

# Simplex Integration

By Domain Subdivision

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# Teil I.

## Theory

### 1. Problem

Total Energy in the System:

$$\Pi = \int_{\Omega} g(\phi)\psi(\mathbf{u}) \, d\Omega + \frac{G_c}{2l} \int_{\Omega} \phi^2 + l^2 \nabla \phi \cdot \nabla \phi \, d\Omega \rightarrow \min \quad [1.1]$$

Degradation Function

$$g(\phi) = \left(1 - \phi^2\right)^2 + k \quad [1.2]$$

mit

- $k$  ... being a small but finite scalar such as  $10^{-6}$
- $G_c$  ... critical energy release rate, material parameter
- $l$  ... *width* of phase field
- $\psi(\mathbf{u})$  ... strain energy density function
- $\mathbf{u}$  ... displacement function
- $\phi$  ... phase field parameter, ansatz function discussed below
- $\nabla \phi$  ... gradient of phase field parameter

$$\delta_{\mathbf{u}} \Pi = \int_{\Omega} g(\phi) \sigma(\mathbf{u}) \frac{\partial \varepsilon}{\partial \mathbf{u}} \delta \mathbf{u} \, d\Omega = 0 \quad [1.3]$$

$$\delta_{\phi} \Pi = \int_{\Omega} 2(\phi - 1) \delta \phi \psi(\mathbf{u}) \, d\Omega + \frac{G_c}{l} \int_{\Omega} \phi \delta \phi + l^2 \nabla \phi \cdot \nabla \delta \phi \, d\Omega = 0 \quad [1.4]$$

#### 1.1. Ansatz functions

$$\mathbf{u} = \sum_i N_i \mathbf{u}_i + \sum_i N_i F \mathbf{a}_i \quad [1.5]$$

mit

- $N_i$  ... are quadratic lagrange (standard) shape functions for tetrahedrons
- $\mathbf{u}_i = \mathbf{u}_i, \mathbf{a}_i$  ... are nodal degrees of freedom for displacement function
- $F$  ... is an enrichment function (sigmoid like, depends on  $\phi$ , later)

$$f_{\text{base}} = \sum_i N_i \phi_i \quad [1.6]$$

$$\zeta = \frac{f_{\text{base}}}{\sqrt[4]{f_{\text{base}}^2 + k_{\text{res}}}} \quad [1.7]$$

mit

$k_{\text{reg}} \dots$  small but finite parameter

$$\phi = \exp(-\varsigma) \quad [1.8]$$

$$\phi = \exp\left(-\frac{\varsigma}{l}\right) \quad [1.9]$$

we need to be able to integrate the residual vectors and the stiffness matrices efficiently and accurately

$$\delta_{\mathbf{u}_i} \Pi = \int_{\Omega} g(\phi) \sigma(\mathbf{u}) \frac{\partial \varepsilon}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{U}} d\Omega \cdot \delta \mathbf{U}_i \quad [1.10]$$

$$\delta_{\phi_i} \Pi = \int_{\Omega} 2(\phi - 1) \frac{\partial \phi}{\partial \phi_i} \psi(\mathbf{u}) d\Omega \delta \phi_i + \frac{G_c}{l} \int_{\Omega} \phi \frac{\partial \phi}{\partial \phi_i} + l^2 \nabla \phi \cdot \frac{\partial \nabla \phi}{\partial \phi_i} d\Omega \delta \phi_i = 0 \quad [1.11]$$

$$\Delta_{\mathbf{u}_i} \delta_{\mathbf{u}_i} \Pi = \mathbf{U}_i \cdot \int_{\Omega} g(\phi) \frac{\partial \varepsilon}{\partial \mathbf{U}_i} \cdot \mathbb{C} \cdot \frac{\partial \varepsilon}{\partial \mathbf{U}_i} d\Omega \cdot \delta \mathbf{U}_i \quad [1.12]$$

$$\Delta_{\phi_j} \delta_{\phi_i} \Pi = \phi_j \int_{\Omega} 2 \left( \frac{\partial \phi}{\partial \phi_i} \right)^2 \psi(\mathbf{u}) d\Omega \delta \phi_i + \phi_j \int_{\Omega} 2(\phi - 1) \frac{\partial^2 \phi}{\partial \phi_i^2} \phi(\mathbf{u}) d\Omega \delta \phi_i \quad [1.13]$$

