第十六章　偏导数与全微分

§1 偏导数与全微分的概念

1．求下列函数的偏导数：

（1）$u=x^2\ln(x^2+y^2)$；

（2）$u=(x+y)\cos(xy)$；

（3）$u=\arctan\dfrac{y}{x}$；

（4）$u=xy+\dfrac{x}{y}$；

（5）$u=xye^{\sin(xy)}$；

（6）$u=x^y+y^x$.

解（1）$\dfrac{\partial u}{\partial x}=2x\ln(x^2+y^2)+x^2\dfrac{2x}{x^2+y^2}=2x[\ln(x^2+y^2)+\dfrac{x^2}{x^2+y^2}]$；

$\dfrac{\partial u}{\partial x}=x^2\dfrac{2y}{x^2+y^2}=\dfrac{2x^2y}{x^2+y^2}$.

（2）$\dfrac{\partial u}{\partial x}=\cos(xy)+(x+y)(-\sin(xy))y=\cos(xy)-y(x+y)\sin(xy)$；由 x， y 的对称性，$\dfrac{\partial u}{\partial y}=\cos(xy)-x(x+y)\sin(xy)$.

（3）$\dfrac{\partial u}{\partial x}=\dfrac{1}{1+(\dfrac{y}{x})^2}(-\dfrac{y}{x^2})=-\dfrac{y}{x^2+y^2}$；　$\dfrac{\partial u}{\partial y}=\dfrac{1}{1+(\dfrac{y}{x})^2}\dfrac{1}{x}=\dfrac{x}{x^2+y^2}$.

（4）$\dfrac{\partial u}{\partial x}=y+\dfrac{1}{y}$，　$\dfrac{\partial u}{\partial y}=x-\dfrac{x}{y^2}$.

（5）$\dfrac{\partial u}{\partial x}=ye^{\sin(xy)}+xye^{\sin(xy)}\cos(xy)y=y(1+xy\cos(xy))e^{\sin(xy)}$，根据 x， y 的对称性，$\dfrac{\partial u}{\partial y}=x(1+xy\cos(xy))e^{\sin(xy)}$.

（6）$\dfrac{\partial u}{\partial x}=yx^{y-1}+y^x\ln y$；　$\dfrac{\partial u}{\partial y}=x^y\ln x+xy^{x-1}$.

2．设

$$f(x,y)=\begin{cases} y\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0. \end{cases}$$

考察函数在 $(0,0)$ 点的偏导数．

解 $\lim\limits_{\Delta x\to 0}\dfrac{\Delta_x f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x}=0$， 即 $f_x(0,0)=0$， 而

$\lim\limits_{\Delta y\to 0}\dfrac{\Delta_y f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\dfrac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\dfrac{\Delta y\sin\dfrac{1}{\Delta y^2}-0}{\Delta y}=\lim\limits_{\Delta y\to 0}\sin\dfrac{1}{(\Delta y)^2}$ 不存在，

$f_y(0,0)$ 不存在．

3．证明函数 $u=\sqrt{x^2+y^2}$ 在 $(0,0)$ 点连续但偏导数不存在．

证明 显然 $u=\sqrt{x^2+y^2}$ 在 $(0,0)$ 点连续，但

$$\lim_{\Delta x\to 0}\frac{\Delta_x u(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{(\Delta x)^2}-0}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}$$

不存在，由对称性 $\lim\limits_{\Delta y\to 0}\dfrac{\Delta_y u(0,0)}{\Delta y}$ 不存在，因而 $u=\sqrt{x^2+y^2}$ 在 $(0,0)$ 点的两个偏导数均不存在．

4．求下列函数的全微分：

（1）$u=\sqrt{x^2+y^2+z^2}$；

（2）$u=xe^{yz}+e^{-x}+y$．

解（1）$du=d\sqrt{x^2+y^2+z^2}=\dfrac{1}{2\sqrt{x^2+y^2+z^2}}d(x^2+y^2+z^2)$

$$=\frac{1}{\sqrt{x^2+y^2+z^2}}(xdx+ydy+zdz)$$

$$=\frac{x}{\sqrt{x^2+y^2+z^2}}dx+\frac{y}{\sqrt{x^2+y^2+z^2}}dy+\frac{z}{\sqrt{x^2+y^2+z^2}}dz\,.$$

（2）$du=d(xe^{yz}+e^{-x}+y)=e^{yz}dx+xe^{yz}(zdy+ydz)-e^{-x}dx+dy$

$$=(e^{yz}-e^{-x})dx+(xze^{yz}+1)dy+xye^{yz}dz\,.$$

5．求下列函数在给定点的全微分：

（1）$u=\dfrac{x}{\sqrt{x^2+y^2}}$ 在点 $(1,0)$ 和 $(0,1)$；

（2）$u=\ln(x+y^2)$ 在点 $(0,1)$ 和 $(1,1)$；

（3）$u=\sqrt[z]{\dfrac{x}{y}}$ 在点 $(1,1,1)$；

（4）$u=x+(y-1)\arcsin\sqrt{\dfrac{x}{y}}$ 在点 $(0,1)$．

解（1）$du=\dfrac{dx}{\sqrt{x^2+y^2}}+xd(\dfrac{1}{\sqrt{x^2+y^2}})=\dfrac{dx}{\sqrt{x^2+y^2}}-x\dfrac{1}{2(\sqrt{x^2+y^2})^3}d(x^2+y^2)$

$$=\frac{dx}{\sqrt{x^2+y^2}}-\frac{x}{(\sqrt{x^2+y^2})^3}(xdx+ydy)=\frac{y^2dx-xydy}{(x^2+y^2)\sqrt{x^2+y^2}},$$

所以，在点 $(1,0)$， $du=0$，在点 $(0,1)$， $du=dx$．

（2）$du=\dfrac{1}{x+y^2}(dx+2ydy)=\dfrac{1}{x+y^2}dx+\dfrac{2y}{x+y^2}dy$，在点 $(0,1)$，$du=dx+2dy$；

在点 $(1,1)$， $du=\dfrac{1}{2}dx+dy$．

（3）$\dfrac{\partial u}{\partial x}=\dfrac{1}{yz}(\dfrac{x}{y})^{\frac{1}{z}-1}$， $\dfrac{\partial u}{\partial y}=-\dfrac{x}{y^2z}(\dfrac{x}{y})^{\frac{1}{z}-1}$， $\dfrac{\partial u}{\partial z}=-\dfrac{1}{z^2}(\dfrac{x}{y})^{\frac{1}{z}}\ln\dfrac{x}{y}$， 所以，

$$du=\frac{1}{yz}(\frac{x}{y})^{\frac{1}{z}-1}dx-\frac{x}{y^2z}(\frac{x}{y})^{\frac{1}{z}-1}dy-\frac{1}{z^2}(\frac{x}{y})^{\frac{1}{z}}\ln\frac{x}{y}dz,$$

故在 $(1,1,1)$ 有， $du=dx-dy$．

（4）函数的定义域为 $\{(x,y):0\le x\le y\ \ or\ \ 0\le y\le x\}$．当 $x\ne0$ 时，有

$$du=dx+\arcsin\sqrt{\frac{x}{y}}dy+(y-1)\frac{1}{\sqrt{1-\frac{x}{y}}}\frac{1}{2\sqrt{\frac{x}{y}}}\frac{ydx-xdy}{y^2}$$

$$=(1+\frac{(y-1)\operatorname{sgn}y}{2\sqrt{xy-x^2}})dx+(\arcsin\sqrt{\frac{x}{y}}+\frac{x(1-y)\operatorname{sgn}y}{2y\sqrt{xy-x^2}})dy,$$

而当$x=0$时，由于$\lim\limits_{x\to0^+}\dfrac{f(x,y)-f(0,y)}{x}=\lim\limits_{x\to0^+}(1+\dfrac{y-1}{\sqrt{x}\sqrt{y}}\dfrac{\arcsin\sqrt{x/y}}{\sqrt{x/y}})$不存在，所以

在$(0,y)$，$f_x(0,y)$不存在，虽然$f_y(0,y)=\lim\limits_{\Delta y\to0}\dfrac{f(0,y+\Delta y)-f(0,y)}{\Delta y}=\lim\limits_{\Delta y\to0}\dfrac{0}{\Delta y}=0$，

但在点$(0,y)$，du不存在，因而$u=x+(y-1)\arcsin\sqrt{\dfrac{x}{y}}$在点$(0,1)$不可微.

6. 考虑函数$f(x,y)$在$(0,0)$点的可微性，其中

$$f(x,y)=\begin{cases}xy\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq0,\\ 0, & x^2+y^2=0.\end{cases}$$

解 因为$\lim\limits_{\Delta x\to0}\dfrac{\Delta_x f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{0-0}{\Delta x}=0$，所以$f_x(0,0)=0$，

由对称性，$f_y(0,0)=0$．若函数$f(x,y)$在$(0,0)$可微，则按可微的定义，应有

$$f(\Delta x,\Delta y)-f(0,0)-[f_x(0,0)\Delta x+f_y(0,0)\Delta y]=\Delta x\Delta y\sin\frac{1}{\Delta x^2+\Delta y^2},$$

是比$\rho=\sqrt{\Delta x^2+\Delta y^2}$更高阶的无穷小，为此考察极限

$$\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta x\Delta y\sin\frac{1}{\Delta x^2+\Delta y^2}}{\rho}=\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\sin\frac{1}{\Delta x^2+\Delta y^2},$$

由于

$$\left|\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\sin\frac{1}{\Delta x^2+\Delta y^2}\right|\le\frac{|\Delta x||\Delta y|}{\sqrt{\Delta x^2+\Delta y^2}}\le\frac{\frac{1}{2}(\Delta x^2+\Delta y^2)}{\sqrt{\Delta x^2+\Delta y^2}},$$

所以，$\lim\limits_{\substack{\Delta x\to0\\ \Delta y\to0}}\dfrac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\sin\dfrac{1}{\Delta x^2+\Delta y^2}=0$，故$f(x,y)$在$(0,0)$可微.

7. 证明函数

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq0,\\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$点连续且偏导数存在，但在此点不可微.

证明 因为 $\left|\dfrac{x^2y}{x^2+y^2}\right| = |x|\cdot\left|\dfrac{xy}{x^2+y^2}\right| \le \dfrac{1}{2}|x|$，所以 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0=f(0,0)$，即 $f(x,y)$ 在点 $(0,0)$ 点连续，又

$$\lim_{\Delta x\to 0}\frac{\Delta_x f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$\lim_{\Delta y\to 0}\frac{\Delta_y f(0,0)}{\Delta y}=\lim_{\Delta y\to o}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0,$$

所以，$f_x(0,0)=f_y(0,0)=0$.

若函数 $f(x,y)$ 在 $(0,0)$ 可微，则应有

$$f(\Delta x,\Delta y)-f(0,0)-[f_x(0,0)\Delta x+f_y(0,0)\Delta y]=\frac{\Delta x^2\Delta y}{(\Delta x)^2+(\Delta y)^2}$$

是比 $\rho=\sqrt{\Delta x^2+\Delta y^2}$ 更高阶的无穷小量，为此考察极限

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{1}{\rho}\frac{\Delta x^2\Delta y}{\Delta x^2+\Delta y^2}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x^2\Delta y}{(\sqrt{\Delta x^2+\Delta y^2})^3},$$

令 $\Delta y=\Delta x$，当 $(\Delta x,\Delta y)$ 沿直线 $\Delta y=\Delta x$ 趋于 $(0,0)$ 时，$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y=\Delta x}}\dfrac{\Delta x^2\Delta y}{(\sqrt{\Delta x^2+\Delta y^2})^3}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta x}{|\Delta x|}$ 不存在，即 $\dfrac{\Delta x^2\Delta y}{(\Delta x)^2+(\Delta y)^2}$ 不是比 ρ 更高阶的无穷小量，因此 $f(x,y)$ 在 $(0,0)$ 不可微.

8．证明函数

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\ne 0,\\ 0, & x^2+y^2=0\end{cases}$$

的偏导数存在，但偏导数在 $(0,0)$ 点不连续，且在 $(0,0)$ 点的任何邻域中无界，而 f 在原点 $(0,0)$ 可微.

证明 $f_x(x,y)=\begin{cases}2x(\sin\dfrac{1}{x^2+y^2}-\dfrac{1}{x^2+y^2}\cos\dfrac{1}{x^2+y^2}), & x^2+y^2\ne 0,\\ 0, & x^2+y^2=0,\end{cases}$

$$f_y(x,y)=\begin{cases}2y(\sin\frac{1}{x^2+y^2}-\frac{1}{x^2+y^2}\cos\frac{1}{x^2+y^2}), & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

而 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_x(x,y)$ 不存在，$\lim\limits_{\substack{x\to 0\\ y\to 0}} f_y(x,y)$ 也不存在，因而 $f_x(x,y), f_y(x,y)$ 在 $(0,0)$ 点均不连续.

$\forall\delta>0, \forall M>0, \exists n$，使 $\frac{1}{\sqrt{2n\pi}}<\delta$ 且 $2\sqrt{2n\pi}>M$，但 $P_n(\frac{1}{\sqrt{2n\pi}},0)\in O(P,\delta)$，

$P_n'(0,\frac{1}{\sqrt{2n\pi}})\in O(P,\delta)$ 时，而由于

$$|f_x(P_n)|=\left|f_x(\frac{1}{\sqrt{2n\pi}},0)\right|=2\sqrt{2n\pi}>M\text{，}$$

$$|f_y(P_n')|=\left|f_x(0,\frac{1}{\sqrt{2n\pi}})\right|=2\sqrt{2n\pi}>M\text{，}$$

所以，$f_x(x,y)$，$f_y(x,y)$ 在 $(0,0)$ 点的任何邻域中均无界.

但由于

$$\frac{f(\Delta x,\Delta y)-f(0,0)-[f_x(0,0)\Delta x+f_y(0,0)\Delta y]}{\rho}$$

$$=\sqrt{(\Delta x)^2+(\Delta y)^2}\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}\to 0\ ((\Delta x,\Delta y)\to(0,0))\text{，}$$

所以，$f(x,y)$ 在 $(0,0)$ 可微，且在 $(0,0)$ 的微分 $df(0,0)=0$.

9．设

$$f(x,y)=\begin{cases}\frac{2xy^4}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

证明 $\frac{\partial f}{\partial x}$ 和 $\frac{\partial f}{\partial y}$ 在 $(0,0)$ 点连续.

证明 $$\frac{\partial f}{\partial x}=\begin{cases}\frac{2xy^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

因为 $\frac{|2xy^4|}{|x^2+y^2|^2}\le\frac{|y|^3}{x^2+y^2}\le|y|$，所以 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\partial f}{\partial x}=0=f_x(0,0)$，即 $f_x(x,y)$ 在 $(0,0)$ 连续，

由对称性，$\frac{\partial f}{\partial y}$ 亦在 $(0,0)$ 点连续.

10．设

$$f(x,y)=\begin{cases}\dfrac{1-e^{x(x^2+y^2)}}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

证明 $f(x,y)$ 在点 $(0,0)$ 可微，并求 $df(0,0)$.

证明 $f_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{1-e^{\Delta x^3}}{\Delta x^3}=\lim\limits_{\Delta x\to 0}\dfrac{-\Delta x^3+o(\Delta x^3)}{\Delta x^3}=-1$，$f_y(0,0)=\lim\limits_{\Delta y\to 0}0=0$，

$$\frac{f(\Delta x,\Delta y)-f(0,0)-[f_x(0,0)\Delta x+f_y(0,0)\Delta y]}{\rho}$$

$$=\frac{1}{\rho}\left(\frac{1-e^{\Delta x(\Delta x^2+\Delta y^2)}}{\Delta x^2+\Delta y^2}+\Delta x\right)=\frac{1+\Delta x(\Delta x^2+\Delta y^2)-e^{\Delta x(\Delta x^2=\Delta y^2)}}{(\Delta x^2+\Delta y^2)^{\frac{3}{2}}}$$

$$=\frac{-\frac{1}{2}\Delta x^2(\Delta x^2+\Delta y^2)^2+o(\Delta x^2(\Delta x^2+\Delta y^2)^2}{(\Delta x^2+\Delta y^2)^{\frac{3}{2}}}$$

$$=-\frac{1}{2}\Delta x^2(\Delta x^2+\Delta y^2)^{\frac{1}{2}}+o(\Delta x^2(\Delta x^2+\Delta y^2)^{\frac{1}{2}})\to 0\ ((\Delta x,\Delta y)\to(0,0)),$$

所以，$f(x,y)$ 在 $(0,0)$ 可微，且 $df(0,0)=-\Delta x=-dx$.

