how to compute $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$.

1.2.1 Equivalence to the block of a finite group of Lie type

Recall that a depth-zero regular supercuspidal representation is of the form

$$\pi = \text{c-Ind}_{G_x}^{G(F)} \rho$$

for some representation ρ of the parahoric subgroup G_x corresponding to a vertex x in the Bruhat-Tits building of G over F. Moreover, ρ is the inflation of some regular supercuspidal representation $\overline{\rho}$ of the finite group of Lie type $\overline{G_x} := G_x/G_x^+$.

Let $\mathcal{A}_{x,1}$ denote the block $\mathrm{Rep}_\Lambda(\overline{G_x})_{[\overline{\rho}]}$ of $\mathrm{Rep}_\Lambda(\overline{G_x})$ containing $\overline{\rho}$. Similarly, let

$$\mathcal{B}_{x,1} := \mathrm{Rep}_\Lambda(G_x)_{[\rho]}, \qquad \mathcal{C}_{x,1} := \mathrm{Rep}_\Lambda(G(F))_{[\pi]}.$$

Assume that the residue field of F is $\mathbb{F}_q$. For simplicity, we assume that q is greater than the Coxeter number of G (See Theorem 3.2.4 for reason). Then $\mathcal{A}_{x,1}$ is equivalent to a block of a finite torus via Broué's equivalence 3.2.4. And it is not hard to show that the inflation induces an equivalence of categories $\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}$.

The main theorem we will prove for the representation side is Theorem 3.1.2: Assume that q is greater than the Coxeter number of $\overline{G_x}$. Then the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}.$$

Once this is proven, $\mathcal{C}_{x,1}$ is equivalent to $\mathcal{A}_{x,1}$, hence admits an explicit description. The proof of the Theorem 3.1.2 occupies the most part of Chapter 3.

1.2.2 Proof of the main theorem for the representation side

In the rest of the section, let us briefly explain the idea of the proof of Theorem 3.1.2. The fully faithfulness of

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$$

is a usual computation by Frobenius reciprocity and Mackey's formula. Since a similar computation will be used later, we record it in Theorem 3.1.4. The key point is that

$$\mathrm{Hom}_G\left(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_y}^{G(F)} \rho_2\right)$$

could be computed explicitly assuming that one of ρ_1, ρ_2 has supercuspidal reduction (i.e. $\overline{\rho_1}$ or $\overline{\rho_2}$ is supercuspidal).

There is a little subtlety that we want not only ρ to have supercuspidal reduction, but also any representation $\rho' \in \mathcal{B}_{x,1}$ to have supercuspidal reduction. This subtlety is dealt with in Theorem 3.1.3. And this is why we need the **regular** supercuspidal assumption.

The difficulty for proving Theorem 3.1.2 lies in proving that

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$$

is essentially surjective. For this, we prove that the compact induction

$$\Pi_{x,1} := \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$$

is a projective generator of $\mathcal{C}_{x,1}$.

The first key is that $\Pi_{x,1}$ is a summand of a projective generator of a larger category. Indeed, $\Pi_{x,1}$ is a summand of

$$\Pi := \text{c-Ind}_{G_x^+}^{G(F)} \Lambda,$$

which is known to be a projective generator of the category $\text{Rep}_\Lambda(G(F))_0$ of depth-zero representations, i.e.,

$$\Pi = \Pi_{x,1} \oplus \Pi^{x,1}.$$

The second key is that the complement $\Pi^{x,1}$ doesn't interfere with $\Pi_{x,1}$. More precisely, we could compute using Theorem 3.1.4 that

$$\text{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = \text{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = 0.$$

The above two keys allow us to conclude that $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$.

1.3 The example of GL_n

To illustrate the theory, we do the example of GL_n in Chapter 4 (Although GL_n is not simply connected, the theory still works without much change). It is quite concise once we have the theories developed in Chapter 2 and 3, so let us don't say anything more here. However, the readers are encouraged to do the example of GL_2 throughout the paper, which will definitely help to understand the theories in Chapter 2 and 3. Actually, the author always do the example of GL_2 first, and then figure out the theory for general groups.

1.4 The categorical local Langlands conjecture for GL_n

As an application, we could deduce the categorical local Langlands conjecture in Fargues-Scholze's form (See [FS21, Conjecture X.3.5]) for depth-zero supercuspidal blocks of GL_n [1] in Chapter 5.

[1] Notice that supercuspidal implies regular supercuspidal automatically in the GL_n case.

The idea is that we could unravel both sides of the categorical conjecture explicitly using our computation in Chapter 4. They both turn out to be

$$\bigoplus_{\mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu).$$

We want to show that the spectral action gives an equivalence. This reduces to the degree-zero part (of the $\mathbb{Z}$-grading) by compatibility of the spectral action with the $\mathbb{Z}$-grading (See Proposition 5.1.2). And the degree-zero part reduces to the theory of local Langlands in families (See [HM18]). Fortunately, several technical results we need to do the reductions are already available by [Zou22].

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Chapter 2

TRSELP components of the stack of L-parameters

Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a tame regular semisimple elliptic L-parameter. In this chapter, we compute the connected component $[X_\varphi/\hat{G}]$ of the stack of L-parameters $Z^1(W_F, \hat{G})/\hat{G}$ containing φ. In Section 2.1, following the theory developed in [Dat22, Section 3, 4], we first compute the connected component X_φ of the space of 1-cocycles $Z^1(W_F, \hat{G})$ (without modulo out the $\hat{G}$-action). The result turns out to be very explicit:

$$X_\varphi \cong (\hat{G} \times T \times \mu)/T, \tag{2.0.1}$$

see Theorem 2.1.7 for details. In Section 2.2, we use (2.0.1) to obtain a particular simple description of $[X_\varphi/\hat{G}]$ under mild conditions (see Theorem 2.2.3):

$$[X_\varphi/\hat{G}] \cong [*/S_\varphi] \times \mu. \tag{2.0.2}$$

2.1 The connected component X_φ containing a TRSELP φ

The goal of this section is to compute the connected component X_φ containing a TRSELP φ. In 2.1.1, we recall the theory of moduli space of Langlands parameters. In 2.1.2, we define the class of L-parameters that we are interested in – tame regular semisimple elliptic L-parameters (TRSELP for short). In 2.1.3, we compute X_φ explicitly as $(\hat{G} \times T \times \mu)/T$ using the theory of moduli space of Langlands parameters. In 2.1.4, we spell out the T-action on $(\hat{G} \times T \times \mu)$ to prepare for the next section.

2.1.1 Recollections on the moduli space of Langlands parameters

Since our computation uses heavily the theory of moduli space of Langlands parameters, we recollect some basic facts here. For more sophisticated knowledge that will be used, we refer to [Dat22, Section 3 and Section 4], or [DHKM20, Section 2 and Section 4].

Let us first fix some notations.

- Let $p \neq 2$ be a fixed prime number and $\ell \neq 2$ be a prime number different from p.
- Let F be a non-archimedean local field with residue field $\mathbb{F}_q$, where $q = p^r$ for some $r \in \mathbb{Z}_{\geq 1}$.

- Let W_F be the Weil group of F, $I_F \subseteq W_F$ be the inertia subgroup, $P_F \subseteq W_F$ be the wild inertia subgroup.
- Let $W_t := W_F/P_F$ be the tame Weil group.
- Let $I_t := I_F/P_F$ be the tame inertia subgroup.

Fix $\mathrm{Fr} \in W_F$ any lift of the arithmetic Frobenius. We will abuse the notation and also denote by Fr the image of Fr in W_t. Then $W_t \cong I_t \rtimes \langle \mathrm{Fr} \rangle$. Here I_t is non-canonically isomorphic to $\prod_{p' \neq p} \mathbb{Z}_{p'}$, which is procyclic. We fix such an isomorphism

$$I_t \cong \prod_{p' \neq p} \mathbb{Z}_{p'}. \tag{2.1.1}$$

This gives rise to a topological generator s_0 of I_t, which corresponds to $(1, 1, ...)$ under the above isomorphism (2.1.1). Let us recall the following important relation in I_F/P_F:

$$\mathrm{Fr}\, s_0\, \mathrm{Fr}^{-1} = s_0^q. \tag{2.1.2}$$

In fact, this is true for any $s \in I_t$ instead of s_0.

Let

$$W_t^0 := \langle s_0, \mathrm{Fr} \rangle = \mathbb{Z}[1/p]^{s_0} \rtimes \mathbb{Z}^{\mathrm{Fr}}$$

be the subgroup of W_t generated by s_0 and Fr. Let W_F^0 denote the preimage of W_t^0 under the natural projection $W_F \to W_t$. W_F^0 is referred to as the discretization of the Weil group. To summarize, W_t^0 is generated by two elements Fr and s_0 with a single relation

$$W_t^0 = \left\langle \mathrm{Fr}, s_0 \mid \mathrm{Fr}\, s_0\, \mathrm{Fr}^{-1} = s_0^q \right\rangle. \tag{2.1.3}$$

Let G be a connected split reductive group over F. Let $\hat{G}$ be its dual group over $\mathbb{Z}$. Then the space of cocycles from the discretization

$$Z^1(W_t^0, \hat{G}) = \underline{\mathrm{Hom}}(W_t^0, \hat{G}) = \{(x, y) \in \hat{G} \times \hat{G} \mid yxy^{-1} = x^q\} \tag{2.1.4}$$

is an explicit closed subscheme of $\hat{G} \times \hat{G}$ (See [Dat22, Section 3]). An important fact (See [Dat22, Proposition 3.9]) is that over a $\mathbb{Z}_\ell$-algebra R (the cases $R = \overline{\mathbb{F}_\ell}, \overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}$ are most relevant for us), the restriction from W_t to W_t^0 induces an isomorphism

$$Z^1(W_t, \hat{G}) \cong Z^1(W_t^0, \hat{G}).$$

Therefore, we can compute $Z^1(W_t, \hat{G})$ using the explicit formula 2.1.4 above. This is fundamental for the study of the moduli space of Langlands parameters $Z^1(W_t, \hat{G})$. we refer the readers to [Dat22, Section 3, 4] for the precise definition and properties of $Z^1(W_t, \hat{G})$.

Example 2.1.1. *If $G = GL_1$,*

$$\begin{aligned} &Z^1(W_t, \hat{G}) \cong Z^1(W_t^0, \hat{G}) \\ =\ &\{(x, y) \in GL_1 \times GL_1 \mid yxy^{-1} = x^q\} \\ =\ &\{(x, y) \in GL_1 \times GL_1 \mid x = x^q\} \cong \mu_{q-1} \times \mathbb{G}_m. \end{aligned} \tag{2.1.5}$$

More generally, let $G = T$ be a (possibly non-split) torus. Then $\hat{T}$ is equipped with a W_F-action. We could compute that

$$Z^1(W_t, \hat{T}) \cong \hat{T} \times \hat{T}^{\mathrm{Fr}=(-)^q}, \tag{2.1.6}$$

where $\hat{T}^{\mathrm{Fr}=(-)^q}$ is the subscheme of $\hat{T}$ on which Fr *acts by raising to q-th power. See [Dat22, Example 3.14] for details.*

Let I_F^ℓ be the prime-to-ℓ inertia subgroup of W_F, i.e., $I_F^\ell := \ker(t_\ell)$, where

$$t_\ell : I_F \to I_F/P_F \cong \prod_{p' \neq p} \mathbb{Z}_{p'} \to \mathbb{Z}_\ell$$

is the composition. In other words, it is the maximal subgroup of I_F with pro-order prime to ℓ. This property makes I_F^ℓ important when determining the connected components of $Z^1(W_F, \hat{G})$ over $\overline{\mathbb{Z}_\ell}$ (See [Dat22, Theorem 4.2 and Subsection 4.6]).

2.1.2 Tame regular semisimple elliptic L-parameters

We want to define a class of L-parameters, called TRSELP, which roughly corresponds to depth-zero regular supercuspidal representations. Before that, let us define the concept of schematic centralizer, which will be used throughout the article.

Definition 2.1.2 (Schematic centralizer)**.** *Let H be an affine algebraic group over a ring R, Γ be a finite group. Let $u \in Z^1(\Gamma, H(R'))$ be a 1-cocycle for some R-algebra R'. Let*

$$\alpha_u : H_{R'} \longrightarrow Z^1(\Gamma, H)_{R'} \qquad h \longmapsto hu(-)h^{-1}$$

be the orbit morphism. Then the schematic centralizer $C_H(u)$ is defined to be the fiber of α_u at u.

$$\begin{tikzcd} C_H(u) \arrow[r] \arrow[d] & H_{R'} \arrow[d, "\alpha_u"] \\ R' \arrow[r, "u"] & Z^1(\Gamma, H)_{R'} \end{tikzcd}$$

One can show that $C_H(u)(R'') = C_{H(R'')}(u)$ is the set-theoretic centralizer for all R'-algebra R'', see for example [DHKM20, Appendix A].

Remark 2.1.3. Note this is enough for our applications where Γ is more generally taken as a profinite group, because $u : \Gamma \to H$ will factor through a finite quotient Γ' of Γ in practice.

Let us now define a tame, regular semisimple, elliptic Langlands parameter (TRSELP for short) over $\overline{\mathbb{F}_\ell}$, roughly in the sense of [DR09, Section 3.4, 4.1], but with $\overline{\mathbb{F}_\ell}$-coefficients instead of $\mathbb{C}$-coefficients.

Definition 2.1.4. *A **tame regular semisimple elliptic L-parameter (TRSELP) over $\overline{\mathbb{F}_\ell}$** is a homomorphism $\varphi : W_F \to \hat{G}(\overline{\mathbb{F}_\ell})$ such that:*

1. *(smooth) $\varphi(I_F)$ is a finite subgroup of $\hat{G}(\overline{\mathbb{F}_\ell})$.*
2. *(Frobenius semisimple) $\varphi(\mathrm{Fr})$ is a semisimple element of $\hat{G}(\overline{\mathbb{F}_\ell})$.*

3. *(tame) The restriction of φ to P_F is trivial.*

4. *(elliptic) The identity component $C_{\hat{G}}(\varphi)^0$ of the centralizer $C_{\hat{G}}(\varphi)$ is equal to the identity component $Z(\hat{G})^0$ of the center $Z(\hat{G})$.*

5. *(regular semisimple) The centralizer of the inertia $C_{\hat{G}}(\varphi|_{I_F})$ is a torus (in particular, connected).*

Concretely, a TRSELP consists of the following data:

1. The restriction to the inertia $\varphi|_{I_F}$, which is essentially a direct sum of characters of some $\mathbb{F}_{q^n}^*$. Indeed, $I_F/P_F \cong \varprojlim \mathbb{F}_{q^n}^*$ and that

$$\mathrm{Hom}_{\mathrm{Cont}}(I_F/P_F, \overline{\mathbb{F}_\ell}^*) \cong \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*) \cong \varinjlim \mathrm{Hom}_{\mathrm{Cont}}(\mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*).$$

   In particular, it factors through (the $\overline{\mathbb{F}_\ell}$-points of) some maximal torus, say S. Then regular semisimple means that $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

2. The image of the Frobenius $\varphi(\mathrm{Fr})$, which turns out to be an element of the normalizer $N_{\hat{G}(\overline{\mathbb{F}_\ell})}(S)$ (Since $\mathrm{Fr}.s.\mathrm{Fr}^{-1} = s^q \in I_t$ for any $s \in I_t$ implies that $\varphi(\mathrm{Fr})$ normalizes $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.). And "elliptic" means that the center $Z(\hat{G})$ has finite index in the centralizer $C_{\hat{G}}(\varphi)$. As we will see later, ellipticity implies that $\hat{G}(\overline{\mathbb{F}_\ell})$ acts transitively on the connected component $X_\varphi(\overline{\mathbb{F}_\ell})$ of the moduli space of L-parameters containing φ (See the proof of Lemma 2.2.4), which is essential for the description (roughly, see Theorem 2.2.3 for the precise statement)

$$[X_\varphi/\hat{G}] \cong [*/\underline{S_\varphi}],$$

   where $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is the centralizer of the whole L-parameter φ.