$$+ \phi_j \frac{G_c}{l} \int_{\Omega} \frac{\partial \phi}{\partial \phi_i} + \frac{\partial^2 \phi}{\partial \phi_i^2} + l^2 \frac{\partial \nabla \phi}{\partial \phi_j} \cdot \frac{\partial \nabla \phi}{\partial \phi_i} + l^2 \nabla \phi \cdot \frac{\partial^2 \nabla \phi}{\partial \phi_i^2} d\Omega \delta \phi_i \quad [1.14]$$

## 2. Element Description

Shape Functions in barycentric coordinates

$$N_1(\xi_1, \xi_2, \xi_3) = \xi_1 \quad [2.1]$$

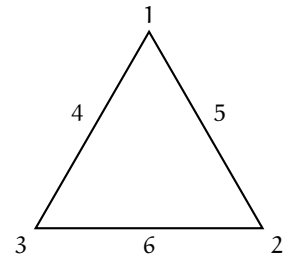
$$N_2(\xi_1, \xi_2, \xi_3) = \xi_2 \quad [2.2]$$

$$N_3(\xi_1, \xi_2, \xi_3) = \xi_3 \quad [2.3]$$

$$N_4(\xi_1, \xi_2, \xi_3) = 4\xi_1 \xi_3 \quad [2.4]$$

$$N_5(\xi_1, \xi_2, \xi_3) = 4\xi_1 \xi_2 \quad [2.5]$$

$$N_6(\xi_1, \xi_2, \xi_3) = 4\xi_2 \xi_3 \quad [2.6]$$



Barycentric Interpolation Formula  $P : \mathbb{B}^3 \rightarrow \mathbb{R}^2$

$$P(\xi_1, \xi_2, \xi_3) = p_1 \xi_1 + p_2 \xi_2 + p_3 \xi_3 \quad [2.7]$$

mit  $p_i \in \mathbb{R}^2$ ,  $\xi_i \in [0, 1]$

$\xi$ - $\eta$ -Transformation

$$\xi_1 := 1 - \xi - \eta \quad [2.8]$$

$$\xi_2 := \xi \quad [2.9]$$

$$\xi_3 := \eta \quad [2.10]$$

mit  $\xi \in [0, 1]$ ,  $\eta \in [0, 1]$

Es gilt:

$$T(\xi, \eta) = \begin{bmatrix} 1 - \xi - \eta \\ \xi \\ \eta \end{bmatrix} \quad [2.11]$$

$$T^{-1}(\xi_1, \xi_2, \xi_3) = \xi_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \xi_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad [2.12]$$

### 3. Simplex Integration in $n = 2$

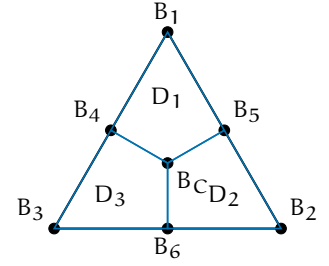
#### 3.1. Simplex Integration

Characteristic points in barycentric coordinates

$$B_C = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad [3.1]$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad [3.2]$$

$$B_4 = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} \quad [3.3]$$



Domain of a Simplex  $\Delta$  can be decomposed into three disjunct subdomains:

$$\Delta = D_1 \cup D_2 \cup D_3 \quad [3.4]$$

Therefore the double-Integral

$$\iint_{\Delta} F d\Delta = \iint_{D_1} F dD_1 + \iint_{D_2} F dD_2 + \iint_{D_3} F dD_3 \quad [3.5]$$

