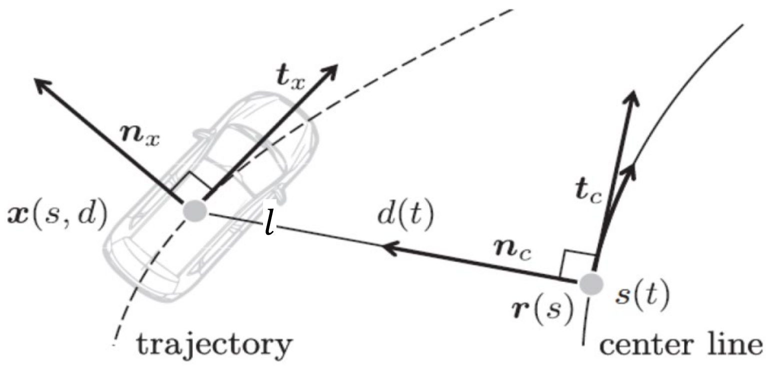
Transformation between Cartesian coordinate and Frenet coordinate



1. Transform $[x, y]$ in Cartesian coordinate into $[s, l]$ in Frenet coordinate

Find the reference point on the reference line. Definition of the reference point is a point on the reference line whose normal line passes through the vehicle position.

Let $\boldsymbol{x} = [x, y]$ be the location of the vehicle in Cartesian coordinate and $\boldsymbol{r} = [x_r, y_r]$ be the location of the reference point in Cartesian coordinate.

$$\boldsymbol{x} = \boldsymbol{r} + l\boldsymbol{n_c} \tag{1}$$

$$l = \sqrt{(x - x_r)^2 + (y - y_r)^2}$$

Positive direction of l is defined to be on the left along the path direction.

Thus, in Cartesian coordinate,

$$\boldsymbol{n_c} = \frac{1}{l}[x - x_r, y - y_r]$$

$$\boldsymbol{t_c} = [cos\theta_r, sin\theta_r]$$

Cross product in z direction is:

$$\boldsymbol{n_c} \times \boldsymbol{t_c} = (y - y_r)cos\theta_r - (x - x_r)sin\theta_r$$

Therefore,

$$l = sign((y - y_r)cos\theta_r - (x - x_r)sin\theta_r)\sqrt{(x - x_r)^2 + (y - y_r)^2}$$

2. Transform $[x, y, \theta_x, \kappa, v, a]$ in Cartesian coordinate into $[s, \dot{s}, \ddot{s}, l, l', l'']$ in Frenet coordinate

$$l' = \frac{dl}{ds} = \frac{dl/dt}{ds/dt} = \frac{\dot{l}}{\dot{s}}$$

From equation (1):

$$l = \boldsymbol{n_c^T}(\boldsymbol{x} - \boldsymbol{r}) = (\boldsymbol{x} - \boldsymbol{r})^T\boldsymbol{n_c}$$

Differentiate w.r.t. time, we have

$$\dot{l} = (\dot{\boldsymbol{x}} - \dot{\boldsymbol{r}})^T \boldsymbol{n_c} + (\boldsymbol{x} - \boldsymbol{r})^T \dot{\boldsymbol{n}}_{\boldsymbol{c}} = (\dot{\boldsymbol{x}} - \dot{\boldsymbol{r}})^T \dot{\boldsymbol{n}}_{\boldsymbol{c}} = (\dot{\boldsymbol{x}} - \dot{\boldsymbol{r}}) \boldsymbol{n_c} + (l\boldsymbol{n_c})^T \dot{\boldsymbol{n}}_{\boldsymbol{c}} \tag{2}$$

According to the definition of speed:

$$\dot{x} = v\boldsymbol{t_x}, \dot{r} = \dot{s}\boldsymbol{t_c} \tag{3}$$

Apply chain differentiation rule,

$$\dot{\boldsymbol{n}}_{\boldsymbol{c}} = \frac{d\boldsymbol{n_c}}{dt} = \frac{d\boldsymbol{n_c}}{ds}\frac{ds}{dt} = \dot{s}\frac{d\boldsymbol{n_c}}{ds}$$

According to Frenet formula:

$$\frac{d\boldsymbol{n_c}}{ds} = -\kappa \boldsymbol{t_c}$$

$$\dot{\boldsymbol{n}}_{\boldsymbol{c}} = -\kappa\dot{s}\boldsymbol{t_c} \tag{4}$$

Substitute (3)(4) into (2), we have:

$$\dot{l} = (v\boldsymbol{t_x} - \dot{s}\boldsymbol{t_c})^T \boldsymbol{n_c} - (l\boldsymbol{n_c})^T \kappa\dot{s}\boldsymbol{t_c}$$

Since t_c and n_c are perpendicular, the above equation can be simplified into:

$$\dot{l} = v\boldsymbol{t_x^T}\boldsymbol{n_c}$$

Substitute T_x and n_c into the above equation, we have:

$$\dot{l} = v sin(\theta_x - \theta_r) \tag{5}$$

Hence, we have linked $\dot{l}$ and v. This can also be obtained by geometric relationships.