Example 2.1.5. *For $G = GL_n$, a TRSELP is same as an irreducible tame L-parameter. See Section 4.1 for irreducible tame L-parameters of GL_n expressed in explicit matrices.*

Remark 2.1.6.

1. Let $R \in \{\overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. It is important for our purpose to distinguish between the set-theoretic centralizer (for example, $C_{\hat{G}(R)}(\varphi(W_F))$) and the schematic centralizer (for example, $C_{\hat{G}}(\varphi)$). However, we might still use $\hat{G}$ to mean $\hat{G}(R)$ sometimes by abuse of notation, for which we hope the readers could recognize. One reason for doing this is that we assume throughout that $\hat{G}$ is split over R, hence $\hat{G}$ is completely determined by its R-points. And many statements can either be phrased in terms of the R-scheme or its R-points (for example, 4 and 5 in Definition 2.1.4).

2. As we will see later in Theorem 2.1.7, $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ turns out to be the $\overline{\mathbb{F}_\ell}$-points of the split torus $T = C_{\hat{G}}(\psi|_{I_F^\ell})$ for some lift ψ of φ over $\overline{\mathbb{Z}_\ell}$.

2.1.3 Description of the component

Now let us fix a TRSELP $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$. Pick any lift $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ of φ, whose existence is ensured by the flatness of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ (See Lemma 2.1.11). Let $\psi_\ell := \psi|_{I_F^\ell}$ denote the restriction of ψ to the prime-to-ℓ inertia I_F^ℓ. Note that $\psi \in Z^1(W_F, \hat{G})$ factors through $N_{\hat{G}}(\psi_\ell)$ (Since I_F^ℓ is normal in W_F). Let $\overline{\psi}$ denote the image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Let X_φ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ. Note that X_φ also contains ψ since ψ specializes to φ. So we sometimes also denote X_φ as X_ψ.

Theorem 2.1.7. *Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a TRSELP over $\overline{\mathbb{F}_\ell}$. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of φ. Assume that the center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$. Then the connected component $X_\varphi = X_\psi$ of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ is isomorphic to*

$$\left(\hat{G} \times C_{\hat{G}}(\psi_\ell)^0 \times \mu\right) / \, C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where

1. *$C_{\hat{G}}(\psi_\ell)^0$ is the identity component of the schematic centralizer $C_{\hat{G}}(\psi_\ell)$. And $C_{\hat{G}}(\psi_\ell) = C_{\hat{G}}(\psi_\ell)^0$ is a split torus T over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$.*

2. *$\mu := \left(T^{\mathrm{Fr}=(-)^q}\right)^0$ is the identity component of $T^{\mathrm{Fr}=(-)^q}$ [1] containing 1, which is a product of some $\mu_{\ell^{k_i}}$ (the group scheme of ℓ^{k_i}-th roots of unity over $\overline{\mathbb{Z}_\ell}$), $k_i \in \mathbb{Z}_{\geq 0}$.* [2]

3. *$C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is the (schematic) stabilizer of $\overline{\psi}$ in $C_{\hat{G}}(\psi_\ell)$.*

In other words, we have the following isomorphism of schemes over $\overline{\mathbb{Z}_\ell}$:

$$X_\varphi \cong \left(\hat{G} \times T \times \mu\right) /T.$$

And we will specify in the next subsection what the T-action on $\left(\hat{G} \times T \times \mu\right)$ is.

Proof. First, recall by [Dat22, Subsection 4.6],

$$X_\psi \cong \left(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}\right) /C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$ denotes the space of cocycles whose restriction to I_F^ℓ equals ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\overline{\psi}$. More precisely, recall (See [Dat22, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the L-parameters whose restriction to I_F^ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $(\psi_\ell, \overline{\psi})$. Hence

$$X_\varphi \cong (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \qquad g\eta(-)g^{-1} \leftarrow\!\shortmid (g, \eta),$$

[1] This is the subscheme of T on which Fr acts by raising to q-th power, see Equation 2.1.6. See also [Dat22, Example 3.14]

[2] Note that μ could be trivial, depending on $\hat{G}$ and some congruence relations between q, ℓ.

with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$ by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ and $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$.

Second, $\eta.\psi \hookleftarrow \eta$ defines an isomorphism

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \cong Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_{1_{I_F^\ell}} =: Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$$

where $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell))$ means the space of cocycles with W_F acting on $N_{\hat{G}}(\psi_\ell)$ via conjugacy action through ψ, and the subscript $1_{I_F^\ell}$ or 1 means the cocycles whose restriction to I_F^ℓ is trivial. Indeed, this is clear by unraveling the definitions: two cocycles whose restriction to I_F^ℓ are both ψ_ℓ differ by something whose restriction to I_F^ℓ is trivial; two cocycles whose pushforward to $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ are both $\overline{\psi}$ differ by something whose pushforward to $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is trivial, i.e., which factors through the identity component $N_{\hat{G}}(\psi_\ell)^0$.

Next, we show that $C_{\hat{G}}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$. By [Dat22, Subsection 3.1], the centralizer $C_{\hat{G}}(\psi_\ell)$ is generalized reductive (See Lemma 2.1.12), hence split over $\overline{\mathbb{Z}_\ell}$, and $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0$. So we can determine $C_{\hat{G}}(\psi_\ell)$ by computing its $\overline{\mathbb{F}_\ell}$-points. Indeed,

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)),$$

where the last equality follows since I_F/I_F^ℓ does not contribute to the image of φ (See Lemma 2.1.14). Therefore, $C_{\hat{G}}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$. Denote $T = C_{\hat{G}}(\psi_\ell)$. In particular, $C_{\hat{G}}(\psi_\ell)$ is connected, hence

$$N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell) = T. \tag{2.1.7}$$

Now we could compute

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0) = Z^1_{Ad(\psi)}(W_F, T) \cong T \times T^{\mathrm{Fr}=(-)^q},$$

where the last isomorphism is given by $\eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$, where $s_0 \in W_t^0$ is the topological generator of I_t fixed before (See [Dat22, Example 3.14]).

Then we show that the identity component of $T^{\mathrm{Fr}=(-)^q}$ gives μ in the statement of the theorem. Note that $T^{\mathrm{Fr}=(-)^q}$ is a diagonalizable group scheme over $\overline{\mathbb{Z}_\ell}$ of dimension zero (This can be seen either by $\dim Z^1(W_F/P_F, T) = \dim T$, or by noticing that $\eta(s_0) \in T^{\mathrm{Fr}=(-)^q}$ is semisimple with finitely many possible eigenvalues), hence of the form $\prod_i \mu_{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$. Hence its identity component $(T^{\mathrm{Fr}=(-)^q})^0$ over $\overline{\mathbb{Z}_\ell}$ is of the form $\prod_i \mu_{\ell^{k_i}}$, with k_i maximal such that ℓ^{k_i} divides n_i. Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong (T \times T^{\mathrm{Fr}=(-)^q})^0 \cong T \times (T^{\mathrm{Fr}=(-)^q})^0 \cong T \times \mu,$$

(See Lemma 2.1.15 for the first isomorphism) where μ is of the form $\prod_i \mu_{\ell^{k_i}}$.

Finally, we show that $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$. Recall that $C_{\hat{G}}(\psi_\ell)$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))$ by conjugation, inducing an action of $C_{\hat{G}}(\psi_\ell)$ on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. And $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is by definition the stabilizer of $\overline{\psi} \in Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ in $C_{\hat{G}}(\psi_\ell)$. Now $C_{\hat{G}}(\psi_\ell) = T$

is connected, hence acts trivially on the component group $\pi_0(N_{\hat{G}}(\psi_\ell))$, hence also acts trivially on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Therefore, the stabilizer $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$.

Above all, we have

$$X_\varphi \cong (\hat{G} \times Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \cong (\hat{G} \times T \times \mu)/T. \tag{2.1.8}$$

$\square$

Example 2.1.8. *Let $p = q = 11, \ell = 5, G = GL_2$.*[3] *Let F_2 be the unique degree 2 unramified extension of F. Then the Weil group of F_2 is $W_{F_2} \cong I_F \rtimes \langle \mathrm{Fr}^2 \rangle$.*

We define a tame character $\eta : W_{F_2}/P_F \to \overline{\mathbb{F}_\ell}^*$ as follows. Let

$$\eta|_{I_F} : I_F/P_F \cong \prod_{p' \neq 11} \mathbb{Z}_{p'} \to \mathbb{Z}_3 \to \mathbb{Z}/3\mathbb{Z} \to \overline{\mathbb{F}_5}^*$$

be the composition, where the last map is a non-trivial character $\chi : \mathbb{Z}/3\mathbb{Z} \to \overline{\mathbb{F}_5}^*$. Let $\eta(\mathrm{Fr}^2) := 1$.

Let $\varphi := \mathrm{Ind}_{W_{F_2}}^{W_F} \eta$. $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ is an irreducible tame L-parameter, hence a TRSELP of $G = GL_2$.

To compute the connected component of $Z^1(W_F, \hat{G})$ containing φ over $\overline{\mathbb{Z}_\ell}$, let us choose a lift ψ of φ, as follows. First, let us define a lift $\tilde{\eta} : W_{F_2}/P_F \to \overline{\mathbb{Z}_\ell}^*$ of η, as follows. Let

$$\tilde{\eta}|_{I_F} : I_F/P_F \cong \prod_{p' \neq 11} \mathbb{Z}_{p'} \to \mathbb{Z}_3 \to \mathbb{Z}/3\mathbb{Z} \to \overline{\mathbb{Z}_5}^*$$

be the composition, where the last map is a non-trivial character $\tilde{\chi} : \mathbb{Z}/3\mathbb{Z} \to \overline{\mathbb{Z}_5}^*$ lifting χ. Let $\tilde{\eta}(\mathrm{Fr}^2) := 1$. Next, define $\psi := \mathrm{Ind}_{W_{F_2}}^{W_F} \tilde{\eta}$.

Under a nice basis, we could express $\psi : W_F \to GL_2(\overline{\mathbb{Z}_\ell})$ in terms of matrices, as follows:

$$\psi(s_0) = \begin{pmatrix} \tilde{\chi}(1) & 0 \\ 0 & \tilde{\chi}^q(1) \end{pmatrix} = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix} \qquad \psi(\mathrm{Fr}) = \begin{pmatrix} 0 & 1 \\ \tilde{\eta}(\mathrm{Fr}^2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ζ_3 is a primitive 3-rd root of unity of $\overline{\mathbb{Z}_\ell}$.

Recall we have that

$$X_\varphi \cong (\hat{G} \times Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}.$$

We see that

$$\psi(I_F^\ell) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}, \begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3 \end{pmatrix}\}.$$

In this case, $T = C_{\hat{G}}(\psi_\ell)$ is the diagonal torus of GL_2, $N_{\hat{G}}(\psi_\ell)$ is the normalizer of T, $N_{\hat{G}}(\psi_\ell)^0 = T$, and $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = T$ since $T = C_{\hat{G}}(\psi_\ell)$ fixes $\overline{\psi}$.

It remains to compute $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong Z^1_{Ad(\psi)}(W_F, T)_1$. We first compute $Z^1_{Ad(\psi)}(W_F, T)$ (without the subscription 1). Indeed, by Equation 2.1.6,

$$Z^1_{Ad(\psi)}(W_F, T) \cong T \times T^{\mathrm{Fr}=(-)^q},$$

[3] They are chosen such that μ turns out to be non-trivial.

where $T^{\mathrm{Fr}=(-)^q} := \{x \in T \mid \mathrm{Fr}.x = x^q\}$. *In the case of* $Z^1_{Ad(\psi)}(W_F, T)$, *the* W_F*-action on* T *is conjugation through* ψ, *so*

$$T^{\mathrm{Fr}=(-)^q} = \{x \in T \mid \psi(\mathrm{Fr})x\psi(\mathrm{Fr})^{-1} = x^q\} = \{\begin{pmatrix} t & 0 \\ 0 & t^q \end{pmatrix} \mid t^{q^2-1} = 1\} \cong \mu_{q^2-1} = \mu_{120}.$$

$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$ *turns out to be the connected component of* $Z^1_{Ad(\psi)}(W_F, T) \cong T \times T^{\mathrm{Fr}=(-)^q}$ *containing* 1. *In our case,*

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong (T \times \mu_{120})^0 \cong T \times \mu_5.$$

Above all,

$$X_\varphi \cong (\hat{G} \times T \times \mu_5)/T,$$

where T *is the diagonal torus of* $\hat{G} = GL_2$.

2.1.4 The T-action on $(\hat{G} \times T \times \mu)$

The goal of this subsection is to specify the T-action on $(\hat{G} \times T \times \mu)$. Before that, let us record a lemma on several equivalent definitions of T.

Lemma 2.1.9. $T := C_{\hat{G}}(\psi|_{I_F^\ell}) = C_{\hat{G}}(\psi|_{I_F^\ell})^0 \cong C_{\hat{G}}(\psi|_{I_F})$.

Proof. We have seen the first equality in Equation 2.1.7. To see that $C_{\hat{G}}(\psi|_{I_F^\ell}) = C_{\hat{G}}(\psi|_{I_F})$, we first note that $C_{\hat{G}}(\psi|_{I_F}) \subseteq C_{\hat{G}}(\psi|_{I_F^\ell}) =: T$ is included in a commutative group scheme. But ψ is tame, hence a character from I_F/P_F, which is abelian, so

$$\psi(I_F) \subseteq C_{\hat{G}}(\psi|_{I_F}) \subseteq T.$$

Therefore, $C_{\hat{G}}(\psi|_{I_F}) \supseteq T$ since T is commutative, and hence

$$C_{\hat{G}}(\psi|_{I_F}) = T.$$

□

Now let us make it explicit the T-action on $(\hat{G} \times T \times \mu)$.

Recall (See [Dat22, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the L-parameters ψ' such that $(\psi'_\ell, \overline{\psi'})$ is $\hat{G}$-conjugate to $(\psi_\ell, \overline{\psi})$. Hence X_φ is isomorphic to

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$$

via $g\eta(-)g^{-1} \leftarrow\!\shortmid (g, \eta)$, with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$ by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \cong T$ and $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$.

Next, recall that

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \cong Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong T \times \mu \quad \eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0)).$$

Let us focus on the isomorphism $\eta.\psi \leftarrow\!\shortmid \eta$:

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \cong Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1.$$

Recall that $T \subseteq \hat{G}$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}}$ via conjugation. Hence the above isomorphism induces an T-action on $Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$, by

$$(t, \eta) \mapsto (t(\eta\psi(-))t^{-1})\psi^{-1}.$$

Hence in $(\hat{G} \times T \times \mu)/T$, we compute by tracking the above isomorphisms that

1. T acts on $\hat{G}$ via $(t, g) \mapsto gt^{-1}$.
2. $T = C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acts on $T \subseteq (T \times \mu)$ (corresponding to $\eta(\mathrm{Fr})$) by twisted conjugacy (due to the isomorphisms $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$), i.e.,
$$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$
where $n = \psi(\mathrm{Fr})$; Note that here n, a prior lies in $\hat{G}$, actually lies in $N_{\hat{G}}(T)$ (Since $\mathrm{Fr}\,.s.\,\mathrm{Fr}^{-1} = s^q$ implies that $\psi(\mathrm{Fr})$ normalizes $C_{\hat{G}}(\psi|_{I_F^\ell}) = T$). To summarize, $t \in T$ acts on T via multiplication by $tnt^{-1}n^{-1}$.
3. T acts trivially on μ. This is because $\eta(s_0) \in T$ and $\psi(s_0) \in T$. So the conjugacy action is trivial.