11．设

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

（1）$x=x(t),\ y=y(t)$ 是通过原点的任意可微曲线（即 $x^2(0)+y^2(0)=0,\ t\neq 0$ 时，$x^2(t)+y^2(t)\neq 0,\ x(t),\ y(t)$ 可微）．求证 $f(x(t),y(t))$ 可微；

（2）$f(x,y)$ 在 $(0,0)$ 不可微.

证明（1）设 $\varphi(t)=f(x(t),y(t))=\begin{cases}\dfrac{x^3(t)}{x^2(t)+y^2(t)}, & t\neq 0,\\ 0, & t=0,\end{cases}$，所以

$$\varphi'(t)=\frac{x^2(t)[x^2(t)x'(t)+3y^2(t)x'(t)-2x(t)y(t)y'(t)]}{[x^2(t)+y^2(t)]^2},\ t\neq 0,$$

而在$t=0$，由于$\varphi'(0)=\lim\limits_{t\to 0}\dfrac{\varphi(t)-\varphi(0)}{t}=\lim\limits_{t\to 0}\dfrac{x^3(t)}{t(x^2(t)+y^2(t))}$，若$x'(0)\neq 0$, 则

$$\varphi'(0)=\lim_{t\to 0}\left(\frac{x(t)}{t}\right)^3\frac{1}{\left(\frac{x(t)}{t}\right)^2+\left(\frac{y(t)}{t}\right)^2}=[x'(0)]^3\frac{1}{[x'(0)]^2+[y'(0)]^2},$$

若$x'(0)=0$, 则由于$\left|\dfrac{x^3(t)}{t(x^3(t)+y^2(t))}\right|\leq\left|\dfrac{x(t)}{t}\right|$，而$\lim\limits_{t\to 0}\dfrac{x(t)}{t}=x'(0)=0$，

所以，$\lim\limits_{t\to 0}\dfrac{x^3(t)}{t(x^2(t)+y^2(t))}=0$，即$\varphi'(0)=0$．故$f(x(t),y(t))$可微.

（2） $f_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=1$；

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0,$$

若函数$f(x,y)$在$(0,0)$可微，则按可微的定义，应有

$$f(\Delta x,\Delta y)-f(0,0)-(f_x(0,0)\Delta x+f_y(0,0)\Delta y)$$

$$=\frac{\Delta x^3}{\Delta x^2+\Delta y^2}-\Delta x=-\frac{\Delta x\Delta y^2}{\Delta x^2+\Delta y^2},$$

是比$\rho=\sqrt{\Delta x^2+\Delta y^2}$更高阶的无穷小，为此考察极限

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{1}{\rho}\left(-\frac{\Delta x\Delta y^2}{\Delta x^2+\Delta y^2}\right)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{-\Delta x\Delta y^2}{(\Delta x^2+\Delta y^2)^{\frac{3}{2}}},$$

设$\Delta y=\Delta x$，则有

$$\lim_{\substack{\Delta x\to 0\\ \Delta y=\Delta x}}\frac{-\Delta x\Delta y^2}{(\Delta x^2+\Delta y^2)^{\frac{3}{2}}}=\lim_{\Delta x\to 0}\frac{-\Delta x^3}{(2\Delta x^2)^{\frac{3}{2}}}=-\frac{1}{2\sqrt{2}}\lim_{\Delta x\to 0}\frac{\Delta x}{|\Delta x|},$$

该极限不存在，因而$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{1}{\rho}\left(\dfrac{-\Delta x\Delta y^2}{\Delta x^2+\Delta y^2}\right)$不是比$\rho$更高阶的无穷小量，因此$f(x,y)$在$(0,0)$不可微.

12．设$|x|,|y|$很小，利用全微分推出下列各式的近似公式：

（1）$(1+x)^m(1+y)^n$；

（2）$\arctan\dfrac{x+y}{1+xy}$.

解（1） $f_x(x,y)=m(1+x)^{m-1}(1+y)^n$， $f_y(x,y)=n(1+x)^m(1+y)^{n-1}$，因而，

$$f_x(0,0)=m, \quad f_y(0,0)=n,$$

$$(1+x)^m(1+y)^n = f(0,0)+f_x(0,0)x+f_y(0,0)y+o(\rho)=1+mx+ny+o(\sqrt{x^2+y^2}),$$

因此，当$|x|,|y|$很小时，$(1+x)^m(1+y)^n \approx 1+mx+ny$.

（2） $f_x(x,y)=\dfrac{1}{1+(\dfrac{x+y}{1+xy})^2}\dfrac{1-y^2}{(1+xy)^2}=\dfrac{1-y^2}{(1+xy)^2+(x+y)^2}$，由对称性，

$$f_y(x,y)=\frac{1-x^2}{(1+xy)^2+(x+y)^2},$$

所以，$f_x(0,0)=1=f_y(0,0)$，而 $f(0,0)=\arctan 0=0$，故$\arctan\dfrac{x+y}{1+xy}=x+y+o(\rho)$，

因此，当$|x|,|y|$很小时，$\arctan\dfrac{x+y}{1+xy}\approx x+y$.

13．设$u=f(x,y)$在矩形：$a<x<b, c<y<d$ 内可微，且全微分 du 恒为零，问 $f(x,y)$在该矩形内是否应取常数值？证明你的结论.

解　$f(x,y)$在该矩形内应取常数值．证明如下：

由于$u=f(x,y)$在矩形内可微，故$\forall(x,y)\in(a,b)\times(c,d)$，因为

$$du=f_x(x,y)dx+f_y(x,y)dy\equiv 0,$$

所以，

$$f_x(x,y)\equiv 0, \quad f_y(x,y)\equiv 0,$$

故取定$P_0(x_0,y_0)\in$该矩形，有

$$f(x,y)-f(x_0,y_0)=[f(x,y)-f(x_0,y)]+[f(x_0,y)-f(x_0,y_0)]$$

$$=f_x(x_0+\theta_1(x-x_0),y)(x-x_0)+f_y(x_0,y_0+\theta_2(y-y_0))(y-y_0)$$

$= 0 \quad (0<\theta_1<1, 0<\theta_2<1)$，

所以，$f(x,y)=f(x_0,y_0)\equiv C$，即 $f(x,y)$ 取常数值 $C=f(x_0,y_0)$.

14．设 $\dfrac{\partial f}{\partial x}$ 在 (x_0,y_0) 存在，$\dfrac{\partial f}{\partial y}$ 在 (x_0,y_0) 连续，求证 $f(x,y)$ 在 (x_0,y_0) 可微.

证明 $f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$

$$=f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)+f(x_0+\Delta x,y_0)-f(x_0,y_0),$$

由于 $\dfrac{\partial f}{\partial y}$ 在 (x_0,y_0) 连续，因而在 $P_0(x_0,y_0)$ 存在，由一元函数的 Lagrange 中值定理，知 $\exists\theta:0<\theta<1$，使得

$$f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)=f_y(x_0+\Delta x,y_0+\theta\Delta y)\Delta y,$$

由于 $\dfrac{\partial f}{\partial y}$ 在 (x_0,y_0) 连续，故 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_y(x_0+\Delta x,y_0+\theta\Delta y)=f_y(x_0,y_0)$，所以

$$f_y(x_0+\Delta x,y_0+\theta\Delta y)=f_y(x_0,y_0)+\beta,\ \text{其中}\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\beta=0.$$

而对 $f(x_0+\Delta x,y_0)-f(x_0,y_0)$，设 $\Phi(x)=f(x,y_0)$，则

$$f(x_0+\Delta x,y_0)-f(x_0,y_0)=\Phi(x_0+\Delta x)-\Phi(x_0),$$

由于 $\Phi'(x_0)=f_x(x_0,y_0)$，故 $\Phi(x)$ 在 x_0 可导，因而可微，即

$$f(x_0+\Delta x,y_0)-f(x_0,y_0)=\Phi(x_0+\Delta x)-\Phi(x_0)$$
$$=\Phi'(x_0)\Delta x+\alpha=f_x(x_0,y_0)\Delta x+\alpha,$$

其中 $\alpha=o(\Delta x)\ (\Delta x\to 0)$，所以，

$$f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f_y(x_0,y_0)\Delta y+f_x(x_0,y_0)\Delta x+\beta\Delta y+\alpha,$$

其中 $|\beta\Delta y+\alpha|/\rho\le|\beta|+|\alpha/\rho|\to 0\ (\rho\to 0)$，所以 $f(x,y)$ 在 (x_0,y_0) 可微.

15．求下列函数的所有二阶偏导数：

（1）$u=\ln\sqrt{x^2+y^2}$；

（2）$u=xy+\dfrac{y}{x}$；

（3）$u=x\sin(x+y)+y\cos(x+y)$；

（4）$u=e^{xy}$；

解 $u=\dfrac{1}{2}\ln(x^2+y^2)$，$\dfrac{\partial u}{\partial x}=\dfrac{1}{2}\dfrac{2x}{x^2+y^2}=\dfrac{x}{x^2+y^2}$，由对称性，$\dfrac{\partial u}{\partial y}=\dfrac{y}{x^2+y^2}$，

$$\frac{\partial^2 u}{\partial x^2}=\frac{y^2-x^2}{(x^2+y^2)^2},\qquad \frac{\partial^2 u}{\partial x\partial y}=-\frac{2xy}{(x^2+y^2)},$$

由对称性，

$$\frac{\partial^2 u}{\partial y\partial x}=-\frac{2xy}{(x^2+y^2)^2},\qquad \frac{\partial^2 u}{\partial y^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

（2）$\dfrac{\partial u}{\partial x}=y-\dfrac{y}{x^2}$，$\dfrac{\partial u}{\partial x}=x+\dfrac{1}{x}$，

$$\frac{\partial^2 u}{\partial x^2}=\frac{2y}{x^3},\quad \frac{\partial^2 u}{\partial x\partial y}=1-\frac{1}{x^2},\quad \frac{\partial^2 u}{\partial y\partial x}=1-\frac{1}{x^2},\quad \frac{\partial^2 u}{\partial y^2}=0.$$

（3）$\dfrac{\partial u}{\partial x}=\sin(x+y)+x\cos(x+y)-y\sin(x+y)$，

$$\frac{\partial u}{\partial y}=x\cos(x+y)+\cos(x+y)-y\sin(x+y),$$

$$\frac{\partial^2 u}{\partial x^2}=2\cos(x+y)-x\sin(x+y)-y\cos(x+y),$$

$$\frac{\partial^2 u}{\partial x\partial y}=\cos(x+y)-x\sin(x+y)-\sin(x+y)-y\cos(x+y),$$

$$\frac{\partial^2 u}{\partial y\partial x}=\cos(x+y)-x\sin(x+y)-\sin(x+y)-y\cos(x+y),$$

$$\frac{\partial^2 u}{\partial y^2}=-x\sin(x+y)-2\sin(x+y)-y\cos(x+y).$$

（4）$\dfrac{\partial u}{\partial x}=ye^{xy}$，$\dfrac{\partial u}{\partial y}=xe^{xy}$，$\dfrac{\partial^2 u}{\partial x^2}=y^2e^{xy}$，$\dfrac{\partial^2 u}{\partial x\partial y}=e^{xy}+xye^{xy}$，$\dfrac{\partial^2 u}{\partial y\partial x}=e^{xy}+xye^{xy}$，

$$\frac{\partial^2 u}{\partial y^2}=x^2e^{xy}.$$

16．求下列函数指定阶的偏导数：

（1）$u=x^3\sin y+y^3\sin x$，求 $\dfrac{\partial^6 u}{\partial x^3\partial y^3}$；

（2）$u=\arctan\dfrac{x+y}{1-xy}$，求所有三阶偏导数；

（3）$u=\sin(x^2+y^2)$，求 $\dfrac{\partial^3 u}{\partial x^3},\dfrac{\partial^3 u}{\partial y^3}$；

（4）$u=xyze^{x+y+z}$，求 $\dfrac{\partial^{p+q+r}u}{\partial x^r\partial y^q\partial z^r}$；

（5）$u=\dfrac{x+y}{x-y}(x\neq y)$，求 $\dfrac{\partial^{m+n}u}{\partial x^m\partial y^n}$；

（6）$u=\ln(ax+by)$，求 $\dfrac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

解（1）$\dfrac{\partial u}{\partial x}=3x^2\sin y+y^3\cos x$，$\dfrac{\partial^2 u}{\partial x^2}=6x\sin y-y^3\sin x$，

$$\frac{\partial^3 u}{\partial x^3}=6\sin y-y^3\cos x,\quad \frac{\partial^4 u}{\partial x^3\partial y}=6\cos y-3y^2\cos x,$$

$$\frac{\partial^5 u}{\partial x^3\partial y^2}=-6\sin y-6y\cos x,\quad \frac{\partial^2 u}{\partial x^3\partial y^3}=-6\cos y-6\cos x.$$

（2）$$\frac{\partial u}{\partial x}=\frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2}\frac{1+y^2}{(1-xy)^2}=\frac{1+y^2}{(1-xy)^2+(x+y)^2}=\frac{1}{1+x^2};$$

$$\frac{\partial u}{\partial y}=\frac{1}{1+y^2},\quad \frac{\partial^2 u}{\partial x^2}=-\frac{2x}{(1+x^2)^2},\quad \frac{\partial^2 u}{\partial y^2}=-\frac{2y}{(1+y^2)^2},\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=0,$$

$$\frac{\partial^3 u}{\partial x^3}=\frac{2(3x^2-1)}{(1+x^2)^3},\quad \frac{\partial^3 u}{\partial y^3}=\frac{2(3y^2-1)}{(1+y^2)^3},\quad \frac{\partial^3 u}{\partial x^2\partial y}=\frac{\partial^3 u}{\partial x\partial y^2}=0.$$

（3）$\dfrac{\partial u}{\partial x}=2x\cos(x^2+y^2)$，$\dfrac{\partial^2 u}{\partial x^2}=2\cos(x^2+y^2)-4x^2\sin(x^2+y^2)$，

$$\frac{\partial^3 u}{\partial x^3}=-12x\sin(x^2+y^2)-8x^3\cos(x^2+y^2),$$

由对称性，$\dfrac{\partial^3 u}{\partial y^3}=-12y\sin(x^2+y^2)-8y^3\cos(x^2+y^2)$.

（4）$\frac{\partial u}{\partial x}=yze^{x+y+z}+xyze^{x+y+z}=(x+1)yze^{x+y+z}$，

$$\frac{\partial^2 u}{\partial x^2}=yze^{x+y+z}+(x+1)yze^{x+y+z}=(x+2)yze^{x+y+z},$$

由归纳法不难知道，$\frac{\partial^p u}{\partial x^p}=(x+p)yze^{x+y+z}$.

$$\frac{\partial^{p+1} u}{\partial x^p\partial y}=(x+p)ze^{x+y+z}+(x+p)yze^{x+y+z}=(x+p)(y+1)ze^{x+y+z},$$

不难用归纳法知道，$\frac{\partial^{p+q} u}{\partial x^p\partial y^q}=(x+p)(y+q)ze^{x+y+z}$.

$$\frac{\partial^{p+q+1} u}{\partial x^p\partial y^q\partial z}=(x+p)(y+q)e^{x+y+z}+(x+p)(y+q)ze^{x+y+z}$$

$$=(x+p)(y+q)(z+1)e^{x+y+z},$$

同样用归纳法不难知道，$\frac{\partial^{p+q+r} u}{\partial x^p\partial y^q\partial z^r}=(x+p)(y+q)(z+r)e^{x+y+z}$.

