Chapter 1

Depth-zero regular supercuspidal blocks

The goal of this chapter is to describe the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$ (denoted $\mathcal{C}_{x,1}$ later) of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$ containing a depth-zero regular supercuspidal representation π.

Recall that a depth-zero regular supercuspidal representation π is of the form

$$\pi = \text{c-Ind}_{G_x}^{G(F)} \rho,$$

where ρ is a representation of G_x whose reduction $\overline{\rho}$ to the finite reductive group $\overline{G_x} = G_x/G_x^+$ is supercuspidal.

In the end, assuming that G is simply connected, the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$ would be equivalent to the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\overline{G_x})_{[\overline{\rho}]}$ (denoted $\mathcal{A}_{x,1}$ later) of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\overline{G_x})$ containing $\overline{\rho}$. And $\mathcal{A}_{x,1}$ has an explicit description via the Broué equivalence 1.2.4.

Indeed, let $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G_x)_{[\rho]}$ (denoted $\mathcal{B}_{x,1}$ later) be the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G_x)$ containing ρ. It is not hard to see that the inflation along $G_x \to \overline{G_x}$ induces an equivalence of categories $\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}$. The main theorem we prove in this chapter is that the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}.$$

The proof of this main theorem 1.1.2 would occupy most of this chapter, from Section 1.1 to 1.4. The proof relies on three theorems. In Section 1.1, we prove the main theorem modulo the three theorems. And the proofs of the three theorems are given in Sections 1.2, 1.3, 1.4, respectively.

1.1 The compact induction induces an equivalence

In this section, we prove the Main Theorem 1.1.2 modulo Theorem 1.1.3 1.1.4 1.1.5.

Let G be a split reductive group scheme over $\mathbb{Z}$, which is simply connected. Let F be a non-archimedean local field, with ring of integers $\mathcal{O}_F$ and residue field $k_F \cong \mathbb{F}_q$ of characteristic p. For simplicity, we assume that q is greater than the Coxeter number of G (See Theorem 1.2.4 for reason).

Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(G,F)$. Let G_x be the parahoric subgroup associated to x, and G_x^+ be its pro-unipotent radical. Recall that $\overline{G_x} := G_x/G_x^+$ is a generalized Levi subgroup of $G(k_F)$ with root system Φ_x, see [Rab03, Theorem 3.17].

Let $\Lambda = \overline{\mathbb{Z}_\ell}$, with $\ell \neq p$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is regular supercuspidal. Here **regular supercuspidal** (See Definition 1.2.7 for precise definition.) means ρ is supercuspidal and lies in a **regular block** of $\mathrm{Rep}_\Lambda(\overline{G_x})$, in the sense of [Bro90]. The reason we want the regularity assumption is that we want to work with a block of $\mathrm{Rep}_\Lambda(\overline{G_x})$ which consists purely of supercuspidal representations. See Section 1.2 for details. We make this a definition for later use.

Definition 1.1.1. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$. We say ρ* ***has supercuspidal reduction*** *(resp.* ***has regular supercuspidal reduction****), if ρ is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is supercuspidal (resp. regular supercuspidal). Let's denote the reduction of ρ modulo G_x^+ by $\overline{\rho} \in \mathrm{Rep}_\Lambda(\overline{G_x})$.*

Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Let $\mathcal{C}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G(F))$ containing $\pi := \text{c-Ind}_{G_x}^{G(F)}\rho$. Now we can state the Main Theorem of this chapter.

Theorem 1.1.2 (Main Theorem)**.** *Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(G,F)$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Let $\mathcal{C}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G(F))$ containing $\pi := \text{c-Ind}_{G_x}^{G(F)}\rho$. Then the compact induction $\text{c-Ind}_{G_x}^{G(F)}$ induces an equivalence of categories $\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}$.*

As mentioned before, the reason we want the regular supercuspidal assumption is the following Theorem.

Theorem 1.1.3. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Then any $\rho' \in \mathcal{B}_{x,1}$ has supercuspidal reduction.*

The proof of the Main Theorem 1.1.2 basically splits into two parts – fully faithfulness and essentially surjectivity. It is convenient to have the following theorem available at an early stage, which implies fully faithfulness immediately and is also used in the proof of essentially surjectivity.

Theorem 1.1.4. *Let x, y be two vertices of the Bruhat-Tits building $\mathcal{B}(G,F)$. Let ρ_1 be a representation of the parahoric G_x which is trivial on the pro-unipotent radical G_x^+. Let ρ_2 be a representation of G_y which is trivial on G_y^+. Assume one of them has supercuspidal reduction. Then exactly one of the following happens:*

1. *If there exists an element $g \in G(F)$ such that $g.x = y$, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)}\rho_1, \text{c-Ind}_{G_y}^{G(F)}\rho_2) = \mathrm{Hom}_{G_x}(\rho_1, {}^g\rho_2).$$

2. *If there is no elements $g \in G(F)$ such that $g.x = y$, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)}\rho_1, \text{c-Ind}_{G_y}^{G(F)}\rho_2) = 0.$$

The proof of the above Theorem is basically a computation using Mackey's formula. See Section 1.3.

Proof of Theorem 1.1.2. Now we proceed by steps towards our goal: The compact induction c-$\mathrm{Ind}_{G_x}^{G(F)}$ induces an equivalence of categories $\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}$.

First, we show that c-$\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is well-defined. We need to show that the image of $\mathcal{B}_{x,1}$ under c-$\mathrm{Ind}_{G_x}^{G(F)}$ lies in $\mathcal{C}_{x,1}$. By Theorem 1.1.3 and Theorem 1.1.4 above,

$$\text{c-Ind}_{G_x}^{G(F)}|_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$$

is fully faithful (See Lemma 1.1.6, note here we used Theorem 1.1.3 that any representation in $\mathcal{B}_{x,1}$ has supercuspidal reduction, so that we can apply Theorem 1.1.4), hence an equivalence onto the essential image. Since $\mathcal{B}_{x,1}$ is indecomposable as an abelian category, so is its essential image (See Lemma 1.1.7), hence its essential image is contained in a single block of $\mathrm{Rep}_\Lambda(G(F))$. But such a block must be $\mathcal{C}_{x,1}$ since c-$\mathrm{Ind}_{G_x}^{G(F)}$ maps ρ to $\pi \in \mathcal{C}_{x,1}$. Therefore, c-$\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is well-defined.