On the other hand, recall we have the natural $\hat{G}$-action on $Z^1(W_F, \hat{G})$ by conjugation, hence the $\hat{G}$-action on this component X_φ. Under the isomorphism $X_\varphi \cong (\hat{G} \times T \times \mu)/T$, the $\hat{G}$-action becomes

$$(g', (g, t, m)) \mapsto (g'g, t, m), \text{ for any } g' \in \hat{G} \text{ and } (g, t, m) \in (\hat{G} \times T \times \mu)/T.$$

Note that the T-action and the $\hat{G}$-action on $(\hat{G} \times T \times \mu)$ commute with each other, we thus have the following:

Proposition 2.1.10.

$$[X_\varphi/\hat{G}] = \left[\left((\hat{G} \times T \times \mu)/T\right)/\hat{G}\right] \cong \left[\left((\hat{G} \times T \times \mu)/\hat{G}\right)/T\right] \cong [(T \times \mu)/T],$$

with $t \in T$ acting on T via multiplication by $tnt^{-1}n^{-1}$, and $t \in T$ acting trivially on μ.

2.1.5 Some lemmas

Lemma 2.1.11. *Let $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$. Then there exists $\psi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{Z}_\ell}))$ such that ψ' is a lift of φ'.*

Proof. In the statement, $Z^1(W_t, \hat{G})$ is the abbreviation for $Z^1(W_t, \hat{G})_{\overline{\mathbb{Z}_\ell}}$. Recall that $Z^1(W_t, \hat{G}) \to \overline{\mathbb{Z}_\ell}$ is flat (See [Dat22, Proposition 3.3]), hence generalizing (See [SP, Stack, Tag 01U2]). Therefore, given $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$, there exists $\xi \in Z^1(W_t, \hat{G}(\overline{\mathbb{Q}_\ell}))$ such that ξ specializes to φ'. In other words, $\ker(\xi) \subseteq \ker(\varphi')$. We are going to show that $\xi : W_t \to \hat{G}(\overline{\mathbb{Q}_\ell})$ factors through $\hat{G}(\overline{\mathbb{Z}_\ell})$.

This is true by the following more general statement: Let $Y = \mathrm{Spec}(R)$ be an affine scheme over $\overline{\mathbb{Z}_\ell}$, let $y_\eta \in Y(\overline{\mathbb{Q}_\ell})$ specializing to $y_s \in Y(\overline{\mathbb{F}_\ell})$. Then $y_\eta \in Y(\overline{\mathbb{Q}_\ell}) = \mathrm{Hom}(R, \overline{\mathbb{Q}_\ell})$ factors through $\overline{\mathbb{Z}_\ell}$.

To prove the above statement, let $\mathfrak{p} := \ker(y_\eta)$ and $\mathfrak{q} := \ker(y_s)$ be the corresponding prime ideas. Then "y_η specializes to y_s" translates to "$\mathfrak{p} \subseteq \mathfrak{q}$". Recall that we are going to show that $y_\eta : R \to \overline{\mathbb{Q}_\ell}$ factors through $\overline{\mathbb{Z}_\ell}$. We argue by contradiction. Otherwise there is some element $f \in R$ mapping to $\ell^{-m}u$ for some $m \in \mathbb{Z}_{\geq 1}$ and $u \in \overline{\mathbb{Z}_\ell}^*$. Hence

$$\ell^m u^{-1} f - 1 \in \ker(y_\eta) \subseteq \ker(y_s). \tag{2.1.9}$$

However, $\ell \in \ker(y_s)$ since y_s lives on the special fiber. This together with equation 2.1.9 implies that $1 \in \ker(y_s)$. Contradiction! □

Lemma 2.1.12. *The schematic centralizer $C_{\hat{G}}(\psi_\ell)$ is a generalized reductive group scheme over $\overline{\mathbb{Z}_\ell}$.*

Proof. We are going to invoke [Dat22, Lemma 3.2]. We first show that

$$C_{\hat{G}}(\psi_\ell) = C_{\hat{G}}(\psi(I_F^\ell)),$$

where $C_{\hat{G}}(\psi(I_F^\ell))$ is the schematic centralizer of the subgroup $\psi(I_F^\ell) \subseteq \hat{G}(\overline{\mathbb{Z}_\ell})$ in $\hat{G}$. This can be checked by Yoneda Lemma on R-valued points for any $\overline{\mathbb{Z}_\ell}$-algebra R.

Then we could conclude by [Dat22, Lemma 3.2]. Indeed, ψ_ℓ factors through some finite quotient Q of I_F^ℓ, which has order invertible in the base $\overline{\mathbb{Z}_\ell}$. So the assumptions of [Dat22, Lemma 3.2] are satisfied (For details, see Remark 2.1.13 below). □

Remark 2.1.13.

1. While [Dat22, Lemma 3.2] is phrased in the setting that R is a normal subring of a number field, it still works for $\overline{\mathbb{Z}_\ell} \subseteq \overline{\mathbb{Q}_\ell}$ instead of $\mathbb{Z} \subseteq \mathbb{Q}$. Indeed, ψ_ℓ factors through some finite quotient Q of I_F^ℓ, say of order $|Q| = N$ (Note that N is coprime to ℓ since Q is a quotient of I_F^ℓ). Then we could use [Dat22, Lemma 3.2] to conclude that $C_{\hat{G}}(\psi_\ell)$ is generalized reductive over $\mathbb{Z}[1/pN]$. Hence $C_{\hat{G}}(\psi_\ell)$ is also generalized reductive over $\overline{\mathbb{Z}_\ell}$ by base change.

2. There is also a small issue that $\overline{\mathbb{Z}_\ell}$ is not finite over $\mathbb{Z}_\ell$, but this can be resolved since everything is already defined over some sufficiently large finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$.

Lemma 2.1.14.

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)).$$

Proof. The first equation is by definition of the schematic centralizer and that $C_{\hat{G}}(\psi_\ell)$ represents the set-theoretic centralizer. See Definition 2.1.2.

For the second equation, note that $\varphi|_{I_t} = \gamma_1 + ... + \gamma_d$ is a direct sum of characters (Since $I_t \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$), so it suffices to show that each γ_i is trivial on the summand $\mathbb{Z}_\ell$ of $I_t \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$. Indeed,

$$\mathrm{Hom}_{\mathrm{Cont}}(\mathbb{Z}_\ell, \overline{\mathbb{F}_\ell}^*) = \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \varinjlim \mathrm{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \{1\}.$$

□

Lemma 2.1.15. $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong (T \times T^{\mathrm{Fr}=(-)^q})^0$.

Proof. We have omitted from the notations but here everything is over $\overline{\mathbb{Z}_\ell}$. Recall that $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = T$ and that

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0) \cong T \times T^{\mathrm{Fr}=(-)^q}.$$

By [Dat22, Section 5.4, 5.5], $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$ is connected (over $\overline{\mathbb{Z}_\ell}$). We need to check here that the assumptions of [Dat22, Section 5.4, 5.5] are satisfied. Indeed, since $N_{\hat{G}}(\psi_\ell)^0 = T$ is a connected torus, the W_t^0-action on T automatically fixes a Borel pair of T. Moreover, s_0 acts trivially on $N_{\hat{G}}(\psi_\ell)^0 = T$ via ψ, so in particular the action of s_0 (which is denoted by s in [Dat22, Section 5.5]) has order a power of ℓ (which is $1 = \ell^0$).

Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \subseteq Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \cong (T \times T^{\mathrm{Fr}=(-)^q})^0.$$

By [Dat22, Section 4.6],

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \hookrightarrow Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)$$

is open and closed. This is done by considering the restriction to the prime-to-ℓ inertia I_F^ℓ, and then use [Dat22, Theorem 4.2].

Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 = Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \cong (T \times T^{\mathrm{Fr}=(-)^q})^0.$$

□

2.2 Main Theorem: description of $[X_\varphi/\hat{G}]$

The goal of this section is to describe $[X_\varphi/\hat{G}]$ explicitly (See Theorem 2.2.3 for the precise statement).

Let F be a non-archimedean local field, G be a connected split reductive group over F. Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a tame, regular semisimple, elliptic L-parameter (TRSELP for short). Recall that this means that the centralizer

$$C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) =: S \subseteq \hat{G}(\overline{\mathbb{F}_\ell})$$

is a maximal torus, and $\varphi(\mathrm{Fr}) \in N_{\hat{G}}(S)$ gives rise to an element $w = \overline{\varphi(\mathrm{Fr})} \in N_{\hat{G}}(S)/S$ in the Weyl group (and that φ is tame and elliptic).

Assume that

1. The center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$.

2. $Z(\hat{G})$ is finite.

Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of φ. Let ψ_ℓ denote the restriction $\psi|_{I_F^\ell}$, and $\overline{\psi}$ denote the image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Recall that the schematic centralizer $C_{\hat{G}}(\psi_\ell) = T$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}}_\ell$-points $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

For later use, we record the following lemma – w can also be defined in terms of ψ instead of φ. This is helpful because we will reduce to a computation on the special fiber later. First, notice that since T is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\ell \neq 2$, we can identify

$$\left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) \cong \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell}),$$

and denote it by Ω. (See Lemma 2.2.6 below)

Remark 2.2.1. Lemma 2.2.6 below shows that $N_{\hat{G}}(T)/T$ is representable by a group scheme which is split over $\overline{\mathbb{Z}_\ell}$. Therefore, we will slightly abuse notations and use

$$\Omega, N_{\hat{G}}(T)/T, N_{\hat{G}}(S)/S$$

interchangeably.

Lemma 2.2.2. *The image of $\varphi(\mathrm{Fr})$ and $\psi(\mathrm{Fr})$ in the Weyl group Ω agree, hence giving a well defined element w in the Weyl group Ω.*

Proof. Let

$$\Omega = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell})$$

as above and $\underline{\Omega}$ be the associated constant group scheme (See Lemma 2.2.6 below). Since ψ is a lift of φ, $\psi(\mathrm{Fr})$ specializes to $\varphi(\mathrm{Fr})$ in $N_{\hat{G}}(T)$. Then the lemma follows since

$$N_{\hat{G}}(T) \to N_{\hat{G}}(T)/T = \underline{\Omega}$$

is a morphism of schemes, hence the following diagram commutes:

$$\begin{tikzcd}
N_{\hat{G}}(T)(\overline{\mathbb{Z}_\ell}) \arrow[r] \arrow[d] & N_{\hat{G}}(T)(\overline{\mathbb{F}_\ell}) \arrow[d] \\
\underline{\Omega}(\overline{\mathbb{Z}_\ell}) = \Omega \arrow[r] & \underline{\Omega}(\overline{\mathbb{F}_\ell}) = \Omega
\end{tikzcd}$$

∎

Our main theorem is the following.

Theorem 2.2.3. *Assume that the center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$, and that $Z(\hat{G})$ is finite. Let $X_\varphi(= X_\psi)$ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ (hence also containing ψ). Then we have isomorphisms of quotient stacks*

$$[X_\varphi/\hat{G}] \cong [(T \times \mu)/T] \cong [*/C_T(n)] \times \mu \cong [*/S_\psi], \tag{2.2.1}$$

where $C_T(n)$ is the schematic centralizer of $n = \psi(\mathrm{Fr})$ in $T = C_{\hat{G}}(\psi|_{I_F^\ell})$, $\mu = \prod_{i=1}^m \mu_{\ell^{k_i}}$ for some $k_i \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}_{\geq 0}$ is a product of group schemes of roots of unity, and $S_\psi := C_{\hat{G}}(\psi)$ is the schematic centralizer of ψ in $\hat{G}$.

If we moreover assume that ℓ does not divide the order of $w = \overline{\varphi(\mathrm{Fr})}$ in the Weyl group $N_{\hat{G}}(S)/S$, then

$$[X_\varphi/\hat{G}] \cong [(T \times \mu)/T] \cong [*/\underline{S_\varphi(\overline{\mathbb{F}_\ell})}] \times \mu,$$

where $S_\varphi(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$, and $\underline{S_\varphi(\overline{\mathbb{F}_\ell})}$ is the corresponding constant group scheme. By abuse of notation, we sometimes denote $S_\varphi(\overline{\mathbb{F}_\ell})$ simply by S_φ.

Proof. Recall that X_φ is isomorphic to the contracted product

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

and that $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$ induces isomorphisms

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \cong Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \cong T \times \mu.$$

This implies that $[X/\hat{G}] \cong [(T \times \mu)/T]$ with T acting on T by twisted conjugacy:

$$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where $n = \psi(\mathrm{Fr})$. In other words, T acts on T via multiplication by $tnt^{-1}n^{-1}$. And T acts trivially on μ (See Proposition 2.1.10).

So we are reduced to compute $[T/T]$ with respect to a nice action of the split torus T on T, which should be and turns out to be very explicit.

For clarification, let us denote the source torus T as $T^{(1)}$ and the target torus T as $T^{(2)}$. Consider the morphism

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \qquad s \longmapsto sns^{-1}n^{-1}.$$

This is surjective on $\overline{\mathbb{F}_\ell}$-points by our assumption 2 that $Z(\hat{G})$ is finite and φ is elliptic (See Lemma 2.2.4 below). Hence f is an epimorphism in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes (See the same Lemma 2.2.4 below). Therefore, f induces an isomorphism

$$T^{(1)}/\ker(f) \cong T^{(2)} \tag{2.2.2}$$

as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Moreover, if we let $t \in T$ act on $T^{(1)}$ by left multiplication by t, and on $T^{(2)}$ via multiplication by $(tnt^{-1}n^{-1})$, this isomorphism induced by f is T-equivariant.

Note that $T^{(1)} = T$ is commutative, so the T-action (via multiplication by $tnt^{-1}n^{-1}$) and the $\ker(f)$-action (via left multiplication) on T commute with each other. Hence by the T-equivariant isomorphism (2.2.2), we have

$$[T/T] = [T^{(2)}/T] \cong \left[\left(T^{(1)}/\ker(f)\right)/T\right] \cong \left[\left(T^{(1)}/T\right)/\ker(f)\right] \cong [*/\ker(f)] = [*/C_T(n)].$$

Moreover, recall we have $T := C_{\hat{G}}(\psi|_{I_F^\ell}) \cong C_{\hat{G}}(\psi|_{I_F})$ (See Lemma 2.1.9). So

$$C_T(n) \cong C_{\hat{G}}(\psi(I_F), \psi(\mathrm{Fr})) \cong C_{\hat{G}}(\psi) =: S_\psi.$$

For the second part of the theorem, see Lemma 2.2.5 below.

$\square$

Lemma 2.2.4. *The morphism*

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \qquad s \longmapsto sns^{-1}n^{-1}$$

is epimorphic in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. And it induces an isomorphism $T^{(1)}/\ker(f) \cong T^{(2)}$ as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes.