（5）$\frac{\partial u}{\partial x}=-\frac{2y}{(x-y)^2}\Rightarrow\frac{\partial^m u}{\partial x^m}=\frac{2(-1)^m m!\,y}{(x-y)^{m+1}}$（使用数学归纳法），

$$\frac{\partial^{m+1} u}{\partial x^m\partial y}=2(-1)^m m!\frac{x+my}{(x-y)^{m+2}},$$

$$\frac{\partial^{m+2} u}{\partial x^m\partial y^2}=2(-1)^m m!\frac{(m+1)(2x+my)}{(x-y)^{m+3}},$$

$$\frac{\partial^{m+3} u}{\partial x^m\partial y^3}=2(-1)^m m!\frac{(m+1)(m+2)(3x+my)}{(x-y)^{m+4}},$$

用归纳法，不难计算，

$$\frac{\partial^{m+n} u}{\partial x^m\partial y^n}=2(-1)^m (m+n-1)!\frac{nx+my}{(x-y)^{m+n+1}}.$$

（6）$\frac{\partial u}{\partial x}=\frac{a}{ax+by}=\frac{1}{x+(b/a)y}\ (a\neq 0)$，

$$\frac{\partial x^m u}{\partial x^m}=\frac{(-1)^{m-1}(m-1)!}{(x+\frac{b}{a}y)^m}=\frac{(-1)^{m-1}(m-1)!a^m}{(ax+by)^m}=\frac{(-1)^{m-1}(m-1)!a^m}{b^m(y+\frac{a}{b}x)^m}\ (b\neq 0)$$

$$\frac{\partial^{m+n}u}{\partial x^m\partial y^n}=\frac{(-1)^{m-1}(m-1)!a^m m(m+1)\cdots(m+n-1)(-1)^n}{b^m(y+\frac{a}{b}x)^{m+n}}$$

$$=\frac{(-1)^{m+n-1}(m+n-1)!a^m b^n}{(ax+by)^{m+n}}.$$

17．验证下列函数满足

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

（1）$u=\ln(x^2+y^2)$；

（2）$u=x^2-y^2$；

（3）$u=e^x\cos y$；

（4）$u=\arctan\frac{y}{x}$.

证明（1）由 15（1），知 $\frac{\partial^2 u}{\partial x^2}=\frac{2(y^2-x^2)}{(x^2+y^2)^2}$，$\frac{\partial^2 u}{\partial y^2}=\frac{2(x^2-y^2)}{(x^2+y^2)^2}$，所以 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

（2）$\frac{\partial u}{\partial x}=2x$，$\frac{\partial^2 u}{\partial x^2}=2$，$\frac{\partial u}{\partial y}=-2y$，$\frac{\partial^2 u}{\partial y^2}=-2$ 所以 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

（3）$\frac{\partial u}{\partial x}=e^x\cos y$，$\frac{\partial^2 u}{\partial x^2}=e^x\cos y$，$\frac{\partial u}{\partial y}=-e^x\sin y$，$\frac{\partial^2 u}{\partial y^2}=-e^x\cos y$，所以，

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

（4）$\frac{\partial u}{\partial x}=\frac{1}{1+(\frac{y}{x})^2}(-\frac{y}{x^2})=\frac{-y}{x^2+y^2}$，$\frac{\partial^2 u}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}$，

$\frac{\partial u}{\partial y}=\frac{1}{1+(\frac{y}{x})^2}\frac{1}{x}=\frac{x}{x^2+y^2}$，$\frac{\partial^2 u}{\partial y^2}=-\frac{2xy}{(x^2+y^2)^2}$，

所以，$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

18．设函数 $u=\varphi(x+\psi(y))$，证明

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}.$$

证明 $\frac{\partial u}{\partial x}=\varphi'(x+\psi(y)),\frac{\partial u}{\partial y}=\varphi'(x+\psi(y))\psi'(y).$

$$\frac{\partial^2 u}{\partial x^2}=\varphi''(x+\psi(y)),\frac{\partial^2 u}{\partial x\partial y}=\varphi''(x+\psi(y))\psi'(y);$$

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}=\varphi'(x+\psi(y))\varphi''(x+\psi(y))\psi'(y);$$

$$\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}=\varphi'(x+\psi(y))\varphi'(y)\varphi''(x+\psi(y));$$

即 $\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}$.

19．设 f_x,f_y 在点 (x_0,y_0) 的某邻域内存在且在点 (x_0,y_0) 点可微，则有

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0).$$

证明 像定理 16. 4 的证明过程中一样计算，知 $f_{xy}(x_0,y_0)$ 与 $f_{yx}(x_0,y_0)$ 是函数

$$\frac{W}{\Delta x\Delta y}=\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)-f(x_0+\Delta x,y_0)+f(x_0,y_0)}{\Delta x\Delta y}$$

的两个累次极限．我们利用 f_x',f_y' 在 (x_0,y_0) 处的可微性，下面证明 $\frac{W}{\Delta x\Delta y}$ 可改写成

$$\frac{W}{\Delta x\Delta y}=f_{yx}''(x_0,y_0)+\varepsilon_1+\varepsilon_2\theta\frac{\Delta y}{\Delta x}-\varepsilon_3\theta\frac{\Delta y}{\Delta x}, \qquad (*)$$

和 $$\frac{W}{\Delta x\Delta y}=f_{xy}''(x_0,y_0)+\varepsilon_4+\varepsilon_5\theta_1\frac{\Delta x}{\Delta y}-\varepsilon_6\theta_1\frac{\Delta x}{\Delta y}, \qquad (**)$$

二者对充分小的 $\Delta x,\Delta y$ 同时成立，且当 $\Delta x\to 0,\Delta y\to 0$ 时，$\varepsilon_i\to 0\,(i=1,\cdots,6)$，$0<\theta,\theta_1<1$．于是令 $\Delta x=\Delta y\to 0$ 可得，

$$f_{yx}''(x_0,y_0)=f_{xy}''(x_0,y_0). \qquad (\#)$$

可见，问题归结为证明（*），（**）成立．为此取 $\Delta x,\Delta y$ 充分小，引入辅助函数

$$\varphi(y)=f(x_0+\Delta x,y)-f(x_0,y),$$

式 $\frac{W}{\Delta x\Delta y}$ 可改写为

$$\frac{W}{\Delta x\Delta y}=\frac{1}{\Delta x\Delta y}[\varphi(y_0+\Delta y)-\varphi(y_0)]=\frac{1}{\Delta x}\varphi_y'(y_0+\theta\Delta y)$$

$$=\frac{1}{\Delta x}[f_y(x_0+\Delta x,y_0+\theta\Delta y)-f_y(x_0,y_0+\theta\Delta y)]，（0<\theta_1<1），$$

由于 f_y 在 (x_0,y_0) 处可微，故

$$f_y(x_0+\Delta x,y_0+\theta\Delta y)=f_y(x_0,y_0)+f_{yx}(x_0,y_0)\Delta x+f_{yy}(x_0,y_0)\theta\Delta y$$

$$+\varepsilon_1\Delta x+\varepsilon_2\theta\Delta y,$$

其中 $\varepsilon_1,\varepsilon_2\to 0$（当 $\Delta x,\Delta y\to 0$ 时），

$$f_y(x_0,y_0+\theta\Delta y)=f_y(x_0,y_0)+f_{yy}(x_0,y_0)\theta\Delta y+\varepsilon_3\theta\Delta y,$$

其中 $\varepsilon_3\to 0$（当 $\Delta y\to 0$ 时），因此得

$$\frac{W}{\Delta x\Delta y}=\frac{1}{\Delta x}\{f_y(x_0+\Delta x,y_0+\theta\Delta y)-f_y(x_0,y_0+\theta\Delta y)\}$$

$$=\frac{1}{\Delta x}\{f_y(x_0,y_0)+f_{yx}(x_0,y_0)\Delta x+f_{yy}(x_0,y_0)\theta\Delta y+\varepsilon_1\Delta x+\varepsilon_2\theta\Delta y$$

$$-f_y(x_0,y_0)-f_{yy}(x_0,y_0)\theta\Delta y-\varepsilon_3\theta\Delta y\}$$

$$=f_{yx}(x_0,y_0)+\varepsilon_1+\varepsilon_2\theta\frac{\Delta y}{\Delta x}-\varepsilon_3\theta\frac{\Delta y}{\Delta x},$$

这正是（*）式．同样，令 $\psi(x)=f(x,y_0+\Delta y)-f(x,y_0)$, 则

$$\frac{W}{\Delta x\Delta y}=\frac{1}{\Delta x\Delta y}[\psi(x_0+\Delta x)-\psi(x_0)]=\frac{1}{\Delta y}\psi'(x_0+\theta_1\Delta x)$$

$$=\frac{1}{\Delta y}\{f_x(x_0+\theta_1\Delta x,y_0+\Delta y)-f_x(x_0+\theta_1\Delta x,y_0)\}，（0<\theta_1<1），$$

因 f_x 在 (x_0,y_0) 处可微，故

$$f_x(x_0+\theta_1\Delta x,y_0+\Delta y)=f_x(x_0,y_0)+f_{xx}(x_0,y_0)\theta_1\Delta x+f_{xy}(x_0,y_0)\Delta y$$

$$+\varepsilon_4\theta_1\Delta x+\varepsilon_5\Delta y,$$

其中$\varepsilon_4,\varepsilon_5\to 0$（当$\Delta x,\Delta y\to 0$时），

$$f_x(x_0+\theta_1\Delta x,y_0)=f_x(x_0,y_0)+f_{xx}(x_0,y_0)\theta_1\Delta x+\varepsilon_6\theta_1\Delta x,$$

所以

$$\frac{W}{\Delta x\Delta y}=\frac{1}{\Delta y}\{f_x(x_0+\theta_1\Delta x,y_0+\Delta y)-f_x(x_0+\theta_1\Delta x,y_0)\}$$

$$=\frac{1}{\Delta y}\{f_x(x_0,y_0)+f_{xx}(x_0,y_0)\theta_1\Delta x+f_{xy}(x_0,y_0)\Delta y+\varepsilon_4\theta_1\Delta x+\varepsilon_5\Delta y$$

$$-f_x(x_0,y_0)-f_{xx}(x_0,y_0)\theta_1\Delta x-\varepsilon_6\theta_1\Delta x\}$$

$$=f_{xy}(x_0,y_0)+\varepsilon_4\theta_1\frac{\Delta x}{\Delta y}+\varepsilon_5-\varepsilon_6\theta_1\frac{\Delta x}{\Delta y},$$

这正是（**）式.

§2 复合函数与隐函数微分法

1．求下列函数的所有二阶偏导数：

（1）$u=f(ax,by)$；

（2）$u=f(x+y,x-y)$；

（3）$u=f(xy^2,x^2y)$；

（4）$u=f(\frac{x}{y},\frac{y}{z})$；

（5）$u=f(x^2+y^2+z^2)$；

（6）$u=f(x+y,xy,\frac{x}{y})$.

解（1）$\frac{\partial u}{\partial x}=f_1(ax,by)\cdot a=af_1(ax,by)$，$\frac{\partial u}{\partial y}=f_2(ax,by)\cdot b=bf_2(ax,by)$；

$$\frac{\partial^2 u}{\partial x^2}=a^2f_{11}(ax,by),\quad \frac{\partial^2 u}{\partial x\partial y}=abf_{12}(ax,by),$$

$$\frac{\partial^2 u}{\partial y \partial x} = abf_{21}(ax,by)，\quad \frac{\partial^2 u}{\partial y^2} = b^2 f_{22}(ax,by)．$$

（2）$\frac{\partial u}{\partial x} = f_1(x+y,x-y) + f_2(x+y,x-y)$，

$$\frac{\partial u}{\partial y} = f_1(x+y,x-y) - f_2(x+y,x-y)；$$

$$\frac{\partial^2 u}{\partial x^2} = f_{11}(x+y,x-y) + f_{12}(x+y,x-y) + f_{21}(x+y,x-y) + f_{22}(x+y,x-y)$$

$$= f_{11}(x+y,x-y) + 2f_{12}(x+y,x-y) + f_{22}(x+y,x-y)，$$

$$\frac{\partial^2 u}{\partial x \partial y} = f_{11}(x+y,x-y) - f_{12}(x+y,x-y) + f_{21}(x+y,x-y) - f_{22}(x+y,x-y)$$

$$= f_{11}(x+y,x-y) - f_{22}(x+y,x-y)，$$

$$\frac{\partial^2 u}{\partial y \partial x} = f_{11}(x+y,x-y) + f_{12}(x+y,x-y) - f_{21}(x+y,x-y) - f_{22}(x+y,x-y)$$

$$= f_{11}(x+y,x-y) - f_{22}(x+y,x-y)，$$

$$\frac{\partial^2 u}{\partial y^2} = f_{11}(x+y,x-y) - f_{12}(x+y,x-y) - f_{21}(x+y,x-y) + f_{22}(x+y,x-y)$$

$$= f_{11}(x+y,x-y) - 2f_{12}(x+y,x-y) + f_{22}(x+y,x-y)．$$

（3）$\frac{\partial u}{\partial x} = y^2 f_1(xy^2,x^2y) + 2xyf_2(xy^2,x^2y)$，

$$\frac{\partial u}{\partial y} = 2xyf_1(xy^2,x^2y) + x^2 f_2(xy^2,x^2y)；$$

$$\frac{\partial^2 u}{\partial x^2} = y^2[y^2 f_{11}(xy^2,x^2y) + 2xyf_{12}(xy^2,x^2y)] + 2yf_2(xy^2,x^2y)$$

$$+ 2xy[y^2 f_{21}(xy^2,x^2y) + 2xyf_{22}(xy^2,x^2y)]$$

$$= y^4 f_{11}(xy^2,x^2y) + 4xy^3 f_{12}(xy^2,x^2y) + 4x^2y^2 f_{22}(xy^2,x^2y) + 2yf_2(xy^2,x^2y)，$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2yf_1(xy^2,x^2y) + y^2[2xyf_{11}(xy^2,x^2y) + x^2 f_{12}(xy^2,x^2y)]$$

$$+ 2xf_2(xy^2,x^2y) + 2xy[2xyf_{21}(xy^2,x^2y) + x^2 f_{22}(xy^2,x^2y)]$$

$$=2xy^3 f_{11}(xy^2,x^2y)+5x^2y^2 f_{12}(xy^2,x_2y)+2x^3yf_{22}(xy^2,x^2y)$$

$$+2yf_1(xy^2,x^2y)+2xf_2(xy^2,x^2y),$$

$$\frac{\partial^2 u}{\partial y\partial x}=2yf_1(xy^2,x^2y)+2xy[y^2 f_{11}(xy^2,x^2y)+2xyf_{12}(xy^2,x^2y)]$$

$$+2xf_2(xy^2,x^2y)+x^2[y^2 f_{21}(xy^2,x^2y)+2xyf_{22}(xy^2,x^2y)]$$

$$=2xy^3 f_{11}(xy^2,x^2y)+5x^2y^2 f_{12}(xy^2,x_2y)+2x^3yf_{22}(xy^2,x^2y)$$

$$+2yf_1(xy^2,x^2y)+2xf_2(xy^2,x^2y),$$

$$\frac{\partial^2 u}{\partial y^2}=2xf_1(xy^2,x^2y)+2xy[2xyf_{11}(xy^2,x^2y)+x^2 f_{12}(xy^2,x^2y)]$$

$$+x^2[2xyf_{21}(xy^2,x^2y)+x^2 f_{22}(xy^2,x^2y)]$$

$$=4x^2y^2 f_{11}(xy^2,x^2y)+4x^3yf_{12}(xy^2,x^2y)+x^4 f_{22}(xy^2,x^2y)+2xf_1(xy^2,x^2y).$$

（4）$\frac{\partial u}{\partial x}=\frac{1}{y}f_1(\frac{x}{y},\frac{y}{z})$，$\frac{\partial u}{\partial y}=-\frac{x}{y^2}f_1(\frac{x}{y},\frac{y}{z})+\frac{1}{z}f_2(\frac{x}{y},\frac{y}{z})$，$\frac{\partial u}{\partial z}=-\frac{y}{z^2}f_2(\frac{x}{y},\frac{y}{z})$；

