This is only an example

X Y

Born in New York, U.S.A.

Master's Thesis Mathematics

Advisor: Prof. Dr. X Y

Second Advisor: Prof. Dr. X Y

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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Chapter 1

MoLP

1.1 Description of the connected component X_φ containing a TRSELP φ

1.1.1 Recollections on the moduli space of Langlands parameters

I assume the readers to be familiar with the theory of the moduli space of Langlands parameters, see for example [7, Section 3 and Section 4], or [9, Section 2 and Section 4]. (I could also recollect the theory in the appendix.)

Let us first fix some notations.

- Let $p \neq 2$ be a fixed prime number and $\ell \neq 2$ be a prime number different from p.
- Let F be a non-archimedean local field with residue characteristic $q = p^r$ for some $r \in \mathbb{Z}_{\geq 1}$.
- Let W_F be the Weil group of F, $I_F \subset W_F$ be the inertia subgroup, P_F be the wild inertia subgroup.
- Let $W_t := W_F/P_F$ be the tame Weil group.
- Let $I_t := I_F/P_F$ be the tame inertia subgroup.

Fix $\mathrm{Fr} \in W_F$ any lift of the arithmetic Frobenius. I will abuse the notation and also denote Fr the image of Fr in W_t. Then $W_t \simeq I_t \rtimes \langle \mathrm{Fr} \rangle$. Here I_t is non-canonically isomorphic to $\prod_{p' \neq p} \mathbb{Z}_{p'}$, which is procyclic. We fix such an isomorphism

$$I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}. \tag{1.1.1}$$

This gives rise to a topological generator s_0 of I_t, which correspond to $(1, 1, ...)$ under the above isomorphism (1.1.1). Let us recall the following important relation in I_F/P_F:

$$\mathrm{Fr}\,.s_0.\,\mathrm{Fr}^{-1} = s_0^q. \tag{1.1.2}$$

In fact, this is true for any $s \in I_t$ instead of s_0.

Let

$$W_t^0 := \langle s_0, \mathrm{Fr} \rangle = \mathbb{Z}[1/p]^{s_0} \rtimes \mathbb{Z}^{\mathrm{Fr}}$$

be the subgroup of W_t generated by s_0 and Fr. Denote $W_F^0 \subset W_F$ the preimage of W_t^0 under the natural projection $W_F \to W_t$. This is known as the discretization of the Weil group. To summarize, W_t^0 is generated by two elements Fr and s_0 with a single relation $\mathrm{Fr}\,.s_0.\,\mathrm{Fr}^{-1} = s_0^q$.

Let G be a connected split reductive group over F. Let $\hat{G}$ be its dual group over $\mathbb{Z}$. Then the space of cocycles from the discretization

$$Z^1(W_t^0, \hat{G}) = \underline{\mathrm{Hom}}(W_t^0, \hat{G}) = \{(x,y) \in \hat{G} \times \hat{G} \mid yxy^{-1} = x^q\} \tag{1.1.3}$$

is an explicit closed subscheme of $\hat{G} \times \hat{G}$ (See [7, Section 3]). An important fact (See [7, Proposition 3.9]) is that over a $\mathbb{Z}_\ell$-algebra R (the cases $R = \overline{\mathbb{F}_\ell}, \overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}$ are most relevant for us), the restriction from W_t to W_t^0 induces an isomorphism

$$Z^1(W_t, \hat{G}) \simeq Z^1(W_t^0, \hat{G}).$$

Therefore, we can compute $Z^1(W_t, \hat{G})$ using the explicit formula (1.1.3) above. This is fundamental for the study of the moduli space of Langlands parameters $Z^1(W_t, \hat{G})$. I refer the readers to [7, Section 3 and Section 4] for the precise definition and properties of $Z^1(W_t, \hat{G})$.

(maybe add an example here)

Let I_F^ℓ be the prime-to-ℓ inertia subgroup of W_F, i.e., $I_F^\ell := \ker(t_\ell)$, where

$$t_\ell : I_F \to I_F/P_F \simeq \prod_{p' \neq p} \mathbb{Z}_{p'} \to \mathbb{Z}_\ell$$

is the composition. In other words, it is the maximal subgroup of I_F with pro-order prime to ℓ. This property makes I_F^ℓ important when determining the connected components of $Z^1(W_F, \hat{G})$ over $\overline{\mathbb{Z}_\ell}$ (See [7, Theorem 4.2 and Subsection 4.6]).

1.1.2 Tame regular semisimple elliptic L-parameters

I want to define a class of L-parameters, called TRSELP, which roughly corresponds to depth-zero regular supercuspidal representations. Before that, let me define the concept of schematic centralizer, which will be used throughout the article.

Definition 1.1.1 (Schematic centralizer)**.** *Let H be an affine algebraic group over a ring R, Γ be a finite group. Let $u \in Z^1(\Gamma, H(R'))$ be a 1-cocycle for some R-algebra R'. Let*

$$\alpha_u : H_{R'} \longrightarrow Z^1(\Gamma, H)_{R'} \qquad h \longmapsto hu(-)h^{-1}$$

be the orbit morphism. Then the schematic centralizer $C_H(u)$ is defined to be the fiber of α_u at u.

$$\begin{array}{ccc} C_H(u) & \longrightarrow & H_{R'} \\ \downarrow & & \downarrow{\scriptstyle \alpha_u} \\ R' & \xrightarrow{u} & Z^1(\Gamma, H)_{R'} \end{array}$$

One can show that $C_H(u)(R'') = C_{H(R'')}(u)$ is the set-theoretic centralizer for all R'-algebra R'', see for example [9, Appendix A].

Remark 1.1.2. Note this is enough for our applications where Γ is more generally taken as a profinite group, because $u : \Gamma \to H$ usually factors through a finite quotient Γ' of Γ.

Let me now define a tame, regular semisimple, elliptic Langlands parameter (TRSELP for short) over $\overline{\mathbb{F}_\ell}$, roughly in the sense of [10, Section 3.4 and Section 4.1] in the case that G is F-split, but with $\overline{\mathbb{F}_\ell}$-coefficients instead of $\mathbb{C}$-coefficients.

Definition 1.1.3. *A* ***tame regular semisimple elliptic*** L***-parameter (TRSELP) over*** $\overline{\mathbb{F}_\ell}$ *is a homomorphism* $\varphi : W_F \to \hat{G}(\overline{\mathbb{F}_\ell})$ *such that:*

1. *(smooth)* $\varphi(I_F)$ *is a finite subgroup of* $\hat{G}(\overline{\mathbb{F}_\ell})$*.*
2. *(Frobenius semisimple)* $\varphi(\mathrm{Fr})$ *is a semisimple element of* $\hat{G}(\overline{\mathbb{F}_\ell})$*.*
3. *(tame) The restriction of* φ *to* P_F *is trivial.*
4. *(elliptic) The identity component of the centralizer* $C_{\hat{G}}(\varphi)^0$ *is equal to the identity component of the center* $Z(\hat{G})^0$*.*
5. *(regular semisimple) The centralizer of the inertia* $C_{\hat{G}}(\varphi|_{I_F})$ *is a torus (in particular, connected).*

Concretely, a TRSELP consists of the following data:

1. The restriction to the inertia $\varphi|_{I_F}$, which is a direct sum of characters of finite abelian groups since $I_F/P_F \simeq \varprojlim \mathbb{F}_{q^n}^*$ is compact abelian and that
$$\mathrm{Hom}_{\mathrm{Cont}}(I_F/P_F, \overline{\mathbb{F}_\ell}^*) \simeq \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*) \simeq \varinjlim \mathrm{Hom}_{\mathrm{Cont}}(\mathbb{F}_{q^n}^*, \overline{\mathbb{F}_\ell}^*).$$
In particular, it factors through (the $\overline{\mathbb{F}_\ell}$-points of) some maximal torus, say S. Then regular semisimple means that $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

2. The image of Frobenius $\varphi(\mathrm{Fr})$, which turns out to be an element of the normalizer $N_{\hat{G}(\overline{\mathbb{F}_\ell})}(S)$ (Since $\mathrm{Fr}\,.s.\,\mathrm{Fr}^{-1} = s^q \in I_t$ for any $s \in I_t$ implies that $\varphi(\mathrm{Fr})$ normalizes $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.). And "elliptic" means that the center $Z(\hat{G})$ has finite index in the centralizer $C_{\hat{G}}(\varphi)$. As we will see later, ellipticity implies that $\hat{G}(\overline{\mathbb{F}_\ell})$ acts transitively on the connected component $X_\varphi(\overline{\mathbb{F}_\ell})$ of the moduli space of L-parameters containing φ (See the proof of Lemma 1.2.4), which is essential for the description (roughly, see Theorem 1.2.3 for the precise statement)
$$[X_\varphi/\hat{G}] \simeq [*/\underline{S_\varphi}]$$
where $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is the centralizer of the whole L-parameter φ.

(Maybe add an example here)

Remark 1.1.4. 1. Let $\overline{\Lambda} \in \{\overline{\mathbb{Z}_\ell}, \overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. It is important for my purpose to distinguish between the set-theoretic centralizer (for example, $C_{\hat{G}(\overline{\Lambda})}(\varphi(I_F))$) and the schematic centralizer (for example, $C_{\hat{G}}(\varphi)$). However, I might still use $\hat{G}$ to mean $\hat{G}(\overline{\Lambda})$ sometimes by abuse of notation, for which I hope the readers could recognize. One reason for doing this is that $\hat{G}$ is split over $\overline{\Lambda}$, hence $\hat{G}$ is completely determined by its $\overline{\Lambda}$-points. And many statements can either be phrased in terms of the $\overline{\Lambda}$-scheme or its $\overline{\Lambda}$-points (for example, 4 and 5 in Definition 1.1.3).

2. As we will see later in Theorem 1.1.5, $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ turns out to be the $\overline{\mathbb{F}_\ell}$-points of the split torus $T = C_{\hat{G}}(\psi|_{I_F^\ell})$ for some lift ψ of φ over $\overline{\mathbb{Z}_\ell}$.

1.1.3 Description of the component

Now given a TRSELP $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$. Pick any lift $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ of φ, whose existence is ensured by the flatness of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ (See Lemma 1.1.7). Let $\psi_\ell := \psi|_{I_F^\ell}$ denotes the restriction of ψ to the prime-to-ℓ inertia. Note that $\psi \in Z^1(W_F, \hat{G})$ factors through $N_{\hat{G}}(\psi_\ell)$ (Since I_F^ℓ is normal in W_F). Let $\overline{\psi}$ denotes the image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Let X_φ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ. Note X_φ also contains ψ since ψ specializes to φ. So we sometimes also denote X_φ as X_ψ.

Theorem 1.1.5. *Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a TRSELP over $\overline{\mathbb{F}_\ell}$. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of φ. Then at least when the center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$, the connected component $X_\varphi = X_\psi$ of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ is isomorphic to*

$$\left(\hat{G} \times C_{\hat{G}}(\psi_\ell)^0 \times \mu\right) / \, C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where

1. *$C_{\hat{G}}(\psi_\ell)^0$ is the identity component of the schematic centralizer $C_{\hat{G}}(\psi_\ell)$, which turns out to be a split torus T over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$.*

2. *μ is the connected component of $T^{\mathrm{Fr}=(-)^q}$ (the subscheme of T on which Fr acts by raising to q-th power) containing 1 (See [7, Example 3.14]), which is a product of some $\mu_{\ell^{k_i}}$ (the group scheme of ℓ^{k_i}-th roots of unity over $\overline{\mathbb{Z}_\ell}$), $k_i \in \mathbb{Z}_{\geq 0}$. Note that μ could be trivial, depending on $\hat{G}$ and some congruence relations between q, ℓ.*

3. *$C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ is the (schematic) stabilizer (definition see Appendix. Could be defined by Yoneda) of $\overline{\psi}$ in $C_{\hat{G}}(\psi_\ell)$.*

In other words, we have the following isomorphism of schemes over $\overline{\mathbb{Z}_\ell}$:

$$X_\varphi \simeq \left(\hat{G} \times T \times \mu\right) / T.$$

And we will specify in the next subsection what the T-action on $\left(\hat{G} \times T \times \mu\right)$ is.

Proof. First, recall by [7, Subsection 4.6],

$$X_\psi \simeq \left(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}\right) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

where $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$ denotes the space of cocycles whose restriction to I_F^ℓ equals ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\overline{\psi}$. Explanation: Recall (See [7, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the L-parameters whose restriction to I_F^ℓ is $\hat{G}$-conjugate to ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $\overline{\psi}$. Hence

$$X_\varphi \cong (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}) / C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \qquad g\eta(-)g^{-1} \leftarrow\!\shortmid (g, \eta),$$

with $C_{\hat G}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat G \times Z^1(W_F, N_{\hat G}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$ by

$$(t,(g,\psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat G}(\psi_\ell)_{\overline{\psi}}$ and $(g,\psi') \in (\hat G \times Z^1(W_F, N_{\hat G}(\psi_\ell))_{\psi_\ell,\overline{\psi}})$.

Second, $\eta.\psi \leftarrow\!\shortmid \eta$ defines an isomorphism

$$Z^1(W_F, N_{\hat G}(\psi_\ell))_{\psi_\ell,\overline{\psi}} \simeq Z^1_{Ad(\psi)}(W_F, N_{\hat G}(\psi_\ell)^0)_{1_{I_F^\ell}} =: Z^1_{Ad(\psi)}(W_F, N_{\hat G}(\psi_\ell)^0)_1$$

where $Z^1_{Ad(\psi)}(W_F, N_{\hat G}(\psi_\ell))$ means the space of cocycles with W_F acting on $N_{\hat G}(\psi_\ell)$ via conjugacy action through ψ, and the subscript $1_{I_F^\ell}$ or 1 means the cocycles whose restriction to I_F^ℓ is trivial. Explanation: This is clear by unraveling the definitions: two cocycles whose restriction to I_F^ℓ are both ψ_ℓ differ by something whose restriction to I_F^ℓ is trivial; two cocycles whose pushforward to $Z^1(W_F, \pi_0(N_{\hat G}(\psi_\ell)))$ are both $\overline{\psi}$ differ by something whose pushforward to $Z^1(W_F, \pi_0(N_{\hat G}(\psi_\ell)))$ is trivial, i.e., which factors through the identity component $N_{\hat G}(\psi_\ell)^0$.