Proof. Recall that T is a split torus over $\overline{\mathbb{Z}_\ell}$, hence a diagonalizable $\overline{\mathbb{Z}_\ell}$-group scheme. Notice that f is a morphism of $\overline{\mathbb{Z}_\ell}$-group schemes, hence a morphism of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Recall that the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes is equivalent to the category of abelian groups (See [BCO14, p70, Section 5] or [Con14]) via

$$D \mapsto \mathrm{Hom}_{\overline{\mathbb{Z}_\ell}-GrpSch}(D, \mathbb{G}_m),$$

and the inverse is given by

$$\overline{\mathbb{Z}_\ell}[M] \leftarrow\!\shortmid M,$$

where $\overline{\mathbb{Z}_\ell}[M]$ is the group algebra of M with $\overline{\mathbb{Z}_\ell}$-coefficients.

Therefore, we could argue in the category of abelian groups via the above category equivalence: f is epimorphic if and only if the map f^* in the category of abelian groups is monomorphic. Note ellipticity and $Z(\hat{G})$ finite imply that S_φ is finite, hence

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_T(n)(\overline{\mathbb{F}_\ell}) = S_\varphi(\overline{\mathbb{F}_\ell})$$

is finite (where the first equality is by definition of f, and the second equality is because $T(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ and $n = \psi(\mathrm{Fr})$ maps to $\varphi(\mathrm{Fr}) \in \hat{G}(\overline{\mathbb{F}_\ell})$ by Lemma 2.2.2), hence $\mathrm{coker}(f^*)$ is finite. Therefore,

$$f^* : \mathrm{Hom}(T^{(2)}, \mathbb{G}_m) \to \mathrm{Hom}(T^{(1)}, \mathbb{G}_m)$$

is injective (i.e., monomorphism). Indeed, otherwise $\ker(f^*)$ would be a nonzero sub-$\mathbb{Z}$-module of the finite free $\mathbb{Z}$-module $\mathrm{Hom}(T^{(2)}, \mathbb{G}_m)$, hence a free $\mathbb{Z}$-module of positive rank, which contradicts with $\mathrm{coker}(f^*)$ being finite.

The statement on the quotient follows from the corresponding result in the category of abelian groups: f^* induces an isomorphism

$$\mathrm{Hom}(T^{(1)}, \mathbb{G}_m)/\mathrm{Hom}(T^{(2)}, \mathbb{G}_m) \cong \mathrm{coker}(f^*)$$

(See [BCO14, p71, Subsection 5.3]). ☐

Lemma 2.2.5. *Assume that ℓ does not divide the order of w. Then $\ker(f) \cong \underline{S_\varphi(\overline{\mathbb{F}_\ell})}$ is the constant group scheme of the finite abelian group $S_\varphi(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$.*

Proof. We recall the following fact: Let H be a smooth affine group scheme over some ring R, let Γ be a finite group whose order is invertible in R. Then the fixed point functor H^Γ is representable and is smooth over R.

For a proof of the above fact, see [Edi92, Proposition 3.4] or [DHKM20, Lemma A.1, A.13].

In our case, let $H = T$, $\Gamma = \langle w \rangle$ be the subgroup of the Weyl group $W_{\hat{G}}(T)$ generated by w. Hence

$$\ker(f) = C_T(n) = H^\Gamma$$

is smooth over $\overline{\mathbb{Z}_\ell}$. Therefore, $\ker(f)$ is finite étale over $\overline{\mathbb{Z}_\ell}$ (Because it is smooth of relative dimension 0 over $\overline{\mathbb{Z}_\ell}$, which can be checked on $\overline{\mathbb{F}_\ell}$-points). Hence $\ker(f)$ is a constant group scheme over $\overline{\mathbb{Z}_\ell}$, since $\overline{\mathbb{Z}_\ell}$ has no non-trivial finite étale cover.

Since $\ker(f)$ is constant, we can determine it by computing its $\overline{\mathbb{F}_\ell}$-points:

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_{T(\overline{\mathbb{F}_\ell})}(n) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F)), \tag{2.2.3}$$

where the middle equality follows by noticing $T(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ and $n = \psi(\mathrm{Fr})$ maps to $\varphi(\mathrm{Fr}) \in \hat{G}(\overline{\mathbb{F}_\ell})$ by Lemma 2.2.2.

Finally, note by our TRSELP assumption, $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ is (the $\overline{\mathbb{F}_\ell}$-points of) a torus. Hence $S_\varphi(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is abelian, hence finite abelian as we have noticed in the proof of the previous lemma that $S_\varphi(\overline{\mathbb{F}_\ell})$ is finite (by ellipticity and $Z(\hat{G})$ finite). □

Lemma 2.2.6. *Let $\hat{G}$ be a connected reductive group over $\overline{\mathbb{Z}_\ell}$, and T a maximal torus of $\hat{G}$. Then the Weyl group $N_{\hat{G}}(T)/T$ is split over $\overline{\mathbb{Z}_\ell}$.*

Proof. By [Con14, Proposition 3.2.8], the Weyl group $N_{\hat{G}}(T)/C_{\hat{G}}(T)$ is finite étale over $\overline{\mathbb{Z}_\ell}$ and hence split over $\overline{\mathbb{Z}_\ell}$. In our case, $C_{\hat{G}}(T) = T$ since $\hat{G}$ is connected (For example, use the third paragraph of the proof of [Con14, Proposition 3.1.12]). □

Chapter 3

Depth-zero regular supercuspidal blocks

The goal of this chapter is to describe the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$ (denoted $\mathcal{C}_{x,1}$ later) of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$ containing a depth-zero regular supercuspidal representation π.

Recall that a depth-zero regular supercuspidal representation π is of the form

$$\pi = \text{c-Ind}_{G_x}^{G(F)} \rho,$$

where ρ is a representation of G_x whose reduction $\overline{\rho}$ to the finite reductive group $\overline{G_x} = G_x/G_x^+$ is supercuspidal.

In the end, assuming that G is simply connected, the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$ would be equivalent to the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\overline{G_x})_{[\overline{\rho}]}$ (denoted $\mathcal{A}_{x,1}$ later) of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\overline{G_x})$ containing $\overline{\rho}$. And $\mathcal{A}_{x,1}$ has an explicit description via the Broué equivalence 3.2.4.

Indeed, let $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G_x)_{[\rho]}$ (denoted $\mathcal{B}_{x,1}$ later) be the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G_x)$ containing ρ. It is not hard to see that the inflation along $G_x \to \overline{G_x}$ induces an equivalence of categories $\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}$. The main theorem we prove in this chapter is that the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}.$$

The proof of this main theorem 3.1.2 would occupy most of this chapter, from Section 3.1 to 3.4. The proof relies on three theorems. In Section 3.1, we prove the main theorem modulo the three theorems. And the proofs of the three theorems are given in Sections 3.2, 3.3, 3.4, respectively.

3.1 The compact induction induces an equivalence

In this section, we prove the Main Theorem 3.1.2 modulo Theorem 3.1.3 3.1.4 3.1.5.

Let G be a split reductive group scheme over $\mathbb{Z}$, which is simply connected. Let F be a non-archimedean local field, with ring of integers $\mathcal{O}_F$ and residue field $k_F \cong \mathbb{F}_q$ of characteristic p. For simplicity, we assume that q is greater than the Coxeter number of G (See Theorem 3.2.4 for reason).

Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(G,F)$. Let G_x be the parahoric subgroup associated to x, and G_x^+ be its pro-unipotent radical. Recall that $\overline{G_x} := G_x/G_x^+$ is a generalized Levi subgroup of $G(k_F)$ with root system Φ_x, see [Rab03, Theorem 3.17].

Let $\Lambda = \overline{\mathbb{Z}_\ell}$, with $\ell \neq p$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is regular supercuspidal. Here **regular supercuspidal** (See Definition 3.2.7 for precise definition.) means ρ is supercuspidal and lies in a **regular block** of $\mathrm{Rep}_\Lambda(\overline{G_x})$, in the sense of [Bro90]. The reason we want the regularity assumption is that we want to work with a block of $\mathrm{Rep}_\Lambda(\overline{G_x})$ which consists purely of supercuspidal representations. See Section 3.2 for details. We make this a definition for later use.

Definition 3.1.1. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$. We say ρ **has supercuspidal reduction** (resp. **has regular supercuspidal reduction**), if ρ is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is supercuspidal (resp. regular supercuspidal). Let's denote the reduction of ρ modulo G_x^+ by $\overline{\rho} \in \mathrm{Rep}_\Lambda(\overline{G_x})$.*

Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Let $\mathcal{C}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G(F))$ containing $\pi := \text{c-Ind}_{G_x}^{G(F)} \rho$. Now we can state the main theorem of this chapter.

Theorem 3.1.2 (Main Theorem)**.** *Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(G,F)$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be a representation which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Let $\mathcal{C}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G(F))$ containing $\pi := \text{c-Ind}_{G_x}^{G(F)} \rho$. Then the compact induction $\text{c-Ind}_{G_x}^{G(F)}$ induces an equivalence of categories $\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}$.*

As mentioned before, the reason we want the regular supercuspidal assumption is the following theorem.

Theorem 3.1.3. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Then any $\rho' \in \mathcal{B}_{x,1}$ has supercuspidal reduction.*

The proof of the Main Theorem 3.1.2 basically splits into two parts – fully faithfulness and essentially surjectivity. It is convenient to have the following theorem available at an early stage, which implies fully faithfulness immediately and is also used in the proof of essentially surjectivity.

Theorem 3.1.4. *Let x, y be two vertices of the Bruhat-Tits building $\mathcal{B}(G,F)$. Let ρ_1 be a representation of the parahoric G_x which is trivial on the pro-unipotent radical G_x^+. Let ρ_2 be a representation of G_y which is trivial on G_y^+. Assume that one of them has supercuspidal reduction. Then exactly one of the following happens:*

1. *If there exists an element $g \in G(F)$ such that $g.x = y$, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_y}^{G(F)} \rho_2) = \mathrm{Hom}_{G_x}(\rho_1, {}^g\rho_2).$$

2. *If there is no elements $g \in G(F)$ such that $g.x = y$, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_y}^{G(F)} \rho_2) = 0.$$

The proof of the above theorem is basically a computation using Mackey's formula. See Section 3.3.

Proof of Theorem 3.1.2. Now we proceed by steps towards our goal: The compact induction $\text{c-Ind}_{G_x}^{G(F)}$ induces an equivalence of categories $\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}$.

First, we show that $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is well-defined. We need to show that the image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$ lies in $\mathcal{C}_{x,1}$. By Theorem 3.1.3 and Theorem 3.1.4 above,

$$\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \text{Rep}_\Lambda(G(F))$$

is fully faithful (See Lemma 3.1.6, note that here we used Theorem 3.1.3 that any representation in $\mathcal{B}_{x,1}$ has supercuspidal reduction, so that we can apply Theorem 3.1.4), hence an equivalence onto the essential image. Since $\mathcal{B}_{x,1}$ is indecomposable as an abelian category, so is its essential image (See Lemma 3.1.7), hence its essential image is contained in a single block of $\text{Rep}_\Lambda(G(F))$. But such a block must be $\mathcal{C}_{x,1}$ since $\text{c-Ind}_{G_x}^{G(F)}$ maps ρ to $\pi \in \mathcal{C}_{x,1}$. Therefore, $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is well-defined.

Second, we show that $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is fully faithful. This is already noticed in the proof of "well-defined" in the last paragraph. Indeed,

$$\text{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_x}^{G(F)} \rho_2) = \text{Hom}_{G_x}(\rho_1, \rho_2)$$

by Theorem 3.1.3 and Theorem 3.1.4 (See Lemma 3.1.6.). Therefore, $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is fully faithful.

Finally, we show that $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective. This will occupy the rest of this section.

The idea is to find a projective generator of $\mathcal{C}_{x,1}$ and show that it is in the essential image. Fix a vertex x of the Bruhat-Tits building $\mathcal{B}(G, F)$ as before. Let V be the set of equivalence classes of vertices of the Bruhat-Tits building $\mathcal{B}(G, F)$ up to $G(F)$-action. For $y \in V$, let $\sigma_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$. Let $\Pi := \bigoplus_{y \in V} \Pi_y$ where $\Pi_y := \text{c-Ind}_{G_y^+}^{G(F)} \Lambda$. Then Π is a projective generator of the category of depth-zero representations $\text{Rep}_\Lambda(G(F))_0$, see [Dat09, Appendix]. Let $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xhookrightarrow{summand} \text{Rep}_\Lambda(G_x)$ be the $\mathcal{B}_{x,1}$-summand of σ_x. And let $\Pi_{x,1} := \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$. Note $\Pi_{x,1}$ is a summand of $\Pi_x = \text{c-Ind}_{G_x}^{G(F)} \sigma_x$, hence a summand of Π. Using Theorem 3.1.4, one can show that the rest of the summands of Π don't interfere with $\Pi_{x,1}$ (See Lemma 3.4.2 and Lemma 3.4.3 for precise meaning), hence $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$. Let us state it as a Theorem, see Section 3.4 for details.

Theorem 3.1.5. $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ *is a projective generator of* $\mathcal{C}_{x,1}$*.*

Now we've found a projective generator $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ of $\mathcal{C}_{x,1}$, and it is clear that $\Pi_{x,1}$ is in the essential image of $\text{c-Ind}_{G_x}^{G(F)}$. We now deduce from this that $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective. Indeed, for any $\pi' \in \mathcal{C}_{x,1}$, we can resolve π' by some copies of $\Pi_{x,1}$:

$$\Pi_{x,1}^{\oplus I} \xrightarrow{f} \Pi_{x,1}^{\oplus J} \to \pi' \to 0.$$

Using Theorem 3.1.4 and $\text{c-Ind}_{G_x}^{G(F)}$ commutes with arbitrary direct sums (See Lemma 3.1.8) we see that $f \in \mathrm{Hom}_G(\Pi_{x,1}^{\oplus I}, \Pi_{x,1}^{\oplus J})$ comes from a morphism $g \in \mathrm{Hom}_{G_x}(\sigma_{x,1}^{\oplus I}, \sigma_{x,1}^{\oplus J})$. Using $\text{c-Ind}_{G_x}^{G(F)}$ is exact we see that π' is the image of $\mathrm{coker}(g) \in \mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$. Therefore, $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective.

□

Lemma 3.1.6. $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ *is fully faithful.*

Proof. Let $\rho_1, \rho_2 \in \mathcal{B}_{x,1}$. By the regular supercuspidal assumption and Theorem 3.1.3, ρ_1, ρ_2 has supercuspidal reduction. Hence the assumption of Theorem 3.1.4 is satisfied and we compute using the first case of Theorem 3.1.4 that

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_x}^{G(F)} \rho_2) \cong \mathrm{Hom}_{G_x}(\rho_1, \rho_2).$$

In other words, $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ is fully faithful. □

Lemma 3.1.7. *The image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$ is indecomposable as an abelian category.*

Proof. The point is that $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ is not only fully faithful, i.e., an equivalence of categories onto the essential image, but also an equivalence of **abelian** categories onto the essential image. Indeed, it suffices to show that $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ preserves kernels, cokernels, and finite (bi-)products. But this follows from the next Lemma 3.1.8.

Assume otherwise that the essential image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$ is decomposable, then so is $\mathcal{B}_{x,1}$. But $\mathcal{B}_{x,1}$ is a block, hence indecomposable, contradiction! □

Lemma 3.1.8. $\text{c-Ind}_{G_x}^{G(F)}$ *is exact and commutes with arbitrary direct sums.*

Proof. For the statement that $\text{c-Ind}_{G_x}^{G(F)}$ is exact, we refer to [Vig96, I.5.10].