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{y^2}f_{11}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{y^2}f_1(\frac{x}{y},\frac{y}{z})+\frac{1}{y}[-\frac{x}{y^2}f_{11}(\frac{x}{y},\frac{y}{z})+\frac{1}{z}f_{12}(\frac{x}{y},\frac{y}{z})]$$

$$=-\frac{1}{y^2}f_1(\frac{x}{y},\frac{y}{z})+-\frac{x}{y^3}f_{11}(\frac{x}{y},\frac{y}{z})+\frac{1}{yz}f_{12}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial x\partial z}=\frac{1}{y}(-\frac{y}{z^2})f_{12}(\frac{x}{y},\frac{y}{z})=-\frac{1}{z^2}f_{12}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial y\partial x}=-\frac{1}{y^2}f_1(\frac{x}{y},\frac{y}{z})-\frac{x}{y^2}\cdot\frac{1}{y}f_{11}(\frac{x}{y},\frac{y}{z})+\frac{1}{z}\cdot\frac{1}{y}f_{21}(\frac{x}{y},\frac{y}{z})$$

$$=-\frac{1}{y^2}f_1(\frac{x}{y},\frac{y}{z})-\frac{x}{y^3}f_{11}(\frac{x}{y},\frac{y}{z})+\frac{1}{yz}f_{12}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{2x}{y^3}f_1(\frac{x}{y},\frac{y}{z})-\frac{x}{y^2}[-\frac{x}{y^2}f_{11}(\frac{x}{y},\frac{y}{z})+\frac{1}{z}f_{12}(\frac{x}{y},\frac{y}{z})]$$

$$+\frac{1}{z}[-\frac{x}{y^2}f_{21}(\frac{x}{y},\frac{y}{z})+\frac{1}{z}f_{22}(\frac{x}{y},\frac{y}{z})]$$

$$=\frac{2x}{y^3}f_1(\frac{x}{y},\frac{y}{z})+\frac{x^2}{y^4}f_{11}(\frac{x}{y},\frac{y}{z})-\frac{2x}{y^2z}f_{12}(\frac{x}{y},\frac{y}{z})+\frac{1}{z^2}f_{22}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial y\partial z}=-\frac{x}{y^2}f_{12}(\frac{x}{y},\frac{y}{z})\cdot(-\frac{y}{z^2})+(-\frac{1}{z^2})f_2(\frac{x}{y},\frac{y}{z})+\frac{1}{z}f_{22}(\frac{x}{y},\frac{y}{z})\cdot(-\frac{y}{z^2})$$

$$=\frac{x}{yz^2}f_{12}(\frac{x}{y},\frac{y}{z})-\frac{1}{z^2}f_2(\frac{x}{y},\frac{y}{z})-\frac{y}{z^3}f_{22}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial z\partial x}=-\frac{y}{z^2}\cdot\frac{1}{y}f_{21}(\frac{x}{y},\frac{y}{z})=-\frac{1}{z^2}f_{12}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial z\partial y}=-\frac{1}{z^2}f_2(\frac{x}{y},\frac{y}{z})-\frac{y}{z^2}[-\frac{x}{y^2}f_{21}(\frac{x}{y},\frac{y}{z})+\frac{1}{z}f_{22}(\frac{x}{y},\frac{y}{z})]$$

$$=-\frac{1}{z^2}f_2(\frac{x}{y},\frac{y}{z})+\frac{x}{yz^2}f_{12}(\frac{x}{y},\frac{y}{z})-\frac{y}{z^3}f_{22}(\frac{x}{y},\frac{y}{z}),$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{2y}{z^3}f_2(\frac{x}{y},\frac{y}{z})-\frac{y}{z^2}f_{22}(\frac{x}{y},\frac{y}{z})\cdot(-\frac{y}{z^2})=\frac{2z}{y^3}f_2(\frac{x}{y},\frac{y}{z})+\frac{y^2}{z^4}f_{22}(\frac{x}{y},\frac{y}{z}).$$

（5）$\frac{\partial u}{\partial x}=2xf'(x^2+y^2+z^2)$，$\frac{\partial u}{\partial y}=2yf'(x^2+y^2+z^2)$，$\frac{\partial u}{\partial z}=2zf'(x^2+y^2+z^2)$；

$$\frac{\partial^2 u}{\partial x^2}=2f'(x^2+y^2+z^2)+4x^2f''(x^2+y^2+z^2),$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=4xyf''(x^2+y^2+z^2),\quad \frac{\partial^2 u}{\partial x\partial z}=\frac{\partial^2 u}{\partial z\partial x}=4xzf''(x^2+y^2+z^2),$$

$$\frac{\partial^2 u}{\partial y^2}=2f'(x^2+y^2+z^2)+4y^2f''(x^2+y^2+z^2),$$

$$\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial^2 u}{\partial z\partial y}=4yzf''(x^2+y^2+z^2),$$

$$\frac{\partial^2 u}{\partial z^2}=2f'(x^2+y^2+z^2)+4z^2f''(x^2+y^2+z^2).$$

（6）$\frac{\partial u}{\partial x}=f_1(x+y,xy,\frac{x}{y})+yf_2(x+y,xy,\frac{x}{y})+\frac{1}{y}f_3(x+y,xy,\frac{x}{y})$，

$$\frac{\partial u}{\partial y}=f_1(x+y,xy,\frac{x}{y})+xf_2(x+y,xy,\frac{x}{y})-\frac{x}{y^2}f_3(x+y,xy,\frac{x}{y}),$$

$$\frac{\partial^2 u}{\partial x^2}=f_{11}(x+y,xy,\frac{x}{y})+yf_{12}9x+y,xy,\frac{x}{y})+\frac{1}{y}f_{13}(x+y,xy,\frac{x}{y})$$

$$+y[f_{21}(x+y,xy,\frac{x}{y})+yf_{22}(x+y,xy,\frac{x}{y})+\frac{1}{y}f_{23}(x+y,xy,\frac{x}{y})]$$

$$+\frac{1}{y}[f_{31}(x+y,xy,\frac{x}{y})+yf_{32}(x+y,xy,\frac{x}{y})+\frac{1}{y}f_{33}(x+y,xy,\frac{x}{y})]$$

$$=f_{11}(x+y,xy,\frac{x}{y})+2yf_{12}(x+y,xy,\frac{x}{y})+\frac{2}{y}f_{13}(x+y,xy,\frac{x}{y})$$

$$+y^2f_{22}(x+y,xy,\frac{x}{y})+2f_{23}(x+y,xy,\frac{x}{y})+\frac{1}{y^2}f_{33}(x+y,xy,\frac{x}{y}),$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=f_{11}(x+y,xy,\frac{x}{y})+xf_{12}(x+y,xy,\frac{x}{y})-\frac{x}{y^2}f_{13}(x+y,xy,\frac{x}{y})$$

$$+f_2(x+y,xy,\frac{x}{y})+y[f_{21}(x+y,xy,\frac{x}{y})+xf_{22}(x+y,xy,\frac{x}{y})$$

$$-\frac{x}{y^2}f_{23}(x+y,xy,\frac{x}{y})]+(-\frac{1}{y^2})f_3(x+y,xy,\frac{x}{y})+\frac{1}{y}[f_{31}(x+y,xy,\frac{x}{y})$$

$$+xf_{32}(x+y,xy,\frac{x}{y})-\frac{x}{y^2}f_{33}(x+y,xy,\frac{x}{y})]$$

$$=f_2(x+y,xy,\frac{x}{y})-\frac{1}{y^2}f_3(x+y,xy,\frac{x}{y})+f_{11}(x+y,xy,\frac{x}{y})$$

$$+(x+y)f_{12}(x+y,xy,\frac{x}{y})+\frac{y-x}{y^2}f_{13}(x+y,xy,\frac{x}{y})$$

$$+xyf_{22}(x+y,xy,\frac{x}{y})-\frac{x}{y^3}f_{33}(x+y,xy,\frac{x}{y}),$$

$$\frac{\partial^2 u}{\partial y^2}=f_{11}(x+y,xy,\frac{x}{y})+xf_{12}(x+y,xy,\frac{x}{y})-\frac{x}{y^2}f_{13}(x+y,xy,\frac{x}{y})$$

$$+x[f_{21}(x+y,xy,\frac{x}{y})+xf_{22}(x+y,xy,\frac{x}{y})-\frac{x}{y^2}f_{23}(x+y,xy,\frac{x}{y})]$$

$$+\frac{2x}{y^3}f_3(x+y,xy,\frac{x}{y})-\frac{x}{y^2}[f_{31}(x+y,xy,\frac{x}{y})+xf_{32}(x+y,xy,\frac{x}{y})$$

$$-\frac{x}{y^2}f_{33}(x+y,xy,\frac{x}{y})]$$

$$=\frac{2x}{y^3}f_3(x+y,xy,\frac{x}{y})+f_{11}(x+y,xy,\frac{x}{y})+2xf_{12}(x+y,xy,\frac{x}{y})$$

$$-\frac{2x}{y^2}f_{13}(x+y,xy,\frac{x}{y})+x^2f_{22}(x+y,xy,\frac{x}{y})-\frac{2x^2}{y^2}f_{23}(x+y,xy,\frac{x}{y})$$

$$+\frac{x^4}{y^4}f_{33}(x+y,xy,\frac{x}{y}).$$

（在以上各题中，都假设 f 对各自变量的二阶混合偏导数与求导次序无关）.

2．设 $z=\dfrac{y}{f(x^2-y^2)}$，其中 f 是可微函数，验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}.$$

证明 $$\frac{\partial z}{\partial x}=\frac{-y}{f^2(x^2-y^2)}f'(x^2-y^2)\cdot 2x=-\frac{2xyf'(x^2-y^2)}{f^2(x^2-y^2)},$$

$$\frac{\partial z}{\partial y}=\frac{-y}{f^2(x^2-y^2)}f'(x^2-y^2)(-2y)+\frac{1}{f(x^2-y^2)}$$

$$=\frac{2y^2f'(x^2-y^2)}{f^2(x^2-y^2)}+\frac{1}{f(x^2-y^2)},$$

所以，

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=-\frac{2yf(x^2-y^2)}{f^2(x^2+y^2)}+\frac{2yf'(x^2+y^2)}{f^2(x^2+y^2)}+\frac{1}{yf(x^2-y^2)}$$

$$=\frac{1}{y^2}\frac{y}{f(x^2-y^2)}=\frac{z}{y^2}.$$

3．设 $v=\dfrac{1}{r}g(t-\dfrac{r}{c})$，$c$ 为常数，函数 g 二阶可导，$r=\sqrt{x^2+y^2+z^2}$．证明

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}.$$

证明 $\dfrac{\partial v}{\partial t}=\dfrac{1}{r}g'(t-\dfrac{r}{c})$，$\dfrac{\partial^2 v}{\partial t^2}=\dfrac{1}{r}g''(t-\dfrac{r}{c})$，

$$\frac{\partial v}{\partial x}=-\frac{1}{r^2}\frac{2x}{2\sqrt{x^2+y^2+z^2}}g(t-\frac{r}{c})+\frac{1}{r}g'(t-\frac{r}{c})(-\frac{1}{c}\frac{2x}{2\sqrt{x^2+y^2+z^2}})$$

$$=-\frac{x}{r^3}g(t-\frac{r}{c})-\frac{x}{cr^2}g'(t-\frac{r}{c}),$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{-r^3+x\cdot 3r^2\cdot\frac{x}{r}}{r^6}g(t-\frac{r}{c})-\frac{r^2-x\cdot 2r\cdot\frac{x}{r}}{cr^4}g'(t-\frac{r}{c})$$

$$-\frac{x}{r^3}g'(t-\frac{r}{c})\cdot(-\frac{x}{cr})-\frac{x}{cr^2}g''(t-\frac{r}{c})\cdot(-\frac{x}{cr})$$

$$=\frac{3x^2-r^2}{r^5}g(t-\frac{r}{c})+\frac{3x^2-r^2}{cr^4}g'(t-\frac{r}{c})+\frac{x^2}{c^2r^3}g''(t-\frac{r}{c}),$$

由函数v关于x,y,z的对称性知，

$$\frac{\partial^2 v}{\partial y^2}=\frac{3y^2-r^2}{r^5}g(t-\frac{r}{c})+\frac{3y^2-r^2}{cr^4}g'(t-\frac{r}{c})+\frac{y^2}{c^2r^3}g''(t-\frac{r}{c}),$$

$$\frac{\partial^2 v}{\partial z^2}=\frac{3z^2-r^2}{r^5}g(t-\frac{r}{c})+\frac{3z^2-r^2}{cr^4}g'(t-\frac{r}{c})+\frac{z^2}{c^2r^3}g''(t-\frac{r}{c}),$$

所以，

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{3(x^2+y^2+z^2)-3r^2}{r^5}g(t-\frac{r}{c})$$

$$+\frac{3(x^2+y^2+z^2)-3r^2}{cr^4}g'(t-\frac{r}{c})+\frac{x^2+y^2+z^2}{cr^3}g''(t-\frac{r}{c})$$

$$=\frac{1}{c^2r}g''(t-\frac{r}{c})=\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}.$$

4．若函数$f(x,y,z)$对任意的正实数 t 满足关系

$$f(tx,ty,tz)=t^n f(x,y,z),$$

则称$f(x,y,z)$为n次齐次函数．设$f(x,y,z)$可微，试证明$f(x,y)$为n次齐次函数的充要条件是

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z).$$

证明 必要性．由于$f(x,y)$为n次齐次函数，因此$f(tx,ty,tz)=t^n f(x,y,z)$，两边

对 t 求导，有

$$xf_1(tx,ty,tz)+yf_2(tx,ty,tz)+zf_3(tx,ty,tz)=nt^{n-1}f(x,y,z),$$

令 $tx=\xi, ty=\eta, tz=\zeta$，则有

$$\frac{\xi}{t}f_1(\xi,\eta,\zeta)+\frac{\eta}{t}f_2(\xi,\eta,\zeta)+\frac{\zeta}{t}f_3(\xi,\eta,\zeta)=nt^{n-1}f(\frac{\xi}{t},\frac{\eta}{t},\frac{\zeta}{t})=\frac{nt^{n-1}}{t^n}f(\xi,\eta,\zeta),$$

再把 ξ,η,ζ 用 x, y, z 替代，就有 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z)$.

充分性．设 $f(x,y,z)$ 满足 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z)$，任意固定定义域中一点 (x,y,z)，考察下面的 t 的函数 $F(t)=\frac{f(tx,ty,tz)}{t^n}$，$(t>0)$.

它在 $t>0$ 时有定义且是可微的，对 t 求导，得

$$\begin{aligned}F'(t)&=\frac{1}{t^n}\{xf_x(tx,ty,tz)+yf_y(tx,ty,tz)+zf_z(tx,ty,tz)\}-\frac{n}{t^{n+1}}f(tx,ty,tz)\\&=\frac{1}{t^{n+1}}\{txf_x(tx,ty,tz)+tyf_y(tx,ty,tz)+tzf_z(tx,ty,tz)-nf(tx,ty,tz)\}\\&=0,\end{aligned}$$

从而当 $t>0$ 时，$F(t)=c$（与 t 无关的常数）．在函数 $F(t)$ 的等式中令 $t=1$，得

$$c=F(1)=f(x,y,z),$$

于是，

$$F(t)=\frac{f(tx,ty,tz)}{t^n}=f(x,y,z),$$

即 $f(tx,ty,tz)=t^n f(x,y,z)$，从而 $f(x,y,z)$ 为 n 次齐次函数．

5．验证下列各式：

（1）$u=\varphi(x^2+y^2)$，则 $y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0$；

（2）$u=y\varphi(x^2-y^2)$，则 $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=\frac{xu}{y}$；

（3）$u=x\varphi(x+y)+y\psi(x+y)$，则 $\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$；

（4）$u=x\varphi(\frac{y}{x})+\psi(\frac{y}{x})$，则 $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$.