Next, I show that $C_{\hat G}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$. By [7, Subsection 3.1], the centralizer $C_{\hat G}(\psi_\ell)$ is generalized reductive (See Lemma 1.1.8), hence split over $\overline{\mathbb{Z}_\ell}$, and $N_{\hat G}(\psi_\ell)^0 = C_{\hat G}(\psi_\ell)^0$. So we can determine $C_{\hat G}(\psi_\ell)$ by computing its $\overline{\mathbb{F}_\ell}$-points. Indeed,

$$C_{\hat G}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat G(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat G(\overline{\mathbb{F}_\ell})}(\varphi(I_F)),$$

where the last equality follows since I_F/I_F^ℓ doesn't contribute to the image of φ (See Lemma 1.1.9). Therefore, $C_{\hat G}(\psi_\ell)$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $S = C_{\hat G(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$. Denote $T = C_{\hat G}(\psi_\ell)$. In particular, $C_{\hat G}(\psi_\ell)$ is connected, hence

$$N_{\hat G}(\psi_\ell)^0 = C_{\hat G}(\psi_\ell)^0 = C_{\hat G}(\psi_\ell) = T.$$

Now we could compute

$$Z^1_{Ad(\psi)}(W_F, N_{\hat G}(\psi_\ell)^0) = Z^1_{Ad(\psi)}(W_F, T) \simeq T \times T^{\mathrm{Fr}=(-)^q},$$

where the last isomorphism is given by $\eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$, where $s_0 \in W_t^0$ is the topological generator of I_t fixed before (See [7, Example 3.14]).

Then we show that the identity component of $T^{\mathrm{Fr}=(-)^q}$ gives μ in the statement of the theorem. Note $T^{\mathrm{Fr}=(-)^q}$ is a diagonalizable group scheme over $\overline{\mathbb{Z}_\ell}$ of dimension zero (This can be seen either by $\dim Z^1(W_F/P_F, T) = \dim T$, or by noticing that $\eta(s_0) \in T^{\mathrm{Fr}=(-)^q}$ is semisimple with finitely many possible eigenvalues), hence of the form $\prod_i \mu_{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$. Hence its connected component $(T^{\mathrm{Fr}=(-)^q})^0$ over $\overline{\mathbb{Z}_\ell}$ is of the form $\prod_i \mu_{\ell^{k_i}}$, with k_i maximal such that ℓ^{k_i} divides n_i. Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat G}(\psi_\ell)^0)_1 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0 \simeq T \times (T^{\mathrm{Fr}=(-)^q})^0 \simeq T \times \mu,$$

(See Lemma 1.1.10 for the first isomorphism[1]) where μ is of the form $\prod_i \mu_{\ell^{k_i}}$.

Finally, we show that $C_{\hat G}(\psi_\ell)_{\overline{\psi}} = C_{\hat G}(\psi_\ell)$. Recall $C_{\hat G}(\psi_\ell)$ acts on $Z^1(W_F, N_{\hat G}(\psi_\ell))$ by conjugation, inducing an action of $C_{\hat G}(\psi_\ell)$ on $Z^1(W_F, \pi_0(N_{\hat G}(\psi_\ell)))$. And $C_{\hat G}(\psi_\ell)_{\overline{\psi}}$ is

[1] This can be rewritten more elegantly if I have time, with as less dependence on [7] as possible.

by definition the stabilizer of $\overline{\psi} \in Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ in $C_{\hat{G}}(\psi_\ell)$. Now $C_{\hat{G}}(\psi_\ell) = T$ is connected, hence acts trivially on the component group $\pi_0(N_{\hat{G}}(\psi_\ell))$ [2], hence also acts trivially on $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Therefore, the stabilizer $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} = C_{\hat{G}}(\psi_\ell)$ (Check).

Above all, we have

$$X_\varphi \simeq (\hat{G} \times Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1)/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \simeq (\hat{G} \times T \times \mu)/T.$$

□

1.1.4 The T-action on $(\hat{G} \times T \times \mu)$

For later use, let me make it explicit the T-action on $(\hat{G} \times T \times \mu)$.

Recall (See [7, Subsection 4.6]) first that the component $X_\varphi = X_\psi$ morally consists of the L-parameters whose restriction to I_F^ℓ is $\hat{G}$-conjugate to ψ_ℓ and whose image in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$ is $\hat{G}$-conjugate to $\overline{\psi}$. Hence X_φ is isomorphic to

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$$

via $g\eta(-)g^{-1} \leftarrow\!\shortmid (g, \eta)$, with $C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acting on $(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$ by

$$(t, (g, \psi')) \mapsto (gt^{-1}, t\psi'(-)t^{-1}),$$

where $t \in C_{\hat{G}}(\psi_\ell)_{\overline{\psi}} \simeq T$ and $(g, \psi') \in (\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})$.

Next, recall that

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq T \times \mu \quad \eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0)).$$

Let's focus on the isomorphism $\eta.\psi \leftarrow\!\shortmid \eta$:

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1.$$

Recall that $T \subset \hat{G}$ acts on $Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}}$ via conjugation. Hence the above isomorphism induces an T-action on $Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$, by

$$(t, \eta) \mapsto (t(\eta\psi(-))t^{-1})\psi^{-1}.$$

Hence in $(\hat{G} \times T \times \mu)/T$, we compute by tracking the above isomorphisms that

1. T acts on $\hat{G}$ via $(t, g) \mapsto gt^{-1}$.

2. $T = C_{\hat{G}}(\psi_\ell)_{\overline{\psi}}$ acts on $T \subset (T \times \mu)$ (corresponds to $\eta(\mathrm{Fr})$) by twisted conjugacy (due to the isomorphisms $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$), i.e.,

$$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where $n = \psi(\mathrm{Fr})$; Note here n, a prior lies in $\hat{G}$, actually lies in $N_{\hat{G}}(T)$ (Since $\mathrm{Fr}\,.s.\,\mathrm{Fr}^{-1} = s^q$ implies that $\psi(\mathrm{Fr})$ normalizes $C_{\hat{G}}(\psi|_{I_F^\ell}) = T$. Check!). To summarize, $t \in T$ acts on T via multiplication by $tnt^{-1}n^{-1}$.

[2] Explain this if have time

3. T acts trivially on μ. This is because $\eta(s_0) \in T$ and $\psi(s_0) \in T$ (Check!).

On the other hand, recall we have the natural $\hat{G}$-action on $Z^1(W_F, \hat{G})$ by conjugation, hence the $\hat{G}$-action on this component X_φ. Under the isomorphism $X_\varphi \simeq (\hat{G} \times T \times \mu)/T$, the $\hat{G}$-action becomes

$$(g', (g, t, m)) \mapsto (g'g, t, m), \text{ for any } g' \in \hat{G} \text{ and } (g, t, m) \in (\hat{G} \times T \times \mu)/T.$$

Note that the T-action and the $\hat{G}$-action on $(\hat{G} \times T \times \mu)$ commute with each other, we thus have the following:

Proposition 1.1.6.

$$[X_\varphi/\hat{G}] = \left[\left((\hat{G} \times T \times \mu)/T\right)/\hat{G}\right] \simeq \left[\left((\hat{G} \times T \times \mu)/\hat{G}\right)/T\right] \simeq [(T \times \mu)/T],$$

with $t \in T$ acting on T via multiplication by $tnt^{-1}n^{-1}$, and $t \in T$ acting trivially on μ.

1.1.5 Some lemmas

Lemma 1.1.7. *Let $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$. Then there exists $\psi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{Z}_\ell}))$ such that ψ' is a lift of φ'.*

Proof. In the statement, $Z^1(W_t, \hat{G})$ is the abbreviation for $Z^1(W_t, \hat{G})_{\overline{\mathbb{Z}_\ell}}$. Recall that $Z^1(W_t, \hat{G}) \to \overline{\mathbb{Z}_\ell}$ is flat (See [7, Proposition 3.3]), hence generalizing (See Stack Project, 01U2). Therefore, given $\varphi' \in Z^1(W_t, \hat{G}(\overline{\mathbb{F}_\ell}))$, there exists $\xi \in Z^1(W_t, \hat{G}(\overline{\mathbb{Q}_\ell}))$ such that ξ specializes to φ'. In other words, $\ker(\xi) \subset \ker(\varphi')$. I'm going to show that $\xi : W_t \to \hat{G}(\overline{\mathbb{Q}_\ell})$ factors through $\hat{G}(\overline{\mathbb{Z}_\ell})$.

This is true by the following more general statement: Let $Y = \mathrm{Spec}(R)$ be an affine scheme over $\overline{\mathbb{Z}_\ell}$, let $y_\eta \in Y(\overline{\mathbb{Q}_\ell})$ specializing to $y_s \in Y(\overline{\mathbb{F}_\ell})$. Then $y_\eta \in Y(\overline{\mathbb{Q}_\ell}) = \mathrm{Hom}(R, \overline{\mathbb{Q}_\ell})$ factors through $\overline{\mathbb{Z}_\ell}$.

To prove the above statement, let $\mathfrak{p} := \ker(y_\eta)$ and $\mathfrak{q} := \ker(y_s)$ be the corresponding prime ideas. Then "y_η specializes to y_s" translates to "$\mathfrak{p} \subset \mathfrak{q}$". Recall that we are going to show that $y_\eta : R \to \overline{\mathbb{Q}_\ell}$ factors through $\overline{\mathbb{Z}_\ell}$. We argue by contradiction. Otherwise there is some element $f \in R$ mapping to $\ell^{-m}u$ for some $m \in \mathbb{Z}_{\geq 1}$ and $u \in \overline{\mathbb{Z}_\ell}^*$. Hence

$$\ell^m u^{-1} f - 1 \in (y_\eta) \subset \ker(y_s). \tag{1.1.4}$$

However, $\ell \in \ker(y_s)$ since y_s lives on the special fiber. This together with equation 1.1.4 implies that $1 \in \ker(y_s)$. Contradiction! □

Lemma 1.1.8. *The schematic centralizer $C_{\hat{G}}(\psi_\ell)$ is a generalized reductive group scheme over $\overline{\mathbb{Z}_\ell}$.*

Proof. To invoke [7, Lemma 3.2], I first show that

$$C_{\hat{G}}(\psi_\ell) = C_{\hat{G}}(\psi(I_F^\ell)),$$

where $C_{\hat{G}}(\psi(I_F^\ell))$ is the schematic centralizer of the subgroup $\psi(I_F^\ell) \subset \hat{G}(\overline{\mathbb{Z}_\ell})$ in $\hat{G}$. This can be checked by Yoneda Lemma on R-valued points for any $\overline{\mathbb{Z}_\ell}$-algebra R.

Then we could conclude by [7, Lemma 3.2]. Indeed, ψ_ℓ factors through some finite quotient Q of I_F^ℓ, which has order invertible in the base $\overline{\mathbb{Z}_\ell}$. So the assumptions of [7, Lemma 3.2] are satisfied.

Some explanations to use [7, Lemma 3.2]:

1. While [7, Lemma 3.2] is phrased in the setting that R is a normal subring of a number field, it still works for $\overline{\mathbb{Z}_\ell} \subset \overline{\mathbb{Q}_\ell}$ instead of $\mathbb{Z} \subset \mathbb{Q}$. Indeed, ψ_ℓ factors through some finite quotient Q of I_F^ℓ, say of order $|Q| = N$ (Note N is coprime to ℓ since Q is a quotient of I_F^ℓ). Then we could use [7, Lemma 3.2] to conclude that $C_{\hat{G}}(\psi_\ell)$ is generalized reductive over $\mathbb{Z}[1/pN]$ (Check!). Hence $C_{\hat{G}}(\psi_\ell)$ is also generalized reductive over $\overline{\mathbb{Z}_\ell}$ by base change.

2. There is also a small issue that $\overline{\mathbb{Z}_\ell}$ is not finite over $\mathbb{Z}_\ell$, but this can be resolved since everything is already defined over some sufficiently large finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$.

□

Lemma 1.1.9.

$$C_{\hat{G}}(\psi_\ell)(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F^\ell)) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)).$$

Proof. The first equation is by definition (and that $C_{\hat{G}}(\psi_\ell)$ represents the set-theoretic centralizer).

For the second equation, note that $\varphi|_{I_t} = \gamma_1 + ... + \gamma_d$ is a direct sum of characters (Since $I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}$), so it suffices to show that each γ_i is trivial on the summand $\mathbb{Z}_\ell$ of $I_t \simeq \prod_{p' \neq p} \mathbb{Z}_{p'}$. Indeed,

$$\mathrm{Hom}_{\mathrm{Cont}}(\mathbb{Z}_\ell, \overline{\mathbb{F}_\ell}^*) = \mathrm{Hom}_{\mathrm{Cont}}(\varprojlim \mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \varinjlim \mathrm{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}^*) = \{1\}.$$

□

Lemma 1.1.10. $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0$.

Proof. I have omitted from the notations but here everything is over $\overline{\mathbb{Z}_\ell}$. Recall that $N_{\hat{G}}(\psi_\ell)^0 = C_{\hat{G}}(\psi_\ell)^0 = T$ and that

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0) \simeq T \times T^{\mathrm{Fr}=(-)^q}.$$

By [7, Section 5.4, 5.5], $Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1$ is connected (over $\overline{\mathbb{Z}_\ell}$). I check here that the assumptions of [7, Section 5.4, 5.5] are satisfied. Indeed, since $N_{\hat{G}}(\psi_\ell)^0 = T$ is a connected torus, the W_t^0-action on T automatically fixes a Borel pair of T. Moreover, s_0 acts trivially on $N_{\hat{G}}(\psi_\ell)^0 = T$ via ψ, so in particular s_0 (which is denoted by s in [7, Section 5.5]) has order a power of ℓ (which is $1 = \ell^0$).

Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \subset Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0.$$

By [7, Section 4.6],

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \hookrightarrow Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)$$

is open and closed. This is done by considering the restriction to the prime-to-ℓ inertia I_F^ℓ, and then use [7, Theorem 4.2].