We show that $\text{c-Ind}_{G_x}^{G(F)}$ commutes with arbitrary direct sums. Indeed, $\text{c-Ind}_{G_x}^{G(F)}$ is a left adjoint (See [Vig96, I.5.7]), hence commutes with arbitrary colimits. In particular, it commutes with arbitrary direct sums. □

3.2 Regular supercuspidal blocks for finite groups of Lie type

In this section, we prove Theorem 3.1.3. As mentioned before, we made the **regular** assumption in order that the conclusion of Theorem 3.1.3 – all representations in such a block have supercuspidal reduction – is true. So the readers are welcome to skip this section for a first reading and pretend that we begin with a block in which all representations have supercuspidal reduction.

The main body of this section is to define regular supercuspidal blocks with $\Lambda = \overline{\mathbb{Z}_\ell}$-coefficients of a finite group of Lie type, and to show that a regular supercuspidal block consists purely of supercuspidal representations.

Let $\Lambda := \overline{\mathbb{Z}_\ell}$ be the coefficients of representations. Fix a prime number p. Let ℓ be a prime number different from p. Let q be a power of p.

Definition 3.2.1 ([Vig96, I.4.1])**.** *Let Λ' be any ring.*

1. *Let H be a profinite group, a* ***representation of*** H ***with*** Λ'***-coefficients*** (π, V) *is a Λ'-module V, together with a H-action $\pi : H \to GL_{\Lambda'}(V)$.*

2. *A representation of H with Λ'-coefficients is called* ***smooth*** *if for any $v \in V$, the stabilizer $Stab_H(v) \subseteq H$ is open.*

From now on, all representations are assumed to be smooth. The category of smooth representations of H with Λ'-coefficients is denoted by $\mathrm{Rep}_{\Lambda'}(H)$.

3.2.1 Regular blocks

The following notations are used in this subsection only. Let $\mathcal{G}$ be a split reductive group scheme over $\mathbb{Z}$. Let $\mathbb{G} := \mathcal{G}(\overline{\mathbb{F}_q})$, $G := \mathbb{G}^F = \mathcal{G}(\mathbb{F}_q)$, where F is the Frobenius. By abuse of notation, we sometimes identify the group scheme $\mathcal{G}_{\overline{\mathbb{F}_q}}$ with its $\overline{\mathbb{F}_q}$-points $\mathbb{G}$. Let $\mathbb{G}^*$ be the dual group (over $\overline{\mathbb{F}_q}$) of $\mathbb{G}$, and F^* the dual Frobenius (See [Car85, Section 4.2]). Fix an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$.

The definition of regular supercuspidal blocks and regular supercuspidal representations of a finite group of Lie type Γ involves modular Deligne-Lusztig theory and block theory. We refer to [DL76], [Car85], and [DM20] for Deligne-Lusztig theory, [BM89] and [Bro90] for modular Deligne-Lusztig theory, and [Bon11, Appendix B] for generalities on blocks.

First, let us recall a result in Deligne-Lusztig theory (See [DM20, Proposition 11.1.5]).

Proposition 3.2.2. *The set of $\mathbb{G}^F$-conjugacy classes of pairs $(\mathbb{T}, \theta)$, where $\mathbb{T}$ is a F-stable maximal torus of $\mathbb{G}$ and $\theta \in \widehat{\mathbb{T}^F}$, is in non-canonical bijection to the set of ${\mathbb{G}^*}^{F^*}$-conjugacy classes of pairs $(\mathbb{T}^*, s)$, where s is a semisimple element of $\mathbb{G}^*$ and $\mathbb{T}^*$ is a F^*-stable maximal torus of $\mathbb{G}^*$ such that $s \in {\mathbb{T}^*}^{F^*}$. Moreover, we could and will fix a compatible system of isomorphisms $\mathbb{F}_{q^n}^* \cong \mathbb{Z}/(q^n-1)\mathbb{Z}$ to pin down this bijection.*

Now let s be a **strongly regular semisimple** element of $G^* = {\mathbb{G}^*}^{F^*}$ (note that we require s to be fixed by F^* here), i.e., the centralizer $C_{\mathbb{G}^*}(s)$ is a F^*-stable maximal torus, denoted $\mathbb{T}^*$. Let $\mathbb{T}$ be the dual torus of $\mathbb{T}^*$. Let $T = \mathbb{T}^F$ and $T^* = {\mathbb{T}^*}^{F^*}$. Let T_ℓ denote the ℓ-part of T.

Recall for s strongly regular semisimple, the (rational) Lusztig series $\mathcal{E}(G, (s))$ consists of only one element, namely, $\pm R_T^G(\hat{s})$, where $\hat{s} = \theta$ is such that $(\mathbb{T}, \theta)$ corresponds to $(\mathbb{T}^*, s)$ via the previous bijection in Proposition 3.2.2. Here and after the sign $\pm$ is taken such that it is an honest representation (See [Car85, Section 7.5]).

From now on, we assume moreover that $s \in {\mathbb{G}^*}^{F^*}$ **has order prime to** ℓ**.** In other words, we assume that $s \in G^* = {\mathbb{G}^*}^{F^*}$ is a **strongly regular semisimple** ℓ'**-element**. We are going to define regular blocks, we refer to [Bon11, Appendix B] for generalities on blocks.

Define the ℓ**-Lusztig series**

$$\mathcal{E}_\ell(G, (s)) := \{\pm R_T^G(\hat{s}\eta) \mid \eta \in \widehat{T_\ell}\}.$$

Note the notation $\mathcal{E}_\ell(T, (s))$ also makes sense by putting $G = T$.

By [BM89], $\mathcal{E}_\ell(G, (s))$ is a union of ℓ-blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(G)$. Such a block (or more precisely, a union of blocks) is called a **(ℓ-)regular block**. Let $e_s^G \in \overline{\mathbb{Z}_\ell}G$ denotes the corresponding central idempotent. Note e_s^T also makes sense by putting $G = T$. We shall see later that a regular block is indeed a block, i.e., indecomposible. (This follows from, for example, Broué's equivalence. See Theorem 3.2.4 below.)

Definition 3.2.3 (Regular blocks)**.** *Let $s \in G^* = \mathbb{G}^{*F^*}$ be a strongly regular semisimple ℓ'-element. We call the block $\overline{\mathbb{Z}_\ell}Ge_s^G$ of the group algebra $\overline{\mathbb{Z}_\ell}G$ corresponding to the central idempotent e_s^G the* ***regular block*** *associated to s. Let $\mathcal{A}_s := \overline{\mathbb{Z}_\ell}Ge_s^G$-Mod be the corresponding category of modules, this is also referred to as a regular block, by abuse of notation.*

Thanks to [Bro90], we understand the category $\mathcal{A}_s = \overline{\mathbb{Z}_\ell}Ge_s^G$-Mod quite well. Roughly speaking, it is equivalent to the category of representations of a torus, via Deligne-Lusztig induction. This is what we are going to explain now.

Let $\mathbb{B} \subseteq \mathbb{G}$ be a Borel subgroup containing our torus $\mathbb{T}$, let $\mathbb{U}$ be the unipotent radical of $\mathbb{B}$. Let $X_{\mathbb{U}}$ be the Deligne-Lusztig variety defined by

$$X_{\mathbb{U}} := \{g \in \mathbb{G} \mid g^{-1}F(g) \in \mathbb{U}\}.$$

The main result of [Bro90] is the following: The Deligne-Lusztig induction

$$\pm R_T^G : \overline{\mathbb{Z}_\ell}T\text{-Mod} \to \overline{\mathbb{Z}_\ell}G\text{-Mod}$$

induces an equivalence of categories between the blocks $\overline{\mathbb{Z}_\ell}Te_s^T$-Mod and $\overline{\mathbb{Z}_\ell}Ge_s^G$-Mod. In particular, one could deduce that the irreducible objects in $\overline{\mathbb{F}_\ell}Ge_s^G$-Mod lifts to $\overline{\mathbb{Z}_\ell}$. More precisely, let us state it as the following theorem.

Theorem 3.2.4 (Broué's equivalence, [Bro90, Theorem 3.3])**.** *With the previous assumptions and notations, assume that $X_{\mathbb{U}}$ is affine of dimension d (which is the case if q is greater than the Coxeter number of $\mathbb{G}$.). Then the cohomology complex $R\Gamma_c(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell}) = R\Gamma_c(X_{\mathbb{U}}, \mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\overline{\mathbb{Z}_\ell}$ is concentrated in degree $d = dimX_{\mathbb{U}}$. And the $(\overline{\mathbb{Z}_\ell}Ge_s^G, \overline{\mathbb{Z}_\ell}Te_s^T)$-bimodule $e_s^GH_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell})e_s^T$ induces an equivalence of categories*

$$e_s^GH_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell})e_s^T \otimes_{\overline{\mathbb{Z}_\ell}Te_s^T} - : \overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod} \to \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}.$$

From now on, we assume that the above theorem holds for all finite groups of Lie type we encountered in this paper. we hope this is not a severe restriction. This is the case at least when q is greater than the Coxeter number of $\mathbb{G}$.

Note also that the category $\overline{\mathbb{Z}_\ell}Te_s^T$-Mod is equivalent to the category $\overline{\mathbb{Z}_\ell}T_\ell$-Mod, where T_ℓ is the order-ℓ-part of T, this is essentially the category of representations of some product of $\mathbb{Z}/\ell^{k_i}\mathbb{Z}$. In particular, it has a unique irreducible representation (simple object), which is already defined over $\overline{\mathbb{F}_\ell}$. Let us denote its corresponding character by $\theta_s : T \to \overline{\mathbb{F}_\ell}^*$. Accordingly, $\overline{\mathbb{Z}_\ell}Ge_s^G$-Mod has a unique simple object $\pm R_T^G(\theta_s)$.

3.2.2 Regular supercuspidal blocks

Let us first recall the definition of supercuspidal representations.

Definition 3.2.5. *1. An irreducible representation is called* ***supercuspidal*** *if it does not occur as a subquotient of any proper parabolic induction.*

2. *A representation is called **supercuspidal** if all its irreducible subquotients are supercuspidal.*

Now let us define regular supercuspidal blocks and regular supercuspidal representations.

Definition 3.2.6. *By a **regular supercuspidal block**, we mean a regular block $\mathcal{A}_s$ whose unique simple object $\pm R_T^G(\theta_s)$ (See the explanations after Theorem 3.2.4 for definition) is supercuspidal.*

Definition 3.2.7.

1. *An irreducible representation is called **regular supercuspidal** if it lies in a regular supercuspidal block.*

2. *A representation is called **regular supercuspidal** if all its irreducible subquotients are regular supercuspidal.*

It is clear from the definitions that we have the following proposition.

Proposition 3.2.8. *Let $\mathcal{A}_s$ be a regular supercuspidal block. Then any representation in this block is supercuspidal.*

Proof. By definition of supercuspidality, it suffices to check that any irreducible representation in this block is supercuspidal. But as we noted before in the explanations after Theorem 3.2.4, $\mathcal{A}_s$ has only one irreducible representation – $\pm R_T^G(\theta_s)$, which we assumed to be supercuspidal in the definition of regular supercuspidal block. So we win! $\square$

3.2.3 Proof of Theorem 3.1.3 on supercuspidal reduction

We now apply the previous results on finite groups of Lie type to representations of the parahoric subgroups of a p-adic group. For this, we show that the inflation induces an equivalence of categories between (certain summand of) the category of representations of a finite reductive group and the corresponding parahoric subgroup (See Subsection 3.2.4).

Let us get back to the notation at the beginning of this chapter.

Let G be a split reductive group scheme over $\mathbb{Z}$, which is simply connected. Let F be a non-archimedean local field, with ring of integers $\mathcal{O}_F$ and residue field $k_F \cong \mathbb{F}_q$ of residue characteristic p. Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(G,F)$, G_x the parahoric subgroup associated to x, G_x^+ its pro-unipotent radical. Recall that $\overline{G_x} := G_x/G_x^+$ is a generalized Levi subgroup of $G(k_F)$ with root system Φ_x, see [Rab03, Theorem 3.17].

Let $\Lambda = \overline{\mathbb{Z}_\ell}$, with $\ell \neq p$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is regular supercuspidal.

In other words, we start with an irreducible representation $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the ($\overline{\mathbb{Z}_\ell}$-)block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. We can now prove Theorem 3.1.3, which we restate as follows.

Theorem 3.2.9. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the $\overline{\mathbb{Z}_\ell}$-block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Then any $\rho' \in \mathcal{B}_{x,1}$ has supercuspidal reduction.*

Proof. Let $\overline{\rho} \in \mathrm{Rep}_\Lambda(\overline{G_x})$ be the reduction of ρ modulo G_x^+. $\overline{\rho}$ is irreducible (since ρ is) and regular supercuspidal by assumption, so it is of the form $\pm R_T^G(\theta_s)$, for some strongly regular semisimple ℓ'-element s of the finite dual group $\overline{G_x}^*$ (See Definition 3.2.7).

Let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consists of representations of G_x that are trivial on G_x^+. The key observation is that $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand (as abelian category) of $\mathrm{Rep}_\Lambda(G_x)$ (See Lemma 3.2.10).

Then since $\rho \in \mathrm{Rep}_\Lambda(G_x)_0$, its block $\mathcal{B}_{x,1}$ is a summand of $\mathrm{Rep}_\Lambda(G_x)_0$.

On the other hand, notice that the inflation induces an equivalence of categories between $\mathrm{Rep}_\Lambda(\overline{G_x})$ and $\mathrm{Rep}_\Lambda(G_x)_0$, with inverse the reduction modulo G_x^+. So the blocks of $\mathrm{Rep}_\Lambda(\overline{G_x})$ and $\mathrm{Rep}_\Lambda(G_x)_0$ are in one-one correspondence. Let $\mathcal{A}_{x,1}$ be the corresponding block of $\mathrm{Rep}_\Lambda(\overline{G_x})$ to $\mathcal{B}_{x,1}$. Then $\mathcal{A}_{x,1}$ is the regular supercuspidal block $\mathcal{A}_s$ corresponding to s (recall $\overline{\rho} = \pm R_T^G(\theta_s)$). By Theorem 3.2.8, $\mathcal{A}_s$ consists purely of supercuspidal representation. Therefore, $\mathcal{B}_{x,1}$ consists purely of representations that have supercuspidal reductions. □

3.2.4 Inflation induces an equivalence

Lemma 3.2.10. *Let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consists of representations of G_x that are trivial on G_x^+. Then $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand as abelian category of $\mathrm{Rep}_\Lambda(G_x)$.*

Remark 3.2.11. A similar proof as [Dat09, Appendix] should work. Nevertheless, I include here an alternative computational proof.

Proof. Note G_x^+ is pro-p (See [Vig96, II.5.2.(b)]), in particular, it has pro-order invertible in Λ. So we have a normalized Haar measure μ on G_x such that $\mu(G_x^+) = 1$ (See [Vig96, I.2.4]). The characteristic function $e := 1_{G_x^+}$ is an idempotent of the Hecke algebra $\mathcal{H}_\Lambda(G_x)$ under convolution with respect to the Haar measure μ. We shall show that $e = 1_{G_x^+}$ cuts out $\mathrm{Rep}_\Lambda(G_x)_0$ as a summand of $\mathrm{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)$-Mod.

Let's first check that $e = 1_{G_x^+}$ is central. This can be done by an explicit computation. Recall that we have a descending filtration $\{G_{x,r} | r \in \mathbb{R}_{>0}\}$ of G_x such that

1. $\forall r \in \mathbb{R}_{>0}, G_{x,r}$ is an open compact pro-p subgroup of G_x.