Mapping functions from the  $[-1, 1] \times [-1, 1]$  X-Y-Unit Square

$$g_1(X) = \frac{X}{2} + \frac{1}{2} \quad g_2(X) = -\frac{X}{2} + \frac{1}{2} \quad [3.6]$$

$$\frac{\partial g_1}{\partial X} = \frac{1}{2} \quad \frac{\partial g_2}{\partial X} = -\frac{1}{2} \quad [3.7]$$

$$g_1(Y) = \frac{Y}{2} + \frac{1}{2} \quad g_2(Y) = -\frac{Y}{2} + \frac{1}{2} \quad [3.8]$$

$$\frac{\partial g_1}{\partial Y} = \frac{1}{2} \quad \frac{\partial g_2}{\partial Y} = -\frac{1}{2} \quad [3.9]$$

$$G_1(X, Y) = g_1(X)g_1(Y) \quad G_2(X, Y) = g_1(X)g_2(Y) \quad [3.10]$$

$$G_3(X, Y) = g_2(X)g_1(Y) \quad G_4(X, Y) = g_2(X)g_2(Y) \quad [3.11]$$

to barycentric coordinates of the  $D_1, D_2, D_3$  Quadrilaterals

$$B_{D_1}(X, Y) = B_1 \cdot G_1(X, Y) + B_5 \cdot G_2(X, Y) + B_4 \cdot G_3(X, Y) + B_C \cdot G_4(X, Y) \quad [3.12]$$

$$B_{D_2}(X, Y) = B_2 \cdot G_1(X, Y) + B_6 \cdot G_2(X, Y) + B_5 \cdot G_3(X, Y) + B_C \cdot G_4(X, Y) \quad [3.13]$$

$$B_{D_3}(X, Y) = B_3 \cdot G_1(X, Y) + B_4 \cdot G_2(X, Y) + B_6 \cdot G_3(X, Y) + B_C \cdot G_4(X, Y) \quad [3.14]$$

### 3.2. Simplex Subdivision

The Integral over a parent Simplex  $\Delta_p$ , can be expressed as an Integral over 4 Child Simplexes  $\Delta_i$ :

$$\iint_{\Delta_p} F d\Delta_p = \iint_{\Delta_1} F d\Delta_1 + \iint_{\Delta_2} F d\Delta_2 + \iint_{\Delta_3} F d\Delta_3 + \iint_{\Delta_4} F d\Delta_4 \quad [3.15]$$

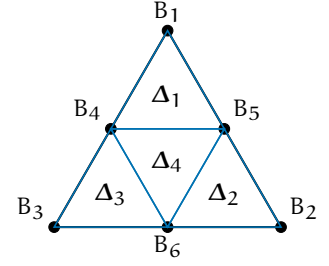
The Coordinates of a Child Simplex  $\Delta_i$  can be expressed in local-barycentric coordinates  $\xi'_{i,1}, \xi'_{i,2}, \xi'_{i,3}$

The corresponding Transformation from the local coordinate System into the global is given by

$$T_{lg}(\xi'_{i,1}, \xi'_{i,2}, \xi'_{i,3}) = B_{i,1}\xi'_{i,1} + B_{i,2}\xi'_{i,2} + B_{i,3}\xi'_{i,3} \quad [3.16]$$

where  $B_{i,j}$  are the coordinates of the Vertices of the Child Simplex  $\Delta_i$ .

This can be done recursively, to get a desired accuracy.



### 3.3. Numerical Integration Scheme

The Integration is done on the Square  $[-1, 1] \times [-1, 1]$ , which allows for Gaussian Integration to be used:

$$\iint_{[-1,1] \times [-1,1]} F(X, Y) d(X, Y) \approx \sum_i \sum_j F(X_i, X_j) w_i w_j \quad [3.17]$$