Second, we show that c-$\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is fully faithful. This is already noticed in the proof of "well-defined" in the last paragraph. Indeed,

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_x}^{G(F)} \rho_2) = \mathrm{Hom}_{G_x}(\rho_1, \rho_2)$$

by Theorem 1.1.3 and Theorem 1.1.4 (See Lemma 1.1.6.). Therefore, c-$\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is fully faithful.

Finally, we show that c-$\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective. This will occupy the rest of this section.

The idea is to find a projective generator of $\mathcal{C}_{x,1}$ and show that it is in the essential image. Fix a vertex x of the Bruhat-Tits building $\mathcal{B}(G, F)$ as before. Let V be the set of equivalence classes of vertices of the Bruhat-Tits building $\mathcal{B}(G, F)$ up to $G(F)$-action. For $y \in V$, let $\sigma_y :=$ c-$\mathrm{Ind}_{G_y^+}^{G_y} \Lambda$. Let $\Pi := \bigoplus_{y \in V} \Pi_y$ where $\Pi_y :=$ c-$\mathrm{Ind}_{G_y^+}^{G(F)} \Lambda$. Then Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G(F))_0$, see [Dat09, Appendix]. Let $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xhookrightarrow{\textit{summand}} \mathrm{Rep}_\Lambda(G_x)$ be the $\mathcal{B}_{x,1}$-summand of σ_x. And let $\Pi_{x,1} :=$ c-$\mathrm{Ind}_{G_x}^{G(F)} \sigma_{x,1}$. Note $\Pi_{x,1}$ is a summand of $\Pi_x =$ c-$\mathrm{Ind}_{G_x}^{G(F)} \sigma_x$, hence a summand of Π. Using Theorem 1.1.4, one can show that the rest of the summands of Π don't interfere with $\Pi_{x,1}$ (See Lemma 1.4.2 and Lemma 1.4.3 for precise meaning), hence $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$. Let us state it as a Theorem, see Section 2 for details.

Theorem 1.1.5. $\Pi_{x,1} =$ c-$\mathrm{Ind}_{G_x}^{G(F)} \sigma_{x,1}$ *is a projective generator of* $\mathcal{C}_{x,1}$.

Now we've found a projective generator $\Pi_{x,1} =$ c-$\mathrm{Ind}_{G_x}^{G(F)} \sigma_{x,1}$ of $\mathcal{C}_{x,1}$, and it is clear that $\Pi_{x,1}$ is in the essential image of c-$\mathrm{Ind}_{G_x}^{G(F)}$. We now deduce from this that c-$\mathrm{Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective. Indeed, for any $\pi' \in \mathcal{C}_{x,1}$, we can resolve π' by some copies of $\Pi_{x,1}$:

$$\Pi_{x,1}^{\oplus I} \xrightarrow{f} \Pi_{x,1}^{\oplus J} \to \pi' \to 0.$$

Using Theorem 1.1.4 and $\text{c-Ind}_{G_x}^{G(F)}$ commutes with arbitrary direct sums (See Lemma 1.1.8) we see that $f \in \mathrm{Hom}_G(\Pi_{x,1}^{\oplus I}, \Pi_{x,1}^{\oplus J})$ comes from a morphism $g \in \mathrm{Hom}_{G_x}(\sigma_{x,1}^{\oplus I}, \sigma_{x,1}^{\oplus J})$. Using $\text{c-Ind}_{G_x}^{G(F)}$ is exact we see that π' is the image of $\mathrm{coker}(g) \in \mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$. Therefore, $\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective.

□

Lemma 1.1.6. $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ *is fully faithful.*

Proof. Let $\rho_1, \rho_2 \in \mathcal{B}_{x,1}$. By the regular supercuspidal assumption and Theorem 1.1.3, ρ_1, ρ_2 has supercuspidal reduction. Hence the assumption of Theorem 1.1.4 is satisfied and we compute using the first case of Theorem 1.1.4 that

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)} \rho_1, \text{c-Ind}_{G_x}^{G(F)} \rho_2) \cong \mathrm{Hom}_{G_x}(\rho_1, \rho_2).$$

In other words, $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ is fully faithful. □

Lemma 1.1.7. *The image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$ is indecomposable as an abelian category.*

Proof. The point is that $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ is not only fully faithful, i.e., an equivalence of categories onto the essential image, but also an equivalence of **abelian** categories onto the essential image. Indeed, it suffices to show that $\text{c-Ind}_{G_x}^{G(F)} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G(F))$ preserves kernels, cokernels, and finite (bi-)products. But this follows from the next Lemma 1.1.8.

Assume otherwise that the essential image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G(F)}$ is decomposable, then so is $\mathcal{B}_{x,1}$. But $\mathcal{B}_{x,1}$ is a block, hence indecomposable, contradiction! □

Lemma 1.1.8. $\text{c-Ind}_{G_x}^{G(F)}$ *is exact and commutes with arbitrary direct sums.*

Proof. For $\text{c-Ind}_{G_x}^{G(F)}$ is exact, we refer to [Vig96, I.5.10].

We show that $\text{c-Ind}_{G_x}^{G(F)}$ commutes with arbitrary direct sums. Indeed, $\text{c-Ind}_{G_x}^{G(F)}$ is a left adjoint (See [Vig96, I.5.7]), hence commutes with arbitrary colimits. In particular, it commutes with arbitrary direct sums. □

1.2 Regular supercuspidal blocks for finite groups of Lie type

In this section, we prove Theorem 1.1.3. As mentioned before, we made the **regular** assumption in order that the conclusion of Theorem 1.1.3 – all representations in such a block have supercuspidal reduction – is true. So the readers are welcome to skip this section for a first reading and pretend that we begin with a block in which all representations have supercuspidal reduction.

