Simplex Integration

By Domain Subdivision

Inhaltsverzeichnis

Teil I.
Theory

1. Problem

Total Energy in the System:

$$\Pi = \int_\Omega g(\phi)\psi(\mathbf{u})\,d\Omega + \frac{G_c}{2l}\int_\Omega \phi^2 + l^2\nabla\phi\cdot\nabla\phi\,d\Omega \rightarrow \min \qquad [1.1]$$

Degradation Function

$$g(\phi) = \left(1-\phi^2\right)^2 + k \qquad [1.2]$$

mit

k ... being a small but finite scalar such as 10 6
G_c ... critical energy release rate, material parameter
l ... *width* of phase field
$\psi(u)$ ... strain energy density function
u ... displacement function
ϕ ... phase field parameter, ansatz function discussed below
$\nabla\phi$ ... gradient of phase field parameter

$$\delta_u\Pi = \int_\Omega g(\phi)\sigma(\mathbf{u})\frac{\partial\varepsilon}{\partial\mathbf{u}}\,\delta\mathbf{u} = 0 \qquad [1.3]$$

$$\delta_\phi\Pi = \int_\Omega 2(\phi-1)\,\delta\phi\psi(\mathbf{u})\,d\Omega + \frac{G_c}{l}\int_\Omega \phi\,\delta\phi + l^2\nabla\phi\cdot\nabla\,\delta\phi\,d\Omega = 0 \qquad [1.4]$$

1.1. Ansatz functions

$$\mathbf{u} = \sum_i N_i\mathbf{u}_i + \sum_i N_i F\mathbf{a}_i \qquad [1.5]$$

mit

N_i ... are quadratic lagrange (standard) shape functions for tetrahedrons
$\mathbf{U}_i = \mathbf{u}_i, \mathbf{a}_i$ ... are nodal degrees of freedom for displacement function
F ... is an enrichment function (sigmoid like, depends on ϕ, later)

$$f_{base} = \sum_i N_i\phi_i \qquad [1.6]$$

$$\varsigma = \frac{f_{base}}{\sqrt[4]{f_{base}^2 + k_{res}}} \qquad [1.7]$$

mit

k_{reg} . . . small but finite parameter

$$\phi = \exp(-\varsigma) \tag{1.8}$$

$$\phi = \exp(-\frac{\varsigma}{l}) \tag{1.9}$$

we need to be able to integrate the residual vectors and the stiffness matrices efficiently and accurately

$$\delta_{\mathbf{U}_i}\Pi = \int_\Omega g(\phi)\boldsymbol{\sigma}(\mathbf{u})\frac{\partial\varepsilon}{\partial\mathbf{u}}\cdot\frac{\partial\mathbf{u}}{\partial\mathbf{U}}\,d\Omega\cdot\delta\mathbf{U}_i \tag{1.10}$$

$$\delta_{\phi_i}\Pi = \int_\Omega 2(\phi-1)\frac{\partial\phi}{\partial\phi_i}\psi(\mathbf{u})\,d\Omega\,\delta\phi_i + \frac{G_c}{l}\int_\Omega \phi\frac{\partial\phi}{\partial\phi_i} + l^2\nabla\phi\cdot\frac{\partial\nabla\phi}{\partial\phi_i}\,d\Omega\,\delta\phi_i = 0 \tag{1.11}$$

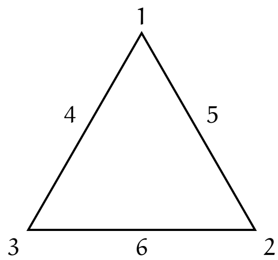
$$\Delta_{U_i}\,\delta_{U_i}\Pi = U_i\cdot\int_\Omega g(\phi)\frac{\partial\varepsilon}{\partial U_i}\cdot\mathbb{C}\cdot\frac{\partial\varepsilon}{\partial U_i}\,d\Omega\cdot\delta U_i \tag{1.12}$$

$$\Delta_{\phi_j}\,\delta_{\phi_i}\Pi = \phi_j\int_\Omega 2\left(\frac{\partial\phi}{\partial\phi_i}\right)^2\psi(u)\,d\Omega\,\delta\phi_i + \phi_j\int_\Omega 2(\phi-1)\frac{\partial^2\phi}{\partial\phi_i}\phi(u)\,d\Omega\,\delta\phi_i \tag{1.13}$$

$$+\,\phi_j\frac{G_c}{l}\int_\Omega \frac{\partial\phi}{\partial\phi_i} + \frac{\partial^2\phi}{\partial\phi_i} + l^2\frac{\partial\nabla\phi}{\partial\phi_j}\cdot\frac{\partial\nabla\phi}{\partial\phi_i} + l^2\nabla\phi\cdot\frac{\partial^2\nabla\phi}{\partial\phi_i^2}\,d\Omega\,\delta\phi_i \tag{1.14}$$

2. Element Description

Shape Functions in barycentric coordinates



$$N_1(\xi_1,\xi_2,\xi_3) = \xi_1 \tag{2.1}$$

$$N_2(\xi_1,\xi_2,\xi_3) = \xi_2 \tag{2.2}$$

$$N_3(\xi_1,\xi_2,\xi_3) = \xi_3 \tag{2.3}$$

$$N_4(\xi_1,\xi_2,\xi_3) = 4\xi_1\xi_3 \tag{2.4}$$

$$N_5(\xi_1,\xi_2,\xi_3) = 4\xi_1\xi_2 \tag{2.5}$$

$$N_6(\xi_1,\xi_2,\xi_3) = 4\xi_2\xi_3 \tag{2.6}$$

Barycentric Interpolation Formula $P : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$P(\xi_1,\xi_2,\xi_3) = p_1\xi_1 + p_2\xi_2 + p_3\xi_3 \tag{2.7}$$

mit $p_i \in \mathbb{R}^2$, $\xi_i \in [0,1]$

ξ-η-Transformation

$$\xi_1 := 1 - \xi - \eta \tag{2.8}$$

$$\xi_2 := \xi \tag{2.9}$$

$$\xi_3 := \eta \tag{2.10}$$

mit $\xi \in [0,1]$, $\eta \in [0,1]$

Es gilt:

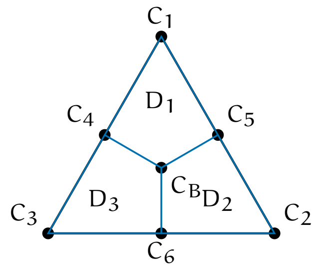
$$T(\xi,\eta) = \begin{bmatrix} 1-\xi-\eta \\ \xi \\ \eta \end{bmatrix} \tag{2.11}$$

$$T^{-1}(\xi_1,\xi_2,\xi_3) = \xi_1\begin{bmatrix}0\\0\end{bmatrix} + \xi_2\begin{bmatrix}1\\0\end{bmatrix} + \xi_3\begin{bmatrix}0\\1\end{bmatrix} \tag{2.12}$$

3. Simplex Integration in $n = 2$

3.1. Simplex Integration

Characteristic points in barycentric coordinates



$$C_B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad [3.1]$$

$$C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad C_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad [3.2]$$

$$C_4 = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} \qquad C_5 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \qquad C_6 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} \quad [3.3]$$

Domain of a Simplex $\boldsymbol{\Delta}$ can be decomposed into three disjunct subdomains:

$$\boldsymbol{\Delta} = D_1 \cup D_2 \cup D_3 \quad [3.4]$$

Therefore the double-Integral

$$\iint_{\boldsymbol{\Delta}} F \, d\boldsymbol{\Delta} = \iint_{D_1} F \, dD_1 + \iint_{D_2} F \, dD_2 + \iint_{D_3} F \, dD_3 \quad [3.5]$$

Mapping functions from the $[-1,1] \times [-1,1]$ X-Y-Unit Square

$$g_1(X) = \frac{X}{2} + \frac{1}{2} \qquad g_2(X) = -\frac{X}{2} + \frac{1}{2} \quad [3.6]$$

$$\frac{\partial g_1}{\partial X} = \frac{1}{2} \qquad \frac{\partial g_2}{\partial X} = -\frac{1}{2} \quad [3.7]$$

$$g_1(Y) = \frac{Y}{2} + \frac{1}{2} \qquad g_2(Y) = -\frac{Y}{2} + \frac{1}{2} \quad [3.8]$$

$$\frac{\partial g_1}{\partial Y} = \frac{1}{2} \qquad \frac{\partial g_2}{\partial Y} = -\frac{1}{2} \quad [3.9]$$

to barycentric coordinates of the D_1, D_2, D_3 Quadrilaterials

$$C_{D_1}(X,Y) = C_1 \cdot g_1(X)g_1(Y) + C_5 \cdot g_1(X)g_2(Y) + C_B \cdot g_2(X)g_2(Y) + C_4 \cdot g_2(X)g_1(Y) \quad [3.10]$$

$$C_{D_2}(X,Y) = C_2 \cdot g_1(X)g_1(Y) + C_6 \cdot g_1(X)g_2(Y) + C_B \cdot g_2(X)g_2(Y) + C_5 \cdot g_2(X)g_1(Y) \quad [3.11]$$

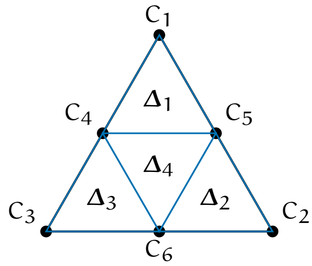
$$C_{D_3}(X,Y) = C_3 \cdot g_1(X)g_1(Y) + C_4 \cdot g_1(X)g_2(Y) + C_B \cdot g_2(X)g_2(Y) + C_6 \cdot g_2(X)g_1(Y) \quad [3.12]$$

3.2. Integraltransformation

3.3. Simplex Subdivision

The Integral over a parent Simplex $\boldsymbol{\Delta}_p$, can be expressed as an Integral over 4 Child Simpleces $\boldsymbol{\Delta}_i$:

$$\iint_{\Delta_p} F\, d\boldsymbol{\Delta}_p = \iint_{\Delta_1} F\, d\boldsymbol{\Delta}_1 + \iint_{\Delta_2} F\, d\boldsymbol{\Delta}_2 + \iint_{\Delta_3} F\, d\boldsymbol{\Delta}_3 + \iint_{\Delta_4} F\, d\boldsymbol{\Delta}_4 \quad [3.13]$$



The Coordinates of a Child $\boldsymbol{\Delta}_i$ can be expressed in local-barycentric coordinates $\xi'_{i,1}, \xi'_{i,2}, \xi'_{i,3}$

The corresponding Transformation from the local coordinate System into the global is given by

$$T_{lg}(\xi'_{i,1}, \xi'_{i,2}, \xi'_{i,3}) = C_{i,1}\xi'_{i,1} + C_{i,2}\xi'_{i,2} + C_{i,3}\xi'_{i,3} \quad [3.14]$$

where $C_{i,j}$ are the coordinates of the Verteces of the Child Simplex $\boldsymbol{\Delta}_i$.