Therefore,

$$Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 = Z^1_{Ad(\psi)}(W_F, N_{\hat{G}}(\psi_\ell)^0)^0 \simeq (T \times T^{\mathrm{Fr}=(-)^q})^0.$$

□

1.2 Main Theorem: description of $[X_\varphi/\hat{G}]$

Let F be a non-archimedean local field, G be a connected split reductive group over F. Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be a tame, regular semisimple, elliptic L-parameter (TRSELP for short). Recall that this means that the centralizer

$$C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) =: S \subset \hat{G}(\overline{\mathbb{F}_\ell})$$

is a maximal torus, and $\varphi(\mathrm{Fr}) \in N_{\hat{G}}(S)$ gives rise to an element $w = \overline{\varphi(\mathrm{Fr})} \in N_{\hat{G}}(S)/S$ in the Weyl group (and that φ is tame and elliptic).

Assume that

1. The center $Z(\hat{G})$ is smooth over $\overline{\mathbb{Z}_\ell}$.
2. $Z(\hat{G})$ is finite.

Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lifting of φ. Let ψ_ℓ denotes the restriction $\psi|_{I_F^\ell}$, and $\overline{\psi}$ denotes the image of ψ in $Z^1(W_F, \pi_0(N_{\hat{G}}(\psi_\ell)))$. Recall that the schematic centralizer $C_{\hat{G}}(\psi_\ell) = T$ is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\overline{\mathbb{F}_\ell}$-points $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F)) = S$.

For later use, I record the following lemma – w can also be defined in terms of ψ instead of φ. This is helpful because we will reduce to a computation on the special fiber later. First, notice that since T is a split torus over $\overline{\mathbb{Z}_\ell}$ with $\ell \neq 2$, we can identify

$$\left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) \simeq \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell}),$$

and denote it by Ω. (See Lemma 1.2.6 below)

Remark 1.2.1. Lemma 1.2.6 below shows that $N_{\hat{G}}(T)/T$ is representable by a group scheme which is split over $\overline{\mathbb{Z}_\ell}$. Therefore, we will slightly abuse notations and use $\Omega, N_{\hat{G}}(T)/T, N_{\hat{G}}(S)/S$ interchangeably.

Lemma 1.2.2. *The image of $\varphi(\mathrm{Fr})$ and $\psi(\mathrm{Fr})$ in the Weyl group Ω agree, hence giving a well defined element w in the Weyl group Ω. (Check carefully!)*

Proof. Let

$$\Omega = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{Z}_\ell}) = \left(N_{\hat{G}}(T)/T\right)(\overline{\mathbb{F}_\ell})$$

as above and $\underline{\Omega}$ be the associated constant group scheme. Since ψ is a lift of φ, $\psi(\mathrm{Fr})$ specializes to $\varphi(\mathrm{Fr})$ in $N_{\hat{G}}(T)$. Then the lemma follows since

$$N_{\hat{G}}(T) \to N_{\hat{G}}(T)/T = \underline{\Omega}$$

is a morphism of schemes, hence the following diagram commutes:

$$\begin{tikzcd}
N_{\hat{G}}(T)(\overline{\mathbb{Z}_\ell}) \arrow[r] \arrow[d] & N_{\hat{G}}(T)(\overline{\mathbb{F}_\ell}) \arrow[d] \\
\underline{\Omega}(\overline{\mathbb{Z}_\ell}) = \Omega \arrow[r] & \underline{\Omega}(\overline{\mathbb{F}_\ell}) = \Omega
\end{tikzcd}$$

∎

Our main theorem is the following.

Theorem 1.2.3. *Let $X_\varphi (= X_\psi)$ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ (hence also containing ψ). The we have isomorphisms of quotient stacks*

$$[X_\varphi/\hat{G}] \simeq [(T \times \mu)/T] \simeq [*/C_T(n)] \times \mu,$$

where $C_T(n)$ is the schematic centralizer of $n = \psi$ in $T = C_{\hat{G}}(\psi|_{I_F^\ell})$, and $\mu = \prod_{i=1}^m \mu_{\ell^{k_i}}$ for some $k_i \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}_{\geq 0}$ is a product of group schemes of roots of unity.

If we moreover assume that ℓ doesn't divide the order of $w = \overline{\varphi(\mathrm{Fr})}$ in the Weyl group $N_{\hat{G}}(S)/S$, then

$$[X_\varphi/\hat{G}] \simeq [(T \times \mu)/T] \simeq [*/\underline{S_\varphi}] \times \mu,$$

where $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$, and $\underline{S_\varphi}$ is the corresponding constant group scheme.

Proof. Recall that X_φ is isomorphic to the contracted product

$$(\hat{G} \times Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}})/C_{\hat{G}}(\psi_\ell)_{\overline{\psi}},$$

and that $\eta.\psi \leftarrow\!\shortmid \eta \mapsto (\eta(\mathrm{Fr}), \eta(s_0))$ induces isomorphisms

$$Z^1(W_F, N_{\hat{G}}(\psi_\ell))_{\psi_\ell, \overline{\psi}} \simeq Z^1_{Ad\psi}(W_F, N_{\hat{G}}(\psi_\ell)^0)_1 \simeq T \times \mu.$$

This implies that $[X/\hat{G}] \simeq [(T \times \mu)/T]$ with T acting on T by twisted conjugacy:

$$(t, t') \mapsto \left(t(t'n)t^{-1}\right) n^{-1} = tt'(nt^{-1}n^{-1}) = t(nt^{-1}n^{-1})t' = (tnt^{-1}n^{-1})t',$$

where $n = \psi(\mathrm{Fr})$. In other words, T acts on T via multiplication by $tnt^{-1}n^{-1}$. And T acts trivially on μ (See Proposition 1.1.6).

So we are reduced to compute $[T/T]$ with respect to a nice action of the split torus T on T, which should be and turns out to be very explicit.

For clarification, let me denote the source torus T by $T^{(1)}$ and the target torus T by $T^{(2)}$. Consider the morphism

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \qquad s \longmapsto sns^{-1}n^{-1}.$$

This is surjective on $\overline{\mathbb{F}_\ell}$-points by our assumption 2 that $Z(\hat{G})$ is finite and φ is elliptic (See Lemma 1.2.4 below). Hence f is an epimorphism in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes (See the same Lemma 1.2.4 below)(maybe add an appendix on diagonalizable group schemes?). Therefore, f induces an isomorphism

$$T^{(1)}/\ker(f) \simeq T^{(2)} \tag{1.2.1}$$

as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Moreover, if we let $t \in T$ act on $T^{(1)}$ by left multiplication by t, and on $T^{(2)}$ via multiplication by $(tnt^{-1}n^{-1})$, this isomorphism induced by f is T-equivariant.

Note $T^{(1)} = T$ is commutative, so the T-action (via multiplication by $tnt^{-1}n^{-1}$) and the $\ker(f)$-action (via left multiplication) on T commutes with each other. Hence by the T-equivariant isomorphism (1.2.1), we have

$$[T/T] = [T^{(2)}/T] \simeq \left[\left(T^{(1)}/\ker(f)\right)/T\right] \simeq \left[\left(T^{(1)}/T\right)/\ker(f)\right] \simeq [*/\ker(f)] = [*/C_T(n)].$$

For the last assertion, see Lemma 1.2.5 below.

∎

Does $C_T(n) \simeq C_{\hat{G}}(\psi)$ holds?

Some lemmas

Lemma 1.2.4. *The morphism*

$$f : T^{(1)} = T \longrightarrow T = T^{(2)} \qquad s \longmapsto sns^{-1}n^{-1}$$

is epimorphic in the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. And it induces an isomorphism $T^{(1)}/\ker(f) \simeq T^{(2)}$ as diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes.

Proof. Recall that T is a split torus over $\overline{\mathbb{Z}_\ell}$, hence a diagonalizable $\overline{\mathbb{Z}_\ell}$-group scheme. Notice that f is a morphism of $\overline{\mathbb{Z}_\ell}$-group schemes, hence a morphism of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes. Recall that the category of diagonalizable $\overline{\mathbb{Z}_\ell}$-group schemes is equivalent to the category of abelian groups (See [3, p70, Section 5] or [6]) via

$$D \mapsto \mathrm{Hom}_{\overline{\mathbb{Z}_\ell}-GrpSch}(D, \mathbb{G}_m),$$

and the inverse is given by

$$\overline{\mathbb{Z}_\ell}[M] \leftarrow\!\shortmid M,$$

where $\overline{\mathbb{Z}_\ell}[M]$ is the group algebra of M with $\overline{\mathbb{Z}_\ell}$-coefficients.

Therefore, we could argue in the category of abelian groups via the above category equivalence: f is epimorphic if and only if the map f^* in the category of abelian groups is monomorphic. Note ellipticity and $Z(\hat{G})$ finite imply that S_φ is finite, hence

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_T(n) = S_\varphi$$

is finite (See Equation (1.2.2) below), hence $\mathrm{coker}(f^*)$ is finite. Therefore,

$$f^* : \mathrm{Hom}(T^{(2)}, \mathbb{G}_m) \to \mathrm{Hom}(T^{(1)}, \mathbb{G}_m)$$

is injective (i.e., monomorphism). Indeed, otherwise $\ker(f^*)$ would be a nonzero sub-$\mathbb{Z}$-module of the finite free $\mathbb{Z}$-module $\mathrm{Hom}(T^{(2)}, \mathbb{G}_m)$, hence a free $\mathbb{Z}$-module of positive rank, which contradicts with $\mathrm{coker}(f^*)$ being finite.

The statement on the quotient follows from the corresponding result in the category of abelian groups: f^* induces an isomorphism

$$\mathrm{Hom}(T^{(1)}, \mathbb{G}_m)/\mathrm{Hom}(T^{(2)}, \mathbb{G}_m) \simeq \mathrm{coker}(f^*)$$

(See [3, p71, Subsection 5.3].) $\square$

Lemma 1.2.5. *Assume that ℓ doesn't divide the order of w. $\ker(f) \simeq \underline{S_\varphi}$ is the constant group scheme of the finite abelian group $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$.*

Proof. We recall the following fact: Let H be a smooth affine group scheme over some ring R, let Γ be a finite group whose order is invertible in R. Then the fixed point functor H^Γ is representable and is smooth over R.

For a proof of the above fact, see [13, Proposition 3.4] or [9, Lemma A.1, A.13].

In our case, let $H = T$, $\Gamma =< w >$ the subgroup of the Weyl group $W_{\hat{G}}(T)$ generated by w. Hence

$$\ker(f) = C_T(n) = H^\Gamma$$

(See Lemma ? for the last equalityShould be able to check by Yoneda.) is smooth over $\overline{\mathbb{Z}_\ell}$. Therefore, $\ker(f)$ is finite etale over $\overline{\mathbb{Z}_\ell}$ (Because it is smooth of relative dimension 0 over $\overline{\mathbb{Z}_\ell}$, by Equation (1.2.2) below). Hence $\ker(f)$ is a constant group scheme over $\overline{\mathbb{Z}_\ell}$, since $\overline{\mathbb{Z}_\ell}$ has no non-trivial finite etale cover.

Since $\ker(f)$ is constant, we can determine it by computing its $\overline{\mathbb{F}_\ell}$-points:

$$\ker(f)(\overline{\mathbb{F}_\ell}) = C_{T(\overline{\mathbb{F}_\ell})}(n) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F)), \tag{1.2.2}$$

where the middle equality follows by noticing $T(\overline{\mathbb{F}_\ell}) = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ and $n = \varphi(\mathrm{Fr})$.

Finally, note by our TRSELP assumption, $C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(I_F))$ is (the $\overline{\mathbb{F}_\ell}$-points of) a torus. Hence $S_\varphi = C_{\hat{G}(\overline{\mathbb{F}_\ell})}(\varphi(W_F))$ is abelian, hence finite abelian as we've noticed in the proof of the previous lemma that S_φ is finite (by ellipticity and $Z(\hat{G})$ finite). □

Lemma 1.2.6. *Let $\hat{G}$ be a connected reductive group over $\overline{\mathbb{Z}_\ell}$, and T a maximal torus of $\hat{G}$. Then the Weyl group $N_{\hat{G}}(T)/T$ is split over $\overline{\mathbb{Z}_\ell}$.*

Proof. By [6, Proposition 3.2.8], the Weyl group $N_{\hat{G}}(T)/C_{\hat{G}}(T)$ is finite etale over $\overline{\mathbb{Z}_\ell}$ and hence split over $\overline{\mathbb{Z}_\ell}$. In our case, $C_{\hat{G}}(T) = T$ since $\hat{G}$ is connected (For example, use the proof of [6, Theorem 3.1.12]). □

Chapter 2

Rep

2.1 Proof of the Main Theorem 2.1.2 modulo Theorem 2.1.3 2.1.4 2.1.5

Let $\mathcal{G}$ be a split reductive group scheme over $\mathbb{Z}$, which is simply connected. Let $G := \mathcal{G}(\mathbb{Q}_p)$. For simplicity, I assume that p is greater than the Coxeter number of $\mathcal{G}$ (See Theorem 2.2.5 for reason).

Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(\mathcal{G}, \mathbb{Q}_p)$, G_x the parahoric subgroup associated to x, G_x^+ its pro-unipotent radical. Recall that $\overline{G_x} := G_x/G_x^+$ is a generalized Levi subgroup of $\mathcal{G}(\mathbb{F}_p)$ with root system Φ_x, see [16, Theorem 3.17].

Let $\Lambda = \overline{\mathbb{Z}_\ell}$, with $\ell \neq p$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is regular cuspidal. Here **regular cuspidal** (See Definition 2.2.6 for precise definition.) means ρ is cuspidal (Which I think follows from regularity? No, it doesn't. For example, the irreducible principal series $\mathrm{Ind}_B^G \chi$ for $G = GL_2$.) and lies in a **regular block** of $\mathrm{Rep}_\Lambda(\overline{G_x})$, in the sense of [4]. The reason we want the regularity assumption is that we want to work with a block of $\mathrm{Rep}_\Lambda(\overline{G_x})$ which consists purely of cuspidal representations. See Section 2.2 for details. We make this a definition for later use.