2. $\forall r \in \mathbb{R}_{>0}, G_{x,r}$ is a normal subgroup of G_x.

3. $G_{x,r}$ form a neighborhood basis of 1 inside G_x.

(See [Vig96, II.5.1].) Therefore, to check $e*f = f*e$, for all $f \in \mathcal{H}_\Lambda(G_x)$, it suffices to check for all f of the form $1_{gG_{x,r}}$, the characteristic function of the (both left and right) coset $gG_{x,r}(= G_{x,r}g$, by normality) for some $g \in G(F)$ and $r \in \mathbb{R}_{>0}$. Indeed, one can compute that $(e * 1_{gG_{x,r}})(y) = \mu(G_x^+ \cap G_{x,r}yg^{-1})$ and that $(1_{gG_{x,r}} * e)(y) = \mu(gG_{x,r} \cap yG_x^+)$, for any $y \in G_x$. Note that $G_{x,r} \subseteq G_x^+$, we get that $\mu(G_x^+ \cap G_{x,r}yg^{-1}) = \mu(G_{x,r})$ if $yg^{-1} \in G_x^+$ and 0 otherwise. Same for $\mu(gG_{x,r} \cap yG_x^+)$. Therefore, e is central.

Next, under the isomorphism $\mathrm{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)$-Mod, $\mathrm{Rep}_\Lambda(G_x)_0$ corresponds to the subcategory $\mathcal{H}_\Lambda(G_x, G_x^+)$-Mod $= e\mathcal{H}_\Lambda(G_x)e$-Mod corresponding to the central idempotent $e := 1_{G_x^+} \in \mathcal{H}_\Lambda(G_x)$ of $\mathcal{H}_\Lambda(G_x)$-Mod.

Finally, note that G_x is compact, so its Hecke algebra $\mathcal{H}(G_x)$ is unital with unit 1 the normalized characteristic function of G_x. Hence

$$\mathcal{H}_\Lambda(G_x)\text{-Mod} \cong e\mathcal{H}_\Lambda(G_x)e\text{-Mod} \oplus (1-e)\mathcal{H}_\Lambda(G_x)(1-e)\text{-Mod}.$$

Therefore, $\mathrm{Rep}_\Lambda(G_x)_0 \cong e\mathcal{H}_\Lambda(G_x)e\text{-Mod}$ is a summand of $\mathrm{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)\text{-Mod}$. □

Lemma 3.2.12. *The inflation induces an equivalence of categories between* $\mathrm{Rep}_\Lambda(\overline{G_x})$ *and* $\mathrm{Rep}_\Lambda(G_x)_0$*. In particular, let* ρ *be as in Theorem 3.2.9 and let* $\mathcal{A}_{x,1}$ *be the block of* $\mathrm{Rep}_\Lambda(\overline{G_x})$ *containing* $\overline{\rho}$*, then the inflation induces an equivalence of categories*

$$\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}.$$

Proof. The inverse functor is given by the reduction modulo G_x^+. One could check by hand that they are equivalences of categories. □

3.3 Hom between compact inductions

Let's now prove Theorem 3.1.4 which computes the Hom between compact inductions of ρ_1 and ρ_2, assuming that one of them has supercuspidal reduction.

Proof of Theorem 3.1.4.

$$\begin{aligned}
&\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)}\rho_1, \text{c-Ind}_{G_y}^{G(F)}\rho_2)\\
=\ &\mathrm{Hom}_{G_x}\left(\rho_1, (\text{c-Ind}_{G_y}^{G(F)}\rho_2)|_{G_x}\right)\\
=\ &\mathrm{Hom}_{G_x}\left(\rho_1, \bigoplus_{g\in G_y\backslash G(F)/G_x} \text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x}\rho_2(g-g^{-1})\right)
\end{aligned}$$

Recall that $g^{-1}G_yg = G_{g^{-1}.y}$. So it suffices to show that for $g \in G(F)$ with $G_x \cap g^{-1}G_yg \neq G_x$, or equivalently, for $g \in G(F)$ with $g.x \neq y$ (since x and y are vertices), it holds that

$$\mathrm{Hom}_{G_x}\left(\rho_1, \text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x}\rho_2(g-g^{-1})\right) = 0.$$

Note $G_x/(G_x \cap g^{-1}G_yg)$ is compact, hence $\text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x} = \mathrm{Ind}_{G_x\cap g^{-1}G_yg}^{G_x}$, and we have Frobenius reciprocity in the other direction

$$\mathrm{Hom}_{G_x}\left(\rho_1, \text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x}\rho_2(g-g^{-1})\right) \cong \mathrm{Hom}_{G_x\cap g^{-1}G_yg}\left(\rho_1, \rho_2(g-g^{-1})\right).$$

So it suffices to show that for $g \in G(F)$ with $g.x \neq y$,

$$\mathrm{Hom}_{G_x\cap g^{-1}G_yg}\left(\rho_1, \rho_2(g-g^{-1})\right) = 0.$$

Note now this expression is symmetric with respect to ρ_1 and ρ_2, so is the following argument.

First, if ρ_2 has supercuspidal reduction (denoted $\overline{\rho_2}$),

$$\mathrm{Hom}_{G_x\cap g^{-1}G_yg}\left(\rho_1, \rho_2(g-g^{-1})\right)$$

$$
\begin{aligned}
&= \mathrm{Hom}_{G_x \cap G_{g^{-1}.y}} \left(\rho_1, \rho_2(g - g^{-1})\right) && \\
&\subseteq \mathrm{Hom}_{G_x^+ \cap G_{g^{-1}.y}} \left(\rho_1, \rho_2(g - g^{-1})\right) && \\
&= \mathrm{Hom}_{G_x^+ \cap G_{g^{-1}.y}} (1^{\oplus d_1}, \rho_2(g - g^{-1})) && \rho_1 \text{ is trivial on } G_x^+ \\
&= \mathrm{Hom}_{G_{g.x}^+ \cap G_y} (1^{\oplus d_1}, \rho_2) && \text{Conjugate by } g^{-1} \\
&= \mathrm{Hom}_{U_y(g.x)} (1^{\oplus d_1}, \overline{\rho_2}) && \text{Reduction modulo } G_y^+ \text{. See below.} \\
&= 0 && \overline{\rho_2} \text{ is supercuspidal. See below.}
\end{aligned}
$$

The last two equalities need some explanation.

The former one uses the following consequence from Bruhat-Tits theory: If x_1 and x_2 are two different vertices of the Bruhat-Tits building, then $\overline{G_{x_i}} := G_{x_i}/G_{x_i}^+$ is a generalized Levi subgroup of $\overline{G} = G(\mathbb{F}_q)$, for $i = 1, 2$. Moreover, $G_{x_1} \cap G_{x_2}$ projects onto a proper parabolic subgroup $P_{x_1}(x_2)$ of $\overline{G_{x_1}}$ under the reduction map $G_{x_1} \to \overline{G_{x_1}}$. And $G_{x_1} \cap G_{x_2}^+$ projects onto $U_{x_1}(x_2)$, the unipotent radical of $P_{x_1}(x_2)$, under the reduction map $G_{x_1} \to \overline{G_{x_1}}$. For details, see Lemma 3.3.1 below. Note that the assumption of Lemma 3.3.1 is satisfied since without loss of generality we may assume that $x_1 = x$ and $x_2 = y$ lie in the closure of a common alcove (since G acts simply transitively on the set of alcoves).

The latter one uses that for a supercuspidal representation ρ of a finite group of Lie type Γ,

$$\mathrm{Hom}_U(1, \rho|_U) = \mathrm{Hom}_U(\rho|_U, 1) = 0,$$

for the unipotent radical U of P, where P is any proper parabolic subgroup of Γ. For details, see Lemma 3.3.2 below.

Symmetrically, a similar argument works if ρ_1 has supercuspidal reduction. Indeed, if ρ_1 has supercuspidal reduction (denoted $\overline{\rho_1}$),

$$
\begin{aligned}
&\mathrm{Hom}_{G_x \cap g^{-1} G_y g} \left(\rho_1, \rho_2(g - g^{-1})\right) && \\
&= \mathrm{Hom}_{g G_x g^{-1} \cap G_y} \left(\rho_1(g^{-1} - g), \rho_2\right) && \text{Conjugate by } g^{-1} \\
&\subseteq \mathrm{Hom}_{g G_x g^{-1} \cap G_y^+} \left(\rho_1(g^{-1} - g), \rho_2\right) && \\
&= \mathrm{Hom}_{g G_x g^{-1} \cap G_y^+} (\rho_1(g^{-1} - g), 1^{\oplus d_2}) && \rho_2 \text{ is trivial on } G_y^+ \\
&= \mathrm{Hom}_{G_x \cap g^{-1} G_y^+ g} (\rho_1, 1^{\oplus d_2}) && \text{Conjugate by } g \\
&= \mathrm{Hom}_{G_x \cap G_{g^{-1}.y}^+} (\rho_1, 1^{\oplus d_2}) && \\
&= \mathrm{Hom}_{U_x(g^{-1}.y)} (\overline{\rho_1}, 1^{\oplus d_2}) && \text{Reduction modulo } G_x^+ \\
&= 0 && \overline{\rho_1} \text{ is supercuspidal.}
\end{aligned}
$$

□

Lemma 3.3.1. *Let x_1 and x_2 be two points of the Bruhat-Tits building $\mathcal{B}(G, F)$. Assume that they lie in the closure of a same alcove.*

(i) The image of $G_{x_1} \cap G_{x_2}$ in $\overline{G_{x_1}}$ is a parabolic subgroup of $\overline{G_{x_1}}$. Let's denote it by $P_{x_1}(x_2)$. Moreover, the image of $G_{x_1} \cap G_{x_2}^+$ in $\overline{G_{x_1}}$ is the unipotent radical of $P_{x_1}(x_2)$. Let's denote it by $U_{x_1}(x_2)$.

(ii) Assume moreover that x_1 and x_2 are two different vertices of the building. Then $P_{x_1}(x_2)$ is a proper parabolic subgroup of $\overline{G_{x_1}}$.

Proof. (i) is [Vig96, II.5.1.(k)].

Let's prove (ii). It suffices to show that $G_{x_1} \neq G_{x_2}$. Assume otherwise that $G_{x_1} = G_{x_2}$, then x_1 and x_2 lie in the same facet, which contradicts with the assumption that x_1 and x_2 are two different vertices. □

Lemma 3.3.2. *Let $\overline{\rho}$ be a supercuspidal representation of a finite group of Lie type Γ. Let P be a proper parabolic subgroup of Γ, with unipotent radical U. Then*

$$Hom_U(1_U, \overline{\rho}) = Hom_U(\overline{\rho}, 1_U) = 0.$$

Proof. $\mathrm{Hom}_U(\overline{\rho}|_U, 1_U) = \mathrm{Hom}_\Gamma(\overline{\rho}, Ind_P^\Gamma(\sigma)) = 0$, where $\sigma = Ind_U^P(1_U)$. The last equality holds because $\overline{\rho}$ is assumed to be supercuspidal. A similar argument shows that $Hom_U(1_U, \overline{\rho}) = 0$.

□

3.4 $\Pi_{x,1}$ is a projective generator

In this subsection, we prove Theorem 3.1.5: $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$. Before doing this, let us recall the setting. Fix a vertex x of the building of G. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be a representation which is trivial on G_x^+ and whose reduction to $\overline{G_x} = G_x/G_x^+$ is regular supercuspidal, $\pi = \text{c-Ind}_{G_x}^{G(F)} \rho$ as before. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ, and $\mathcal{C}_{x,1}$ the block of $\mathrm{Rep}_\Lambda(G(F))$ containing π.

Let V be the set of equivalence classes of vertices of the Bruhat-Tits building $\mathcal{B}(G, F)$ up to $G(F)$-action. For $y \in V$, let $\sigma_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$. Let $\Pi := \bigoplus_{y \in V} \Pi_y$ where $\Pi_y := \text{c-Ind}_{G_y^+}^{G(F)} \Lambda$. Then Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G(F))_0$, see [Dat09, Appendix]. Let $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xhookrightarrow{summand} \mathrm{Rep}_\Lambda(G_x)$ be the $\mathcal{B}_{x,1}$-summand of σ_x. And let $\Pi_{x,1} := \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$.

Let's summarize the setting in the following diagram.

$$\begin{array}{ccc}
\mathrm{Rep}_\Lambda(G_x) & \xrightarrow{\text{c-Ind}_{G_x}^{G(F)}} & \mathrm{Rep}_\Lambda(G(F)) \\
\cup\!\mid & & \cup\!\mid \\
\mathrm{Rep}_\Lambda(G_x)_0 & \longrightarrow & \mathrm{Rep}_\Lambda(G(F))_0 \\
\cup\!\mid & & \cup\!\mid \\
\mathcal{B}_{x,1} & \longrightarrow & \mathcal{C}_{x,1}
\end{array}$$

Theorem 3.4.1. $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ *is a projective generator of* $\mathcal{C}_{x,1}$.

Proof. First, let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consisting of representations that are trivial on G_x^+ (Don't confuse with $\mathrm{Rep}_\Lambda(G(F))_0$, the depth-zero category of G). Note that $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand of $\mathrm{Rep}_\Lambda(G_x)$ (see Lemma 3.2.10).

Second, note that $\mathrm{Rep}_\Lambda(G_x)_0 \cong \mathrm{Rep}_\Lambda(\overline{G_x})$. We may assume that

$$\mathrm{Rep}_\Lambda(G_x)_0 = \mathcal{B}_{x,1} \oplus ... \oplus \mathcal{B}_{x,m}$$

is its block decomposition. So that $\sigma_x = \sigma_{x,1} \oplus ... \oplus \sigma_{x,m}$ accordingly. Write $\sigma_x^1 := \sigma_{x,2} \oplus ... \oplus \sigma_{x,m}$. Then $\sigma_x = \sigma_{x,1} \oplus \sigma_x^1$, and $\Pi_x = \Pi_{x,1} \oplus \Pi_x^1$ accordingly, where $\Pi_x^1 := \text{c-Ind}_{G_x}^{G(F)} \sigma_x^1$. And

$$\Pi = \Pi_{x,1} \oplus \Pi_x^1 \oplus \Pi^x,$$

where $\Pi^x := \bigoplus_{y\in V, y\neq x} \Pi_y$. Let $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$, then we have

$$\Pi = \Pi_{x,1} \oplus \Pi^{x,1}.$$

Recall that Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G(F))_0$. This implies that

$$\mathrm{Hom}_G(\Pi, -) : \mathrm{Rep}_\Lambda(G(F))_0 \to \text{Mod-}\,\mathrm{End}_G(\Pi)$$

is an equivalence of categories. See [Ber92, Lemma 22].

Next, it is not hard to see that Theorem 3.1.4 implies that

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0,$$

see Lemma 3.4.2. This implies that

$$\text{Mod-}\,\mathrm{End}_G(\Pi) \cong \text{Mod-}\,\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\,\mathrm{End}_G(\Pi^{x,1})$$

is an equivalence of categories.

Now we can combine the above to show that $\Pi^{x,1}$ does not interfere with $\Pi_{x,1}$, i.e.,

$$\mathrm{Hom}_G(\Pi^{x,1}, X) = 0,$$

for any object $X \in \mathcal{C}_{x,1}$ (see Importent Lemma 3.4.3).