解（1） $\frac{\partial u}{\partial x}=\varphi'(x^2+y^2)\cdot 2x=2x\varphi'(x^2+y^2)$， $\frac{\partial u}{\partial y}=2y\varphi'(x^2+y^2)$，所以，

$$y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=2xy\varphi'(x^2+y^2)-2xy\varphi'(x^2+y^2)=0.$$

（2） $\frac{\partial u}{\partial x}=2xy\varphi'(x^2-y^2)$， $\frac{\partial u}{\partial y}=\varphi(x^2-y^2)-2y^2\varphi'(x^2-y^2)$，所以，

$$y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=2xy^2\varphi'(x^2-y^2)+x\varphi(x^2-y^2)-2xy^2\varphi'(x^2-y^2)=\frac{xu}{y}.$$

（3） $\frac{\partial u}{\partial x}=\varphi(x+y)+x\varphi'(x+y)+y\psi'(x+y)$，

$$\frac{\partial u}{\partial y}=x\varphi'(x+y)+\psi(x+y)+y\psi'(x+y),$$

$$\frac{\partial^2 u}{\partial x^2}=2\varphi'(x+y)+x\varphi''(x+y)+y\psi''(x+y),$$

$$\frac{\partial^2 u}{\partial x\partial y}=\varphi'(x+y)+x\varphi''(x+y)+\psi'(x+y)+y\psi''(x+y),$$

$$\frac{\partial^2 u}{\partial y^2}=x\varphi''(x+y)+2\psi'(x+y)+y\psi''(x+y),$$

所以，

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}&=(2\varphi'(x+y)+x\varphi''(x+y)+y\psi''(x+y))\\&\quad-2(\varphi'(x+y)+x\varphi''(x+y)+\psi'(x+y)+y\psi'(x+y))\\&\quad+(x\varphi''(x+y)+2\psi'(x+y)+y\psi''(x+y)\\&=0.\end{aligned}$$

（4） $\frac{\partial u}{\partial x}=\varphi(\frac{y}{x})+x\varphi'(\frac{y}{x})(-\frac{y}{x^2})+\psi'(\frac{y}{x})(-\frac{y}{x^2})=\varphi(\frac{y}{x})-\frac{y}{x}\varphi'(\frac{y}{x})-\frac{y}{x^2}\psi'(\frac{y}{x})$，

$$\frac{\partial u}{\partial y}=\varphi'(\frac{y}{x})+\frac{1}{x}\varphi'(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{y}{x^2}\varphi'(\frac{y}{x})+\frac{y}{x^2}\varphi'(\frac{y}{x})+\frac{y^2}{x^3}\varphi''(\frac{y}{x})+\frac{2y}{x^3}\psi'(\frac{y}{x})+\frac{y^2}{x^4}\psi''(\frac{y}{x})$$

$$=\frac{y^2}{x^3}\varphi''(\frac{y}{x})+\frac{2y}{x^3}\psi'(\frac{y}{x})+\frac{y^2}{x^4}\psi''(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\frac{y}{x^2}\varphi''(\frac{y}{x})-\frac{1}{x^2}\psi'(\frac{y}{x})-\frac{y}{x^3}\psi''(\frac{y}{x}),\quad \frac{\partial^2 u}{\partial y^2}=\frac{1}{x}\varphi''(\frac{y}{x})+\frac{1}{x^2}\psi''(\frac{y}{x}),$$

所以，

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{y^2}{x}\varphi''(\frac{y}{x})+\frac{2y}{x}\psi'(\frac{y}{x})+\frac{y^2}{x^2}\psi''(\frac{y}{x})$$

$$-\frac{2y^2}{x}\varphi''(\frac{y}{x})-\frac{2y}{x}\psi'(\frac{y}{x})-\frac{2y^2}{x^2}\psi''(\frac{y}{x})+\frac{y^2}{x}\varphi''(\frac{y}{x})+\frac{y^2}{x^2}\psi''(\frac{y}{x})=0.$$

6．设 $u=f(x,y)$ 可微，在坐标变换

$$x=r\cos\theta,\quad y=r\sin\theta,$$

下，证明

$$(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial\theta})^2=(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2,$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

证明 $\frac{\partial u}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta,$

$$\frac{\partial u}{\partial\theta}=\frac{\partial f}{\partial x}\cdot(-r\sin\theta)+\frac{\partial f}{\partial y}(r\cos\theta)=-\frac{\partial f}{\partial x}r\sin\theta+r\cos\theta\frac{\partial f}{\partial y},$$

所以，

$$(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial\theta})^2=(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)^2+\frac{1}{r^2}(-\frac{\partial f}{\partial x}r\sin\theta+r\cos\theta\frac{\partial f}{\partial y})^2$$

$$=(\frac{\partial f}{\partial x})^2\cos^2\theta+(\frac{\partial f}{\partial y})^2\sin^2\theta+2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\sin\theta\cos\theta$$

$$+(\frac{\partial f}{\partial x})^2\sin^2\theta-2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\sin\theta\cos\theta+(\frac{\partial f}{\partial y})^2\cos^2\theta$$

$$=(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2=(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2.$$

$$\frac{\partial^2 u}{\partial r^2}=(\frac{\partial^2 f}{\partial x^2}\cos\theta+\frac{\partial^2 f}{\partial x\partial y}\sin\theta)\cos\theta+(\frac{\partial^2 f}{\partial y\partial x}\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin\theta)\sin\theta$$

$$=\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta,$$

$$\frac{\partial^2 u}{\partial \theta^2}=[\frac{\partial^2 f}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 f}{\partial x\partial y}r\cos\theta](-r\sin\theta)+\frac{\partial f}{\partial x}(-r\cos\theta)$$

$$+[\frac{\partial^2 f}{\partial y\partial x}(-r\sin\theta)+\frac{\partial^2 f}{\partial y^2}r\cos\theta]r\cos\theta+\frac{\partial f}{\partial y}(-r\sin\theta)$$

$$=\frac{\partial^2 f}{\partial x^2}r^2\sin^2\theta-2\frac{\partial^2 f}{\partial x\partial y}r^2\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}r^2\cos^2\theta$$

$$-\frac{\partial f}{\partial x}r\cos\theta-\frac{\partial f}{\partial y}r\sin\theta,$$

因此，

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$=\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta+\frac{1}{r}\frac{\partial f}{\partial x}\cos\theta+\frac{1}{r}\frac{\partial f}{\partial y}\sin\theta$$

$$+\frac{\partial^2 f}{\partial x^2}\sin^2\theta-2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\cos^2\theta-\frac{1}{r}\frac{\partial f}{\partial x}\cos\theta-\frac{1}{r}\frac{\partial f}{\partial y}\sin\theta$$

$$=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

7．设 $z=f(x,y)$ 可微，在坐标旋转变换

$$x=u\cos\theta-v\sin\theta,\quad y=u\sin\theta+v\cos\theta$$

下（其中旋转角 θ 是常数），证明：

$$(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2=(\frac{\partial z}{\partial u})^2+(\frac{\partial z}{\partial v})^2.$$

这时称 $(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2$ 是一个形式不变量.

证明 $\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta$，$\frac{\partial z}{\partial v}=-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta$，

所以，

$$(\frac{\partial z}{\partial u})^2+(\frac{\partial z}{\partial x})^2=(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)^2+(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)^2$$

$$=(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2=(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2.$$

8．设函数$u=f(x,y)$满足 Laplace 方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

证明在下列变换下形式保持不变，即仍有$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=0$.

（1）$x=\frac{s}{s^2+t^2}$， $y=\frac{t}{s^2+t^2}$；

（2）$x=e^s\cos t$， $y=e^s\sin t$；

（3）$x=\varphi(s,t)$， $y=\psi(s,t)$ 满足 $\frac{\partial\varphi}{\partial s}=\frac{\partial\psi}{\partial t}$，$\frac{\partial\varphi}{\partial t}=-\frac{\partial\psi}{\partial s}$． 这组方程称为 Cauchy-Riemann 方程.

解（1）$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{t^2-s^2}{(s^2+t^2)^2}+\frac{\partial u}{\partial y}\frac{-2st}{(s^2+t^2)^2}$， $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{-2st}{(s^2+t^2)^2}+\frac{\partial u}{\partial y}\frac{s^2-t^2}{(s^2+t^2)^2}$，

$$\frac{\partial^2 u}{\partial s^2}=[\frac{\partial^2 u}{\partial x^2}\frac{t^2-s^2}{(t^2+s^2)^2}+\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{-2st}{(s^2+t^2)^2}]\frac{t^2-s^2}{(s^2+t^2)^2}+\frac{\partial u}{\partial x}\frac{2s(s^2-3t^2)}{(s^2+t^2)^3}$$

$$+[\frac{\partial^2 u}{\partial y\partial x}\frac{t^2-s^2}{(t^2+s^2)^2}+\frac{\partial^2 u}{\partial y^2}\frac{-2st}{(s^2+t^2)^2}]\frac{-2st}{(s^2+t^2)^2}+\frac{\partial u}{\partial y}\frac{2t(3s^2-t^2)}{(s^2+t^2)^3}$$

$$=\frac{\partial^2 u}{\partial x^2}\frac{(t^2-s^2)^2}{(t^2+s^2)^4}-\frac{\partial^2 u}{\partial x\partial y}\frac{4st(t^2-s^2)}{(s^2+t^2)^4}+\frac{\partial^2 u}{\partial y^2}\frac{4s^2t^2}{(s^2+t^2)^2}$$

$$+\frac{\partial u}{\partial x}\frac{2s(s^2-3t^2)}{(s^2+t^2)^3}+\frac{\partial u}{\partial y}\frac{2t(3s^2-t^2)}{(s^2+t^2)^2},$$

$$\frac{\partial^2 u}{\partial t^2}=[\frac{\partial^2 u}{\partial x^2}\frac{-2st}{(s^2+t^2)^2}+\frac{\partial^2 u}{\partial x\partial y}\frac{s^2-t^2}{(s^2+t^2)^2}]\frac{-2st}{(s^2+t^2)^2}+\frac{\partial u}{\partial x}\frac{2s(3t^2-s^2)}{(s^2+t^2)^3}$$

$$+[\frac{\partial^2 u}{\partial y\partial x}\frac{-2st}{(s^2+t^2)^2}+\frac{\partial^2 u}{\partial y^2}\frac{s^2-t^2}{(s^2+t^2)^2}]\frac{s^2-t^2}{(s^2+t^2)^2}+\frac{\partial u}{\partial y}\frac{2t(t^2-3s^2)}{(s^2+t^2)^2}$$

$$=\frac{\partial^2 u}{\partial x^2}\frac{4s^2t^2}{(s^2+t^2)^4}-\frac{\partial^2 u}{\partial x\partial y}\frac{4st(s^2-t^2)}{(s^2+t^2)^4}+\frac{\partial^2 u}{\partial y^2}\frac{(s^2-t^2)}{(s^2+t^2)^2}$$

$$+\frac{\partial u}{\partial x}\frac{2s(3t^2-s^2)}{(s^2+t^2)^3}+\frac{\partial u}{\partial y}\frac{2t(t^2-3s^2)}{(s^2+t^2)^3},$$

所以，

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}\frac{(t^2-s^2)^2+4s^2t^2}{(t^2+s^2)^4}+\frac{\partial^2 u}{\partial y^2}\frac{4s^2t^2+(s^2-t^2)^2}{(s^2+t^2)^2}$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

（2） $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t$， $\frac{\partial u}{\partial t}=-\frac{\partial u}{\partial x}e^s\sin t+\frac{\partial u}{\partial y}e^s\cos t$，

$$\frac{\partial^2 u}{\partial s^2}=[\frac{\partial^2 u}{\partial x^2}e^s\cos t+\frac{\partial^2 u}{\partial x\partial y}e^s\sin t]e^s\cos t+\frac{\partial u}{\partial x}e^s\cos t$$

$$+[\frac{\partial^2 u}{\partial y\partial x}e^s\cos t+\frac{\partial^2 u}{\partial y^2}e^s\sin t]e^s\sin t+\frac{\partial u}{\partial y}e^s\sin t$$

$$=\frac{\partial^2 u}{\partial x^2}e^{2s}\cos^2 t+2\frac{\partial^2 u}{\partial x\partial y}e^{2s}\sin t\cos t+\frac{\partial^2 u}{\partial y^2}e^{2s}\sin^2 t$$

$$+\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t$$

$$\frac{\partial^2 u}{\partial t^2}=-(-\frac{\partial^2 u}{\partial x^2}e^s\sin t+\frac{\partial^2 u}{\partial x\partial y}e^s\cos t)e^s\sin t-\frac{\partial u}{\partial x}e^s\cos t$$

$$+(-\frac{\partial^2 u}{\partial y\partial x}e^s\sin t+\frac{\partial^2 u}{\partial y^2}e^s\cos t)e^s\cos t-\frac{\partial u}{\partial y}e^s\sin t$$

$$=\frac{\partial^2 u}{\partial x^2}e^{2s}\sin^2 t-2\frac{\partial^2 u}{\partial x\partial y}e^{2s}\sin t\cos t+\frac{\partial^2 u}{\partial y^2}e^{2s}\cos^2 t$$

$$-\frac{\partial u}{\partial x}e^s\cos t-\frac{\partial u}{\partial y}e^s\sin t,$$

所以，

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})e^{2s}=0.$$

（3）$\dfrac{\partial u}{\partial s}=\dfrac{\partial u}{\partial x}\dfrac{\partial \varphi}{\partial s}+\dfrac{\partial u}{\partial y}\dfrac{\partial \psi}{\partial s}=\dfrac{\partial u}{\partial x}\dfrac{\partial \psi}{\partial t}-\dfrac{\partial u}{\partial y}\dfrac{\partial \varphi}{\partial t}$，

$\dfrac{\partial u}{\partial t}=\dfrac{\partial u}{\partial x}\dfrac{\partial \varphi}{\partial t}+\dfrac{\partial u}{\partial y}\dfrac{\partial \psi}{\partial t}=-\dfrac{\partial u}{\partial x}\dfrac{\partial \psi}{\partial s}+\dfrac{\partial u}{\partial y}\dfrac{\partial \varphi}{\partial s}$，

$$\frac{\partial^2 u}{\partial s^2}=(\frac{\partial^2 u}{\partial x^2}\frac{\partial \varphi}{\partial s}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial \psi}{\partial s})\frac{\partial \psi}{\partial t}+\frac{\partial u}{\partial x}\frac{\partial^2 \psi}{\partial t\partial s}-(\frac{\partial^2 u}{\partial y\partial x}\frac{\partial \varphi}{\partial s}+\frac{\partial^2 u}{\partial y^2}\frac{\partial \psi}{\partial s})\frac{\partial \varphi}{\partial t}-\frac{\partial u}{\partial y}\frac{\partial^2 \varphi}{\partial t\partial s},$$

$$\frac{\partial^2 u}{\partial t^2}=-(\frac{\partial^2 u}{\partial x^2}\frac{\partial \varphi}{\partial t}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial \psi}{\partial t})\frac{\partial \psi}{\partial s}-\frac{\partial u}{\partial x}\frac{\partial^2 \psi}{\partial s\partial t}+(\frac{\partial^2 u}{\partial y\partial x}\frac{\partial \varphi}{\partial t}+\frac{\partial^2 u}{\partial y^2}\frac{\partial \psi}{\partial t})\frac{\partial \varphi}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial^2 \varphi}{\partial s\partial t},$$

所以，

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})[(\frac{\partial \varphi}{\partial s})^2+(\frac{\partial \varphi}{\partial t})^2]=0.$$

9．作自变量的变换，取ξ,η,ζ为新的自变量：

（1）$\xi=x,\eta=x^2+y^2$，变换方程$y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=0$；

（2）$\xi=x,\eta=y-x,\zeta=z-x$，变换方程$\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$.

解（1）$z=f(x,y)=g(\xi,\eta)$，则有

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}=\frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial g}{\partial \xi}+2x\frac{\partial z}{\partial \eta},\quad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}=\frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial y}=2y\frac{\partial z}{\partial \eta},$$

所以，

$$0=y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=y(\frac{\partial z}{\partial \xi}+2x\frac{\partial z}{\partial \eta})-x(2y\frac{\partial z}{\partial \eta})=y\frac{\partial z}{\partial \xi},$$

即$\dfrac{\partial z}{\partial \xi}=0$，所以解出$z=\varphi(\eta),\varphi$是可微函数，亦即$z=\varphi(x^2+y^2)$.

（2）$u=f(x,y,z)=g(\xi,\eta,\zeta)$，则有

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}-\frac{\partial u}{\partial \zeta},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}+\frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y}=\frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial z}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial z}+\frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial z}=\frac{\partial u}{\partial \zeta},$$

所以 $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$，即 $\frac{\partial u}{\partial \xi}=0$．故 $u=\varphi(\eta,\zeta)=\varphi(y-x,z-x),\varphi$ 是可微函数.