Definition 2.1.1. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$. We say ρ* ***has cuspidal reduction*** *(resp.* ***has regualr cuspidal reduction****), if ρ is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is cuspidal (resp. regular cuspidal). Let's denote the reduction of ρ modulo G_x^+ by $\overline{\rho} \in \mathrm{Rep}_\Lambda(\overline{G_x})$.*

Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Let $\mathcal{C}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G)$ containing $\pi := \text{c-Ind}_{G_x}^G \rho$. Now I can state the Main Theorem of this paper.

Theorem 2.1.2 (Main Theorem)**.** *Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(\mathcal{G}, \mathbb{Q}_p)$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which has regular cuspidal reduction. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Let $\mathcal{C}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G)$ containing $\pi := \text{c-Ind}_{G_x}^G \rho$. Then the compact induction $\text{c-Ind}_{G_x}^G$ induces an equivalence of categories $\mathcal{B}_{x,1} \simeq \mathcal{C}_{x,1}$.*

As mentioned before, the reason we want the regular cuspidal assumption is the following Theorem.

Theorem 2.1.3. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which has regular cuspidal reduction. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Then any $\rho' \in \mathcal{B}_{x,1}$ has cuspidal reduction.*

The proof of the Main Theorem 2.1.2 basically splits into two parts – fully faithfulness and essentially surjectivity. It is convenient to have the following Theorem available at an early stage, which implies fully faithfulness immediately and is also used in the proof of essentially surjectivity.

Theorem 2.1.4. *Let x, y be two vertices of the Bruhat-Tits building of G. Let ρ_1 be a representation of the parahoric G_x which is trivial on the pro-unipotent radical G_x^+. Let ρ_2 be a representation of G_y which is trivial on G_y^+. Assume one of them has cuspidal reduction. Then exactly one of the following happens:*

1. *If there exists an element $g \in G$ such that $g.x = y$, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^G \rho_1, \text{c-Ind}_{G_y}^G \rho_2) = \mathrm{Hom}_{G_x}(\rho_1, {}^g\rho_2).$$

2. *If there is no elements $g \in G$ such that $g.x = y$, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^G \rho_1, \text{c-Ind}_{G_y}^G \rho_2) = 0.$$

The proof of the above Theorem is basically a computation using Mackey's formula. see Section 2.3.

Now we proceed by steps towards our goal: The compact induction $\text{c-Ind}_{G_x}^G$ induces an equivalence of categories $\mathcal{B}_{x,1} \simeq \mathcal{C}_{x,1}$.

First, we show that $\text{c-Ind}_{G_x}^G : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is well-defined. We need to show that the image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^G$ lies in $\mathcal{C}_{x,1}$. By Theorem 2.1.3 and Theorem 2.1.4 above,

$$\text{c-Ind}_{G_x}^G |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \mathrm{Rep}_\Lambda(G)$$

is fully faithful (See Lemma 2.1.6, note here we used Theorem 2.1.3 that any representation in $\mathcal{B}_{x,1}$ has cuspidal reduction, so that we can apply Theorem 2.1.4), hence an equivalence onto the essential image. Since $\mathcal{B}_{x,1}$ is indecomposable as an abelian category, so is its essential image (See Lemma 2.1.7), hence its essential image is contained in a single block of $\mathrm{Rep}_\Lambda(G)$. But such a block must be $\mathcal{C}_{x,1}$ since $\text{c-Ind}_{G_x}^G$ maps ρ to $\pi \in \mathcal{C}_{x,1}$. Therefore, $\text{c-Ind}_{G_x}^G : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is well-defined.

Second, we show that $\text{c-Ind}_{G_x}^G : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is fully faithful. This is already noticed in the proof of "well-defined" in the last paragraph. Indeed,

$$\mathrm{Hom}_G(\text{c-Ind}_{G_x}^G \rho_1, \text{c-Ind}_{G_x}^G \rho_2) = \mathrm{Hom}_{G_x}(\rho_1, \rho_2)$$

by Theorem 2.1.3 and Theorem 2.1.4 (See Lemma 2.1.6.). Therefore, $\text{c-Ind}_{G_x}^G : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is fully faithful.

Finally, we show that $\text{c-Ind}_{G_x}^G : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective. This will occupy the rest of this section.

The idea is to find a projective generator of $\mathcal{C}_{x,1}$ and show that it is in the essential image. Fix a vertex x of the Bruhat-Tits building $\mathcal{B}(\mathcal{G}, \mathbb{Q}_p)$ as before. Let V be the set of equivalence classes of vertices of the Bruhat-Tits building $\mathcal{B}(\mathcal{G}, \mathbb{Q}_p)$ up to G-action. For $y \in V$, let $\sigma_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$. Let $\Pi := \bigoplus_{y \in V} \Pi_y$ where $\Pi_y := \text{c-Ind}_{G_y^+}^{G} \Lambda$. Then Π is a projective generator of the category of depth-zero representations $\text{Rep}_\Lambda(G)_0$, see [8, Appendix]. Let $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xhookrightarrow{summand} \text{Rep}_\Lambda(G_x)$ be the $\mathcal{B}_{x,1}$-summand of σ_x. And let $\Pi_{x,1} := \text{c-Ind}_{G_x}^{G} \sigma_{x,1}$. Note $\Pi_{x,1}$ is a summand of $\Pi_x = \text{c-Ind}_{G_x}^{G} \sigma_x$, hence a summand of Π. Using Theorem 2.1.4, one can show that the rest of the summands of Π don't interfere with $\Pi_{x,1}$ (See Lemma 2.4.2 and Lemma 2.4.3 for precise meaning), hence $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$. Let us state it as a Theorem, see Section 2 for details.

Theorem 2.1.5. $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G} \sigma_{x,1}$ *is a projective generator of* $\mathcal{C}_{x,1}$.

Now we've found a projective generator $\Pi_{x,1} = \text{c-Ind}_{G_x}^{G} \sigma_{x,1}$ of $\mathcal{C}_{x,1}$, and it is clear that $\Pi_{x,1}$ is in the essential image of $\text{c-Ind}_{G_x}^{G}$. We now deduce from this that $\text{c-Ind}_{G_x}^{G} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective. Indeed, for any $\pi' \in \mathcal{C}_{x,1}$, we can resolve π' by some copies of $\Pi_{x,1}$:

$$\Pi_{x,1}^{\oplus I} \xrightarrow{f} \Pi_{x,1}^{\oplus J} \to \pi' \to 0.$$

Using Theorem 2.1.4 and $\text{c-Ind}_{G_x}^{G}$ commutes with arbitrary direct sums (See Lemma 2.1.8) we see that $f \in \text{Hom}_G(\Pi_{x,1}^{\oplus I}, \Pi_{x,1}^{\oplus J})$ comes from a morphism $g \in \text{Hom}_{G_x}(\sigma_{x,1}^{\oplus I}, \sigma_{x,1}^{\oplus J})$. Using $\text{c-Ind}_{G_x}^{G}$ is exact we see that π' is the image of $coker(g) \in \mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G}$. Therefore, $\text{c-Ind}_{G_x}^{G} : \mathcal{B}_{x,1} \to \mathcal{C}_{x,1}$ is essentially surjective.

2.1.1 Lemmas

In this subsection I collect some Lemmas used in the proof of the Main Theorem.

Lemma 2.1.6. $\text{c-Ind}_{G_x}^{G} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \text{Rep}_\Lambda(G)$ *is fully faithful.*

Proof. Let $\rho_1, \rho_2 \in \mathcal{B}_{x,1}$. By the regular cuspidal assumption and Theorem 2.1.3, ρ_1, ρ_2 has cuspidal reduction. Hence the assumption of Theorem 2.1.4 is satisfied and we compute using the first case of Theorem 2.1.4 that

$$\text{Hom}_G(\text{c-Ind}_{G_x}^{G} \rho_1, \text{c-Ind}_{G_x}^{G} \rho_2) \simeq \text{Hom}_{G_x}(\rho_1, \rho_2).$$

In other words, $\text{c-Ind}_{G_x}^{G} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \text{Rep}_\Lambda(G)$ is fully faithful. □

Lemma 2.1.7. *The image of* $\mathcal{B}_{x,1}$ *under* $\text{c-Ind}_{G_x}^{G}$ *is indecomposable as an abelian category.*

Proof. The point is that $\text{c-Ind}_{G_x}^{G} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \text{Rep}_\Lambda(G)$ is not only fully faithful, i.e., an equivalence of categories onto the essential image, but also an equivalence of **abelian** categories onto the essential image. Indeed, it suffices to show that $\text{c-Ind}_{G_x}^{G} |_{\mathcal{B}_{x,1}} : \mathcal{B}_{x,1} \to \text{Rep}_\Lambda(G)$ preserves kernels, cokernels, and finite (bi-)products. But this follows from the next Lemma 2.1.8.

Assume otherwise that the essential image of $\mathcal{B}_{x,1}$ under $\text{c-Ind}_{G_x}^{G}$ is decomposable, then so is $\mathcal{B}_{x,1}$. But $\mathcal{B}_{x,1}$ is a block, hence indecomposable, contradiction! □

Lemma 2.1.8. $\text{c-Ind}_{G_x}^{G}$ *is exact and commutes with arbitrary direct sums.*

Proof. For c-$\mathrm{Ind}_{G_x}^G$ is exact, we refer to [18, I.5.10].

We show that c-$\mathrm{Ind}_{G_x}^G$ commutes with arbitrary direct sums. Indeed, c-$\mathrm{Ind}_{G_x}^G$ is a left adjoint (See [18, I.5.7]), hence commutes with arbitrary colimits. In particular, it commutes with arbitrary direct sums. $\square$

2.2 Proof of Theorem 2.1.3

The goal of this section is to define regular blocks and regular cuspidal representations with $\Lambda = \overline{\mathbb{Z}_\ell}$-coefficients of a finite group of Lie type, and to show that a regular block consists purely of cuspidal representations.

Let $\Lambda := \overline{\mathbb{Z}_\ell}$ be the coefficients of representations. Fix a prime number p. Let ℓ be a prime number different from p. For simplicity, let $q = p$.

Definition 2.2.1 ([18, I.4.1])**.** *Let Λ' be any ring.*

1. *Let G be a profinite group, a* ***representation of G with Λ'-coefficients*** *(π, V) is a Λ'-module V, together with a G-action $\pi : G \to GL_{\Lambda'}(V)$.*
2. *A representation of G with Λ'-coefficients is called* ***smooth*** *if for any $v \in V$, the stablizer $Stab_G(v) \subset G$ is open.*

Throughout the article, all representations are assumed to be smooth. The category of smooth representations of G with Λ'-coefficients is denoted by $\mathrm{Rep}_{\Lambda'}(G)$.

2.2.1 Regular blocks and regular cuspidal representations of a finite group of Lie type

The following notations are used in this subsection only. Let $\mathcal{G}$ be a split reductive group scheme over $\mathbb{Z}$. Let $\mathbb{G} := \mathcal{G}(\overline{\mathbb{F}_p})$, $G := \mathbb{G}^F = \mathcal{G}(\mathbb{F}_p)$, where F is the Frobenius. By abuse of notation, I sometimes identify the group scheme $\mathcal{G}_{\overline{\mathbb{F}_p}}$ with its $\overline{\mathbb{F}_p}$-points $\mathbb{G}$. Let $\mathbb{G}^*$ be the dual group (over $\overline{\mathbb{F}_p}$) of $\mathbb{G}$, and F^* the dual Frobenius (See [5, Section 4.2]). Fix an isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$.

The definition of regular blocks and regular cuspidal representations of a finite group of Lie type Γ involves modular Deligne-Lusztig theory and block theory. We refer to [11], [5], and [12] for Deligne-Lusztig theory, [15] and [4] for modular Deligne-Lusztig theory, and [2, Appendix B] for generalities on blocks.

First, let's recall a result in Deligne-Lusztig theory (See [12, Proposition 11.1.5]).

Proposition 2.2.2. *The set of $\mathbb{G}^F$-conjugacy classes of pairs $(\mathbb{T}, \theta)$, where $\mathbb{T}$ is a F-stable maximal torus of $\mathbb{G}$ and $\theta \in \widehat{\mathbb{T}^F}$, is in non-canonical bijection to the set of ${\mathbb{G}^*}^{F^*}$-conjugacy classes of pairs $(\mathbb{T}^*, s)$, where s is a semisimple element of $\mathbb{G}^*$ and $\mathbb{T}^*$ is a F^*-stable maximal torus of $\mathbb{G}^*$ such that $s \in {\mathbb{T}^*}^{F^*}$. Moreover, we could and will fix a compatible system of isomorphisms $\mathbb{F}_{q^n}^* \simeq \mathbb{Z}/(q^n - 1)\mathbb{Z}$ to pin down this bijection.*

Now let s be a **strongly regular semisimple** element of $G^* = {\mathbb{G}^*}^{F^*}$ (note we require s to be fixed by F^* here), i.e., the centralizer $C_{\mathbb{G}^*}(s)$ is a F^*-stable maximal torus, denoted $\mathbb{T}^*$. Let $\mathbb{T}$ be the dual torus of $\mathbb{T}^*$. Let $T = \mathbb{T}^F$ and $T^* = {\mathbb{T}^*}^{F^*}$. Let T_ℓ denote the ℓ-part of T.

Recall for s strongly regular semisimple, the (rational) Lusztig series $\mathcal{E}(G,(s))$ consists of only one element, namely, $R_T^G(\hat{s})$, where $\hat{s} = \theta$ is such that $(\mathbb{T}, \theta)$ corresponds to $(\mathbb{T}^*, s)$ via the previous bijection in Proposition 2.2.2. (This follows from, for example, Broué's equivalence. See Theorem 2.2.5 below.)

From now on, assume moreover that $s \in \mathbb{G}^{*F^*}$ has order prime to ℓ. In other words, assume $s \in G^* = \mathbb{G}^{*F^*}$ is a **strongly regular semisimple ℓ'-element**. We are going to define regular blocks, we refer to [2, Appendix B] for generalities on blocks.