The main body of this section is to define regular supercuspidal blocks with $\Lambda = \overline{\mathbb{Z}_\ell}$-coefficients of a finite group of Lie type, and to show that a regular supercuspidal block consists purely of supercuspidal representations.

Let $\Lambda := \overline{\mathbb{Z}_\ell}$ be the coefficients of representations. Fix a prime number p. Let ℓ be a prime number different from p. Let q be a power of p.

Definition 1.2.1 ([Vig96, I.4.1])**.** *Let Λ' be any ring.*

1. *Let H be a profinite group, a* ***representation of*** H ***with*** Λ'***-coefficients*** (π, V) *is a Λ'-module V, together with a H-action $\pi : H \to GL_{\Lambda'}(V)$.*

2. *A representation of H with Λ'-coefficients is called* ***smooth*** *if for any $v \in V$, the stabilizer $Stab_H(v) \subseteq H$ is open.*

From now on, all representations are assumed to be smooth. The category of smooth representations of H with Λ'-coefficients is denoted by $\mathrm{Rep}_{\Lambda'}(H)$.

1.2.1 Regular blocks

The following notations are used in this subsection only. Let $\mathcal{G}$ be a split reductive group scheme over $\mathbb{Z}$. Let $\mathbb{G} := \mathcal{G}(\overline{\mathbb{F}_q})$, $G := \mathbb{G}^F = \mathcal{G}(\mathbb{F}_q)$, where F is the Frobenius. By abuse of notation, we sometimes identify the group scheme $\mathcal{G}_{\overline{\mathbb{F}_q}}$ with its $\overline{\mathbb{F}_q}$-points $\mathbb{G}$. Let $\mathbb{G}^*$ be the dual group (over $\overline{\mathbb{F}_q}$) of $\mathbb{G}$, and F^* the dual Frobenius (See [Car85, Section 4.2]). Fix an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$.

The definition of regular supercuspidal blocks and regular supercuspidal representations of a finite group of Lie type Γ involves modular Deligne-Lusztig theory and block theory. We refer to [DL76], [Car85], and [DM20] for Deligne-Lusztig theory, [BM89] and [Bro90] for modular Deligne-Lusztig theory, and [Bon11, Appendix B] for generalities on blocks.

First, let us recall a result in Deligne-Lusztig theory (See [DM20, Proposition 11.1.5]).

Proposition 1.2.2. *The set of $\mathbb{G}^F$-conjugacy classes of pairs $(\mathbb{T}, \theta)$, where $\mathbb{T}$ is a F-stable maximal torus of $\mathbb{G}$ and $\theta \in \widehat{\mathbb{T}^F}$, is in non-canonical bijection to the set of ${\mathbb{G}^*}^{F^*}$-conjugacy classes of pairs $(\mathbb{T}^*, s)$, where s is a semisimple element of $\mathbb{G}^*$ and $\mathbb{T}^*$ is a F^*-stable maximal torus of $\mathbb{G}^*$ such that $s \in {\mathbb{T}^*}^{F^*}$. Moreover, we could and will fix a compatible system of isomorphisms $\mathbb{F}_{q^n}^* \cong \mathbb{Z}/(q^n-1)\mathbb{Z}$ to pin down this bijection.*

Now let s be a **strongly regular semisimple** element of $G^* = {\mathbb{G}^*}^{F^*}$ (note we require s to be fixed by F^* here), i.e., the centralizer $C_{\mathbb{G}^*}(s)$ is a F^*-stable maximal torus, denoted $\mathbb{T}^*$. Let $\mathbb{T}$ be the dual torus of $\mathbb{T}^*$. Let $T = \mathbb{T}^F$ and $T^* = {\mathbb{T}^*}^{F^*}$. Let T_ℓ denote the ℓ-part of T.

Recall for s strongly regular semisimple, the (rational) Lusztig series $\mathcal{E}(G, (s))$ consists of only one element, namely, $\pm R_T^G(\hat{s})$, where $\hat{s} = \theta$ is such that $(\mathbb{T}, \theta)$ corresponds to $(\mathbb{T}^*, s)$ via the previous bijection in Proposition 1.2.2. Here and after the sign $\pm$ is taken such that it is an honest representation (See [Car85, Section 7.5]).

From now on, assume moreover that $s \in {\mathbb{G}^*}^{F^*}$ **has order prime to** ℓ**.** In other words, assume $s \in G^* = {\mathbb{G}^*}^{F^*}$ is a **strongly regular semisimple** ℓ'**-element**. We are going to define regular blocks, we refer to [Bon11, Appendix B] for generalities on blocks.

Define the ℓ**-Lusztig series**

$$\mathcal{E}_\ell(G, (s)) := \{\pm R_T^G(\hat{s}\eta) \mid \eta \in \widehat{T_\ell}\}.$$

Note the notation $\mathcal{E}_\ell(T, (s))$ also makes sense by putting $G = T$.

By [BM89], $\mathcal{E}_\ell(G, (s))$ is a union of ℓ-blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(G)$. Such a block (or more precisely, a union of blocks) is called a **(ℓ-)regular block**. Let $e_s^G \in \overline{\mathbb{Z}}_\ell G$ denote the corresponding central idempotent. Note e_s^T also makes sense by putting $G = T$. We shall see later that a regular block is indeed a block, i.e., indecomposible. (This follows from, for example, Broué's equivalence. See Theorem 1.2.4 below.)

Definition 1.2.3 (Regular blocks)**.** *Let $s \in G^* = \mathbb{G}^{*F^*}$ be a strongly regular semisimple ℓ'-element. We call the block $\overline{\mathbb{Z}_\ell}Ge_s^G$ of the group algebra $\overline{\mathbb{Z}_\ell}G$ corresponding to the central idempotent e_s^G the* ***regular block*** *associated to s. Let $\mathcal{A}_s := \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$ be the corresponding category of modules, this is also referred to as a regular block, by abuse of notation.*

Thanks to [Bro90], we understand the category $\mathcal{A}_s = \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$ quite well. Roughly speaking, it is equivalent to the category of representations of a torus, via Deligne-Lusztig induction. This is what we are going to explain now.