Now let's differentiate (1) w.r.t. to time, we have:

$$\dot{x} = \frac{d(r + l\boldsymbol{n_c})}{dt} = \dot{r} + \dot{l}\boldsymbol{n_c} + l\dot{\boldsymbol{n}}_{\boldsymbol{c}}$$

Similarly, substitute (3)(4) into the above equation, we have:

$$\dot{x} = \dot{s}(1 - \kappa_r l)\boldsymbol{t_c} + \dot{l}\boldsymbol{n_c}$$

Since T_r and N_r are perpendicular, take the norm on both sides of the equation

$$v = \sqrt{\dot{s}^2(1 - \kappa_r l)^2 + \dot{l}^2} \tag{6}$$

Note that we assume speed is always positive.

Combining (5) and (6), we have:

$$(\frac{\dot{l}}{\dot{s}})^2 = \tan^2(\theta_x - \theta_r)(1 - \kappa_r l)^2 = l'^2$$

Normally, we assume the curvature of the reference line is much smaller than the vehicle maximum turning angle. It is also assumed that lateral error will be small (typically lane width is smaller than 4m). Hence, we can assume $1 - \kappa_r l > 0$. And assume angular error will not be large, i.e. $|\theta_x - \theta_r| < \frac{\pi}{2}$.

$$l' = \tan(\theta_x - \theta_r)(1 - \kappa_r l) \tag{7}$$

From (5)(7) and $l' = \frac{\dot{l}}{\dot{s}}$, we have:

$$\dot{s} = \frac{v}{1-\kappa_r l} cos\,(\theta_x - \theta_r) \tag{8}$$

Next let's calculate l''. Since $l'' = dl'/ds$, and $\kappa = d\theta/ds$. Let the arc length be s_x

$$\frac{d}{ds} = \frac{ds_x}{ds}\frac{d}{ds_x}$$

Since $v = \frac{ds_x}{dt}$, the above equation can be rewritten as:

$$\frac{d}{ds} = \frac{v}{\dot{s}}\frac{d}{ds_x}$$

Substitute (8) into the above equation

$$\frac{d}{ds} = \frac{1-\kappa_r l}{cos(\theta_x-\theta_r)}\frac{d}{ds} \tag{9}$$

Differentiate (7) using (9), we get:

$$l'' = \frac{dl'}{ds} = \frac{d(1-\kappa_r l)}{ds} tan(\theta_x - \theta_r) + \frac{1-\kappa_r l}{\cos^2(\theta_x-\theta_r)}\frac{d(\theta_x-\theta_r)}{ds} \tag{10}$$

First term

$$\frac{d(1-\kappa_r l)}{ds} = -(d\kappa_r l + \kappa_r l')$$

Second term

$$\frac{d(\theta_x - \theta_r)}{ds} = \frac{d\theta_x}{ds} - \frac{d\theta_r}{ds} = \frac{1-\kappa_r l}{\cos(\theta_x - \theta_r)}\frac{d\theta_x}{ds_x} - \frac{d\theta_r}{ds}$$

From the definition of curvature, we have $\frac{d\theta_x}{ds} = \kappa_x$, $\frac{d\theta_r}{ds} = \kappa_r$. Hence,

$$l'' = -(d\kappa_r l + \kappa_r\, l')\tan(\theta_x - \theta_r) + \frac{1-\kappa_r l}{\cos^2(\theta_x-\theta_r)}\left(\kappa_x \frac{1-\kappa_r l}{\cos(\theta_x-\theta_r)} - \kappa_r\right) \tag{11}$$

Since $\ddot{s}$ is related to acceleration, we differentiate (8) w.r.t. to time:

$$a = \ddot{s}\frac{1-\kappa_r l}{cos\Delta\theta} + \frac{\dot{s}}{cos\Delta\theta}\left((1-\kappa_r l)tan\Delta\theta\frac{d\Delta\theta}{ds} - d\kappa_r l - \kappa_r l'\right), \Delta\theta = \theta_x - \theta_r \tag{12}$$

Therefore, (7)(8)(11)(12) are the second order differentiation transformation.

3. Transform $[s, \dot{s}, \ddot{s}, l, l', l'']$ in Frenet coordinate into $[x, y, \theta_x, \kappa, v, a]$ in Cartesian coordinate.

Find the reference point w.r.t. to s as $[x_r, y_r, \theta_r, \kappa_r, d\kappa_r]$. Since the line between reference point and vehicle is perpendicular to the tangential line:

$$x = x_r - lsin\theta_r$$

$$y = y_r + lcos\theta_r$$

From (7), $l' = \tan(\theta_x - \theta_r)(1 - \kappa_r l)$

we have:

$$\theta_x = arctan2(l', 1 - \kappa_r l) + \theta_r$$

Note the range of θ_x needs to be limited.

From $v = \sqrt{\dot{s}^2(1 - \kappa_r l)^2 + \dot{l}^2}$, we can get v, and κ_x and a can be obtained from (11) and (12)

$$l'' = -(d\kappa_r l + \kappa_r\, l')\tan(\theta_x - \theta_r) + \frac{1 - \kappa_r l}{\cos^2(\theta_x - \theta_r)}(\kappa_x \frac{1 - \kappa_r l}{\cos(\theta_x - \theta_r)} - \kappa_r)$$

$$a = \ddot{s}\frac{1 - \kappa_r l}{cos\Delta\theta} + \frac{\dot{s}}{cos\Delta\theta}\left((1 - \kappa_r l)tan\Delta\theta\frac{d\Delta\theta}{ds} - d\kappa_r l - \kappa_r l'\right), \Delta\theta = \theta_x - \theta_r$$