Chapter 1

Horner's method (Evaluate $P(x)$ at $x = c$)

Set $b_n = a_n$ and compute $b_k = a_k + cb_{k+1}$ for $k = n-1, ..., 1, 0$, then $b_0 = P(c)$

If $Q_0(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + ... + b_3x^2 + b_2x + b_1$

then $P(x) = (x-c)Q_0(x) + R_0$ where $R_0 = b_0 = P(c)$

Absolute Error $E_p = |p - \hat{p}|$ and **Relative Error** $E_r = \left|\frac{p-\hat{p}}{p}\right|$

Significant Digits: $\left|\frac{p-\hat{p}}{p}\right| < \frac{10^{1-d}}{2}$

O(h^n) **Order:** If $\frac{|f(h)-p(h)|}{|h^n|} \le M$, then $f(h) = p(h) + O(h^n)$

Chapter 2

False Position Method: $c = b - \frac{f(b)(b-a)}{f(b)-f(a)}$

Newton-Raphson Theorem: $p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})}$

Eg. Finding Square Roots: $p_k = \frac{P_{k-1}+A/p_{k-1}}{2}$

Secant Method: $p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k)-f(p_{k-1})}$

Accerleration of Newton Iteration: $p_k = p_{k-1} - \frac{Mf(p_{k-1})}{f'(p_{k-1})}$

Chapter 3

LU Fact. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & 6 \\ 3 & 10 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{bmatrix}$

Back Subst.: $x_k = \dfrac{b_k - \sum_{j=k+1}^{N} a_{kj}x_j}{a_{kk}}$, $k = N-1, N-2, ..., 1$

Jacobi Iteration $x_i^{(k+1)} = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}}{a_{ii}}$

Gauss-Seidel $x_i^{(k+1)} = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}}{a_{ii}}$

Chapter 4

Taylor Series Expansions for some Common Functions

$sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \cdots$

$cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$

$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$

$ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$

$arctan(x) = x - x^3/3 + x^5/5 - x^7/7 + \cdots$

$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$

Taylor Polynomial Approximation

$f(x) \approx P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$

for some value c = c(x) that lies in x and x0

Lagrange Polynomial

$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x)$

$L_{N,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_N)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_N)}$

The Divided Differences

$f[x_{k-j}, x_{k-j+1}, \ldots, x_k] = \frac{f[x_{k-j+1},\ldots,x_k] - f[x_{k-j},\ldots,x_{k-1}]}{x_k - x_{k-j}}$

Newton Polynomial

$P_N(x) = a_0 + a_1(x-x_0) + \cdots + a_N(x-x_0)(x-x_1)\cdots(x-x_{N-1}), where\ a_k = f[x_0, x_1, \ldots, x_k], for\ k = 0, 1, \ldots, N$

Lagrange / Newton Polynomial Approximation

$E_N(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_N)f^{(N+1)}(c)}{(N+1)!}$

for some value c = c(x) that lies in the interval [a,b]

Error Bounds for Lagrange Interpolation, Equally Spaced Nodes

$|E_1(x)| \le \frac{h^2 M_2}{8}$ for $x \in [x_0, x_1]$

$|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}}$ for $x \in [x_0, x_2]$

$|E_3(x)| \le \frac{h^4 M_4}{24}$ for $x \in [x_0, x_4]$

Chapter 5

Root-mean-square Error $E_2(f) = \left(\frac{1}{N}\sum_{k=1}^{N} |f(x_k) - y_k|^2\right)^{\frac{1}{2}}$

Least-Squares Line $y = Ax + B$

$\left(\sum_{k=1}^{N} x_k^2\right) A + \left(\sum_{k=1}^{N} x_k\right) B = \sum_{k=1}^{N} x_k y_k$

$\left(\sum_{k=1}^{N} x_k\right) A + NB = \sum_{k=1}^{N} y_k$

Power Fit

$y = Ax^M$, where M is known

$A = \left(\sum_{k=1}^{N} x_k^M y_k\right) / \left(\sum_{k=1}^{N} x_k^{2M}\right)$