Let $\mathbb{B} \subseteq \mathbb{G}$ be a Borel subgroup containing our torus $\mathbb{T}$, let $\mathbb{U}$ be the unipotent radical of $\mathbb{B}$. Let $X_{\mathbb{U}}$ be the Deligne-Lusztig variety defined by

$$X_{\mathbb{U}} := \{g \in \mathbb{G} \mid g^{-1}F(g) \in \mathbb{U}\}.$$

The main result of [Bro90] is the following: The Deligne-Lusztig induction

$$\pm R_T^G : \overline{\mathbb{Z}_\ell}T\text{-Mod} \to \overline{\mathbb{Z}_\ell}G\text{-Mod}$$

induces an equivalence of categories between $\overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod}$ and $\overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$. In particular, one deduce that the irreducible objects in $\overline{\mathbb{F}_\ell}Ge_s^G\text{-Mod}$ lifts to $\overline{\mathbb{Z}_\ell}$. More precisely, let us state it as the following theorem.

Theorem 1.2.4 (Broué's equivalence, [Bro90, Theorem 3.3])**.** *With the previous assumptions and notations, assume $X_{\mathbb{U}}$ is affine of dimension d (which is the case if q is greater than the Coxeter number of $\mathbb{G}$.). The cohomology complex $R\Gamma_c(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell}) = R\Gamma_c(X_{\mathbb{U}}, \mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\overline{\mathbb{Z}_\ell}$ is concentrated in degree $d = dimX_{\mathbb{U}}$. And the $(\overline{\mathbb{Z}_\ell}Ge_s^G, \overline{\mathbb{Z}_\ell}Te_s^T)$-bimodule $e_s^G H_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell})e_s^T$ induces an equivalence of categories*

$$e_s^G H_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell})e_s^T \otimes_{\overline{\mathbb{Z}_\ell}Te_s^T} - : \overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod} \to \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}.$$

From now on, we assume the above Theorem holds for all finite groups of Lie type we encountered in this paper. we hope this is not a severe restriction. This is the case at least when q is greater than the Coxeter number of $\mathbb{G}$.

Note also that the category $\overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod}$ is equivalent to the category $\overline{\mathbb{Z}_\ell}T_\ell\text{-Mod}$, where T_ℓ is the order-ℓ-part of T, this is essentially the category of representations of some product of $\mathbb{Z}/\ell^{k_i}\mathbb{Z}$. In particular, it has a unique irreducible representation (simple object), which is already defined over $\overline{\mathbb{F}_\ell}$. Let us denote its corresponding character by $\theta_s : T \to \overline{\mathbb{F}_\ell}^*$. Accordingly, $\overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$ has a unique simple object $\pm R_T^G(\theta_s)$.

1.2.2 Regular supercuspidal blocks

Let us first recall the definition of supercuspidal representations.

Definition 1.2.5. 1. *An irreducible representation is called* ***supercuspidal*** *if it does not occur as a subquotient of any proper parabolic induction.*

2. *A representation is called **supercuspidal** if all its irreducible subquotients are supercuspidal.*

Now let us define regular supercuspidal blocks and regular supercuspidal representations.

Definition 1.2.6. *By a **regular supercuspidal block**, we mean a regular block $\mathcal{A}_s$ whose unique simple object $\pm R_T^G(\theta_s)$ (See the explanations after Theorem 1.2.4 for definition) is supercuspidal.*

Definition 1.2.7. 1. *An irreducible representation is called **regular supercuspidal** if it lies in a regular supercuspidal block.*

2. *A representation is called **regular supercuspidal** if all its irreducible subquotients are regular supercuspidal.*

It is clear from the definitions that we have the following proposition.

Proposition 1.2.8. *Let $\mathcal{A}_s$ be a regular supercuspidal block. Then any representation in this block is supercuspidal.*

Proof. By definition of supercuspidality, it suffices to check that any irreducible representation in this block is supercuspidal. But as we noted before in the explanations after Theorem 1.2.4, $\mathcal{A}_s$ has only one irreducible representation – $\pm R_T^G(\theta_s)$, which we assumed to be supercuspidal in the definition of regular supercuspidal block. So we win! □

1.2.3 Proof of Theorem 1.1.3 on supercuspidal reduction

We now apply the previous results on finite groups of Lie type to representations of the parahoric subgroups of a p-adic group. For this, we show that the inflation induces an equivalence of categories between (certain summand of) the category of representations of a finite reductive group and the corresponding parahoric subgroup (See Subsection 1.2.4).

Let us get back to the notation at the beginning of this chapter.

Let G be a split reductive group scheme over $\mathbb{Z}$, which is simply connected. Let F be a non-archimedean local field, with ring of integers $\mathcal{O}_F$ and residue field $k_F \cong \mathbb{F}_q$ of residue characteristic p. Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(G, F)$, G_x the parahoric subgroup associated to x, G_x^+ its pro-unipotent radical. Recall that $\overline{G_x} := G_x/G_x^+$ is a generalized Levi subgroup of $G(k_F)$ with root system Φ_x, see [Rab03, Theorem 3.17].

Let $\Lambda = \overline{\mathbb{Z}_\ell}$, with $\ell \neq p$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is regular supercuspidal.

In other words, we start with an irreducible representation $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the ($\overline{\mathbb{Z}_\ell}$-)block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. We can now prove Theorem 1.1.3, which we restate as follows.

Theorem 1.2.9. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which has regular supercuspidal reduction. Let $\mathcal{B}_{x,1}$ be the $\overline{\mathbb{Z}_\ell}$-block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Then any $\rho' \in \mathcal{B}_{x,1}$ has supercuspidal reduction.*

Proof. Let $\overline{\rho} \in \mathrm{Rep}_\Lambda(\overline{G_x})$ be the reduction of ρ modulo G_x^+. $\overline{\rho}$ is irreducible (since ρ is) and regular supercuspidal by assumption, so it is of the form $\pm R_T^G(\theta_s)$, for some strongly regular semisimple ℓ'-element s of the finite dual group $\overline{G_x}^*$ (See Definition 1.2.7.). (problem hyperlink)

Let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consists of representations of G_x that are trivial on G_x^+. The key observation is that $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand (as abelian category) of $\mathrm{Rep}_\Lambda(G_x)$ (See Lemma 1.2.10).