10．作自变量和因变量的变换，取 u,v 为新的自变量，$w=w(u,v)$ 为新的自变量：

（1）$u=x+y,v=\frac{y}{x},w=\frac{z}{x}$，变换方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0;$$

（2）设 $u=\frac{x}{y}$，$v=x,\omega=xz-y$，变换方程

$$y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}=\frac{2}{x}.$$

解（1）由于 $w=\frac{z}{x}$，因此 $z=wx$，函数关系可由下图表示

$$z\begin{cases} w\begin{cases} u\begin{cases} x\\ y\end{cases}\\ v\begin{cases} x\\ y\end{cases}\end{cases}\\ x\end{cases}$$

$$\frac{\partial z}{\partial x}=w+x(\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x})=w+x\frac{\partial w}{\partial u}-\frac{y}{x}\frac{\partial w}{\partial v},$$

$$\frac{\partial z}{\partial y}=x(\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y})=x\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial w}{\partial u}-\frac{y}{x^2}\frac{\partial w}{\partial v}+\frac{\partial w}{\partial u}+x(\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial u\partial v}(-\frac{y}{x^2}))$$

$$+\frac{y}{x^2}\frac{\partial w}{\partial v}-\frac{y}{x}(\frac{\partial^2 w}{\partial v\partial u}+\frac{\partial^2 w}{\partial v^2}(-\frac{y}{x^2}))$$

$$=2\frac{\partial w}{\partial u}+x\frac{\partial^2 w}{\partial u^2}-\frac{2y}{x}\frac{\partial^2 w}{\partial u\partial v}+\frac{y^2}{x^3}\frac{\partial^2 w}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\frac{1}{x}+x(\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial u\partial v}\frac{1}{x})-\frac{1}{x}\frac{\partial w}{\partial v}-\frac{y}{x}(\frac{\partial^2 w}{\partial v\partial u}+\frac{\partial^2 w}{\partial v^2}\frac{1}{x})$$

$$=\frac{\partial w}{\partial u}+x\frac{\partial^2 w}{\partial u^2}+(1-\frac{y}{x})\frac{\partial^2 w}{\partial u\partial v}-\frac{y}{x^2}\frac{\partial^2 w}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial y^2}=x(\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial u\partial v}\frac{1}{x})+(\frac{\partial^2 w}{\partial v\partial u}+\frac{\partial^2 w}{\partial v^2}\frac{1}{x})=x\frac{\partial^2 w}{\partial u^2}+2\frac{\partial^2 w}{\partial u\partial v}+\frac{1}{x}\frac{\partial^2 w}{\partial v^2},$$

所以，

$$0=\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=(\frac{1}{x}+\frac{2y}{x^2}+\frac{y^3}{x^3})\frac{\partial^2 w}{\partial v^2},$$

即 $\frac{\partial^2 w}{\partial v^2}=0$．解得，$\frac{\partial w}{\partial v}=\varphi(u)$，$\varphi$ 是可微函数，于是 $w=\varphi(u)v+\psi(u)$，ψ 是可微函数，

所以，$z=xw=x\varphi(x+y)\frac{y}{x}+\psi(x+y)=y\varphi(x+y)+\psi(x+y)$，其中 φ、ψ 均为任意可微函数．

（2）设 $u=\frac{x}{y},v=x,\omega=xz-y,$ 变换方程 $y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}=\frac{2}{x}$

$w=xz-y\Rightarrow z=\frac{w+y}{x}$，函数关系可由下图表示

$$z\begin{cases} w\begin{cases} u\begin{cases} x\\ y\end{cases}\\ v\ \to x\end{cases}\\ x\\ y\end{cases}$$

$$\frac{\partial z}{\partial y}=\frac{1}{x}\left(\frac{\partial w}{\partial u}\left(-\frac{x}{y^2}\right)+1\right)=-\frac{1}{y^2}\frac{\partial w}{\partial u}+\frac{1}{x},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{2}{y^3}\frac{\partial w}{\partial u}-\frac{1}{y^2}\left(\frac{\partial^2 w}{\partial u^2}\left(-\frac{x}{y^2}\right)\right)=\frac{2}{y^3}\frac{\partial w}{\partial u}+\frac{x}{y^4}\frac{\partial^2 w}{\partial u^4},$$

所以，

$$\frac{2}{x}=y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}=\frac{2}{y^2}\frac{\partial w}{\partial u}+\frac{x}{y^3}\frac{\partial^2 w}{\partial u^2}-\frac{2}{y^2}\frac{\partial w}{\partial u}+\frac{2}{x}=\frac{x}{y^3}\frac{\partial^2 w}{\partial u^2}+\frac{2}{x},$$

即 $\frac{\partial^2 w}{\partial u^2}=0$．解得 $\frac{\partial w}{\partial u}=\varphi(v)$，$\varphi$ 是任意可微函数 $\Rightarrow w=u\varphi(v)+\psi(v)$，$\psi$ 是任意可微函数，所以，

$$z=\frac{w+y}{x}=\frac{1}{x}\left(\frac{x}{y}\varphi(x)+\psi(x)\right)+\frac{y}{x}=\frac{1}{y}\varphi(x)+\frac{1}{x}\psi(x)+\frac{y}{x},$$

其中φ、ψ是任意可微函数.

11．求下列方程所确定的函数$z=f(x,y)$的一阶和二阶偏导数：

（1）$e^{-xy}-2z+e^{z}=0$；

（2）$x+y+z=e^{x+y+z}$；

（3）$xyz=z+y+z$；

（4）$x^2+y^2+z^2-2x+2y-4z-5=0$.

解（1）方程两边分别对x和y求偏导数，得

$$-ye^{-xy}-2\frac{\partial z}{\partial x}+e^{z}\frac{\partial z}{\partial x}=0，\quad -xe^{-xy}-2\frac{\partial z}{\partial y}+e^{z}\frac{\partial z}{\partial y}=0，$$

于是$\dfrac{\partial z}{\partial x}=\dfrac{ye^{-xy}}{e^{z}-2}$，$\dfrac{\partial z}{\partial y}=\dfrac{xe^{-xy}}{e^{z}-2}$.

$$\frac{\partial^2 z}{\partial x^2}=\frac{-y^2e^{-xy}(e^{z}-2)-ye^{-xy}e^{z}\dfrac{\partial z}{\partial x}}{(e^{z}-2)^2}=-\frac{y^2(e^{2z}-4e^{z}+4+e^{z-xy})}{e^{xy}(e^{z}-2)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{(e^{-xy}-xye^{-xy})(e^{z}-2)-ye^{-xy}e^{z}\dfrac{\partial z}{\partial y}}{(e^{z}-2)^2}=\frac{(1+xy)(e^{z}-2)^2-xye^{z-xy}}{(e^{z}-2)^3e^{xy}},$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{x^2(e^{2z}-4e^{z}+4+e^{z-xy})}{e^{xy}(e^{z}-2)^2}.$$

（2）方程两边分别对x和y求偏导数，得

$$1+\frac{\partial z}{\partial x}=e^{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)，\quad 1+\frac{\partial z}{\partial y}=e^{x+y+z}\left(1+\frac{\partial z}{\partial y}\right)，$$

解得，$\dfrac{\partial z}{\partial x}=-1$，$\dfrac{\partial z}{\partial y}=-1$，由此得$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial y^2}=0$.

（3）方程两边分别对x和y求偏导数，得

$$yz+xy\frac{\partial z}{\partial x}=1+\frac{\partial z}{\partial x}，\quad xz+xy\frac{\partial z}{\partial y}=1+\frac{\partial z}{\partial y}，$$

解得，$\dfrac{\partial z}{\partial x}=\dfrac{1-yz}{xy-1}$，$\dfrac{\partial z}{\partial y}=\dfrac{1-xy}{xy-1}$.

$$\frac{\partial^2 z}{\partial x^2}=\frac{-y\frac{\partial z}{\partial x}(xy-1)-(1-yz)y}{(xy-1)^2}=\frac{2y(yz-1)}{(xy-1)^2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\left(-z-y\frac{\partial z}{\partial y}\right)(xy-1)-(1-yz)x}{(xy-1)^2}=\frac{2z}{(xy-1)^2},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{2x(xz-1)}{(xy-1)^2}.$$

（4）方程两边分别对 x 和 y 求偏导数，得

$$2x+2z\frac{\partial z}{\partial x}-2-4\frac{\partial z}{\partial x}=0,\quad 2y+2z\frac{\partial z}{\partial y}+2-4\frac{\partial z}{\partial y}=0,$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{1-x}{z-2},\quad \frac{\partial z}{\partial y}=\frac{-1-y}{z-2}.$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{-(z-2)-(1-x)\frac{\partial z}{\partial x}}{(z-2)^2}=-\frac{(z-2)^2+(1-x)^2}{(z-2)^3},$$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{(1-x)\frac{\partial z}{\partial y}}{(z-2)^2}=-\frac{(1-x)(1+y)}{(z-2)^3}=\frac{\partial^2 z}{\partial y\partial x},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{-(z-2)-(1+y)\frac{\partial z}{\partial y}}{(z-2)^2}=-\frac{(z-2)^2+(1+y)^2}{(z-2)^3}.$$

12．求由下列方程所确定的函数的全微分 dz：

（1）$z=f(xz,z-y)$；

（2）$F(x-y,y-z,z-x)=0$；

（3）$f(x+y+z,x^2+y^2+z^2)=0$；

（4）$f(x,y)+g(y,z)=0$.

解（1）$dz=f_1(xz,z-y)d(xz)+f_2(xz,z-y)d(z-y)$

$$=f_1(xz,z-y)(zdx+xdz)+f_2(xz,z-y)(dz-dy)$$

移项后，解得 $dz=\dfrac{zf_1(xz,z-y)dx-f_2(xz,z-y)dy}{1-xf_1(xz,z-y)-f_2(xz,z-y)}$.

（2）$F_1(x-y,y-z,z-x)(dx-dy)+F_2(x-y,y-z,z-x)(dy-dz)$

$$+F_3(x-y,y-z,z-x)(dz-dx)=0$$

$$\Rightarrow dz=\frac{F_3(x-y,y-z,z-x)-F_1(x-y,y-z,z-x)}{F_3(x-y,y-z,z-x)-F_2(x-y,y-z,z-x)}dx$$

$$+\frac{F_1(x-y,y-z,z-x)-F_2(x-y,y-z,z-x)}{F_3(x-y,y-z,z-x)-F_2(x-y,y-z,z-x)}dy\,.$$

（3） $f_1(x+y+z,x^2+y^2+z^2)(dx+dy+dz)$

$$+f_2(x+y+z,x^2+y^2+z^2)(2xdx+2ydy+2zdz)=0$$

$$\Rightarrow\quad dz=-\frac{f_1(x+y+z,x^2+y^2+z^2)+2xf_2(x+y+z,x^2+y^2+z^2)}{f_1(x+y+z,x^2+y^2+z^2)+2zf_2(x+y+z,x^2+y^2+z^2)}dx$$

$$-\frac{f_1(x+y+z,x^2+y^2+z^2)+2yf_2(x+y+z,x^2+y^2+z^2)}{f_1(x+y+z,x^2+y^2+z^2)+2zf_2(x+y+z,x^2+y^2+z^2)}dy\,.$$

（4） $f_1(x,y)dx+f_2(x,y)dy+g_1(y,z)dy+g_2(y,z)dz=0$，所以，

$$dz=-\frac{f_1(x,y)}{g_2(y,z)}dx-\frac{f_2(x,y)+g_1(y,z)}{g_2(y,z)}dy\,.$$

13．设 $z=z(x,y)$ 由方程

$$x^2+y^2+z^2=yf\left(\frac{z}{y}\right)$$

所确定，证明

$$(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}=2xz\,.$$

证明 方程 $x^2+y^2+z^2=yf\left(\dfrac{z}{y}\right)$ 两边分别对 x,y 求偏导数，得

$$2x+2z\frac{\partial z}{\partial x}=yf'\left(\frac{z}{y}\right)\frac{1}{y}\frac{\partial z}{\partial x},\qquad 2y+2z\frac{\partial z}{\partial y}=f\left(\frac{z}{y}\right)+yf'\left(\frac{z}{y}\right)\frac{\frac{\partial z}{\partial y}y-z}{y^2},$$

所以，$\frac{\partial z}{\partial x}=\frac{2x}{f'\left(\frac{z}{y}\right)-2z}$，$\frac{\partial z}{\partial y}=\frac{2y+\frac{z}{y}f'\left(\frac{z}{y}\right)-f\left(\frac{z}{y}\right)}{f'\left(\frac{z}{y}\right)-2z}$，

因此有，

$$(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}$$

$$=(x^2-y^2-z^2)\frac{2x}{f'\left(\frac{z}{y}\right)-2z}+2xy\frac{2y+\frac{z}{y}f'\left(\frac{z}{y}\right)-f\left(\frac{z}{y}\right)}{f'\left(\frac{z}{y}\right)-2z}$$

$$=\frac{2xzf'\left(\frac{z}{y}\right)-4xz^2}{f'\left(\frac{z}{y}\right)-2z}=2xz.$$

14．设 $z=x^2+y^2$，其中 $y=f(x)$ 为由方程 $x^2-xy+y^2=1$ 所确定的隐函数，求 $\frac{dz}{dx}$ 和 $\frac{d^2z}{dx^2}$．

解 $\frac{dz}{dx}=2x+2y\frac{dy}{dx}$，又在方程 $x^2-xy+y^2=1$ 两边对 x 求导，得

$$2x-y-x\frac{dy}{dx}+2y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{2x-y}{x-2y},$$

所以，$\frac{dz}{dx}=2x+2y\frac{2x-y}{x-2y}=\frac{2(x^2-y^2)}{x-2y}$，

$$\frac{\partial^2 z}{\partial x^2}=\frac{2\left(2x-2y\frac{dy}{dx}\right)(x-2y)-2(x^2-y^2)\left(1-2\frac{dy}{dx}\right)}{(x-2y)^2}$$

$$=\frac{2(5x^3-8x^2y+15xy^2-8y^3)}{(x-2y)^3}.$$

15．设 $u=x^2+y^2+z^2$，其中 $z=f(x,y)$ 为由方程 $x^3+y^3+z^3=3xyz$ 所确定的隐函数，求 $\frac{\partial u}{\partial x}$，$\frac{\partial^2 u}{\partial x^2}$．

解 $\dfrac{\partial u}{\partial x}=2x+2z\dfrac{\partial z}{\partial x}$，而由 $x^3+y^3+z^3=3xyz$ 两边对 x 求导，得

$$3x^2+3z^2\frac{\partial z}{\partial x}=3yz+3xy\frac{\partial z}{\partial x}\Rightarrow\frac{\partial z}{\partial x}=\frac{yz-x^2}{z^2-xy},$$

代入有，$\dfrac{\partial u}{\partial x}=2x+2z\dfrac{yz-x^2}{z^2-xy}=\dfrac{2(xy^2-x^2y+yz^2-x^2z)}{z^2-xy}$．所以，

$$\frac{\partial^2 u}{\partial x^2}=\frac{2\left(z^2+2xz\dfrac{\partial z}{\partial x}-2xy+2yz\dfrac{\partial z}{\partial x}-2xz-x^2\dfrac{\partial z}{\partial x}\right)(z^2-xy)}{(z^2-xy)^2}$$

$$-\frac{2(xz^2-x^2y+yz^2+x^2y)\left(2z\dfrac{\partial z}{\partial x}-y\right)}{(z^2-xy)^2}$$

$$=-\frac{2(x^5y+x^3y^3-2x^3y^2z+x^4z^2-3x^2y^2z^2+3xy^3z^2-4x^2yz^3+3xyz^4)}{(z^2-xy)^3}$$

$$-\frac{2(y^2z^4+2xz^5-z^6)}{(z^2-xy)^3}.$$

16．求下列方程组所确定的函数的导数或偏导数：

（1）$\begin{cases}x^2+y^2+z^2=a^2,\\x^2+y^2=ax,\end{cases}$ 求 $\dfrac{dy}{dx}$，$\dfrac{dz}{dx}$；

（2）$\begin{cases}x-u^2-yv=0,\\y-v^2-xu=0,\end{cases}$ 求 $\dfrac{\partial u}{\partial x}$，$\dfrac{\partial v}{\partial x}$，$\dfrac{\partial u}{\partial y}$，$\dfrac{\partial v}{\partial y}$；

（3）$\begin{cases}u^2-v=3x+y,\\u-2v^2=x-2y,\end{cases}$ 求 $\dfrac{\partial u}{\partial x}$，$\dfrac{\partial v}{\partial x}$，$\dfrac{\partial u}{\partial y}$，$\dfrac{\partial v}{\partial y}$；

（4）$\begin{cases}u=xyz,\\x^2+y^2+z^2=1,\end{cases}$ 求 $\dfrac{\partial^2 u}{\partial x^2}$，$\dfrac{\partial^2 u}{\partial y^2}$，$\dfrac{\partial^2 u}{\partial x\partial y}$．

解（1）$\begin{cases}2x+2yy'+2zz'=0,\\2x+2yy'=a,\end{cases}\Rightarrow\begin{cases}\dfrac{dy}{dx}=\dfrac{a-2x}{2y},\\\dfrac{dz}{dx}=-\dfrac{a}{2y}.\end{cases}$

（2）方程组两边对 x 求偏导数，有

$$\begin{cases} 1-2u\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=0, \\ -2v\dfrac{\partial v}{\partial x}-u-x\dfrac{\partial u}{\partial x}=0, \end{cases} \Rightarrow \frac{\partial u}{\partial x}=\frac{2v+yu}{4uv-xy}, \quad \frac{\partial v}{\partial x}=-\frac{2u^2+x}{4uv-xy};$$

两边对 y 求偏导数，有

$$\begin{cases} -2u\dfrac{\partial u}{\partial y}-v-y\dfrac{\partial v}{\partial y}=0, \\ 1-2v\dfrac{\partial v}{\partial y}-x\dfrac{\partial u}{\partial y}=0, \end{cases} \Rightarrow \frac{\partial u}{\partial y}=-\frac{2v^2+y}{4uv-xy}, \quad \frac{\partial v}{\partial y}=\frac{2u+xv}{4uv-xy}.$$