Least-Squares Parabola $y = Ax^2 + Bx + C$

$\left(\sum_{k=1}^{N} x_k^4\right) A + \left(\sum_{k=1}^{N} x_k^3\right) B + \left(\sum_{k=1}^{N} x_k^2\right) C = \sum_{k=1}^{N} y_k x_k^2$

$\left(\sum_{k=1}^{N} x_k^3\right) A + \left(\sum_{k=1}^{N} x_k^2\right) B + \left(\sum_{k=1}^{N} x_k\right) C = \sum_{k=1}^{N} y_k x_k$

$\left(\sum_{k=1}^{N} x_k^2\right) A + \left(\sum_{k=1}^{N} x_k\right) B + NC = \sum_{k=1}^{N} y_k$

Cubic Splines

$h_k = x_{k+1} - x_k$ for $k = 0, 1, ..., N-1$

$d_k = \frac{y_{k+1}-y_k}{h_k}$ for $k = 0, 1, ..., N-1$

$m_k = S''(x_k)$ for $k = 0, 1, ..., N$

$u_k = 6(d_k - d_{k-1})$ for $k = 1, 2, ..., N-1$

$h_{k-1}m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = u_k$

$s_{k,0} = y_k, s_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}, s_{k,2} = \frac{m_k}{2}, s_{k,3} = \frac{m_{k+1}-m_k}{6h_k}$

$S_k(x) = ((s_{k,3}w + s_{k,2})w + s_{k,1})w + y_k$, where $w = x - x_k$

Natural cubic spline: $m_0 = 0, m_N = 0$

Extrapolate $S''(x)$ **:**

$m_0 = m_1 - \frac{h_0(m_2-m_1)}{h_1}, m_N = m_{N-1} + \frac{h_{N-1}(m_{N-1}-m_{N-2})}{h_{N-2}}$

$S''(x)$ **is constant near endpoints:** $m_0 = m_1, m_N = m_{N-1}$

Chapter 6

Central-Difference Formulas of Order $O(h^2)$

$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$

$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$

$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3}$

$f''''(x_i) = \frac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4}$

Central-Difference Formulas of Order $O(h^4)$

$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$

$f''(x_i) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}$

$f'''(x_i) = \frac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3}$

$f''''(x_i) = (-f(x_{i+3}) + 12f(x_{i+2}) - 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) - f(x_{i-3}))/6h^4$

Error term for central-diff. of order $O(h^2)$ for $f'(x_i)$

$E(f,h) = Eround(f,h) + Etrunc(f,h) = \frac{e_1 - e_{-1}}{2h} - \frac{h^2 f^{(3)}(c)}{6}$

$|E(f,h)| \le \frac{\epsilon}{h} + \frac{Mh^2}{6}$

The value of h that minimizes the right-hand side is $h = (\frac{3\epsilon}{M})^{1/3}$

Error term for central-diff. of order $O(h^4)$ for $f'(x_i)$

$E(f,h) = Eround(f,h) + Etrunc(f,h)$

$= \frac{-e_2 + 8e_1 - 8e_{-1} + e_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$

$|E(f,h)| \le \frac{3\epsilon}{2h} + \frac{Mh^4}{30}$

The value of h that minimizes the right-hand side is $h = (\frac{45\epsilon}{4M})^{1/5}$

Error term for central-diff. of order $O(h^2)$ for $f''(x_i)$

$E(f,h) = Eround(f,h) + Etrunc(f,h) = \frac{e_1 - 2e_0 + e_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$

$|E(f,h)| \le \frac{4\epsilon}{h^2} + \frac{Mh^2}{12}$

And the value of h that minimizes the right-hand side is $h = (\frac{48\epsilon}{M})^{1/4}$