Then since $\rho \in \mathrm{Rep}_\Lambda(G_x)_0$, its block $\mathcal{B}_{x,1}$ is a summand of $\mathrm{Rep}_\Lambda(G_x)_0$.

On the other hand, notice that the inflation induces an equivalence of categories between $\mathrm{Rep}_\Lambda(\overline{G_x})$ and $\mathrm{Rep}_\Lambda(G_x)_0$, with inverse the reduction modulo G_x^+. contained So the blocks of $\mathrm{Rep}_\Lambda(\overline{G_x})$ and $\mathrm{Rep}_\Lambda(G_x)_0$ are in one-one correspondence. Let $\mathcal{A}_{x,1}$ be the corresponding block of $\mathrm{Rep}_\Lambda(\overline{G_x})$ to $\mathcal{B}_{x,1}$. Then $\mathcal{A}_{x,1}$ is in the regular cuspidal block $\mathcal{A}_s$ corresponding to s (recall $\overline{\rho} = \pm R_T^G(\theta_s)$). By Theorem 1.2.8, $\mathcal{A}_s$ consists purely of supercuspidal representation. Therefore, $\mathcal{B}_{x,1}$ consists purely of representations that have supercuspidal reductions. □

1.2.4 Inflation induces an equivalence

Lemma 1.2.10. *Let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consists of representations of G_x that are trivial on G_x^+. Then $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand as abelian category of $\mathrm{Rep}_\Lambda(G_x)$.*

Remark 1.2.11. A similar proof as [Dat09, Appendix] should work. Nevertheless, I include here an alternative computational proof.

Proof. Note G_x^+ is pro-p (See [Vig96, II.5.2.(b)]), in particular, it has pro-order invertible in Λ. So we have a normalized Haar measure μ on G_x such that $\mu(G_x^+) = 1$ (See [Vig96, I.2.4]). The characteristic function $e := 1_{G_x^+}$ is an idempotent of the Hecke algebra $\mathcal{H}_\Lambda(G_x)$ under convolution with respect to the Haar measure μ. We shall show that $e = 1_{G_x^+}$ cuts out $\mathrm{Rep}_\Lambda(G_x)_0$ as a summand of $\mathrm{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)$-Mod.

Let's first check that $e = 1_{G_x^+}$ is central. This can be done by an explicit computation. Recall that we have a descending filtration $\{G_{x,r} | r \in \mathbb{R}_{>0}\}$ of G_x such that

1. $\forall r \in \mathbb{R}_{>0}, G_{x,r}$ is an open compact pro-p subgroup of G_x.

2. $\forall r \in \mathbb{R}_{>0}, G_{x,r}$ is a normal subgroup of G_x.

3. $G_{x,r}$ form a neighborhood basis of 1 inside G_x.

(See [Vig96, II.5.1].) Therefore, to check $e*f = f*e$, for all $f \in \mathcal{H}_\Lambda(G_x)$, it suffices to check for all f of the form $1_{gG_{x,r}}$, the characteristic function of the (both left and right) coset $gG_{x,r}(= G_{x,r}g$, by normality) for some $g \in G(F)$ and $r \in \mathbb{R}_{>0}$. Indeed, one can compute that $(e * 1_{gG_{x,r}})(y) = \mu(G_x^+ \cap G_{x,r}yg^{-1})$ and that $(1_{gG_{x,r}} * e)(y) = \mu(gG_{x,r} \cap yG_x^+)$, for any $y \in G_x$. Note that $G_{x,r} \subseteq G_x^+$, we get that $\mu(G_x^+ \cap G_{x,r}yg^{-1}) = \mu(G_{x,r})$ if $yg^{-1} \in G_x^+$ and 0 otherwise. Same for $\mu(gG_{x,r} \cap yG_x^+)$. Therefore, e is central.

Next, under the isomorphism $\mathrm{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)$-Mod, $\mathrm{Rep}_\Lambda(G_x)_0$ corresponds to the summand $\mathcal{H}_\Lambda(G_x, G_x^+)$-Mod $= e\mathcal{H}_\Lambda(G_x)e$-Mod corresponding to the central idempotent $e := 1_{G_x^+} \in \mathcal{H}_\Lambda(G_x)$ of $\mathcal{H}_\Lambda(G_x)$-Mod.

Finally, note that G_x is compact, so its Hecke algebra $\mathcal{H}(G_x)$ is unital with unit 1 the normalized characteristic function of G_x. Hence

$$\mathcal{H}_\Lambda(G_x)\text{-Mod} \cong e\mathcal{H}_\Lambda(G_x)e\text{-Mod} \oplus (1-e)\mathcal{H}_\Lambda(G_x)(1-e)\text{-Mod}.$$

Therefore, $\mathrm{Rep}_\Lambda(G_x)_0 \cong e\mathcal{H}_\Lambda(G_x)e\text{-Mod}$ is a summand of $\mathrm{Rep}_\Lambda(G_x) \cong \mathcal{H}_\Lambda(G_x)\text{-Mod}$. □

Lemma 1.2.12. *The inflation induces an equivalence of categories between* $\mathrm{Rep}_\Lambda(\overline{G_x})$ *and* $\mathrm{Rep}_\Lambda(G_x)_0$*. In particular, let ρ as in Theorem 1.2.9 and let $\mathcal{A}_{x,1}$ be the block of* $\mathrm{Rep}_\Lambda(\overline{G_x})$ *containing $\overline{\rho}$, then the inflation induces an equivalence of categories*

$$\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1}.$$

Proof. The inverse functor is given by the reduction modulo G_x^+. One could check by hand that they are equivalences of categories. □

1.3 Hom between compact inductions

Let's now prove Theorem 1.1.4 which computes the Hom between compact inductions of ρ_1 and ρ_2, assuming one of them has supercuspidal reduction.