（3）方程组两边对 x 求偏导数，有

$$\begin{cases} 2u\dfrac{\partial u}{\partial x}-\dfrac{\partial v}{\partial x}=3, \\ \dfrac{\partial u}{\partial x}-4v\dfrac{\partial v}{\partial x}=1, \end{cases} \Rightarrow \frac{\partial u}{\partial x}=\frac{12v-1}{8uv-1}, \quad \frac{\partial v}{\partial x}=\frac{3-2u}{8uv-1};$$

两边对 y 求偏导数，有

$$\begin{cases} 2u\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial y}=1, \\ \dfrac{\partial u}{\partial y}-4v\dfrac{\partial v}{\partial y}=-2, \end{cases} \Rightarrow \frac{\partial u}{\partial y}=\frac{4v+2}{8uv-1}, \quad \frac{\partial v}{\partial y}=\frac{4u+1}{8uv-1}.$$

（4）方程组两边对 x 求偏导数，有

$$\begin{cases} \dfrac{\partial u}{\partial x}=yz+xy\dfrac{\partial z}{\partial x}, \\ 2x+2z\dfrac{\partial z}{\partial x}=0, \end{cases} \Rightarrow \frac{\partial z}{\partial x}=-\frac{x}{z}, \quad \frac{\partial u}{\partial x}=\frac{yz^2-x^2y}{z};$$

两边对 y 求偏导数，有

$$\begin{cases} \dfrac{\partial u}{\partial y}=xz+xy\dfrac{\partial z}{\partial y}, \\ 2y+2z\dfrac{\partial z}{\partial y}=0, \end{cases} \Rightarrow \frac{\partial z}{\partial y}=-\frac{y}{z}, \quad \frac{\partial u}{\partial y}=\frac{xz^2-xy^2}{z},$$

所以，$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{(2yz\dfrac{\partial z}{\partial x}-2xy)z-(yz^2-x^2y)\dfrac{\partial z}{\partial x}}{z^2}=-\dfrac{xy(x^2+3z^2)}{z^3}$,

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{(z^2+2yz\dfrac{\partial z}{\partial y}-x^2)z-(yz^2-x^2y)\dfrac{\partial z}{\partial y}}{z^2}=\frac{z^4-x^2z^2-y^2z^2-x^2y^2}{z^3},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{(2xz\frac{\partial z}{\partial y}-2xy)z-(xz^2-xy^2)\frac{\partial z}{\partial y}}{z^2}=-\frac{xy(y^2+3z^2)}{z^3}.$$

17．下列方程组定义 z 为 x,y 的函数，求 $\frac{dz}{dx}$，$\frac{dz}{dy}$．

（1）$\begin{cases}x=\cos\theta\cos\varphi,\\ y=\cos\theta\sin\varphi,\\ z=\sin\theta;\end{cases}$ （2）$\begin{cases}x=u+v,\\ y=u^2+v^2,\\ z=u^3+v^3.\end{cases}$

解（1）方程组两边对 x 求偏导数，有

$$\begin{cases}1=-\sin\theta\cos\varphi\frac{\partial\theta}{\partial x}-\cos\theta\sin\varphi\frac{\partial\varphi}{\partial x},\\ 0=-\sin\theta\sin\varphi\frac{\partial\theta}{\partial x}+\cos\theta\cos\varphi\frac{\partial\varphi}{\partial x},\\ \frac{\partial z}{\partial x}=\cos\theta\frac{\partial\theta}{\partial x},\end{cases}$$

从前两个方程解出 $\frac{\partial\theta}{\partial x}=-\frac{\cos\varphi}{\sin\theta}$，$\frac{\partial\varphi}{\partial x}=-\frac{\sin\varphi}{\cos\theta}$，代入第三个方程得 $\frac{\partial z}{\partial x}=-\cos\varphi\cot\theta$.

方程组两边对 y 求偏导数，有

$$\begin{cases}0=-\sin\theta\cos\varphi\frac{\partial\theta}{\partial y}-\cos\theta\sin\varphi\frac{\partial\varphi}{\partial y},\\ 1=-\sin\theta\sin\varphi\frac{\partial\theta}{\partial y}+\cos\theta\cos\varphi\frac{\partial\varphi}{\partial y},\\ \frac{\partial z}{\partial y}=\cos\theta\frac{\partial\theta}{\partial y},\end{cases}$$

从前两个方程解出 $\frac{\partial\theta}{\partial y}=-\frac{\sin\varphi}{\sin\theta}$，$\frac{\partial\varphi}{\partial y}=-\frac{\cos\varphi}{\cos\theta}$，代入第三个方程得 $\frac{\partial z}{\partial y}=\sin\varphi\cot\theta$.

（2）方程组两边对 x 求偏导数，有

$$\begin{cases}1=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x},\\ 0=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x},\\ \frac{\partial z}{\partial x}=3u^2\frac{\partial u}{\partial x}+3v^2\frac{\partial v}{\partial x},\end{cases}\Rightarrow\begin{cases}\frac{\partial u}{\partial x}=\frac{v}{v-u},\\ \frac{\partial u}{\partial x}=-\frac{u}{v-u},\\ \frac{\partial z}{\partial x}=-3uv=\frac{3}{2}(y-x^2).\end{cases}$$

同样，方程组两边对 y 求导，有

$$\begin{cases} 0 = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y}, \\ 1 = 2u\dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y}, \\ \dfrac{\partial z}{\partial x} = 3u^2\dfrac{\partial u}{\partial y} + 3v^2\dfrac{\partial v}{\partial y}, \end{cases} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{1}{2(v-u)}, \\ \dfrac{\partial v}{\partial y} = \dfrac{1}{2(v-u)}, \\ \dfrac{\partial z}{\partial y} = \dfrac{3}{2}(u+v) = \dfrac{3}{2}x. \end{cases}$$

§3 几何应用

1．求下列曲线在所示点处的切线方程和法平面方程：

（1） $x = a\sin^2 t$, $y = b\sin t\cos t$, $z = c\cos^2 t$, 在点 $t = \dfrac{\pi}{4}$；

（2） $2x^2 + 3y^2 + z^2 = 9$， $z^2 = 3x^2 + y^2$， 在点 $(1,-1,2)$；

（3） $x^2 + y^2 + z^2 = 6$， $x + y + z = 0$， 在点 $(1,-2,1)$；

（4） $x = t - \cos t$， $y = 3 + \sin^2 t$， $z = 1 + \cos t$， 在点 $t = \dfrac{\pi}{2}$.

解（1） $t = \dfrac{\pi}{4}$， 对应的点为 $(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2})$，

$x'(\dfrac{\pi}{4}) = 2a\sin t\cos t\Big|_{t=\frac{\pi}{4}} = a$， $y'(\dfrac{\pi}{4}) = b(\cos^2 t - \sin^2 t)\Big|_{t=\frac{\pi}{4}} = 0$，

$z'(\dfrac{\pi}{4}) = -2c\cos t\sin t\Big|_{t=\frac{\pi}{4}} = -c$，

因此曲线在 $t = \dfrac{\pi}{4}$ 对应的点 $(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2})$ 处的切线方程为

$$\frac{x - \frac{a}{2}}{a} = \frac{y - \frac{b}{2}}{0} = \frac{z - \frac{c}{2}}{-c},$$

法平面方程为： $ax - cz = \dfrac{1}{2}(a^2 - c^2)$.

（2）令 $F(x,y,z) = 2x^2 + 3y^2 + z^2 - 9$， $G(x,y,z) = 3x^2 + y^2 - z^2$， 这时曲线方程为

$$\begin{cases} F(x,y,z) = 0, \\ G(x,y,z) = 0, \end{cases}$$

由 $\begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{pmatrix} = \begin{pmatrix} 4x & 6y & 2z \\ 6x & 2y & -2z \end{pmatrix}$ 知

$$\frac{\partial(F,G)}{\partial(y,z)} = \begin{vmatrix} 6y & 2z \\ 2y & -2z \end{vmatrix} = -16yz, \quad \frac{\partial(F,G)}{\partial(z,x)} = \begin{vmatrix} 2z & 4x \\ -2z & 6x \end{vmatrix} = 20xz,$$

$$\frac{\partial(F,G)}{\partial(x,y)}=\begin{vmatrix}4x & 6y\\6x & 2y\end{vmatrix}=-28xy,$$

因此，曲线在点$(1,-1,2)$处的切向量为

$$\bar{\tau}=(-16yz,20xz,-28xy)\big|_{(1,-1,2)}=4(8,10,7),$$

故曲线在点$(1,-1,2)$处的切线方程为

$$\frac{x-1}{8}=\frac{y+1}{10}=\frac{z-2}{7},$$

法平面方程为$8(x-1)+10(y+1)+7(z-2)=0$，即$8x+10y+7z=12$.

（3）令$F(x,y,z)=x^2+y^2+z^2-6$，$G(x,y,z)=x+y+z$，

由$\begin{pmatrix}F_x & F_y & F_z\\G_x & G_y & G_z\end{pmatrix}=\begin{pmatrix}2x & 2y & 2z\\1 & 1 & 1\end{pmatrix}$，知

$$\frac{\partial(F,G)}{\partial(y,z)}=\begin{vmatrix}2y & 2z\\1 & 1\end{vmatrix}=2(y-z),\quad \frac{\partial(F,G)}{\partial(z,x)}=\begin{vmatrix}2z & 2x\\1 & 1\end{vmatrix}=2(z-x),$$

$$\frac{\partial(F,G)}{\partial(x,y)}=\begin{vmatrix}2x & 2y\\1 & 1\end{vmatrix}=2(x-y),$$

因此曲线在$(1,-2,1)$处的切向量为$\bar{\tau}=2(-3,0,3)=6(-1,0,1)$，故曲线在$(1,-2,1)$处的切线方程为

$$\frac{x-1}{-1}=\frac{y+2}{0}=\frac{z-1}{1},$$

法平面方程为：$-1(x-1)+(z-1)=0$，即$x-z=0$.

（4）$t=\frac{\pi}{2}$对应点为$(\frac{\pi}{2},4,1)$，$x'(\frac{\pi}{2})=(1+\sin t)\big|_{t=\frac{\pi}{2}}=2$，

$$y'(\frac{\pi}{2})=2\sin t\cos t\big|_{t=\frac{\pi}{2}}=0,\quad z'(\frac{\pi}{2})=-3\sin 3t\big|_{t=\frac{\pi}{2}}=3,$$

切向量为$\bar{\tau}=(2,0,3)$，切线方程

$$\frac{x-\frac{\pi}{2}}{2}=\frac{y-4}{0}=\frac{z-1}{3},$$

法平面方程为：$2(x-\frac{\pi}{2})+3(z-1)=0$，即$2x+3z=\pi+3$.

2．求下列曲面在所示点处的切平面方程和法线方程：

（1）$y-e^{2x-z}=0$，在点$(1,1,2)$；

（2）$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$在点$(\dfrac{a}{\sqrt{3}},\dfrac{b}{\sqrt{3}},\dfrac{c}{\sqrt{3}})$；

（3）$z=2x^2+4y^2$在点$(2,1,12)$；

（4）$x=u\cos v,\ y=u\sin v,\ z=av$在点$p_0(u_0,v_0)$.

解（1）令$F(x,y,z)=y-e^{2x-z}$，则法向量

$$\vec{n}=(F_x,F_y,F_z)\big|_{(1,1,2)}=(-2e^{2x-z},1,e^{2x-z})\big|_{(1,1,2)}=(-2,1,1),$$

故所求的切平面方程为：$-2(x-1)+(y-1)+(z-2)=0$ 即$-2x+y+z=1$，法线方程为：$\dfrac{x-1}{-2}=\dfrac{y-1}{1}=\dfrac{z-2}{1}$.

（2）令$F(x,y,z)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1$，就有法向量

$$\vec{n}=(F_x,F_y,F_z)\big|_{(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})}=(\frac{2x}{a^2},\frac{2x}{b^2},\frac{2z}{c^2})\Big|_{(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})}=\frac{2}{\sqrt{3}}(\frac{1}{a},\frac{1}{b},\frac{1}{c}),$$

所求的切平面方程$\dfrac{1}{a}(x-\dfrac{a}{\sqrt{3}})+\dfrac{1}{b}(y-\dfrac{b}{\sqrt{3}})+\dfrac{1}{c}(z-\dfrac{c}{\sqrt{3}})=0$，即$\dfrac{1}{a}x+\dfrac{1}{b}y+\dfrac{1}{c}z=\sqrt{3}$，

法线方程为$\dfrac{x-\frac{a}{\sqrt{3}}}{\frac{1}{a}}=\dfrac{y-\frac{b}{\sqrt{3}}}{\frac{1}{b}}=\dfrac{z-\frac{c}{\sqrt{3}}}{\frac{1}{c}}$.

（3）设$F(x,y,z)=2x^2+4y^2-z$，则法向量$\vec{n}=(4x,8y,-1)\big|_{(2,1,12)}=(8,8,-1)$，所求的切平面方程为$8(x-2)+8(y-1)-(z-12)=0$，即$8x+8y-z=12$ $8x+8y-z=12$，法线方程为$\dfrac{x-2}{8}=\dfrac{y-1}{8}=\dfrac{z-12}{-1}$.

（4）由$\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v}\end{pmatrix}=\begin{pmatrix}\cos v & \sin v & 0\\ -u\sin v & u\cos v & a\end{pmatrix}$，知

$$\left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{p_0}=\left.\begin{vmatrix}\sin v & 0\\ u\cos v & a\end{vmatrix}\right|_{p_0}=a\sin v_0,\quad \left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{p_0}=\left.\begin{vmatrix}0 & \cos v\\ a & -u\sin v\end{vmatrix}\right|_{p_0}=-a\cos v_0,$$

$$\left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{p_0}=\left.\begin{vmatrix}\cos v & \sin v\\ -u\sin v & u\cos v\end{vmatrix}\right|_{p_0}=u_0,$$

从而 $\vec{n}=(a\sin v_0,-a\cos v_0,u_0)$，又 $p_0(u_0,v_0)$ 对应的点

$$(x,y,z)=(u_0\cos v_0,u_0\sin v_0,av_0),$$

故所求的切平面方程为：

$$a\sin v_0(x-u_0\cos v_0)-a\cos v_0(y-u_0\sin v_0)+u_0(z-av_0)=0,$$

即 $ax\sin v_0-ay\cos v_0+u_0z=au_0v_0$，法线方程为：

$$\frac{x-u_0\cos v_0}{a\sin v_0}=\frac{y-u_0\sin v_0}{a\cos v_0}=\frac{z-av_0}{u_0}.$$

3．证明由曲线 $x=ae^t\cos t$，$y=ae^t\sin t$，$z=ae^t$ 与锥面 $x^2+y^2=z^2$ 的母线相交成同一角度.

证明 圆锥 $x^2+y^2=z^2$ 的顶点在原点，过圆锥上任一点 $P(x,y,z)$ 的母线也过原点，因此，母线的方向向量为 $\vec{\tau}_1=(x,y,z)$，曲线在点 P 的切向量

$$\vec{\tau}_2=(x'(t),y'(t),z'(t))=(ae^t(\cos t-\sin t),ae^t(\sin t-\cos t),ae^t)=(x-y,x+y,z),$$

注意到 $x^2+y^2=z^2$，得

$$\cos(\vec{\tau}_1,\vec{\tau}_2)=\frac{\vec{\tau}_1\cdot\vec{\tau}_2}{|\vec{\tau}_1||\vec{\tau}_2|}=\frac{x(x-y)+y(x+y)+z^2}{\sqrt{x^2+y^2+z^2}\sqrt{(x-y)^2+(x+y)^2+z^2}}=\frac{2z^2}{\sqrt{2z^2}\sqrt{3z^2}}=\frac{\sqrt{6}}{3},$$

即交角相同.

4．求平面曲线 $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\ (a>0)$ 上任一点处的切线方程，并证明这些切线被坐标轴所截取的线段等长.

解 在 $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\ (a>0)$ 两边对 x 求导，有 $\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$，

在曲线上任一点 (x_0,y_0)，有切线斜率 $k=-\sqrt[3]{\frac{y_0}{x_0}}\ (x_0\neq 0)$，切线方程

$$y-y_0=-\sqrt[3]{\frac{y_0}{x_0}}(x-x_0).$$

切线与两坐标轴交点为 $A(x_0^{\frac{1}{3}}a^{\frac{2}{3}},0)$，$B(0,y_0^{\frac{1}{3}}a^{\frac{2}{3}})$，故

$$d_{AB}=(x_0^{\frac{2}{3}}a^{\frac{4}{3}}+y_0^{\frac{2}{3}}a^{\frac{4}{3}})^{\frac{1}{2}}=(a^{\frac{4}{3}}a^{\frac{2}{3}})^{\frac{1}{2}}=a,$$

即这些切线被坐标轴所截取的线段等长为 a.