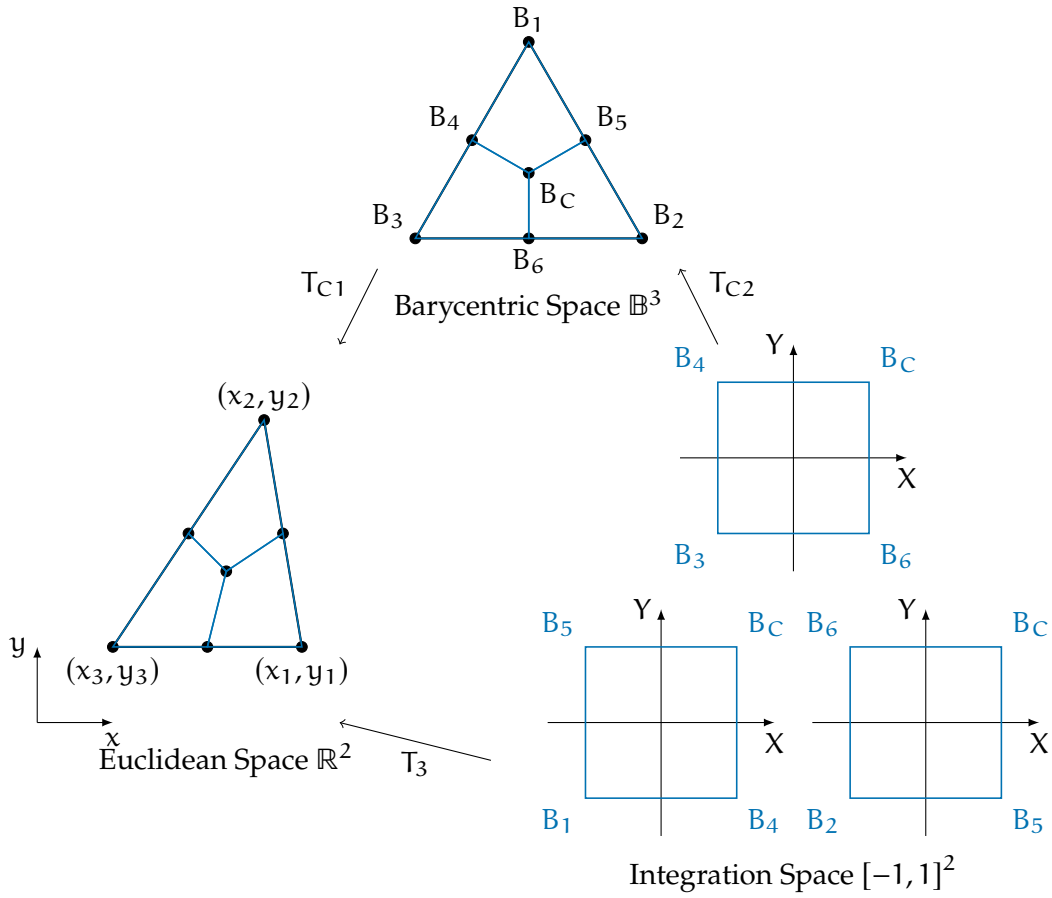
The Gauss-Points  $(X_i, X_j)$  and their weights  $w_i, w_j$  on the Square can be deduced from the one dimensional Gaussian Integration

$$\int_{-1}^1 H(X) dX \approx \sum_i H(X_i) w_i \quad [3.18]$$

The Weights and Points of the 1D Gauss-Legendre Integration are given as:

$n = 1$	$X = 0$	$w = 2$
$n = 2$	$X = \sqrt{\frac{1}{3}}$	$w = 1$
	$X = -\sqrt{\frac{1}{3}}$	$w = 1$
$n = 3$	$X = \sqrt{\frac{3}{5}}$	$w = \frac{5}{9}$
	$X = 0$	$w = \frac{8}{9}$
	$X = -\sqrt{\frac{3}{5}}$	$w = \frac{5}{9}$

### 3.4. Integraltransformation



Let  $S$  denote a Matrix of the coordinates of the vertices of the simplex in the following form

$$S = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad [3.19]$$

The Coordinates  $C_i$  of all characteristic points in Equations [3.1],[3.2] and [3.3] inside the simplex can be expressed in the following form

$$C_C = S \cdot B_C \quad [3.20]$$

$$C_i = S \cdot B_i \quad \forall i \in 1, 2, \dots, 6 \quad [3.21]$$

The transformation from points in the Integration Domain to the points in the bary-

centric given in [3.12],[3.13] and [3.14] can be expressed as

$$B_{D_1}(X, Y) = \begin{bmatrix} B_1 & B_5 & B_4 & B_C \end{bmatrix} \begin{bmatrix} G_1(X, Y) \\ G_2(X, Y) \\ G_3(X, Y) \\ G_4(X, Y) \end{bmatrix} \quad [3.22]$$

$$B_{D_2}(X, Y) = \begin{bmatrix} B_2 & B_6 & B_5 & B_C \end{bmatrix} \begin{bmatrix} G_1(X, Y) \\ G_2(X, Y) \\ G_3(X, Y) \\ G_4(X, Y) \end{bmatrix} \quad [3.23]$$

$$B_{D_3}(X, Y) = \begin{bmatrix} B_3 & B_4 & B_6 & B_C \end{bmatrix} \begin{bmatrix} G_1(X, Y) \\ G_2(X, Y) \\ G_3(X, Y) \\ G_4(X, Y) \end{bmatrix} \quad [3.24]$$