Proof of Theorem 1.1.4.

$$\begin{aligned}
&\mathrm{Hom}_G(\text{c-Ind}_{G_x}^{G(F)}\rho_1, \text{c-Ind}_{G_y}^{G(F)}\rho_2)\\
=\ &\mathrm{Hom}_{G_x}\left(\rho_1, (\text{c-Ind}_{G_y}^{G(F)}\rho_2)|_{G_x}\right)\\
=\ &\mathrm{Hom}_{G_x}\left(\rho_1, \bigoplus_{g\in G_y\backslash G(F)/G_x} \text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x}\rho_2(g-g^{-1})\right)
\end{aligned}$$

Recall that $g^{-1}G_yg = G_{g^{-1}.y}$. So it suffices to show that for $g \in G(F)$ with $G_x \cap g^{-1}G_yg \neq G_x$, or equivalently, for $g \in G(F)$ with $g.x \neq y$ (since x and y are vertices), it holds that

$$\mathrm{Hom}_{G_x}\left(\rho_1, \text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x}\rho_2(g-g^{-1})\right) = 0.$$

Note $G_x/(G_x \cap g^{-1}G_yg)$ is compact, hence $\text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x} = \mathrm{Ind}_{G_x\cap g^{-1}G_yg}^{G_x}$, and we have Frobenius reciprocity in the other direction

$$\mathrm{Hom}_{G_x}\left(\rho_1, \text{c-Ind}_{G_x\cap g^{-1}G_yg}^{G_x}\rho_2(g-g^{-1})\right) \cong \mathrm{Hom}_{G_x\cap g^{-1}G_yg}\left(\rho_1, \rho_2(g-g^{-1})\right).$$

So it suffices to show that for $g \in G(F)$ with $g.x \neq y$,

$$\mathrm{Hom}_{G_x\cap g^{-1}G_yg}\left(\rho_1, \rho_2(g-g^{-1})\right) = 0.$$

Note now this expression is symmetric with respect to ρ_1 and ρ_2, so is the following argument.

First, if ρ_2 has supercuspidal reduction (denoted $\overline{\rho_2}$),

$$\mathrm{Hom}_{G_x\cap g^{-1}G_yg}\left(\rho_1, \rho_2(g-g^{-1})\right)$$

$$
\begin{aligned}
&= \mathrm{Hom}_{G_x \cap G_{g^{-1}.y}}\left(\rho_1, \rho_2(g - g^{-1})\right) && \\
&\subseteq \mathrm{Hom}_{G_x^+ \cap G_{g^{-1}.y}}\left(\rho_1, \rho_2(g - g^{-1})\right) && \\
&= \mathrm{Hom}_{G_x^+ \cap G_{g^{-1}.y}}(1^{\oplus d_1}, \rho_2(g - g^{-1})) && \rho_1 \text{ is trivial on } G_x^+ \\
&= \mathrm{Hom}_{G_{g.x}^+ \cap G_y}(1^{\oplus d_1}, \rho_2) && \text{Conjugate by } g^{-1} \\
&= \mathrm{Hom}_{U_y(g.x)}(1^{\oplus d_1}, \overline{\rho_2}) && \text{Reduction modulo } G_y^+\text{. See below.} \\
&= 0 && \overline{\rho_2} \text{ is supercuspidal. See below.}
\end{aligned}
$$

The last two equations need some explanation.

The former one uses the following consequence from Bruhat-Tits theory: If x_1 and x_2 are two different vertices of the Bruhat-Tits building, then $\overline{G_{x_i}} := G_{x_i}/G_{x_i}^+$ is a generalized Levi subgroup of $\overline{G} = G(\mathbb{F}_q)$, for $i = 1, 2$. Moreover, $G_{x_1} \cap G_{x_2}$ projects onto a proper parabolic subgroup $P_{x_1}(x_2)$ of $\overline{G_{x_1}}$ under the reduction map $G_{x_1} \to \overline{G_{x_1}}$. And $G_{x_1} \cap G_{x_2}^+$ projects onto $U_{x_1}(x_2)$, the unipotent radical of $P_{x_1}(x_2)$, under the reduction map $G_{x_1} \to \overline{G_{x_1}}$. For details, see Lemma 1.3.1 below. Note that the assumption of Lemma 1.3.1 is satisfied since without loss of generality we may assume $x_1 = x$ and $x_2 = y$ lies in the closure of a common alcove (since G acts simply transitively on the set of alcoves).

The latter one uses that for a supercuspidal representation ρ of a finite group of Lie type Γ,

$$\mathrm{Hom}_U(1, \rho|_U) = \mathrm{Hom}_U(\rho|_U, 1) = 0,$$

for the unipotent radical U of P, where P is any proper parabolic subgroup of Γ. For details, see Lemma 1.3.2 below.

Symmetrically, a similar argument works if ρ_1 has supercuspidal reduction. Indeed, if ρ_1 has supercuspidal reduction (denoted $\overline{\rho_1}$),

$$
\begin{aligned}
&\mathrm{Hom}_{G_x \cap g^{-1} G_y g}\left(\rho_1, \rho_2(g - g^{-1})\right) && \\
&= \mathrm{Hom}_{g G_x g^{-1} \cap G_y}\left(\rho_1(g^{-1} - g), \rho_2\right) && \text{Conjugate by } g^{-1} \\
&\subseteq \mathrm{Hom}_{g G_x g^{-1} \cap G_y^+}\left(\rho_1(g^{-1} - g), \rho_2\right) && \\
&= \mathrm{Hom}_{g G_x g^{-1} \cap G_y^+}(\rho_1(g^{-1} - g), 1^{\oplus d_2}) && \rho_2 \text{ is trivial on } G_y^+ \\
&= \mathrm{Hom}_{G_x \cap g^{-1} G_y^+ g}(\rho_1, 1^{\oplus d_2}) && \text{Conjugate by } g \\
&= \mathrm{Hom}_{G_x \cap G_{g^{-1}.y}^+}(\rho_1, 1^{\oplus d_2}) && \\
&= \mathrm{Hom}_{U_x(g^{-1}.y)}(\overline{\rho_1}, 1^{\oplus d_2}) && \text{Reduction modulo } G_x^+ \\
&= 0 && \overline{\rho_1} \text{ is supercuspidal.}
\end{aligned}
$$