Define the **ℓ-Lusztig series**

$$\mathcal{E}_\ell(G,(s)) := \{R_T^G(\hat{s}\eta) | \eta \in \widehat{T}_\ell\}.$$

Note the notation $\mathcal{E}_\ell(T,(s))$ also makes sense by putting $G = T$.

By [15], $\mathcal{E}_\ell(G,(s))$ is a union of ℓ-blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(G)$. Such a block (or more precisely, a union of blocks) is called a **(ℓ-)regular block**. Let $e_s^G \in \overline{\mathbb{Z}_\ell}G$ denote the corresponding central idempotent. Note e_s^T also makes sense by putting $G = T$. We shall see later that a regular block is indeed a block, i.e., indecomposible. (This follows from, for example, Broué's equivalence. See Theorem 2.2.5 below.)

Definition 2.2.3 (Regular blocks). *Let $s \in G^* = \mathbb{G}^{*F^*}$ be a strongly regular semisimple ℓ'-element. We call the block $\overline{\mathbb{Z}_\ell}Ge_s^G$ of the group algebra $\overline{\mathbb{Z}_\ell}G$ corresponding to the central idempotent e_s^G the **regular $\overline{\mathbb{Z}_\ell}$-block** associated to s. Let $\mathcal{A}_s := \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$ be the corresponding category of modules, this is also referred to as a regular block, by abuse of notation.*

Similarly, the block $\overline{\mathbb{F}_\ell}Ge_s^G$ is called a $\overline{\mathbb{F}_\ell}$-block. (However, this notion won't be used later.)

Remark 2.2.4. Above all, "a block" could have three different meanings: ℓ-block, $\overline{\mathbb{Z}_\ell}$-block, and $\overline{\mathbb{F}_\ell}$-block. But they are in one-one correspondence to each other, so I often abuse the notation and simply call it "a block".

Thanks to [4], we understand the category $\mathcal{A}_s = \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$ quite well. Roughly speaking, it is equivalent to the category of representations of a torus, via Deligne-Lusztig induction. This is what I'm going to explain now.

Let $\mathbb{B} \subset \mathbb{G}$ be a Borel subgroup containing our torus $\mathbb{T}$, let $\mathbb{U}$ be the unipotent radical of $\mathbb{B}$. Let $X_\mathbb{U}$ be the Deligne-Lusztig variety defined by

$$X_\mathbb{U} := \{g \in \mathbb{G} | g^{-1}F(g) \in \mathbb{U}\}.$$

The main result of [4] is the following: The Deligne-Lusztig induction

$$R_T^G = R\Gamma_c(X_\mathbb{U}, \overline{\mathbb{Z}_\ell}) \otimes_{\overline{\mathbb{Z}_\ell}T} - : \overline{\mathbb{Z}_\ell}T\text{-Mod} \to \overline{\mathbb{Z}_\ell}G\text{-Mod}$$

induces an equivalence of categories between $\overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod}$ and $\overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}$. In particular, one deduce that the irreducible objects in $\overline{\mathbb{F}_\ell}Ge_s^G\text{-Mod}$ lifts to $\overline{\mathbb{Z}_\ell}$. More precisely, let us state it as the following theorem.

Theorem 2.2.5 (Broué's equivalence, [4, Theorem 3.3]). *With the previous assumptions and notations, assume $X_\mathbb{U}$ is affine of dimension d (which is the case if q is greater than the Coxeter number of $\mathbb{G}$.). The cohomology complex $R\Gamma_c(X_\mathbb{U}, \overline{\mathbb{Z}_\ell}) = R\Gamma_c(X_\mathbb{U}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}_\ell}$*

is concentrated in degree $d = dim X_{\mathbb{U}}$*. And the* $(\overline{\mathbb{Z}_\ell}Ge_s^G, \overline{\mathbb{Z}_\ell}Te_s^T)$*-bimodule* $e_s^G H_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell})e_s^T$ *induces an equivalence of categories*

$$e_s^G H_c^d(X_{\mathbb{U}}, \overline{\mathbb{Z}_\ell})e_s^T \otimes_{\overline{\mathbb{Z}_\ell}Te_s^T} - : \overline{\mathbb{Z}_\ell}Te_s^T\text{-Mod} \to \overline{\mathbb{Z}_\ell}Ge_s^G\text{-Mod}\,.$$

From now on, we assume the above Theorem holds for all finite groups of Lie type we encountered in this paper. I hope this is not a severe restriction. This is the case at least when p (or rather, q. But I assumed $p = q$ for simplicity in this paper.) is greater than the Coxeter number of $\mathbb{G}$.

We now define regular cuspidal representations as those representations that occur in some regular block. The term "cuspidal" in the name "regular cuspidal" shall be justified later by Theorem 2.2.10.

Definition 2.2.6. *Let G be a finite group of Lie type. Let $\Lambda = \overline{\mathbb{Z}_\ell}$. Let $\rho \in \mathrm{Rep}_\Lambda(G)$. Then ρ is called* ***regular cuspidal*** *if each of its irreducible subquotient ρ_i are cuspidal (See Definition 2.2.8) and lies in a regular $\overline{\mathbb{Z}_\ell}$-block $\mathcal{A}_{s_i}$.*

2.2.2 Pure Cuspidality

A digression on cuspidality

Before stating the theorem of pure cuspidality, let's define cuspidality for representations with arbitrary coefficients. Let Λ' be any ring. For example, Λ' can be $\overline{\mathbb{Q}_\ell}$, $\overline{\mathbb{Z}_\ell}$, or $\overline{\mathbb{F}_\ell}$.

First, we define two functors.

Definition 2.2.7 (Parabolic induction and restriction)**.** *Let G be a finite group of Lie type. Let P be a parabolic subgroup and M the corresponding Levi subgroup.*

1. *The* ***parabolic induction functor*** *is defined to be the composition*

$$i_M^G := \mathrm{Ind}_P^G \circ f^*,$$

where

$$f^* : \mathrm{Rep}_{\Lambda'}(M) \to \mathrm{Rep}_{\Lambda'}(P)$$

is the inflation along the natural projection $f : P \to M$.

2. *The* ***parabolic restriction functor*** *is defined to be the composition*

$$r_M^G := (-)_U \circ \mathrm{Res}_P^G,$$

where

$$(-)_U : \mathrm{Rep}_{\Lambda'}(P) \to \mathrm{Rep}_{\Lambda'}(M), V \mapsto V/ < \{u.v - v | u \in U, v \in V\} >_{\Lambda' U\text{-Mod}}$$

is the functor of taking coinvariance.

We recall that r_M^G is left adjoint to i_M^G and they are both exact under our assumption $\ell \neq p$ (See [18, II.2.1]).

Definition 2.2.8 (Cuspidal)**.** *Let G be a finite group of Lie type. Let $\rho \in \mathrm{Rep}_{\Lambda'}(G)$ be a representation of G. Then ρ is called **(Λ'-)cuspidal** if ρ is not a subrepresentation of any proper parabolic induction, i.e.,*

$$\mathrm{Hom}_G(\rho, i_P^G(\sigma)) = 0$$

for any proper parabolic subgroup P of G and any representation $\sigma \in \mathrm{Rep}_{\Lambda'}(M)$, where M is the Levi subgroup corresponding to P.

For example, let $s \in G^*$ strongly regular semisimple, then

$$R_T^G(\hat{s}) = R\Gamma_c(X_{\mathbb{U}}, \overline{\mathbb{Q}_\ell}) \otimes \hat{s}$$

is cuspidal in $\mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(G)$ (See [11, Theorem 8.3]).

I record the following equivalent definition of cuspidality for later use.

Lemma 2.2.9. *[18, II.2.3] $\rho \in \mathrm{Rep}_{\Lambda'}(G)$ is cuspidal if and only if $r_M^G \rho = 0$, for any proper Levi subgroup M of G.*

The theorem of pure cuspidality

We can now state the theorem of pure cuspidality.

As in Broué's paper [4], we fix a finite integral extension $\mathcal{O}$ of $\mathbb{Z}_\ell$, which is big enough. One good thing to work with $\mathcal{O}$ instead of $\overline{\mathbb{Z}_\ell}$ is that $\mathcal{O}$ is a discrete valuation ring, while $\overline{\mathbb{Z}_\ell}$ is not (even not Noetherian). We assume $\mathcal{O}$ to be big enough (for example, $\mathcal{O}$ contains all roots of unity we encounter) so that all things we need to do representation theory are available without change.

Theorem 2.2.10 (Pure Cuspidality)**.** *Let G be a finite group of Lie type. Let $s \in G^* = \mathbb{G}^{*F^*}$ be a strongly regular semisimple ℓ'-element, with corresponding torus $T = \mathbb{T}^F$ and character $\hat{s} \in \hat{T}$ as in Proposition 2.2.2. Assume that $R_T^G(\hat{s})$ is $\overline{\mathbb{Q}_\ell}$-cuspildal. Then the $\overline{\mathbb{Z}_\ell}$-block $\mathcal{A}_s = \overline{\mathbb{Z}_\ell} G e_s^G$-Mod consists purely of cuspidal representations.*

Proof. Recall Broué's equivalence: For $\mathcal{O}$ a finite integral extension of $\mathbb{Z}_\ell$, big enough, we have

$$F := e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T \otimes_{\mathcal{O}Te_s^T} - : \mathcal{O}Te_s^T\text{-Mod} \to \mathcal{O}Ge_s^G\text{-Mod}$$

is an equivalence of categories. This is moreover an equivalence of abelian categories (See Lemma 2.2.11). Let $V := F(\mathcal{O}Te_s^T) = e_s^G H_c^d(X_{\mathbb{U}}, \mathcal{O}) e_s^T$. Then V is a projective generator of $\mathcal{A}_s$, since $\mathcal{O}Te_s^T$ is a projective generator of $\mathcal{O}Te_s^T$-Mod. We first show that V is $\mathcal{O}$-cuspidal.

By classical Deligne-Lusztig theory, $\overline{\mathbb{Q}_\ell} V \simeq \bigoplus_{\eta \in \hat{T}_\ell} R_T^G(\hat{s}\eta)$ is $\overline{\mathbb{Q}_\ell}$-cuspidal (For details, see Lemma below.). In other words,

$$r_{M,\overline{\mathbb{Q}_\ell}}^G(\overline{\mathbb{Q}_\ell} V) := \overline{\mathbb{Q}_\ell} V / < \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} >_{\overline{\mathbb{Q}_\ell} U\text{-Mod}} = 0.$$

However, note

$$< \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} >_{\overline{\mathbb{Q}_\ell} U\text{-Mod}} = < \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell} V\} >_{\mathcal{O}U\text{-Mod}} .$$

So we have

$$r^G_{M,\mathcal{O}}(\overline{\mathbb{Q}_\ell}V) := \overline{\mathbb{Q}_\ell}V/ < \{u.v - v | u \in U, v \in \overline{\mathbb{Q}_\ell}V\} >_{\mathcal{O}U\text{-Mod}} = 0.$$

Note V is finitely presented and projective over $\mathcal{O}Te^T_s$ (See [4, Proof of Theorem 3.3]), hence projective over $\mathcal{O}$ (because the restriction functor $\mathcal{O}T$-Mod $\to \mathcal{O}$-Mod preserves projectivity, since it's left adjoint to an exact functor, the induction functor), which is a local ring , hence V is free over $\mathcal{O}$ (See [17, Theorem 24.4.5]). We thus have an inclusion

$$V \hookrightarrow \overline{\mathbb{Q}_\ell}V := \overline{\mathbb{Q}_\ell} \otimes_{\mathcal{O}} V$$

as $\mathcal{O}G$-modules. Recall that the parabolic restriction $r^G_{M,\mathcal{O}}$ is exact (See [18, II.2.1]), hence

$$r^G_{M,\mathcal{O}}(\overline{\mathbb{Q}_\ell}V) = 0$$

implies that

$$r^G_{M,\mathcal{O}}(V) = 0,$$

i.e., V is $\mathcal{O}$-cuspidal.

Moreover, base change to $\overline{\mathbb{Z}_\ell}$ we see that $\overline{\mathbb{Z}_\ell}V$ is $\overline{\mathbb{Z}_\ell}$-cuspidal. Indeed,

$$r^G_{M,\overline{\mathbb{Z}_\ell}}(\overline{\mathbb{Z}_\ell}V) = \overline{\mathbb{Z}_\ell}V/\overline{\mathbb{Z}_\ell}V(U) = \overline{\mathbb{Z}_\ell} \otimes_{\mathcal{O}} (V/V(U)) = \overline{\mathbb{Z}_\ell} \otimes_{\mathcal{O}} r^G_{M,\mathcal{O}}(V) = 0.$$

For general $V' \in \mathcal{A}_s$, we can resolve it by some direct sum of V's, and we see that

$$r^G_{M,\overline{\mathbb{Z}_\ell}}(V') = 0,$$

(using $r^G_{M,\overline{\mathbb{Z}_\ell}}$ is exact and commutes with arbitrary direct sum) i.e., V' is $\overline{\mathbb{Z}_\ell}$-cuspidal. □

Lemma 2.2.11.

$$F := e^G_s H^d_c(X_{\mathbb{U}}, \mathcal{O})e^T_s \otimes_{\mathcal{O}Te^T_s} - : \mathcal{O}Te^T_s\text{-Mod} \to \mathcal{O}Ge^G_s\text{-Mod}$$

is an equivalence of abelian categories.

Proof. We already know that F is an equivalence of categories. It remains to show that F is exact and commutes with product.

Now $e^G_s H^d_c(X_{\mathbb{U}}, \mathcal{O})e^T_s$ is projective over $\mathcal{O}Te^T_s$ (See [4, Proof of Theorem 3.3]), hence flat over $\mathcal{O}Te^T_s$. Hence $F := e^G_s H^d_c(X_{\mathbb{U}}, \mathcal{O})e^T_s \otimes_{\mathcal{O}Te^T_s} -$ is exact.