Forward-Difference Formulas of Order $O(h^2)$

$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$

$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$

$f'''(x_i) = \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$

$f''''(x_i) = \frac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$

Forward-Difference Formulas of Order $O(h^4)$

$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$

$f''(x_i) = \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$

$f'''(x_i) = \frac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3}$

$f''''(x_i) = \frac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4}$

Backward-Difference Formulas of Order $O(h^2)$

$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$

$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$

$f'''(x_i) = \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$

$f''''(x_i) = \frac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$

Backward-Difference Formulas of Order $O(h^4)$

$f'(x_i) = \frac{3f(x_i)-4f(x_{i-1})+f(x_{i-2})}{2h}$

$f''(x_i) = \frac{2f(x_i)-5f(x_{i-1})+4f(x_{i-2})-f(x_{i-3})}{h^2}$

$f'''(x_i) = \frac{5f(x_i)-18f(x_{i-1})+24f(x_{i-2})-14f(x_{i-3})+3f(x_{i-4})}{2h^3}$

$f''''(x_i) = \frac{3f(x_i)-14f(x_{i-1})+26f(x_{i-2})-24f(x_{i-3})+11f(x_{i-4})-2f(x_{i-5})}{h^4}$

Chapter 7

Newton-Cotes Precision

The trapezoidal rule has degree of precision $n = 1$.

$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12}f^{(2)}(c)$

Simpson's rule has degree of precision $n = 3$.

$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90}f^{(4)}(c)$

Simpson's $\frac{3}{8}$ rule has degree of precision $n = 3$.

$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80}f^{(4)}(c)$

Boole's rule has degree of precision $n = 5$.

$\int_{x_0}^{x_4} f(x)dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8h^7}{945}f^{(6)}(c)$

Composite Trapezoidal Rule

Suppose that the interval [a, b] is subdivided into M subintervals $[x_k, x_{k+1}]$ of width $h = (b-a)/M$, $x_k = a + kh$.

$T(f,h) = \frac{h}{2}\sum_{k=1}^{M}(f(x_{k-1}) + f(x_k)) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{M-2} + 2f_{M-1} + f_M)$

$E_T(f,h) = \frac{-(b-a)f^{(2)}(c)h^2}{12}$

Composite Simpson Rule

Suppose that the interval [a, b] is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ of width $h = (b-a)/(2M)$, $x_k = a + kh$.

$S(f,h) = \frac{h}{3}\sum_{k=1}^{M}(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})) = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M})$

Error: $E_S(f,h) = \frac{-(b-a)f^{(4)}(c)h^4}{180}$

Sequence of Trapezoidal Rules

Define $T(0) = (h/2)(f(a) + f(b))$, which is the trapezoidal rule with step size $h = b - a$. For each $J \geq 1$ define $T(J) = T(f,h)$, where $T(f,h)$ is trapezoidal rule with step size $h = (b-a)/2^J$.

Romberg Integration

$R(J,0) = T(J)$ for $J \geq 0$, is the sequential trapezoidal rule.

$R(J,1) = S(J)$ for $J \geq 0$, is the sequential Simpson rule.

$R(J,2) = B(J)$ for $J \geq 0$, is the sequential Boole's rule.

$R(J,K) = \frac{4^K R(J,K-1) - R(J-1,K-1)}{4^K - 1}$

Precision of Romberg Integration

$\int_a^b f(x)dx = R(J,K) + b_K h^{2K+2} f^{(2K+2)}(c_{J,K})$ (i.e. $O(h^{2K+2})$)

Chapter 9

Euler's Method

$\frac{dy}{dt} \approx \frac{y_{i+1}-y_i}{t_{i+1}-t_i} = f(t_i, y_i) \Rightarrow y_{i+1} = y_i + hf(t_i, y_i)$

$E_a = \frac{f'(t_i,y_i)}{2!}h^2 = O(h^2)$

Precision of Euler's Method:

Assume that $y(t)$ is the solution to the I.V.P. If $y(t) \in C^2[t_0,\ b]$ and $\{(t_k,\ y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Euler's method, then $|e_k| = |y(t_k) - y_k| = O(h)$

$|\epsilon_{k+1}| = |y(t_{k+1})$ - $yk - hf(t_k,\ y_k)| = O(h^2)$

The error at the end of the interval is called the *final global error (F.G.E.)* :$E(y(b),\ h) = |y(b) - y_M| = O(h)$

Heun's Method: $y_{i+1}^0 = y_i + f(t_i, y_i)h$

Corrector (may be applied iteratively)