However, since Π is a projective generator of $\mathrm{Rep}_\Lambda(G(F))_0$, we have

$$\mathrm{Hom}_G(\Pi, X) \neq 0,$$

for any $X \in \mathcal{C}_{x,1}$. This together with the last paragraph implies that

$$\mathrm{Hom}_G(\Pi_{x,1}, X) \neq 0,$$

for any $X \in \mathcal{C}_{x,1}$, i.e. $\Pi_{x,1}$ is a generator of $\mathcal{C}_{x,1}$.

Finally, note that $\Pi_{x,1}$ is projective in $\mathrm{Rep}_\Lambda(G(F))_0$ since it is a summand of the projective object Π. Hence $\Pi_{x,1}$ is projective in $\mathcal{C}_{x,1}$. This together with the last paragraph implies that $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$.

□

Lemma 3.4.2.
$$\mathrm{Hom}_G(\Pi_{x,1},\Pi^{x,1}) = \mathrm{Hom}_G(\Pi^{x,1},\Pi_{x,1}) = 0.$$

Proof. Recall that $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$.

First, we compute

$$\mathrm{Hom}_G(\Pi_{x,1},\Pi_x^1) = \mathrm{Hom}_{G_x}(\sigma_{x,1},\sigma_x^1) = 0,$$

where the first equality is the first case of Theorem 3.1.4 (note that $\sigma_{x,1} \in \mathcal{B}_{x,1}$, hence has supercuspidal reduction by Theorem 3.1.3, and hence the condition of Theorem 3.1.4 is satisfied), and the second equality is because $\sigma_{x,1}$ and σ_x^1 lie in different blocks of $\mathrm{Rep}_\Lambda(G_x)$ by definition.

Second, recall that $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)}\sigma_{x,1}$ with $\sigma_{x,1}$ having supercuspidal reduction, and $\Pi_y = \text{c-Ind}_{G_y}^{G(F)}\sigma_y$. We compute

$$\mathrm{Hom}_G(\Pi_{x,1},\Pi^x) = \bigoplus_{y\in V, y\neq x} \mathrm{Hom}_G(\Pi_{x,1},\Pi_y) = 0,$$

by the second case of Theorem 3.1.4.

Combining the above three paragraphs, we get $\mathrm{Hom}_G(\Pi_{x,1},\Pi^{x,1}) = 0$.

A same argument shows that $\mathrm{Hom}_G(\Pi^{x,1},\Pi_{x,1}) = 0$. □

Lemma 3.4.3 (Important Lemma)**.** $\mathrm{Hom}_G(\Pi^{x,1},X) = 0$, *for any object* $X \in \mathcal{C}_{x,1}$.

Proof. Recall that

$$\mathrm{Hom}_G(\Pi,-) : \mathrm{Rep}_\Lambda(G(F))_0 \to \text{Mod-}\mathrm{End}_G(\Pi) \cong \text{Mod-}\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\mathrm{End}_G(\Pi^{x,1})$$

is an equivalence of categories. It is even an equivalence of abelian categories since $\mathrm{Hom}_G(\Pi,-)$ is exact and commutes with direct product. Hence the image of $\mathcal{C}_{x,1}$ must be indecomposable as $\mathcal{C}_{x,1}$ is indecomposable, i.e.,

$$\mathrm{Hom}_G(\Pi,-) = \mathrm{Hom}_G(\Pi_{x,1},-) \oplus \mathrm{Hom}_G(\Pi^{x,1},-)$$

can map $\mathcal{C}_{x,1}$ nonzeroly to only one of $\text{Mod-}\mathrm{End}_G(\Pi_{x,1})$ and $\text{Mod-}\mathrm{End}_G(\Pi^{x,1})$ (See the diagram below).

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda(G(F))_0 \arrow[rr, "\mathrm{Hom}_G(\Pi,-)"] & & \text{Mod-}\mathrm{End}_G(\Pi) \\
\cup\!\mathrm{I} & & \wr\!\mathrm{I} \\
\mathcal{C}_{x,1} \arrow[r, "\mathrm{Hom}_G(\Pi_{x,1},-)\oplus\mathrm{Hom}_G(\Pi^{x,1},-)"] & \text{Mod-}\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\mathrm{End}_G(\Pi^{x,1}) &
\end{tikzcd}$$

Then it must be $\text{Mod-}\mathrm{End}_G(\Pi_{x,1})$ (that $\mathrm{Hom}_G(\Pi,-)$ maps $\mathcal{C}_{x,1}$ nonzeroly to) since

$$\mathrm{Hom}_G(\Pi_{x,1},\pi) = \mathrm{Hom}_{G_x}(\sigma_{x,1},\rho) = \mathrm{Hom}_{G_x}(\sigma_x,\rho) \neq 0.$$

In other words, $\mathrm{Hom}_G(\Pi^{x,1},-)$ is zero on $\mathcal{C}_{x,1}$.

□

3.5 Application: description of the block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$

Recall we denote $\mathcal{A}_{x,1} = \mathrm{Rep}_\Lambda(\overline{G_x})_{[\overline{\rho}]}$, $\mathcal{B}_{x,1} = \mathrm{Rep}_\Lambda(G_x)_{[\rho]}$, and $\mathcal{C}_{x,1} = \mathrm{Rep}_\Lambda(G(F))_{[\pi]}$. We have proven that the inflation along $G_x \to \overline{G_x}$ induces an equivalence of categories

$$\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1},$$

see Lemma 3.2.12. And we have also proven that the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}.$$

Hence $\mathcal{C}_{x,1} \cong \mathcal{A}_{x,1}$, where the latter is isomorphic to the block of a finite torus via Broué's equivalence 3.2.4.

We will see in the example (See Chapter 4) of GL_n that (up to central characters) such a block of a finite torus corresponds to $\mathrm{QCoh}(\mu)$, where μ is the group scheme of roots of unity appearing in the computation of the L-parameter side (See Theorem 2.2.3).

Chapter 4

Example: $GL_n(F)$

Let's apply the theories in the previous chapters to the example of $GL_n(F)$. Throughout this chapter, $G := GL_n$.

That said, there is a little mismatch between the theories before and the example here. Namely, we assumed for simplicity in the theories that G is simply connected (and in particular, semisimple), while this is not the case for $G = GL_n$. However, there is only some minor difference due to the center $\mathbb{G}_m$ of GL_n. Let us leave it as an exercise for the readers to figure out the details.

4.1 L-parameter side

Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be an irreducible tame L-parameter. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lift of φ. Let C_φ be the connected component of $[Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}/\hat{G}]$ containing φ. By Proposition 2.1.10, we compute that

$$C_\varphi \cong [T/T] \times \mu,$$

where $T = C_{\hat{G}}(\psi_\ell)$ is a maximal torus of GL_n, and $\mu = (T^{Fr=(-)^q})^0$, and the T-action on T is specified in Proposition 2.1.10. To go further, let's choose a nice basis of the Weil group representations φ and ψ.

Indeed, every irreducible tame L-parameter φ with $\overline{\mathbb{F}_\ell}$-coefficients of GL_n are of the form $\varphi = \mathrm{Ind}_{W_E}^{W_F} \eta$, where E is a degree n unramified extension of F, $W_E \cong I_F \rtimes \langle \mathrm{Fr}^n \rangle$ is the Weil group of E, and $\eta : W_E \to \overline{\mathbb{F}_\ell}^*$ is a tame (i.e., trivial on $P_E = P_F$) character of W_E such that $\{\eta, \eta^q, ..., \eta^{q^{n-1}}\}$ are distinct. To find a lift of it with $\overline{\mathbb{Z}_\ell}$-coefficients, we let $\tilde{\eta} : W_E \to \overline{\mathbb{Z}_\ell}^*$ be any lift of η, and let $\psi := \mathrm{Ind}_{W_E}^{W_F} \tilde{\eta}$. Then under a nice basis, we could specify the matrices corresponding to the topological generator s_0 and the Frobenius Fr:

$$\psi(s_0) = \begin{bmatrix} \tilde{\eta}(s_0) & 0 & 0 & \dots & 0 \\ 0 & \tilde{\eta}(s_0)^q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\eta}(s_0)^{q^{n-1}} \end{bmatrix}$$

and

$$\psi(\mathrm{Fr}) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \tilde{\eta}(\mathrm{Fr}^n) & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Under this basis, $T = C_{\hat{G}}(\psi_\ell)$ is the diagonal torus of GL_n, with Fr acting by conjugation via ψ, i.e.,

$$\mathrm{Fr}\,.\,\mathrm{diag}(t_1, t_2, ..., t_{n-1}, t_n) = \mathrm{diag}(t_2, t_3, ..., t_n, t_1).$$

So one could compute that

$$T^{\mathrm{Fr}=(-)^q} \cong \mu_{q^n-1},$$

and that

$$(T^{\mathrm{Fr}=(-)^q})^0 \cong \mu_{\ell^k},$$

where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

To compute the quotient $[T/T]$, we note that T acts on T via twisted conjugation

$$(t, t') \mapsto (tnt^{-1}n^{-1})t',$$

where n is same as $\psi(Fr)$ in effect. So in our case, this action is

$$(t_1, t_2, ..., t_n).(t'_1, t'_2, ..., t'_n) = (t_n^{-1}t_1t'_1, t_1^{-1}t_2t'_2, ..., t_{n-1}^{-1}t_nt'_n).$$

We see that the orbits of this action are determined by the determinants (hence are in bijection with $\mathbb{G}_m$), and the center $\mathbb{G}_m \cong Z \subseteq T$ acts trivially. Therefore,

$$[T/T] \cong [\mathbb{G}_m/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$.

In conclusion, we have that the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ is

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k},$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$, and $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

4.2 Representation side

By modular Deligne-Lusztig theory, the block $\mathcal{A}_{x,1}$ of $GL_n(\mathbb{F}_q)$ containing a supercuspidal representation σ is equivalent to the block of an elliptic torus. Such an elliptic torus is isomorphic to $\mathbb{F}_{q^n}^*$. So this block is equivalent to $\overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)$-Mod, where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

$\mathcal{A}_{x,1}$ inflats to a block of $K := GL_n(\mathcal{O}_F)$ containing the inflation $\tilde{\sigma}$ of σ, and further corresponds to a block $\mathcal{B}_{x,1}$ of KZ containing ρ, an extension of $\tilde{\sigma}$ to KZ, where Z is the center of $GL_n(F)$. We have

$$\mathcal{B}_{x,1} \cong \mathcal{A}_{x,1} \otimes \mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\mathbb{Z}) \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod},$$

because

$$KZ \cong K \times \{\mathrm{diag}(\pi^m, ..., \pi^m) \mid m \in \mathbb{Z}\} \cong K \times \mathbb{Z}.$$

Argue as in the proof of Theorem 3.1.2 we see that the compact induction c-Ind_{KZ}^G induces an equivalence of categories

$$\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1},$$

where $\mathcal{C}_{x,1}$ is the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$ containing $\pi := \text{c-Ind}_{KZ}^G \rho$.

Since every irreducible depth-zero supercuspidal representation π arises as above, we have that the block containing an irreducible depth-zero supercuspidal representation π satisfies

$$\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]} \cong \mathcal{C}_{x,1} \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod},$$

where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

Chapter 5

The categorical local Langlands conjecture

In this chapter, we prove the categorical local Langlands conjecture for depth-zero supercuspidal part of $G = GL_n$ with coefficients $\Lambda = \overline{\mathbb{Z}_\ell}$ in Fargues-Scholze's form (See [FS21, Conjecture X.3.5]).

Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be an irreducible tame L-parameter. Let C_φ be the connected component of $[Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}/\hat{G}]$ containing φ.

The goal is to show that there is an equivalence

$$\mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{Coh,Nilp}}(C_\varphi)$$

of derived categories.

As a first step, let's unravel the definitions of both sides and describe them explicitly.

5.1 Unraveling definitions

5.1.1 L-parameter side

Let's first state a lemma that makes the decorations in $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{Coh,Nilp}}(C_\varphi)$ go away. We postpone its proof to Subsection 5.1.3.

Lemma 5.1.1. $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{Coh,Nilp}}(C_\varphi) \cong \mathcal{D}^{b}_{\mathrm{Coh,Nilp}}(C_\varphi) \cong \mathcal{D}^{b}_{\mathrm{Coh},\{0\}}(C_\varphi) \cong \mathrm{Perf}(C_\varphi)$.

Let's assume the lemma for the moment and continue. By our computation before,

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k} \cong \mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k},$$

where $k \in \mathbb{Z}_{\geq 0}$ is maximal such that ℓ^k divides $q^n - 1$. So

$$\mathrm{Perf}(C_\varphi) \cong \mathrm{Perf}(\mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k}) \cong \mathrm{Perf}(\mathbb{G}_m) \otimes \mathrm{Perf}([*/\mathbb{G}_m]) \otimes \mathrm{Perf}(\mu_{\ell^k}).$$

Here,

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell})\chi \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell}),$$

where χ runs over characters of $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

In conclusion, we have

$$\mathrm{Perf}(C_\varphi) \cong \bigoplus_{\chi} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where χ runs over characters of $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

5.1.2 Bun_G side

Since φ is irreducible,

$$\mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}^{\mathrm{ss}}_G, \overline{\mathbb{Z}_\ell})^\omega.$$

See [FS21, Section X.2].

Since

$$\mathrm{Bun}^{\mathrm{ss}}_G = \bigsqcup_{b \in B(G)_{\mathrm{basic}}} [*/G_b(F)],$$

we have

$$\mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}^{\mathrm{ss}}_G, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega.$$

Let's look closer into each direct summand. In our case $G = GL_n$,

$$B(G)_{\mathrm{basic}} \cong \pi_1(G)_\Gamma \cong \mathbb{Z}.$$

Let's first look at the summand for $b = 1$ (corresponding to $0 \in \mathbb{Z} \cong B(G)_{\mathrm{basic}}$). For $b = 1$, $G_b \cong GL_n$, and

$$\mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]})^\omega,$$

where $\pi \in \mathrm{Rep}_{\overline{\mathbb{F}_\ell}}(GL_n(F))$ is the representation with L-parameter φ, and $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]}$ is the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))$ containing π. And we've computed in Chapter 4 that

$$\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]} \cong \overline{\mathbb{Z}_\ell}[t, t^{-1}] \otimes \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)\text{-Mod} \cong \mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where $k \in \mathbb{Z}_{\geq 0}$ is again maximal such that ℓ^k divides $p^n - 1$. So we have

$$\mathcal{D}^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}(\mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}))^\omega \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

We could get a similar description of $\mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})$ (with arbitrary b) for free by the spectral action and its compatibility with $\pi_1(G)_\Gamma$-grading. For this, we consider the composition

$$q : C_\varphi \cong \mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k} \to [*/\mathbb{G}_m].$$

Recall that

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_{\chi} \mathrm{Perf}(\overline{\mathbb{Z}_\ell})\chi.$$

For any χ, we denote by $\mathcal{M}_\chi$ the corresponding simple object in $\mathrm{Perf}([*/\mathbb{G}_m])$. Moreover, $\mathcal{M}_\chi$ pullbacks to a line bundle on C_φ

$$\mathcal{L}_\chi := q^*\mathcal{M}_\chi.$$

We could now state the key proposition that allows us to get to arbitrary $b \in B(G)_{\mathrm{basic}}$ from the $b = 1$ case, using the spectral action.

Proposition 5.1.2.