It is clear that $F := e^G_s H^d_c(X_{\mathbb{U}}, \mathcal{O})e^T_s \otimes_{\mathcal{O}Te^T_s} -$ commutes with product. □

Lemma 2.2.12. *Let G be a finite group of Lie type. Let $s \in G^* = \mathbb{G}^{*F^*}$ be a strongly regular semisimple ℓ'-element, with corresponding torus $T = \mathbb{T}^F$ and character $\hat{s} \in \hat{T}$ as before. Assume that $R^G_T(\hat{s})$ is $\overline{\mathbb{Q}_\ell}$-cuspildal. Then $R^G_T(\hat{s}\eta)$ is $\overline{\mathbb{Q}_\ell}$-cuspidal for any $\eta \in \hat{T}_\ell$.*

2.2.3 Proof of Theorem 2.1.3

We now apply the previous results on finite group of Lie types to representations of the parahoric subgroups of a p-adic group. The notation, therefore, is different from before.

Let $\mathcal{G}$ be a split, simply connected reductive group scheme over $\mathbb{Z}$. Let $G := \mathcal{G}(\mathbb{Q}_p)$. For simplicity, I assume $p = q$ is greater than the Coxeter number of G (See Theorem 2.2.5 for reason).

Let x be a vertex of the Bruhat-Tits building $\mathcal{B}(\mathcal{G}, \mathbb{Q}_p)$, G_x the parahoric subgroup associated to x, G_x^+ its pro-unipotent radical. Recall that $\overline{G_x} := G_x/G_x^+$ is a generalized Levi subgroup of $\mathcal{G}(\mathbb{F}_p)$ (in particular, a finite group of Lie type) with root system Φ_x, see [16, Theorem 3.17].

Let $\Lambda = \overline{\mathbb{Z}_\ell}$, with $\ell \neq p$. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which is trivial on G_x^+ and whose reduction to the finite group of Lie type $\overline{G_x} = G_x/G_x^+$ is regular cuspidal.

In other words, we start with an irreducible representation $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which has regular cuspidal reduction. Let $\mathcal{B}_{x,1}$ be the ($\overline{\mathbb{Z}_\ell}$-)block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. We can now prove Theorem 2.1.3, which we restate as follows.

Theorem 2.2.13. *Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ be an irreducible representation of G_x, which has regular cuspidal reduction. Let $\mathcal{B}_{x,1}$ be the $\overline{\mathbb{Z}_\ell}$-block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ. Then any $\rho' \in \mathcal{B}_{x,1}$ has cuspidal reduction.*

Proof. Let $\overline{\rho} \in \mathrm{Rep}_\Lambda(\overline{G_x})$ be the reduction of ρ modulo G_x^+. $\overline{\rho}$ is irreducible (since ρ is) and regular cuspidal by assumption, so it is of the form $r_\ell(R_T^G(\hat{s}))$ (i.e., the ℓ-reduction of $R_T^G(\hat{s})$), for some strongly regular semisimple ℓ'-element s of $\overline{G_x}^*$ (See Definition 2.2.6.).

Let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consists of representations of G_x that are trivial on G_x^+. The key observation is that $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand (as abelian category) of $\mathrm{Rep}_\Lambda(G_x)$ (See Lemma 2.2.14).

Then since $\rho \in \mathrm{Rep}_\Lambda(G_x)_0$, its block $\mathcal{B}_{x,1}$ is a summand of $\mathrm{Rep}_\Lambda(G_x)_0$.

On the other hand, notice the inflation induces an equivalence of categories between $\mathrm{Rep}_\Lambda(\overline{G_x})$ and $\mathrm{Rep}_\Lambda(G_x)_0$, with inverse the reduction modulo G_x^+.

So the blocks of $\mathrm{Rep}_\Lambda(\overline{G_x})$ and $\mathrm{Rep}_\Lambda(G_x)_0$ should agree. Let $\mathcal{A}_{x,1}$ be the corresponding block of $\mathrm{Rep}_\Lambda(\overline{G_x})$ to $\mathcal{B}_{x,1}$. Then $\mathcal{A}_{x,1}$ is contained in the regular block $\mathcal{A}_s$ corresponding to s (recall $\overline{\rho} = r_\ell(R_T^G(\hat{s}))$). By Theorem 2.2.10, $\mathcal{A}_s$ consists purely of cuspidal representation. Therefore, $\mathcal{B}_{x,1}$ consists purely of representations that have cuspidal reductions. $\square$

Lemma 2.2.14. *Let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consists of representations of G_x that are trivial on G_x^+. Then $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand as abelian category of $\mathrm{Rep}_\Lambda(G_x)$.*

Proof. Note G_x^+ is pro-p (See [18, II.5.2.(b)]), in particular, it has pro-order invertible in Λ. So we have a normalized Haar measure μ on G_x such that $\mu(G_x^+) = 1$ (See [18, I.2.4]). The characteristic function $e := 1_{G_x^+}$ is an idempotent of the Hecke algebra $\mathcal{H}_\Lambda(G_x)$ under convolution with respect to the Haar measure μ. We shall show that $e = 1_{G_x^+}$ cuts out $\mathrm{Rep}_\Lambda(G_x)_0$ as a summand of $\mathrm{Rep}_\Lambda(G_x) \simeq \mathcal{H}_\Lambda(G_x)$-Mod.

Let's first check that $e = 1_{G_x^+}$ is central. This can be done by an explicit computation. Recall that we have a descending filtration $\{G_{x,r} | r \in \mathbb{R}_{>0}\}$ of G_x such that

1. $\forall r \in \mathbb{R}_{>0}, G_{x,r}$ is an open compact pro-p subgroup of G_x.

2. $\forall r \in \mathbb{R}_{>0}, G_{x,r}$ is a normal subgroup of G_x.

3. $G_{x,r}$ form a neighborhood basis of 1 inside G_x.

(See [18, II.5.1].) Therefore, to check $e * f = f * e$, for all $f \in \mathcal{H}_\Lambda(G_x)$, it suffices to check for all f of the form $1_{gG_{x,r}}$, the characteristic function of the (both left and right) coset $gG_{x,r}(= G_{x,r}g$, by normality) for some $g \in G$ and $r \in \mathbb{R}_{>0}$. Indeed, one can compute that $(e * 1_{gG_{x,r}})(y) = \mu(G_x^+ \cap G_{x,r}yg^{-1})$ and that $(1_{gG_{x,r}} * e)(y) = \mu(gG_{x,r} \cap yG_x^+)$, for any $y \in G_x$. Note that $G_{x,r} \subset G_x^+$, we get that $\mu(G_x^+ \cap G_{x,r}yg^{-1}) = \mu(G_{x,r})$ if $yg^{-1} \in G_x^+$ and 0 otherwise. Same for $\mu(gG_{x,r} \cap yG_x^+)$. Therefore, e is central.

Finally, under the isomorphism $\mathrm{Rep}_\Lambda(G_x) \simeq \mathcal{H}_\Lambda(G_x)$-Mod, $\mathrm{Rep}_\Lambda(G_x)_0$ corresponds to the summand $\mathcal{H}_\Lambda(G_x, G_x^+)$-Mod $= e\mathcal{H}_\Lambda(G_x)e$-Mod corresponding to the central idempotent $e := 1_{G_x^+} \in \mathcal{H}_\Lambda(G_x)$ of $\mathcal{H}_\Lambda(G_x)$-Mod, hence $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand of $\mathrm{Rep}_\Lambda(G_x)$. □

2.3 Proof of Theorem 2.1.4

Let's now prove Theorem 2.1.4.

Proof of Theorem 2.1.4.

$$\begin{aligned}
&\mathrm{Hom}_G(\text{c-Ind}_{G_x}^G \rho_1, \text{c-Ind}_{G_y}^G \rho_2) \\
= &\mathrm{Hom}_{G_x}\left(\rho_1, (\text{c-Ind}_{G_y}^G \rho_2)|_{G_x}\right) \\
= &\mathrm{Hom}_{G_x}\left(\rho_1, \bigoplus_{g \in G_y \backslash G / G_x} \text{c-Ind}_{G_x \cap g^{-1}G_y g}^{G_x} \rho_2(g - g^{-1})\right)
\end{aligned}$$

Recall that $g^{-1}G_y g = G_{g^{-1}.y}$. So it suffices to show that for $g \in G$ with $G_x \cap g^{-1}G_y g \neq G_x$, or equivalently, for $g \in G$ with $g.x \neq y$ (since x and y are vertices), it holds that

$$\mathrm{Hom}_{G_x}\left(\rho_1, \text{c-Ind}_{G_x \cap g^{-1}G_y g}^{G_x} \rho_2(g - g^{-1})\right) = 0.$$

Note $G_x/(G_x \cap g^{-1}G_y g)$ is compact, hence $\text{c-Ind}_{G_x \cap g^{-1}G_y g}^{G_x} = \mathrm{Ind}_{G_x \cap g^{-1}G_y g}^{G_x}$, and we have Frobenius reciprocity in the other direction

$$\mathrm{Hom}_{G_x}\left(\rho_1, \text{c-Ind}_{G_x \cap g^{-1}G_y g}^{G_x} \rho_2(g - g^{-1})\right) \simeq \mathrm{Hom}_{G_x \cap g^{-1}G_y g}\left(\rho_1, \rho_2(g - g^{-1})\right).$$

So it suffices to show that for $g \in G$ with $g.x \neq y$,

$$\mathrm{Hom}_{G_x \cap g^{-1}G_y g}\left(\rho_1, \rho_2(g - g^{-1})\right) = 0.$$

Note now this expression is symmetric with respect to ρ_1 and ρ_2, so is the following argument.

First, if ρ_2 has cuspidal reduction (denoted $\overline{\rho_2}$),

$$\begin{aligned}
&\mathrm{Hom}_{G_x \cap g^{-1}G_y g}\left(\rho_1, \rho_2(g - g^{-1})\right) \\
= &\mathrm{Hom}_{G_x \cap G_{g^{-1}.y}}\left(\rho_1, \rho_2(g - g^{-1})\right)
\end{aligned}$$

$$
\begin{aligned}
&\subseteq \mathrm{Hom}_{G_x^+\cap G_{g^{-1}.y}}\left(\rho_1,\rho_2(g-g^{-1})\right) && \\
&= \mathrm{Hom}_{G_x^+\cap G_{g^{-1}.y}}(1^{\oplus d_1},\rho_2(g-g^{-1})) && \rho_1 \text{ is trivial on } G_x^+ \\
&= \mathrm{Hom}_{G_{g.x}^+\cap G_y}(1^{\oplus d_1},\rho_2) && \text{Conjugate by } g^{-1} \\
&= \mathrm{Hom}_{U_y(g.x)}(1^{\oplus d_1},\overline{\rho_2}) && \text{Reduction modulo } G_y^+\text{. See below.} \\
&= 0 && \overline{\rho_2} \text{ is cuspidal. See below.}
\end{aligned}
$$

The last two equations need some explanation.

The former one uses the following consequence from Bruhat-Tits theory: If x_1 and x_2 are two different vertices of the Bruhat-Tits building, then $\overline{G_{x_i}} := G_{x_i}/G_{x_i}^+$ is a generalized Levi subgroup of $\overline{G} = G(\mathbb{F}_p)$, for $i = 1, 2$. Moreover, $G_{x_1} \cap G_{x_2}$ projects onto a proper parabolic subgroup $P_{x_1}(x_2)$ of $\overline{G_{x_1}}$ under the reduction map $G_{x_1} \to \overline{G_{x_1}}$. And $G_{x_1} \cap G_{x_2}^+$ projects onto $U_{x_1}(x_2)$, the unipotent radical of $P_{x_1}(x_2)$, under the reduction map $G_{x_1} \to \overline{G_{x_1}}$. For details, see Lemma 2.3.1 below. Note that the assumption of Lemma 2.3.1 is satisfied since without loss of generality we may assume $x_1 = x$ and $x_2 = y$ lies in the closure of a common alcove (since G acts simply transitively on the set of alcoves).

The latter one uses that for a cuspidal representation ρ of a finite group of Lie type Γ,

$$\mathrm{Hom}_U(1, \rho|_U) = \mathrm{Hom}_U(\rho|_U, 1) = 0,$$

for the unipotent radical U of P, where P is any proper parabolic subgroup of Γ. For details, see Lemma 2.3.2 below.

Symmetrically, a similar argument works if ρ_1 has cuspidal reduction. Indeed, if ρ_1 has cuspidal reduction (denoted $\overline{\rho_1}$),

$$
\begin{aligned}
&\mathrm{Hom}_{G_x\cap g^{-1}G_y g}\left(\rho_1,\rho_2(g-g^{-1})\right) && \\
=\ &\mathrm{Hom}_{gG_xg^{-1}\cap G_y}\left(\rho_1(g^{-1}-g),\rho_2\right) && \text{Conjugate by } g^{-1} \\
\subseteq\ &\mathrm{Hom}_{gG_xg^{-1}\cap G_y^+}\left(\rho_1(g^{-1}-g),\rho_2\right) && \\
=\ &\mathrm{Hom}_{gG_xg^{-1}\cap G_y^+}(\rho_1(g^{-1}-g),1^{\oplus d_2}) && \rho_2 \text{ is trivial on } G_y^+ \\
=\ &\mathrm{Hom}_{G_x\cap g^{-1}G_y^+g}(\rho_1,1^{\oplus d_2}) && \text{Conjugate by } g \\
=\ &\mathrm{Hom}_{G_x\cap G_{g^{-1}.y}^+}(\rho_1,1^{\oplus d_2}) && \\
=\ &\mathrm{Hom}_{U_x(g^{-1}.y)}(\overline{\rho_1},1^{\oplus d_2}) && \text{Reduction modulo } G_x^+ \\
=\ &0 && \overline{\rho_1} \text{ is cuspidal.}
\end{aligned}
$$

□

2.3.1 Lemmas

Lemma 2.3.1. *Let x_1 and x_2 be two points of the Bruhat-Tits building $\mathcal{B}(\mathcal{G},\mathbb{Q}_p)$. Assume they lie in the closure of a same alcove.*

(i) The image of $G_{x_1} \cap G_{x_2}$ in $\overline{G_{x_1}}$ is a parabolic subgroup of $\overline{G_{x_1}}$. Let's denote it by $P_{x_1}(x_2)$. Moreover, the image of $G_{x_1} \cap G_{x_2}^+$ in $\overline{G_{x_1}}$ is the unipotent radical of $P_{x_1}(x_2)$. Let's denote it by $U_{x_1}(x_2)$.