$y_{i+1} = y_i + \frac{f(t_i,y_i)+f(t_{i+1},y_{i+1}^0)}{2}h$

Precision of Heun's Method

Assume that $y(t)$ is the solution to the I.V.P. (1). If $y(t) \in C^3[t_0,\ b]$ and $\{(t_k,\ y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Heun's method, then

$|e_k| = |y(t_k) - y_k| = O(h^2)$

$|\epsilon_{k+1}| = |y(t_{k+1}) - y_k - h\Phi(t_{k,y_k})| = O(h^3)$

where $\Phi(t_k, y_k) = y_k + (h/2)(f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k)))$

Midpoint Method

$y_{i+1/2} = y_i + f(t_i, y_i)\frac{h}{2}$; $y'_{i+1/2} = f(t_{i+1/2}, y_{i+1/2})$

$y_{i+1} = y_i + f(t_{i+1/2}, y_{i+1/2})h$

Classical Third-order Runge-Kutta Method

$k_1 = f(t_i, y_i)$

$k_2 = f(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$

$k_3 = f(t_i + h, y_i - k_1 h + 2k_2 h)$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$

3rd-Order Heun Method

$$\begin{cases} k_1 = f(t_i, y_i) \\ k_2 = f(t_i + \frac{1}{3}h, y_i + \frac{1}{3}k_1 h) \\ k_3 = f(t_i + \frac{2}{3}h, y_i + \frac{2}{3}k_2 h) \end{cases}$$

$y_{i+1} = y_i + \frac{1}{4}(k_1 + 3k_3)h$

Classical 4th-order Runge-Kutta Method

One-step method

$$\begin{cases} k_1 = f(t_i, y_i) \\ k_2 = f(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h) \\ k_3 = f(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h) \\ k_4 = f(t_i + h, y_i + k_3 h) \end{cases}$$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$

Precision of the Runge-Kutta Method

Assume that $y(t)$ is the solution to the I.V.P. If $y(t) \in C^5[t_0,\ b]$ and $\{(t_k,\ y_k)\}_{k=0}^{M}$ is the sequence of approxi- mations generated by the Runge-Kutta method of order 4, then

$|e_k| = |y(t_k) - y_k| = O(h^4)$

$|\epsilon_{k+1}| = |y(t_{k+1})$ - $y_k - hT_N(t_k,\ y_k)| = O(h^5)$.

In particular, the F.G.E. at the end of the interval will satisfy $E(y(b),\ h) = |y(b) - y_M| = O(h^4)$.

Second-Order ODE

$$\begin{cases} \frac{d^2y}{dt^2} = g(t, y, \frac{dy}{dt}) \\ y(t_0) = \alpha_0, \frac{\mathrm{dy}}{\mathrm{dt}}(\mathrm{t_0}) = \alpha_1 \end{cases}$$

Convert to two first-order ODEs

$$\begin{cases} y_1 = y \\ y_2 = \frac{dy}{dt} \end{cases} \Rightarrow \begin{cases} \frac{dy_1}{dt} = \frac{dy}{dt} = y_2 \\ \frac{dy_2}{dt} = \frac{d^2y}{dt^2} = g(t, y_1, y_2) \end{cases} \quad I.C.s \begin{cases} \mathrm{y_1(t_0)} = \alpha_0 \\ \mathrm{y_2(t_0)} = \alpha_1 \end{cases}$$

Euler Method for Two ODE-IVPs

$$\begin{cases} y_{1,i+1} = y_{1,i} + hf_1(t_i, y_{1,i}, y_{2,i}) \\ y_{2,i+1} = y_{2,i} + hf_2(t_i, y_{1,i}, y_{2,i}) \end{cases}$$