□

Lemma 1.3.1. *Let x_1 and x_2 be two points of the Bruhat-Tits building $\mathcal{B}(G, F)$. Assume they lie in the closure of a same alcove.*

(i) *The image of $G_{x_1} \cap G_{x_2}$ in $\overline{G_{x_1}}$ is a parabolic subgroup of $\overline{G_{x_1}}$. Let's denote it by $P_{x_1}(x_2)$. Moreover, the image of $G_{x_1} \cap G_{x_2}^+$ in $\overline{G_{x_1}}$ is the unipotent radical of $P_{x_1}(x_2)$. Let's denote it by $U_{x_1}(x_2)$.*

(ii) *Assume moreover that x_1 and x_2 are two different vertices of the building. Then $P_{x_1}(x_2)$ is a proper parabolic subgroup of $\overline{G_{x_1}}$.*

Proof. (i) is [Vig96, II.5.1.(k)].

Let's prove (ii). It suffices to show that $G_{x_1} \neq G_{x_2}$. Assume otherwise that $G_{x_1} = G_{x_2}$, then x_1 and x_2 lie in the same facet, which contradicts with the assumption that x_1 and x_2 are two different vertices. □

Lemma 1.3.2. *Let $\overline{\rho}$ be a supercuspidal representation of a finite group of Lie type Γ. Let P be a proper parabolic subgroup of Γ, with unipotent radical U. Then*

$$Hom_U(1_U, \overline{\rho}) = Hom_U(\overline{\rho}, 1_U) = 0.$$

Proof. $\mathrm{Hom}_U(\overline{\rho}|_U, 1_U) = \mathrm{Hom}_\Gamma(\overline{\rho}, Ind_P^\Gamma(\sigma)) = 0$, where $\sigma = Ind_U^P(1_U)$. The last equality holds because $\overline{\rho}$ is assumed to be cuspidal. A similar argument shows that $Hom_U(1_U, \overline{\rho}) = 0$.

□

1.4 $\Pi_{x,1}$ is a projective generator

In this subsection, we prove Theorem 1.1.5: $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$. Before doing this, let us recall the setting. Fix a vertex x of the building of G. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which is trivial on G_x^+ and whose reduction to $\overline{G_x} = G_x/G_x^+$ is regular supercuspidal, $\pi = \text{c-Ind}_{G_x}^{G(F)} \rho$ as before. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ, and $\mathcal{C}_{x,1}$ the block of $\mathrm{Rep}_\Lambda(G(F))$ containing π.

Let V be the set of equivalence classes of vertices of the Bruhat-Tits building $\mathcal{B}(G,F)$ up to $G(F)$-action. For $y \in V$, let $\sigma_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$. Let $\Pi := \bigoplus_{y \in V} \Pi_y$ where $\Pi_y := \text{c-Ind}_{G_y^+}^{G(F)} \Lambda$. Then Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G(F))_0$, see [Dat09, Appendix]. Let $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xhookrightarrow{summand} \mathrm{Rep}_\Lambda(G_x)$ be the $\mathcal{B}_{x,1}$-summand of σ_x. And let $\Pi_{x,1} := \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$.

Let's summarize the setting in the following diagram.

$$\begin{array}{ccc}
\mathrm{Rep}_\Lambda(G_x) & \xrightarrow{\text{c-Ind}_{G_x}^{G(F)}} & \mathrm{Rep}_\Lambda(G(F)) \\
\cup\!| & & \cup\!| \\
\mathrm{Rep}_\Lambda(G_x)_0 & \longrightarrow & \mathrm{Rep}_\Lambda(G(F))_0 \\
\cup\!| & & \cup\!| \\
\mathcal{B}_{x,1} & \longrightarrow & \mathcal{C}_{x,1} \\
\shortparallel & & \shortparallel \\
\text{block of } \rho & & \text{block of } \pi
\end{array}$$

Theorem 1.4.1. $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ *is a projective generator of $\mathcal{C}_{x,1}$.*

Proof. First, let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consisting of representations that are trivial on G_x^+ (Don't confuse with $\mathrm{Rep}_\Lambda(G(F))_0$, the depth-zero category of G). Note $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand of $\mathrm{Rep}_\Lambda(G_x)$ (see Lemma 1.2.10).

Second, note that $\mathrm{Rep}_\Lambda(G_x)_0 \cong \mathrm{Rep}_\Lambda(\overline{G_x})$. We may assume

$$\mathrm{Rep}_\Lambda(G_x)_0 = \mathcal{B}_{x,1} \oplus ... \oplus \mathcal{B}_{x,m}$$

is its block decomposition. So that $\sigma_x = \sigma_{x,1} \oplus ... \oplus \sigma_{x,m}$ accordingly. Write $\sigma_x^1 := \sigma_{x,2} \oplus ... \oplus \sigma_{x,m}$. Then $\sigma_x = \sigma_{x,1} \oplus \sigma_x^1$, and $\Pi_x = \Pi_{x,1} \oplus \Pi_x^1$ accordingly, where $\Pi_x^1 := \text{c-Ind}_{G_x}^{G(F)} \sigma_x^1$. And

$$\Pi = \Pi_{x,1} \oplus \Pi_x^1 \oplus \Pi^x,$$

where $\Pi^x := \bigoplus_{y \in V, y \neq x} \Pi_y$. Let $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$, then we have

$$\Pi = \Pi_{x,1} \oplus \Pi^{x,1}.$$

Recall that Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G(F))_0$. This implies that

$$\mathrm{Hom}_G(\Pi, -) : \mathrm{Rep}_\Lambda(G(F))_0 \to \text{Mod-}\,\mathrm{End}_G(\Pi)$$

is an equivalence of categories. See [Ber92, Lemma 22].

Next, it is not hard to see that Theorem 1.1.4 implies that

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0,$$

see Lemma 1.4.2. This implies that

$$\text{Mod-}\,\mathrm{End}_G(\Pi) \cong \text{Mod-}\,\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\,\mathrm{End}_G(\Pi^{x,1})$$

is an equivalence of categories.