1. *The restriction of the spectral action by $\mathcal{L}_\chi$ to $\mathcal{D}(G_b(F), \overline{\mathbb{Z}_\ell})$ factors through $\mathcal{D}(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell})$.*

$$\begin{tikzcd} \mathcal{L}_\chi * - : & \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell}) \arrow[r] & \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell}) \\ & \mathcal{D}(G_b(F), \overline{\mathbb{Z}_\ell}) \arrow[u, "\cup"] \arrow[r, dashed] & \mathcal{D}(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell}) \arrow[u, "\cup"] \end{tikzcd}$$

2. *$\mathcal{L}_\chi * - : \mathcal{D}(G_b(F), \overline{\mathbb{Z}_\ell}) \to \mathcal{D}(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell})$ is an equivalence of categories, with inverse $\mathcal{L}_{\chi^{-1}} * -$.*

Proof. For the first assertion, see [Zou22, Lemma 5.3.2]. For the second assertion, note that $\mathcal{L}_\chi$ and $\mathcal{L}_{\chi^{-1}}$ are clearly inverse to each other once they are well-defined, since q^* preserves tensor product. □

So we have

$$\mathcal{D}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \mathcal{D}^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{\mathrm{basic}}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

5.1.3 The nilpotent singular support condition

Now we prove Lemma 5.1.1.

The first isomorphism is because C_φ is connected, hence the quasicompact support condition qc is automatic.

The second isomorphism needs some computation. For the definition and properties of the nilpotent singular support condition Nilp, we refer to [FS21, Section VIII.2]. At the end of the day, it boils down to the fact that for any point φ' in C_φ valued in an algebraically closed Λ-field k,

$$\left(x_{\varphi'}^* \operatorname{Sing}_{[Z^1(W_F,\hat{G})/\hat{G}]/\Lambda}\right) \cap \left(\mathcal{N}_{\hat{G}}^* \otimes_{\mathbb{Z}_\ell} k\right) \cong H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1)) \cap \left(\mathcal{N}_{\hat{G}}^* \otimes_{\mathbb{Z}_\ell} k\right) = \{0\},$$

where $\hat{\mathfrak{g}}^*$ is the dual of the adjoint representation of $\hat{G}$, W_F acts by conjugacy on $\hat{\mathfrak{g}}$ through φ' (and then taking dual and Tate twist to get the action on $\hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1)$), and $\mathcal{N}_{\hat{G}}^* \subseteq \hat{\mathfrak{g}}^*$ is the nilpotent cone.

In our case, $\hat{G} = GL_n$, $\hat{\mathfrak{g}} = M_{n\times n}$ is the set of $n\times n$ matrices. Take $\varphi' = \varphi$ for example (the similar argument works for any φ' in C_φ). W_F acts by conjugacy on $\hat{\mathfrak{g}} = M_{n\times n}$ through φ, hence induces an action of W_F on the dual space with Tate twist $\hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1)$. One could use the explicit matrices 4.1 of s_0 to compute that the fixed points $H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1))$ is contained in the (dual of) the diagonal torus of $M^*_{n\times n}$, the dual Lie algebra $\hat{\mathfrak{g}}^*$. On the other hand, the nilpotnet cone $\mathcal{N}^*_{\hat{G}}$ is nothing else than the (dual of) nilpotent matrices in $M^*_{n\times n}$. So we conclude that

$$H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} k(1)) \cap \left(\mathcal{N}^*_{\hat{G}} \otimes_{\mathbb{Z}_\ell} k\right) = \{0\}.$$

The last isomorphism of Lemma 5.1.1 is [FS21, Theorem VIII.2.9].

5.2 The spectral action induces an equivalence of categories

To summarize, we have (abstract) equivalences of categories

$$\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{Coh,Nilp}}(C_\varphi) \cong \bigoplus_{\chi\in\mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong \bigoplus_{b\in\mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong \mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega,$$

where we identified both $X^*(\mathbb{G}_m) \cong X^*(Z(\hat{G}))$ and $B(G)_{\mathrm{basic}} \cong \pi_1(G)_\Gamma$ with $\mathbb{Z}$. The next goal is to show that the spectral action induces an equivalence of categories

$$\mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega \cong \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{Coh,Nilp}}(C_\varphi). \tag{5.2.1}$$

5.2.1 Definition of the functor

Let's first define the functor. For this, let's choose a Whittaker datum consisting of a Borel $B \subseteq G$ and a generic character $\vartheta : U(F) \to \overline{\mathbb{Z}_\ell}^*$, where U is the unipotent radical of B. Let $\mathcal{W}_\vartheta$ be the sheaf concentrated on Bun^1_G corresponding to the representation $W_\vartheta := \text{c-Ind}^{G(F)}_{U(F)} \vartheta$. Let $W_{\vartheta,[\pi]}$ be the restriction of W_ϑ to the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$, and $\mathcal{W}_{\vartheta,[\pi]}$ the corresponding sheaf.

We define our desired functor by spectral acting on $\mathcal{W}_{\vartheta,[\pi]}$:

$$\Theta : \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{Coh,Nilp}}(C_\varphi) \cong \mathrm{Perf}(C_\varphi) \longrightarrow \mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega, \qquad A \mapsto A * \mathcal{W}_{\vartheta,[\pi]}.$$

5.2.2 Equivalence on degree zero part

We now show that Θ induces a derived equivalence on degree zero part. Before that, we do some preparations.

The main input is local Langlands in families (See [HM18]): For $G = GL_n$, there are natural isomorphisms

$$\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda/\hat{G}) \cong \mathcal{Z}_\Lambda(G(F)) \cong \mathrm{End}_G(W_\vartheta),$$

where $\mathcal{Z}_\Lambda(G(F))$ is the Bernstein center of $\mathrm{Rep}_\Lambda(G(F))$; the first map is the unique map between $\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda/\hat{G})$ and $\mathcal{Z}_\Lambda(G(F))$ that is compatible with the classical local Langlands correspondence for GL_n, hence also same as the map defined in [FS21, Section VIII.4]; the second map is given by the action of the Bernstein center on the representation W_ϑ.

We shall also use the following two Lemmas:

Lemma 5.2.1. *The restriction of the Whittaker representation $W_{\vartheta,[\pi]}$ is a finitely generated projective generator of $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$.*

Proof. For projectivity, see [ABS22, Section 4]. Note their argument is with complex coefficients, but still goes through for $\overline{\mathbb{Z}_\ell}$-coefficients, because the Jacquet functor

$$r_{M,G} : \pi \mapsto \pi_U$$

is still exact under the assumption that p is invertible in $\overline{\mathbb{Z}_\ell}$ (See [Vig96, Section II.2.1]).

For being a generator, in the GL_2 case one could argue similarly as the $\overline{\mathbb{Q}_\ell}$-case in [BH06, Section 39]. (Note their definition of Whittaker representation is dual to our definition, as an induction instead of compact induction. But it still go through by taking dual everywhere. See also, [BH03, Section 2.1 and others].) See [BH03] for the GL_n case.

For finitely generation, it's enough to observe that $W_{\vartheta,[\pi]}$ has finitely many irreducible subquotients (by our explicit description of the block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$) with multiplicity one (again, argue similarly as in [BH06, Section 39] for the multiplicity one property). $\square$

Lemma 5.2.2. *The spectral action is compatible with the map*

$$\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda/\hat{G}) \cong \mathcal{Z}_\Lambda(G(F)).$$

Proof. See [Zou22, Section 5]. $\square$

Now we state the main result of this subsection.

By compatibility with $\pi_1(G)_\Gamma$-grading (see Proposition 5.1.2), Θ restricts to a map between degree-0 parts of both sides

$$\Theta_0 := \Theta|_{\mathrm{Perf}(C_\varphi)_{\chi=0}} : \mathrm{Perf}(C_\varphi)_{\chi=0} \longrightarrow \mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0},$$

where $\mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ and

$$\mathcal{D}^{C_\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0} \cong \mathcal{D}(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega.$$

Proposition 5.2.3. *Under the above identifications, the functor*

$$\Theta_0 : \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \longrightarrow \mathcal{D}(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega \qquad A \mapsto A * W_{\vartheta,[\pi]}$$

is an equivalence of derived categories.

Proof. Let's first prove that Θ_0 is fully faithful. The key observation is that fully faithfulness could be checked on generators of the triangulated category $\mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ (See Lemma 5.2.5). In our case, the structure sheaf $\mathcal{O}$ is a generator of $\mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$, hence it suffices to check fully faithfulness on the structure sheaf. Recall this map sends the structure sheaf $\mathcal{O} \in \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ to the restriction of the Whittaker representation $W_{\vartheta,[\pi]}$. So it suffices to show that the map between Hom-sets in the derived category

$$\mathrm{Hom}(\mathcal{O}, \mathcal{O}[n]) \to \mathrm{Hom}(W_{\vartheta,[\pi]}, W_{\vartheta,[\pi]}[n])$$

is a bijection for all $n \in \mathbb{Z}$. The case $n \neq 0$ follows from the vanishing of higher Ext for projective objects ($\mathcal{O}$ and $W_{\vartheta,[\pi]}$). For $n = 0$, $\mathrm{Hom}(\mathcal{O}, \mathcal{O}) \cong \mathcal{O}(C_\varphi)$, and the above map fits into the following commutative diagram by Lemma 5.2.2, hence a bijection.

$$\begin{tikzcd}
\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda/\hat{G}) \arrow[r, "\cong"] & \mathrm{End}_G(W_\vartheta) \\
\mathcal{O}(C_\varphi) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & \mathrm{End}_G(W_{\vartheta,[\pi]}) \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

The essentially surjectivity follows from Lemma 5.2.1 that $W_{\vartheta,[\pi]}$ is a finitely generated projective generator of $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$.

□

Remark 5.2.4. We remark that to use Lemma 5.2.5 in the above proof, we need the fact that the spectral action commutes with direct sums. Indeed, it commutes with colimits. This boils down to the fact that the Hecke operators commutes with colimits, as they are defined using pullback, tensor product, and shriek pushforward, all of which are left adjoints, hence commutes with colimits.

Lemma 5.2.5. *Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a triangulated functor between triangulated categories that commutes with direct sums, and let E be a generator of $\mathcal{D}_1$. Assume that F induces isomorphisms*

$$\mathrm{Hom}(E, E[n]) \cong \mathrm{Hom}(F(E), F(E[n]))$$

for all $n \in \mathbb{Z}$, then F is fully faithful.

Proof. We use the general lemma [SP, Stack, Tag 0ATH] twice.

To check that condition (1) and (3) in the general lemma holds, we use F commutes with direct sums.

To check that condition (2) in the general lemma holds, we use the five lemma.

We first apply it with the property $T = T_1$: an object $M \in \mathcal{D}_1$ has the property T_1 (written $T_1(M)$) if F induces isomorphisms

$$\mathrm{Hom}(M, E[n]) = \mathrm{Hom}(F(M), F(E[n]))$$

for all $n \in \mathbb{Z}$. The assumption implies that condition (4) in the general lemma holds: $T_1(E[n])$ for all $n \in \mathbb{Z}$. Therefore, $T_1(M)$ for all $M \in \mathcal{D}_1$.

We then apply it with the property $T = T_2$: an object $N \in \mathcal{D}_1$ has the property T_2 (written $T_2(M)$) if F induces isomorphisms

$$\mathrm{Hom}(M, N) = \mathrm{Hom}(F(M), F(N))$$

for all $M \in \mathcal{D}_1$. By the last paragraph, $T_1(M)$ for all $M \in \mathcal{D}_1$, i.e., $T_2(E[n])$ for all $n \in \mathbb{Z}$. Therefore, $T_2(N)$ for all $N \in \mathcal{D}_1$. In other words, F is fully faithful. □

5.2.3 The full equivalence

Finally, we use the spectral action to get the full equivalence. Indeed, on the L-parameter side, for any character $\chi' \in X^*(\mathbb{G}_m)$, tensoring with $\mathcal{L}_{\chi'}$ induces an equivalence

$$\mathcal{L}_{\chi'} \otimes - : \mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(C_\varphi)_{\chi=\chi'}.$$

Similarly, on the Bun_G side, by Proposition 5.1.2, spectral acting by $\mathcal{L}_{\chi'}$ induces an equivalence

$$\mathcal{L}_{\chi'} * - : \mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0} \cong \mathcal{D}_{\mathrm{lis}}^{C_\varphi}(\mathrm{Bun}_G, \overline{\mathbb{Z}_\ell})^\omega_{b=-\chi'}.$$

Therefore, we get the full equivalence via the spectral action.

Chapter 6

Conclusions and questions

In this last chapter, let us make some concluding remarks and raise some further questions.

6.1 How do Chapters 2, 3 help?

First, let us reflect on how do Chapters 2, 3 help to prove the categorical conjecture in Chapter 5. It helps to write the L-parameter side as a $\mathbb{Z}$-grading of derived categories of modules over some ring (so that we could reduce to the degree-zero case and that we could check fully faithfulness on the generator). It does not help much thereafter if one accept the local Langlands in families (LLIF)

$$\mathcal{O}(Z^1(W_F, \hat{G})_\Lambda/\hat{G}) \cong \mathcal{Z}_\Lambda(G(F)) \cong \mathrm{End}_G(W_\vartheta).$$

Indeed, the description of the representation side could be reproved using LLIF and that $W_{\vartheta,[\pi]}$ is a projective generator of $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$.

However, we note that (assuming the compatibility of Fargues-Scholze with the usual local Langlands correspondence for GL_n) it's possible to use Chapters 2, 3 to reprove the first isomorphism in LLIF. It boils down to the fact that if you have a morphism

$$f : \mathbb{G}_m \times \mu \longrightarrow \mathbb{G}_m \times \mu$$

over $\overline{\mathbb{Z}_\ell}$, which becomes an isomorphism after base change to $\overline{\mathbb{Q}_\ell}$, then f is an isomorphism over $\overline{\mathbb{Z}_\ell}$.

Moreover, assuming the result in Chapter 3, the second isomorphism in LLIF (when restricted to the block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$) is almost equivalent to the statement that $W_{\vartheta,[\pi]}$ is a projective generator of $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$. The latter could be proven almost by hand as in Lemma 5.2.1.

6.2 Relation to Bernstein's projective generator

In [Ber92, p46, Section 3.3], Bernstein constructed certain projective generator

$$\text{c-Ind}_{G^0}^{G(F)}(\rho|_{G^0})$$

of a supercuspidal block of $G(F)$ by inducing from G^0, the subgroup generated by compact subgroups (for representations with $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$ coefficients). It is interesting to understand the relation between the projective generators constructed in Chapter 3 and Bernstein's projective generators.

6.3 The categorical conjecture for general groups

Since our results in Chapter 2, 3 also work for general reductive groups (other than GL_n), it is expected that they could be used to prove the categorical local Langlands conjectures for the depth-zero supercuspidal blocks of general reductive groups. In particular, the μ occuring in the result of the L-parameter side (See Theorem 2.2.3) should match with the block $\mathcal{A}_{x,1}$ (See Section 3.5) occuring on the representation side: we should have

$$\mathrm{QCoh}(\mu) \cong \mathcal{A}_{x,1}.$$

Indeed, $\mu = (T^{\mathrm{Fr}=(-)^q})^0$ is certain fixed point of a torus (See Theorem 2.1.7), and $\mathcal{A}_{x,1}$ is also a block of some finite torus via Broué's equivalence 3.2.4. And these two finite torus should match (using the identification that $\mathrm{QCoh}(\mu_{n,\Lambda}) \cong \mathrm{Rep}_\Lambda(\mathbb{Z}/n\mathbb{Z})$).

One possible way to do this is via the (so far unknown in general) compatibility of Fargues-Scholze with classical local Langlands correspondences for depth-zero supercuspidal representations, say the work of DeBacker-Reeder [DR09]. Then these two finite torus should be related by local Langlands for tori (See [DR09, Section 4.3]).

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