In general, nth-order ODE

$$\begin{cases} y^{(n)} = f(t, y, y', y'', \cdots, y^{(n-1)}) \\ y(t_0) = \alpha_0, y'(t_0) = \alpha_1, \cdots, y^{(n-1)}(t_0) = \alpha_{n-1} \end{cases}$$

$$let \begin{cases} y_1 = y \\ y_2 = y' \\ y_3 = y'' \\ \vdots \\ y_n = y^{(n-1)} \end{cases} \Rightarrow \begin{cases} y'_1 = y_2, & \mathrm{y_1(t_0)} = \alpha_0 \\ y'_2 = y_3, & \mathrm{y_2(t_0)} = \alpha_1 \\ y'_3 = y_4, & \mathrm{y_3(t_0)} = \alpha_2 \\ \vdots & \\ y'_n = \mathrm{f(t,y_1,y_2,\cdots,y_n)}, & \mathrm{y_n(t_0)} = \alpha_{\mathrm{n-1}} \end{cases}$$

System of Three first-order ODEs Euler's Method

$$\begin{cases} y_1(i+1) = y_1(i) + f_1(t(i), y_1(i), y_2(i), y_3(i))h \\ y_2(i+1) = y_2(i) + f_2(t(i), y_1(i), y_2(i), y_3(i))h \\ y_3(i+1) = y_3(i) + f_3(t(i), y_1(i), y_2(i), y_3(i))h \end{cases}$$

Classical 4th-order RK Method for ODE-IVPs Systems

$$\begin{cases} k_{1,1} = f_1(t(i), y_1(i), y_2(i)) \\ k_{1,2} = f_2(t(i), y_1(i), y_2(i)) \end{cases}$$

$$\begin{cases} \mathrm{k_{2,1}} = \mathrm{f_1}(\mathrm{t}(i) + \mathrm{h}/2, \mathrm{y_1}(i) + \mathrm{k_{1,1}h}/2, \mathrm{y_2}(i) + \mathrm{k_{1,2}h}/2) \\ \mathrm{k_{2,2}} = \mathrm{f_2}(\mathrm{t}(i) + \mathrm{h}/2, \mathrm{y_1}(i) + \mathrm{k_{1,1}h}/2, \mathrm{y_2}(i) + \mathrm{k_{1,2}h}/2) \end{cases}$$

$$\begin{cases} \mathrm{k_{3,1}} = \mathrm{f_1}(\mathrm{t}(i) + \mathrm{h}/2, \mathrm{y_1}(i) + \mathrm{k_{2,1}h}/2, \mathrm{y_2}(i) + \mathrm{k_{2,2}h}/2) \\ \mathrm{k_{3,2}} = \mathrm{f_2}(\mathrm{t}(i) + \mathrm{h}/2, \mathrm{y_1}(i) + \mathrm{k_{2,1}h}/2, \mathrm{y_2}(i) + \mathrm{k_{2,2}h}/2) \end{cases}$$

$$\begin{cases} \mathrm{k_{4,1}} = \mathrm{f_1}(\mathrm{t}(i) + \mathrm{h}, \mathrm{y_1}(i) + \mathrm{k_{3,1}h}, \mathrm{y_2}(i) + \mathrm{k_{3,2}h}) \\ \mathrm{k_{4,2}} = \mathrm{f_2}(\mathrm{t}(i) + \mathrm{h}, \mathrm{y_1}(i) + \mathrm{k_{3,1}h}, \mathrm{y_2}(i) + \mathrm{k_{3,2}h}) \end{cases}$$

$$\begin{cases} y_1(i+1) = y_1(i) + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})h \\ y_2(i+1) = y_2(i) + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2})h \end{cases}$$

Linear Shooting Method

Solve: $x'' = p(t)x'(t) + q(t)x(t) + r(t)$ with $x(a) = \alpha, x(b) = \beta$

$u'' = p(t)u'(t) + q(t)u(t) + r(t)$ with $u(a) = \alpha$ and $u'(a) = 0$

$v'' = p(t)v'(t) + q(t)v(t)$ with $v(a) = 0$ and $v'(a) = 1$

Unique solution $x(t) = u(t) + \frac{\beta - u(b)}{v(b)}v(t)$

Finite-Difference Method

$(\frac{-h}{2}p_j - 1)x_{j-1} + (2 + h^2 q_j)x_j + (\frac{h}{2}p_j - 1)x_{j+1} = -h^2 r_j$ for $j = 1, 2, ..., N-1$, where $x_0 = \alpha$, $x_N = \beta$