Now we can combine the above to show that $\Pi^{x,1}$ does not interfere with $\Pi_{x,1}$, i.e.,

$$\mathrm{Hom}_G(\Pi^{x,1}, X) = 0,$$

for any object $X \in \mathcal{C}_{x,1}$ (see Importent Lemma 1.4.3).

However, since Π is a projective generator of $\mathrm{Rep}_\Lambda(G(F))_0$, we have

$$\mathrm{Hom}_G(\Pi, X) \neq 0,$$

for any $X \in \mathcal{C}_{x,1}$. This together with the last paragraph implies that

$$\mathrm{Hom}_G(\Pi_{x,1}, X) \neq 0,$$

for any $X \in \mathcal{C}_{x,1}$, i.e. $\Pi_{x,1}$ is a generator of $\mathcal{C}_{x,1}$.

Finally, note $\Pi_{x,1}$ is projective in $\mathrm{Rep}_\Lambda(G(F))_0$ since it is a summand of the projective object Π. Hence $\Pi_{x,1}$ is projective in $\mathcal{C}_{x,1}$. This together with the last paragraph implies that $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$.

□

Lemma 1.4.2.
$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0.$$

Proof. Recall that $\Pi^{x,1} := \Pi^1_x \oplus \Pi^x$.

First, we compute

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^1_x) = \mathrm{Hom}_{G_x}(\sigma_{x,1}, \sigma^1_x) = 0,$$

where the first equality is the first case of Theorem 1.1.4 (note $\sigma_{x,1} \in \mathcal{B}_{x,1}$, hence has supercuspidal reduction by Theorem 1.1.3, and hence the condition of Theorem 1.1.4 is satisfied), and the second equality is because $\sigma_{x,1}$ and σ^1_x lie in different blocks of $\mathrm{Rep}_\Lambda(G_x)$ by definition.

Second, recall that $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G(F)} \sigma_{x,1}$ with $\sigma_{x,1}$ having supercuspidal reduction, and $\Pi_y = \text{c-Ind}_{G_y}^{G(F)} \sigma_y$. We compute

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^x) = \bigoplus_{y \in V, y \neq x} \mathrm{Hom}_G(\Pi_{x,1}, \Pi_y) = 0,$$

by the second case of Theorem 1.1.4.

Combining the above three paragraphs, we get $\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = 0$.

A same argument shows that $\mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0$. □

Lemma 1.4.3 (Important Lemma)**.** $\mathrm{Hom}_G(\Pi^{x,1}, X) = 0$, *for any object* $X \in \mathcal{C}_{x,1}$.

Proof. Recall that

$$\mathrm{Hom}_G(\Pi, -) : \mathrm{Rep}_\Lambda(G(F))_0 \to \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi) \cong \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{x,1}) \oplus \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi^{x,1})$$

is an equivalence of categories. It is even an equivalence of abelian categories since $\mathrm{Hom}_G(\Pi, -)$ is exact and commutes with direct product. Hence the image of $\mathcal{C}_{x,1}$ must be indecomposable as $\mathcal{C}_{x,1}$ is indecomposable, i.e.,

$$\mathrm{Hom}_G(\Pi, -) = \mathrm{Hom}_G(\Pi_{x,1}, -) \oplus \mathrm{Hom}_G(\Pi^{x,1}, -)$$

can map $\mathcal{C}_{x,1}$ nonzeroly to only one of $\mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{x,1})$ and $\mathrm{Mod}\text{-}\mathrm{End}_G(\Pi^{x,1})$ (See the diagram below).

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda(G(F))_0 \arrow[rr, "\mathrm{Hom}_G(\Pi,-)"] & & \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi) \\
\mathcal{C}_{x,1} \arrow[u, "\cup"] \arrow[r, "\mathrm{Hom}_G(\Pi_{x,1},-)\oplus\mathrm{Hom}_G(\Pi^{x,1},-)"] & \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{x,1}) \oplus \mathrm{Mod}\text{-}\mathrm{End}_G(\Pi^{x,1}) \arrow[ur, "\cong"] &
\end{tikzcd}$$

Then it must be $\mathrm{Mod}\text{-}\mathrm{End}_G(\Pi_{x,1})$ (that $\mathrm{Hom}_G(\Pi, -)$ maps $\mathcal{C}_{x,1}$ nonzeroly to) since

$$\mathrm{Hom}_G(\Pi_{x,1}, \pi) = \mathrm{Hom}_{G_x}(\sigma_{x,1}, \rho) = \mathrm{Hom}_{G_x}(\sigma_x, \rho) \neq 0.$$

In other words, $\mathrm{Hom}_G(\Pi^{x,1}, -)$ is zero on $\mathcal{C}_{x,1}$.

□

1.5 Application: description of the block $\mathrm{Rep}_\Lambda(G(F))_{[\pi]}$

Recall we denote $\mathcal{A}_{x,1} = \mathrm{Rep}_\Lambda(\overline{G_x})_{[\overline{\rho}]}$, $\mathcal{B}_{x,1} = \mathrm{Rep}_\Lambda(G_x)_{[\rho]}$, and $\mathcal{C}_{x,1} = \mathrm{Rep}_\Lambda(G(F))_{[\pi]}$. We have proven that the inflation along $G_x \to \overline{G_x}$ induces an equivalence of categories

$$\mathcal{A}_{x,1} \cong \mathcal{B}_{x,1},$$

see Lemma 1.2.12. And we have also proven that the compact induction induces an equivalence of categories

$$\text{c-Ind}_{G_x}^{G(F)} : \mathcal{B}_{x,1} \cong \mathcal{C}_{x,1}.$$

Hence $\mathcal{C}_{x,1} \cong \mathcal{A}_{x,1}$, where the latter is isomorphic to the block of a finite torus via Broué's equivalence 1.2.4.

We will see in the example (See Chapter **??**) of GL_n that (up to central characters) such a block of a finite torus corresponds to $\mathrm{QCoh}(\mu)$, where μ is the group scheme of roots of unity appearing in the computation of the L-parameter side (See Theorem **??**).

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