(ii) Assume moreover that x_1 and x_2 are two different vertices of the building. Then $P_{x_1}(x_2)$ is a proper parabolic subgroup of $\overline{G_{x_1}}$.

Proof. (i) is [18, II.5.1.(k)].

Let's prove (ii). It suffices to show that $G_{x_1} \neq G_{x_2}$. Assume otherwise that $G_{x_1} = G_{x_2}$, then x_1 and x_2 lie in the same facet, which contradicts with the assumption that x_1 and x_2 are two different vertices. □

Lemma 2.3.2. *Let $\overline{\rho}$ be a cuspidal representation of a finite group of Lie type Γ. Let P be a proper parabolic subgroup of Γ, with unipotent radical U. Then*

$$Hom_U(1_U, \overline{\rho}) = Hom_U(\overline{\rho}, 1_U) = 0.$$

Proof. $\mathrm{Hom}_U(\overline{\rho}|_U, 1_U) = \mathrm{Hom}_\Gamma(\overline{\rho}, Ind_P^\Gamma(\sigma)) = 0$, where $\sigma = Ind_U^P(1_U)$. The last equality holds because $\overline{\rho}$ is assumed to be cuspidal (Recall Definition 2.2.8). Similar for $\mathrm{Hom}_U(1_U, \rho|_U)$. □

2.4 Proof of Theorem 2.1.5

In this subsection, I prove that $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$. Before doing this, let's recall the setting. Fix a vertex x of the building of G. Let $\rho \in \mathrm{Rep}_\Lambda(G_x)$ which is trivial on G_x^+ and whose reduction to $\overline{G_x} = G_x/G_x^+$ is regular cuspidal, $\pi = \text{c-Ind}_{G_x}^G \rho$ as before. Let $\mathcal{B}_{x,1}$ be the block of $\mathrm{Rep}_\Lambda(G_x)$ containing ρ, and $\mathcal{C}_{x,1}$ the block of $\mathrm{Rep}_\Lambda(G)$ containing π.

Let V be the set of equivalence classes of vertices of the Bruhat-Tits building $\mathcal{B}(\mathcal{G}, \mathbb{Q}_p)$ up to G-action. For $y \in V$, let $\sigma_y := \text{c-Ind}_{G_y^+}^{G_y} \Lambda$. Let $\Pi := \bigoplus_{y \in V} \Pi_y$ where $\Pi_y := \text{c-Ind}_{G_y^+}^{G} \Lambda$. Then Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G)_0$, see [8, Appendix]. Let $\sigma_{x,1} := (\sigma_x)|_{\mathcal{B}_{x,1}} \in \mathcal{B}_{x,1} \xhookrightarrow{summand} \mathrm{Rep}_\Lambda(G_x)$ be the $\mathcal{B}_{x,1}$-summand of σ_x. And let $\Pi_{x,1} := \text{c-Ind}_{G_x}^G \sigma_{x,1}$.

Let's summarize the setting in the following diagram.

$$\begin{array}{ccc} \mathrm{Rep}_\Lambda(G_x) & \xrightarrow{\text{c-Ind}_{G_x}^G} & \mathrm{Rep}_\Lambda(G) \\ \cup & & \cup \\ \mathrm{Rep}_\Lambda(G_x)_0 & \longrightarrow & \mathrm{Rep}_\Lambda(G)_0 \\ \cup & & \cup \\ \mathcal{B}_{x,1} & \longrightarrow & \mathcal{C}_{x,1} \\ \parallel & & \parallel \\ \text{block of } \rho & & \text{block of } \pi \end{array}$$

Theorem 2.4.1. $\Pi_{x,1} = \text{c-Ind}_{G_x}^G \sigma_{x,1}$ *is a projective generator of $\mathcal{C}_{x,1}$.*

Proof. First, let $\mathrm{Rep}_\Lambda(G_x)_0$ be the full subcategory of $\mathrm{Rep}_\Lambda(G_x)$ consisting of representations that are trivial on G_x^+ (Don't confuse with $\mathrm{Rep}_\Lambda(G)_0$, the depth-zero category of G). Note $\mathrm{Rep}_\Lambda(G_x)_0$ is a summand of $\mathrm{Rep}_\Lambda(G_x)$ (see Lemma 2.2.14).

Second, note that $\mathrm{Rep}_\Lambda(G_x)_0 \simeq \mathrm{Rep}_\Lambda(\overline{G_x})$. We may assume

$$\mathrm{Rep}_\Lambda(G_x)_0 = \mathcal{B}_{x,1} \oplus \ldots \oplus \mathcal{B}_{x,m}$$

is its block decomposition. So that $\sigma_x = \sigma_{x,1} \oplus \ldots \oplus \sigma_{x,m}$ accordingly. Write $\sigma_x^1 := \sigma_{x,2} \oplus \ldots \oplus \sigma_{x,m}$. Then $\sigma_x = \sigma_{x,1} \oplus \sigma_x^1$, and $\Pi_x = \Pi_{x,1} \oplus \Pi_x^1$ accordingly, where $\Pi_x^1 := \text{c-Ind}_{G_x}^G \sigma_x^1$. And

$$\Pi = \Pi_{x,1} \oplus \Pi_x^1 \oplus \Pi^x,$$

where $\Pi^x := \bigoplus_{y \neq x} \Pi_y$. Let $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$, then we have

$$\Pi = \Pi_{x,1} \oplus \Pi^{x,1}.$$

Recall that Π is a projective generator of the category of depth-zero representations $\mathrm{Rep}_\Lambda(G)_0$. This implies that

$$\mathrm{Hom}_G(\Pi, -) : \mathrm{Rep}_\Lambda(G)_0 \to \text{Mod-}\,\mathrm{End}_G(\Pi)$$

is an equivalence of categories. See [1, Lemma 22].

Next, it is not hard to see that Theorem 2.1.4 implies that

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0,$$

see Lemma 2.4.2. This implies that

$$\text{Mod-}\,\mathrm{End}_G(\Pi) \simeq \text{Mod-}\,\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\,\mathrm{End}_G(\Pi^{x,1})$$

is an equivalence of categories.

Now we can combine the above to show that $\Pi^{x,1}$ does not interfere with $\Pi_{x,1}$, i.e.,

$$\mathrm{Hom}_G(\Pi^{x,1}, X) = 0,$$

for any object $X \in \mathcal{C}_{x,1}$ (see Importent Lemma 2.4.3).

However, since Π is a projective generator of $\mathrm{Rep}_\Lambda(G)_0$, we have

$$\mathrm{Hom}_G(\Pi, X) \neq 0,$$

for any $X \in \mathcal{C}_{x,1}$. This together with the last paragraph implies that

$$\mathrm{Hom}_G(\Pi_{x,1}, X) \neq 0,$$

for any $X \in \mathcal{C}_{x,1}$, i.e. $\Pi_{x,1}$ is a generator of $\mathcal{C}_{x,1}$.

Finally, note $\Pi_{x,1}$ is projective in $\mathrm{Rep}_\Lambda(G)_0$ since it is a summand of the projective object Π. Hence $\Pi_{x,1}$ is projective in $\mathcal{C}_{x,1}$. This together with the last paragraph implies that $\Pi_{x,1}$ is a projective generator of $\mathcal{C}_{x,1}$.

□

2.4.1 Lemmas

In this subsection, I collect some lemmas used in the proof of Theorem 2.1.5.

Lemma 2.4.2.

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = \mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0.$$

Proof. Recall that $\Pi^{x,1} := \Pi_x^1 \oplus \Pi^x$.

First, we compute

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi_x^1) = \mathrm{Hom}_{G_x}(\sigma_{x,1}, \sigma_x^1) = 0,$$

where the first equality is the first case of Theorem 2.1.4 (note $\sigma_{x,1} \in \mathcal{B}_{x,1}$, hence has cuspidal reduction by Theorem 2.1.3, and hence the condition of Theorem 2.1.4 is satisfied), and the second equality is because $\sigma_{x,1}$ and σ_x^1 lies in different blocks of $\mathrm{Rep}_\Lambda(G_x)$ by definition.

Second, recall that $\Pi_{x,1} = \text{c-Ind}_{G_x} \sigma_{x,1}$ with $\sigma_{x,1}$ having cuspidal reduction, and $\Pi_y = \text{c-Ind}_{G_y} \sigma_y$. We compute

$$\mathrm{Hom}_G(\Pi_{x,1}, \Pi^x) = \bigoplus_{y \neq x} \mathrm{Hom}_G(\Pi_{x,1}, \Pi_y) = 0,$$

by the second case of Theorem 2.1.4.

Combining the above three paragraphs, we get $\mathrm{Hom}_G(\Pi_{x,1}, \Pi^{x,1}) = 0$.

A same argument shows that $\mathrm{Hom}_G(\Pi^{x,1}, \Pi_{x,1}) = 0$. $\square$

Lemma 2.4.3 (Important Lemma)**.** $\mathrm{Hom}_G(\Pi^{x,1}, X) = 0$, *for any object* $X \in \mathcal{C}_{x,1}$.

Proof. Recall that

$$\mathrm{Hom}_G(\Pi, -) : \mathrm{Rep}_\Lambda(G)_0 \to \text{Mod-}\mathrm{End}_G(\Pi) \simeq \text{Mod-}\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\mathrm{End}_G(\Pi^{x,1})$$

is an equivalence of categories. It is even an equivalence of abelian categories since $\mathrm{Hom}_G(\Pi, -)$ is exact and commutes with direct product. Hence the image of $\mathcal{C}_{x,1}$ must be indecomposable as $\mathcal{C}_{x,1}$ is indecomposable, i.e.,

$$\mathrm{Hom}_G(\Pi, -) = \mathrm{Hom}_G(\Pi_{x,1}, -) \oplus \mathrm{Hom}_G(\Pi^{x,1}, -)$$

can map $\mathcal{C}_{x,1}$ nonzeroly to only one of $\text{Mod-}\mathrm{End}_G(\Pi_{x,1})$ and $\text{Mod-}\mathrm{End}_G(\Pi^{x,1})$ (See the diagram below).

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda(G)_0 \arrow[rr, "\mathrm{Hom}_G(\Pi,-)"] & & \text{Mod-}\mathrm{End}_G(\Pi) \\
\mathcal{C}_{x,1} \arrow[u, "\cup"] \arrow[rr, "\mathrm{Hom}_G(\Pi_{x,1},-)\oplus\mathrm{Hom}_G(\Pi^{x,1},-)"] & & \text{Mod-}\mathrm{End}_G(\Pi_{x,1}) \oplus \text{Mod-}\mathrm{End}_G(\Pi^{x,1}) \arrow[u, "\wr"']
\end{tikzcd}$$

Then it must be $\text{Mod-}\mathrm{End}_G(\Pi_{x,1})$ (that $\mathrm{Hom}_G(\Pi, -)$ maps $\mathcal{C}_{x,1}$ nonzeroly to) since

$$\mathrm{Hom}_G(\Pi_{x,1}, \pi) = \mathrm{Hom}_G(\sigma_{x,1}, \rho) = \mathrm{Hom}_G(\sigma_x, \rho) \neq 0.$$

In other words, $\mathrm{Hom}_G(\Pi^{x,1}, -)$ is zero on $\mathcal{C}_{x,1}$.

$\square$

Chapter 3

Example: $GL_n(F)$

Let's apply the theories in the previous chapters to the example of $GL_n(F)$. Throughout this chapter, $G := GL_n$.

That said, there is a little mismatch between the theories before and the example here, namely, we assumed for simplicity in the theories that G is simply connected (and in particular, semisimple), while this is not the case for $G = GL_n$. However, there is only some minor difference due to the center $\mathbb{G}_m$ of GL_n. I leave it as an exercise for the readers to figure out the details.

3.1 L-parameter side

Let $\varphi \in Z^1(W_F, \hat{G}(\overline{\mathbb{F}_\ell}))$ be an irreducible tame L-parameter. Let $\psi \in Z^1(W_F, \hat{G}(\overline{\mathbb{Z}_\ell}))$ be any lift of φ. Let C_φ be the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ. By Need ref?, we compute that

$$C_\varphi \cong [T/T] \times \mu,$$

where $T = C_{\hat{G}}(\psi_\ell)$ is a maximal torus of GL_n, and $\mu = (T^{Fr=(-)^q})^0$, and the T-action on T is specified in Need ref?. To go further, let's choose a nice basis of the Weil group representations φ and ψ.

Indeed, every irreducible tame L-parameter with $\overline{\mathbb{F}_\ell}$-coefficients φ of GL_n are of the form $\varphi = Ind_{W_E}^{W_F}\eta$, where E is a degree n unramified extension of F, $W_E \cong I_F \rtimes \langle \mathrm{Fr}^n \rangle$ is the Weil group of E, and $\eta : W_E \to \overline{\mathbb{F}_\ell}^*$ is a tame (i.e., trivial on $P_E = P_F$) character of W_E such that $\{\eta, \eta^q, ..., \eta^{q^{n-1}}\}$ are distinct. To find a lift of it with $\overline{\mathbb{Z}_\ell}$-coefficients, we let $\tilde{\eta} : W_E \to \overline{\mathbb{Z}_\ell}^*$, and let $\psi := Ind_{W_E}^{W_F}\tilde{\eta}$. Then under a nice basis, we could specify the matrices corresponds to the topological generater s_0 and Fr:

$$\psi(s_0) = \begin{bmatrix} \tilde{\eta}(s_0) & 0 & 0 & \dots & 0 \\ 0 & \tilde{\eta}(s_0)^q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\eta}(s_0)^{q^{n-1}} \end{bmatrix}$$

and

$$\psi(\mathrm{Fr}) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \tilde{\eta}(\mathrm{Fr}^n) & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Under this basis, $T = C_{\hat{G}}(\psi_\ell)$ is the diagonal torus of GL_n, with Fr acting by conjugacy via ψ, i.e.,

$$\mathrm{Fr}\,.\,\mathrm{diag}(t_1, t_1, ..., t_{n-1}, t_n) = \mathrm{diag}(t_2, t_3, ..., t_n, t_1).$$

So one could compute that

$$T^{\mathrm{Fr}=(-)^q} \cong \mu_{q^n-1},$$

and that

$$(T^{\mathrm{Fr}=(-)^q})^0 \cong \mu_{\ell^k},$$

where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

To compute the quotient $[T/T]$, we note that T acts on T via twisted conjugacy

$$(t, t') \mapsto (tnt^{-1}n^{-1})t',$$

where n is same as $\psi(Fr)$ in effect. So in our case, this action is

$$(t_1, t_2, ..., t_n).(t_1', t_2', ..., t_n') = (t_n^{-1}t_1t_1', t_1^{-1}t_2t_2', ..., t_{n-1}^{-1}t_nt_n').$$

We see that the orbits of this action are determined by the determinants (hence are in bijection with $\mathbb{G}_m$), and the center $\mathbb{G}_m \cong Z \subset T$ acts trivially. Therefore,

$$[T/T] \cong [\mathbb{G}_m/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$.