With the Relationship given in [3.20] and [3.21] one can rewrite this as

$$C_{D_1}(X, Y) = S \begin{bmatrix} B_1 & B_5 & B_4 & B_C \end{bmatrix} \begin{bmatrix} G_1(X, Y) \\ G_2(X, Y) \\ G_3(X, Y) \\ G_4(X, Y) \end{bmatrix} \quad [3.25]$$

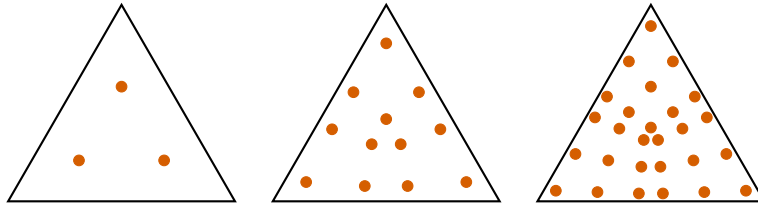
$$C_{D_2}(X, Y) = S \begin{bmatrix} B_2 & B_6 & B_5 & B_C \end{bmatrix} \begin{bmatrix} G_1(X, Y) \\ G_2(X, Y) \\ G_3(X, Y) \\ G_4(X, Y) \end{bmatrix} \quad [3.26]$$

$$C_{D_3}(X, Y) = S \begin{bmatrix} B_3 & B_4 & B_6 & B_C \end{bmatrix} \begin{bmatrix} G_1(X, Y) \\ G_2(X, Y) \\ G_3(X, Y) \\ G_4(X, Y) \end{bmatrix} \quad [3.27]$$

With the Jacobi Matrix of any particular Domain being

$$J(C_{D_i}) = \begin{bmatrix} \frac{\partial C_{D_i}}{\partial X} & \frac{\partial C_{D_i}}{\partial Y} \end{bmatrix} = \begin{bmatrix} C_i & C_j & C_k & C_C \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial G_1}{\partial X} & \frac{\partial G_1}{\partial Y} \\ \frac{\partial G_2}{\partial X} & \frac{\partial G_2}{\partial Y} \\ \frac{\partial G_3}{\partial X} & \frac{\partial G_3}{\partial Y} \\ \frac{\partial G_4}{\partial X} & \frac{\partial G_4}{\partial Y} \end{bmatrix} \quad [3.28]$$

### 3.4.1. Gauss Point Distribution



Gauss Points  $n = 1$    Gauss Points  $n = 2$    Gauss Points  $n = 3$



### 3.5. Simplex Subdivision

The Coordinates of the Verteces of the 4 Child Simplizes  $\Delta_i$  can be calculated with S by

$$S_{\Delta,1} = S \cdot B_{\Delta,1} = S \cdot \begin{bmatrix} B_1 & B_5 & B_4 \end{bmatrix} = S \cdot \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \quad [3.29]$$

$$S_{\Delta,2} = S \cdot B_{\Delta,2} = S \cdot \begin{bmatrix} B_2 & B_6 & B_5 \end{bmatrix} = S \cdot \begin{bmatrix} 0 & 0 & 0.5 \\ 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix} \quad [3.30]$$

$$S_{\Delta,3} = S \cdot B_{\Delta,3} = S \cdot \begin{bmatrix} B_3 & B_4 & B_6 \end{bmatrix} = S \cdot \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 1 & 0.5 & 0.5 \end{bmatrix} \quad [3.31]$$

$$S_{\Delta,4} = S \cdot B_{\Delta,4} = S \cdot \begin{bmatrix} B_4 & B_5 & B_6 \end{bmatrix} = S \cdot \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{bmatrix} \quad [3.32]$$

#### 4. Simplex Integration in $n = 3$