In conclusion, we have that the connected component of $Z^1(W_F, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ is

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k},$$

where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$, and $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

3.2 Representation side

By modular Deligne-Lusztig theory, the block $\mathcal{A}_{x,1}$ of $GL_n(\mathbb{F}_q)$ containing a cuspidal representation σ is equivalent to the block of an elliptic torus, which is isomorphic to $\mathbb{F}_{q^n}^*$. So this block is equivalent to $\overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)$, where $k \in \mathbb{Z}$ is maximal such that ℓ^k divides $q^n - 1$.

$\mathcal{A}_{x,1}$ inflats to a block of $K := GL_n(\mathcal{O}_F)$ containing the inflation $\tilde{\sigma}$ of σ, and further corresponds to a block $\mathcal{B}_{x,1}$ of KZ containing ρ, a extension of $\tilde{\sigma}$ to KZ, where Z is the center of $GL_n(F)$. We have

$$\mathcal{B}_{x,1} \cong \mathcal{A}_{x,1} \otimes \mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(\mathbb{Z}) \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod},$$

because

$$KZ \cong K \times \{\mathrm{diag}(\pi^m, ..., \pi^m | m \in \mathbb{Z})\} \cong K \times \mathbb{Z}.$$

Argue as before (See ?) we see that the compact induction c-Ind_{KZ}^{G} induces an equivalence of categories

$$\mathcal{B}_{x,1} \cong \mathcal{C}_{x,1},$$

where $\mathcal{C}_{x,1}$ is the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))$ containing $\pi := \text{c-Ind}_{KZ}^{G} \rho$.

Since every depth-zero supercuspidal representation π arises as above, we have that the block containing π satisfies

$$\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]} \cong \mathcal{C}_{x,1} \cong \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1) \otimes \overline{\mathbb{Z}_\ell}[t, t^{-1}]\text{-Mod}.$$

Chapter 4

The categorical local Langlands conjecture

In this chapter, I prove the categorical local Langlands conjecture for depth-zero supercuspidal part of $G = GL_n$ with coefficients $\Lambda = \overline{\mathbb{Z}_\ell}$.

Let $\varphi \in Z^1(W_E, \hat{G}(\overline{\mathbb{F}_\ell}))$ be an irreducible tame L-parameter. Let C_φ be the connected component of $Z^1(W_E, \hat{G})_{\overline{\mathbb{Z}_\ell}}$ containing φ.

The goal is to show that there is an equivalence

$$D_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong D_{Coh,Nilp}^{b,qc}(C_\varphi)$$

of derived (?) categories.

As a first step, let's unravel the definition of both sides and describe them explicitly.

4.1 Unraveling definitions

4.1.1 L-parameter side

Let's first state a lemma that makes the decorations in $D_{Coh,Nilp}^{b,qc}(C_\varphi)$ go away. We postpone its proof to a later subsection.

Lemma 4.1.1. $D_{Coh,Nilp}^{b,qc}(C_\varphi) \cong D_{Coh,\{0\}}^{b}(C_\varphi) \cong \mathrm{Perf}(C_\varphi)$.

Let's assume the lemma for the moment and continue. By our computation before,

$$C_\varphi \cong [\mathbb{G}_m/\mathbb{G}_m] \times \mu_{\ell^k} \cong \mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k},$$

where $k \in \mathbb{Z}_{\geq 0}$ is maximal such that ℓ^k divides $q^n - 1$. So

$$\mathrm{Perf}(C_\varphi) \cong \mathrm{Perf}(\mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k}) \simeq \mathrm{Perf}(\mathbb{G}_m) \otimes \mathrm{Perf}([*/\mathbb{G}_m]) \otimes \mathrm{Perf}(\mu_{\ell^k}).$$

Here,

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell})\chi \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell}),$$

where χ runs over characters of $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m | m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

In conclusion, we have

$$\mathrm{Perf}(C_\varphi) \cong \bigoplus_\chi \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where χ runs over characters of $\mathbb{G}_m$

$$X^*(\mathbb{G}_m) = \{t \mapsto t^m | m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

4.1.2 Bun_G side

Since φ is irreducible,

$$D_{lis}^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong D_{lis}^{C_\varphi}(Bun_G^{ss}, \overline{\mathbb{Z}_\ell})^\omega.$$

Since

$$Bun_G^{ss} = \sqcup_{b \in B(G)_{basic}}[*/G_b(F)],$$

we have

$$D_{lis}^{C_\varphi}(Bun_G^{ss}, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{basic}} D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega.$$

Let's look closer into each direct summand. In our case $G = GL_n$,

$$B(G)_{basic} \cong \pi_1(G)_\Gamma \cong \mathbb{Z}$$

.

Let's first look at the summand for $b = 1$ (corresponding to $0 \in \mathbb{Z} \cong B(G)_{basic}$). For $b = 1$, $G_b \cong GL_n$, and

$$D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})^\omega \cong D^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong D(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]})^\omega,$$

where $\pi \in \mathrm{Rep}_{\overline{\mathbb{F}_\ell}}(GL_n(F))$ is the representation with L-parameter φ, and $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]}$ is the block of $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))$ containing π. And we've computed that

$$\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(GL_n(F))_{[\pi]} \cong \overline{\mathbb{Z}_\ell}[t, t^{-1}] \otimes \overline{\mathbb{Z}_\ell}[s]/(s^{\ell^k} - 1)\text{-Mod} \cong \mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}),$$

where $k \in \mathbb{Z}_{\geq 0}$ is again maximal such that ℓ^k divides $p^n - 1$. So we have

$$D^{C_\varphi}(GL_n(F), \overline{\mathbb{Z}_\ell})^\omega \cong D(\mathrm{QCoh}(\mathbb{G}_m \times \mu_{\ell^k}))^\omega \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

We could get a similar description of $D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell})$ for free by the spectral action and the compatibility of Fargues-Scholze with $\pi_1(G)_\Gamma$-grading. For this, we consider the composition

$$q : C_\varphi \cong \mathbb{G}_m \times [*/\mathbb{G}_m] \times \mu_{\ell^k} \to [*/\mathbb{G}_m].$$

Recall that

$$\mathrm{Perf}([*/\mathbb{G}_m]) \cong \bigoplus_\chi \mathrm{Perf}(\overline{\mathbb{Z}_\ell})\chi,$$

we denote by $\mathcal{M}_\chi$ the corresponding simple object in $\mathrm{Perf}([*/\mathbb{G}_m])$. Moreover, $\mathcal{M}_\chi$ pull-backs to a line bundle

$$\mathcal{L}_\chi := q^*\mathcal{M}_\chi.$$

We could now state the key proposition that allows us to get to arbitrary $b \in B(G)_{basic}$ from the $b = 1$ case, using the spectral action.

Proposition 4.1.2. *1. The restriction of the spectral action by $\mathcal{L}_\chi$ to $D(G_b(F), \overline{\mathbb{Z}_\ell})$ factors through $D(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell})$.*

$$\begin{tikzcd} \mathcal{L}_\chi * - : & D_{lis}(Bun_G, \overline{\mathbb{Z}_\ell}) \arrow[r] & D_{lis}(Bun_G, \overline{\mathbb{Z}_\ell}) \\ & D(G_b(F), \overline{\mathbb{Z}_\ell}) \arrow[u, "\subset"] \arrow[r] & D(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell}) \arrow[u, "\subset"'] \end{tikzcd}$$

2. $\mathcal{L}_\chi * - : D(G_b(F), \overline{\mathbb{Z}_\ell}) \to D(G_{b-\chi}(F), \overline{\mathbb{Z}_\ell})$ is an equivalence of categories, with inverse $\mathcal{L}_{\chi^{-1}} * -$.

Proof. For the first assertion, see [19, Lemma 5.3.2]. For the second assertion, note that $\mathcal{L}_\chi$ and $\mathcal{L}_{\chi^{-1}}$ are clearly inverse to each other once they are well-defined, since q^* preserves tensor product. □

So we have

$$D^{C_\varphi}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong \bigoplus_{b \in B(G)_{basic}} D^{C_\varphi}(G_b(F), \overline{\mathbb{Z}_\ell}) \cong \bigoplus_{b \in B(G)_{basic}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}).$$

4.1.3 Proof of Lemma 4.1.1

Now we prove Lemma 4.1.1.

The first isomorphism is because C_φ is connected, hence the quasicompact support condition qc is automatic.

The second isomorphism needs some computation. For the definition and properties of the nilpotent singular support condition $Nilp$, I refer to [14, Section VIII.2]. At the end of the day, it boils to the fact that

$$H^0(W_F, \hat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda(1)) \cap Nilp(\hat{\mathfrak{g}}^*) = \{0\}.$$

(Maybe elaborate more.)

4.2 The spectral action induces an equivalence of categories

To summarize, we have (abstract) equivalence of categories

$$D^{b,qc}_{Coh,Nilp}(C_\varphi) \cong \bigoplus_{\chi \in \mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong \bigoplus_{b \in \mathbb{Z}} \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \cong D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega,$$

where I identified both $X^*(\mathbb{G}_m) \cong X^*(Z(\hat{G}))$ and $B(G)_{basic} \cong \pi_1(G)_\Gamma$ with $\mathbb{Z}$. The next goal is to show that the spectral action induces an equivalence of categories

$$D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega \cong D^{b,qc}_{Coh,Nilp}(C_\varphi). \tag{4.2.1}$$

4.2.1 Definition of the functor

Let's first define the functor. For this, let's choose a Whittaker datum consisting of a Borel $B \subset G$ and a generic character $\vartheta : U(F) \to \overline{\mathbb{Z}_\ell}^*$. Let $\mathcal{W}_\vartheta$ be the sheaf concentrated on Bun_G^1 corresponding to the representation $W_\vartheta := \text{c-Ind}_{U(F)}^{G(F)} \vartheta$. Let $W_{\vartheta,[\pi]}$ be the restriction of W_ϑ to the block $\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]}$, and $\mathcal{W}_{\vartheta,[\pi]}$ the corresponding sheaf.

We define our desired functor by spectral acting on $\mathcal{W}_{\vartheta,[\pi]}$:

$$\Theta : D^{b,qc}_{Coh,Nilp}(C_\varphi) \cong \mathrm{Perf}(C_\varphi) \longrightarrow D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega, \qquad A \mapsto A * \mathcal{W}_{\vartheta,[\pi]}.$$

4.2.2 Equivalence on degree zero part

We now show that Θ induces an equivalence on degree zero part. At the end of the day, this is similar to the following fact: If I have a functor $F : R\text{-Mod} \to R\text{-Mod}$, which is $(R\text{-Mod})$-linear and sends R to R, then F is an equivalence of category.

By compatibility with $\pi_1(G)_\Gamma$-grading, Θ restricts to a map

$$\Theta_0 := \Theta|_{\mathrm{Perf}(C_\varphi)_{\chi=0}} : \mathrm{Perf}(C_\varphi)_{\chi=0} \longrightarrow D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0},$$

where $\mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ and

$$D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0} \cong D(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega \cong D(\mathrm{End}(W_{\vartheta,[\pi]})\text{-Mod})^\omega.$$

By tracking the definition, the structure sheaf $\mathcal{O} \in \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ goes to the Whittaker representation $W_{\vartheta,[\pi]} \in D(\mathrm{Rep}_{\overline{\mathbb{Z}_\ell}}(G(F))_{[\pi]})^\omega$, and further goes to $\mathrm{End}(W_{\vartheta,[\pi]}) \in D(\mathrm{End}(W_{\vartheta,[\pi]})\text{-Mod})$. Moreover, by local Langlands in family (See ?),

$$\mathrm{End}(W_{\vartheta,[\pi]}) \cong \mathcal{Z}(G)_{[\pi]} \cong \mathcal{O}(C_\varphi) \cong \mathcal{O}(\mathbb{G}_m \times \mu_{\ell^k}).$$

Therefore, we have a functor $\Theta_0 : \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k}) \to \mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$ which is $\mathrm{Perf}(\mathbb{G}_m \times \mu_{\ell^k})$-linear and sends the structure sheaf to the structure sheaf, hence an equivalence of categories.

4.2.3 The full equivalence

Finally, we use the spectral action to get the full equivalence. Indeed, on the L-parameter side, for any character $\chi' \in X^*(\mathbb{G}_m)$, tensoring with $\mathcal{L}_{\chi'}$ induces an equivalence

$$\mathcal{L}_{\chi'} \otimes - : \mathrm{Perf}(C_\varphi)_{\chi=0} \cong \mathrm{Perf}(C_\varphi)_{\chi=\chi'}.$$

Similarly, on the Bun_G side, by Proposition 4.1.2, spectral acting by $\mathcal{L}_{\chi'}$ induces an equivalence

$$\mathcal{L}_{\chi'} * - : D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=0} \cong D^{C_\varphi}_{lis}(Bun_G, \overline{\mathbb{Z}_\ell})^\omega_{b=-\chi'}.$$

Therefore, we get the full equivalence via the spectral action.

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