5．求曲面 $x^2+2y^2+3z^2=21$ 的切平面，使它平行于平面 $x+4y+6z=0$.

解 曲面 $x^2+2y^2+3z^2=21$ 上任一点 $P_0(x_0,y_0,z_0)$ 的切平面的法向量为

$$\vec{n}=(2x_0,4y_0,6z_0),$$

要使它平行于平面 $x+4y+6z=0$，即有 $\dfrac{2x_0}{1}=\dfrac{4y_0}{4}=\dfrac{6z_0}{6}$，即：$2x_0=y_0=z_0$，

又 $P_0(x_0,y_0,z_0)$ 在曲面上，故 $x_0^2+2y_0^2+3z_0^2=21$，得到 $P_0(\pm1,\pm2,\pm2)$，故所求切平面方程为

$$1\cdot(x\mp1)+4(y\mp2)+6(z\mp2)=0,$$

或 $x+4y+6z=\pm21$ 为所求切平面方程.

6．证明：曲面 $F(x-az,y-bz)=0$ 的切平面与某一定直线平行，其中 a,b 为常数.

证明 $G(x,y,z)=F(x-az,y-bz)$，则曲面为 $G(x,y,z)=0$，曲面上任意一点 $P_0(x_0,y_0,z_0)$ 处的切平面的法向量为 $\vec{n}=(G_x,G_y,G_z)\big|_{P_0}=(F_1,F_2,-aF_1-bF_2)\big|_{P_0}$.

由于 $\vec{n}\cdot(a,b,1)=0$，故 $\vec{n}\perp(a,b,1)$，故曲面过 P_0 点的切平面平行于方向向量为 $\vec{\tau}=(a,b,1)$ 的直线，因而曲面 $F(x-az,y-bz)=0$ 的切平面与一方向向量为 $\vec{\tau}=(a,b,1)$ 的直线平行.

7．证明曲面 $z=xe^{\frac{x}{y}}$ 的每一切平面都通过原点.

证明 设 $F(x,y,z)=xe^{\frac{x}{y}}-z$，则曲面在任一点 $P_0(x_0,y_0,z_0)$ 处的切平面的法向量为

$$\vec{n}=(F_x,F_y,F_z)\big|_{P_0}=(e^{\frac{x}{y}}+xe^{\frac{x}{y}}\frac{1}{y},xe^{\frac{x}{y}}(-\frac{x}{y^2}),-1)\Big|_{P_0}=(e^{\frac{x_0}{y_0}}+\frac{x_0}{y_0}e^{\frac{x_0}{y_0}},-\frac{x_0^2}{y_0^2}e^{\frac{x_0}{y_0}},-1),$$

切平面为

$$e^{\frac{x_0}{y_0}}(1+\frac{x_0}{y_0})(x-x_0)-\frac{x_0^2}{y_0^2}e^{\frac{x_0}{y_0}}(y-y_0)-(z-z_0)=0,$$

注意到$z_0 = x_0 e^{\frac{x_0}{y_0}}$，化简即得，

$$(1+\frac{x_0}{y_0})e^{\frac{x_0}{y_0}}x - \frac{x_0^2}{y_0^2}e^{\frac{x_0}{y_0}}y - z = 0,$$

所以切平面都通过原点.

8．求两曲面

$$F(x,y,z)=0，\quad G(x,y,z)=0$$

的交线在Oxy平面上的投影曲线的切线方程.

解 空间曲线$\begin{cases}F(x,y,z)=0,\\G(x,y,z)=0\end{cases}$在任一点$P_0(x_0,y_0,z_0)$处的切线方程为:

$$\frac{x-x_0}{\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_{P_0}} = \frac{y-y_0}{\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_{P_0}} = \frac{z-z_0}{\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_{P_0}},$$

因此，两曲面$F(x,y,z)=0$，$G(x,y,z)=0$的交线在Oxy平面上的投影曲线的切线方程为

$$\begin{cases}\dfrac{x-x_0}{\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_{P_0}} = \dfrac{y-y_0}{\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_{P_0}},\\ z=0.\end{cases}$$

§4 方向导数

1．设$f(x,y,z)=x+y^2+z^3$，求f在点$P_0(1,1,1)$沿方向$l=(2,-2,1)$的方向导数.

解 由于$\frac{\partial f}{\partial x}=1$，$\frac{\partial f}{\partial y}=2y$，$\frac{\partial f}{\partial z}=3z^2$，故$\left.(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})\right|_{P_0}=(1,2,3)$，

$l_0=\frac{1}{|l|}l=\frac{1}{3}(2,-2,1)$，所以$\left.\frac{\partial f}{\partial l}\right|_{P_0(1,1,1)}=(1,2,3)\cdot\frac{1}{3}(2,-2,1)=\frac{1}{3}$.

2．求函数$u=xyz$在点$A(5,1,2)$处沿点$B(9,4,14)$的方向$\overrightarrow{AB}$上的方向导数.

解 $\frac{\partial u}{\partial x}=yz$，$\frac{\partial u}{\partial y}=xz$，$\frac{\partial u}{\partial z}=xy$，故

$$\left.(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})\right|_A=(2,10,5)，\quad l_0=\frac{1}{|\overrightarrow{AB}|}\overrightarrow{AB}=\frac{1}{13}(4,3,12),$$

所以，$\left.\dfrac{\partial u}{\partial \overrightarrow{AB}}\right|_{A(5,1,2)}=(2,10,5)\cdot\dfrac{1}{13}(4,3,12)=\dfrac{98}{13}$.

3．求 $\left.\dfrac{\partial u}{\partial l}\right|_{(x_0,y_0)}$：

（1）$u=\ln(x^2+y^2),(x_0,y_0)=(1,1)$，$l$ 与 x 轴正向的夹角为 60^0；

（2）$u=xe^{xy}$，$(x_0,y_0)=(1,1)$，l 与向量 $(1,1)$ 同向.

解（1）$\dfrac{\partial u}{\partial x}=\dfrac{2x}{x^2+y^2}$，$\dfrac{\partial u}{\partial y}=\dfrac{2y}{x^2+y^2}$，所以，

$$\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)_{(x_0,y_0)}=\left(\frac{2x}{x^2+y^2},\frac{2y}{x^2+y^2}\right)_{(x_0,y_0)}=(1,1),$$

$$l_0=(\cos 60^0,\sin 60^0)=(\frac{1}{2},\frac{\sqrt{3}}{2}),$$

所以，

$$\left.\frac{\partial u}{\partial l}\right|_{(x_0,y_0)}=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)_{(1,1)}\cdot l_0=(1,1)\cdot(\frac{1}{2},\frac{\sqrt{3}}{2})=\frac{1+\sqrt{3}}{2}.$$

（2）$\dfrac{\partial u}{\partial x}=e^{xy}(1+xy)$，$\dfrac{\partial u}{\partial y}=x^2e^{xy}$，因此，

$$\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)_{(x_0,y_0)}=\left(e^{xy}(1+xy),x^2e^{xy}\right)_{(1,1)}=(2e,e),$$

$$l_0=\frac{1}{\sqrt{2}}(1,1),$$

所以，

$$\left.\frac{\partial u}{\partial l}\right|_{(1,1)}=(2e,e)\cdot\frac{1}{\sqrt{2}}(1,1)=\frac{3\sqrt{2}}{2}e.$$

4．设函数 $f(x,y)$ 在 (x_0,y_0) 可微，单位向量 $l_1=(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}),l_2=(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}})$，

$\dfrac{\partial f(x_0,y_0)}{\partial l_1}=1,\dfrac{\partial f(x_0,y_0)}{\partial l_2}=0$，确定 l 使得：$\dfrac{\partial f(x_0,y_0)}{\partial l}=\dfrac{7}{5\sqrt{2}}$.

解 由于

$$\begin{cases} 1=\dfrac{\partial f(x_0,y_0)}{\partial l_1}=\dfrac{\partial f(x_0,y_0)}{\partial x}\dfrac{1}{\sqrt{2}}+\dfrac{\partial f(x_0,y_0)}{\partial y}\dfrac{1}{\sqrt{2}}, \\ 0=\dfrac{\partial f(x_0,y_0)}{\partial l_2}=\dfrac{\partial f(x_0,y_0)}{\partial x}(-\dfrac{1}{\sqrt{2}})+\dfrac{\partial f(x_0,y_0)}{\partial y}\dfrac{1}{\sqrt{2}}, \end{cases},$$

解出，$\dfrac{\partial f(x_0,y_0)}{\partial x}=\dfrac{\partial f(x_0,y_0)}{\partial y}=\dfrac{\sqrt{2}}{2}$.

由$\dfrac{\partial f(x_0,y_0)}{\partial l}=\dfrac{7}{5\sqrt{2}}$，即$\dfrac{\sqrt{2}}{2}\alpha+\dfrac{\sqrt{2}}{2}\beta=\dfrac{7}{5\sqrt{2}}\Rightarrow\alpha+\beta=\dfrac{7}{5}$，再由$\alpha^2+\beta^2=1$，推出$(\alpha,\beta)=(\dfrac{4}{5},\dfrac{3}{5})$或$(\alpha,\beta)=(\dfrac{3}{5},\dfrac{4}{5})$，即$l=(\dfrac{4}{5},\dfrac{3}{5})$或$l=(\dfrac{3}{5},\dfrac{4}{5})$.

5．设 f 在 $P_0(2,0)$ 可微，$f(x,y)$ 在 P_0 指向 $P_1=(2,-2)$ 的方向导数是1，指向原点的方向导数是-3，试回答：

（1）指向 $P_2=(2,1)$ 方向导数是多少？

（2）指向 $P_3=(3,2)$ 方向导数是多少？

解 由$\begin{cases} 1=\dfrac{\partial f(2,0)}{\partial x}\cdot 0+\dfrac{\partial f(2,0)}{\partial y}\cdot(-1), \\ -3=\dfrac{\partial f(2,0)}{\partial x}\cdot(-1)+\dfrac{\partial f(2,0)}{\partial y}\cdot 0, \end{cases}\Rightarrow\begin{cases} \dfrac{\partial f(2,0)}{\partial x}=3, \\ \dfrac{\partial f(2,0)}{\partial y}=-1. \end{cases}$

（1）指向 $P_2=(2,1)$ 方向导数为

$$\frac{\partial f(2,0)}{\partial x}\cdot 0+\frac{\partial f(2,0)}{\partial y}\cdot 1=-1.$$

（2）指向 $P_3=(3,2)$ 方向导数为

$$\frac{\partial f(2,0)}{\partial x}\cdot\frac{1}{\sqrt{5}}+\frac{\partial f(2,0)}{\partial y}\cdot\frac{2}{\sqrt{5}}=\frac{\sqrt{5}}{5}.$$

§5 Taylor 公式

1．写出下列函数在指定点的 Taylor 公式：

（1） $f(x,y)=2x^2-xy-y^2-6x-3y+5$ 在 $(1,-2)$ 点；

（2） $f(x,y)=x^2+xy+y^2+3x-2y+4$ 在 $(-1,1)$ 点.

解（1）$\frac{\partial f}{\partial x}=4x-y-6$，$\frac{\partial f}{\partial y}=-x-2y-3$，$\frac{\partial^2 f}{\partial x^2}=4$，$\frac{\partial^2 f}{\partial x\partial y}=-1$，$\frac{\partial^2 f}{\partial y^2}=-2$，高于二阶偏导数均为$0$．在$(1,-2)$点，$f(1,-2)=5$，$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$，$\frac{\partial^2 f}{\partial x^2}=4$，$\frac{\partial^2 f}{\partial x\partial y}=-1$，$\frac{\partial^2 f}{\partial y^2}=-2$，所以，

$$f(x,y)=2(x-1)^2-(x-1)(y+2)-(y+2)^2+5.$$

（2）$\frac{\partial f}{\partial x}=2x+y+3$，$\frac{\partial f}{\partial y}=x+2y-2$，$\frac{\partial^2 f}{\partial x^2}=2$，$\frac{\partial^2 f}{\partial x\partial y}=1$，$\frac{\partial^2 f}{\partial y^2}=2$，高于二阶偏导数均为$0$．在点$(-1,1)$，有$f(-1,1)=0$，$\frac{\partial f}{\partial x}=2$，$\frac{\partial f}{\partial y}=-1$，$\frac{\partial^2 f}{\partial x^2}=2$，$\frac{\partial^2 f}{\partial x\partial y}=1$，$\frac{\partial^2 f}{\partial y^2}=2$，所以，

$$f(x,y)=2(x+1)-(y-1)+(x+1)^2+(x+1)(y-1)+(y-1)^2.$$

2．求函数$f(x,y)=\frac{x}{y}$在$(1,1)$点邻域的n阶带 Lagrange 余项的 Taylor 公式.

解 $f(x,y)=x\frac{1}{1+(y-1)}=[(x-1)+1]\frac{1}{1+(y-1)}$

$$=[(x-1)+1]\{\sum_{k=0}^{n}(-1)^k(y-1)^k+(-1)^{n+1}\frac{1}{[1+\theta(y-1)]^{n+2}}(y-1)^{n+1}\}.$$

3．求函数$f(x,y)=\frac{y^2}{x^2}$在$(1,-1)$点邻域的二阶 Taylor 公式，并写出 Lagrange 余项.

解 $f(x,y)=\frac{y^2}{x^2}=[(y+1)-1]^2\frac{1}{[1+(x-1)]^2}$

$$=[(y+1)-1]^2\{1-2(x-1)+3(x-1)^2-4[1+\theta(x-1)]^{-5}(x-1)^3\},$$

$$0<\theta<1.$$

4．求下列函数在$(0,0)$点邻域的四阶 Taylor 公式：

（1）$f(x,y)=\sin(x^2+y^2)$；

（2） $f(x,y)=e^x\ln(1+y)$；

（3） $f(x,y)=\sqrt{1+x^2+y^2}$；

（4） $f(x,y)=e^x\cos y$.

解（1） $f(x,y)=(x^2+y^2)-\dfrac{\cos[\theta(x^2+y^2)]}{3!}(x^2+y^2)^3$ ， $0<\theta<1$.

（2） $f(x,y)=(1+x+\dfrac{1}{2!}x^2+\dfrac{1}{3!}x^3+o(x^3))(y-\dfrac{y^2}{2}+\dfrac{y^3}{3}-\dfrac{y^4}{4}+o(y^4))$

$=y+(xy-\dfrac{y^2}{2})+(\dfrac{1}{2}x^2y-\dfrac{1}{2}xy^2+\dfrac{y^3}{3})+(\dfrac{1}{6}x^3y-\dfrac{1}{4}x^2y^2+\dfrac{1}{3}xy^3-\dfrac{1}{4}y^4)$

$+o(x^4+x^3y+x^2y^2+xy^2+y^4)$.

（3） $f(x,y)=(1+x^2+y^2)^{-\frac{1}{2}}=1-\dfrac{1}{2}(x^2+y^2)+\dfrac{3}{8}(x^2+y^2)^2+o((x^2+y^2)^2)$.

（4） $f(x,y)=(1+x+\dfrac{1}{2!}x^2+\dfrac{1}{3!}x^3+\dfrac{1}{4!}x^4+o(x^4))(1-\dfrac{y^2}{2!}+\dfrac{y^4}{4!}+o(y^4))$

$=1+x+(\dfrac{1}{2}x^2-\dfrac{1}{2}y^2)+(\dfrac{1}{6}x^3-\dfrac{xy^2}{2})+(\dfrac{1}{24}x^4-\dfrac{1}{4}x^2y^2+\dfrac{1}{24}y^4)$

$+o(x^4+x^3y+x^2y^2+xy^3+y^4)$.

5．证明 Taylor 公式的唯一性：若

$$\sum_{i+j=0}^{n}A_{ij}x^iy^j+o(\rho^n)\ (\rho\to 0),$$

其中 $\rho=\sqrt{x^2+y^2}$. 求证 $A_{ij}=0$（$i,\ j$ 为非负数，$i+j=0,1,\cdots,n$），并利用唯一性求 $f(x,y)=\ln(1+x+y)$ 带 Lagrange 余项的 n 阶 Taylor 展开式.

证明 $f(x,y)=0$，则 $\dfrac{\partial^{i+j}f}{\partial x^i\partial y^j}=0$（$i,\ j$ 为非负数，$i+j=0,1,\cdots,n$），故

$$0=f(x,y)=\sum_{i+j=0}^{n}\frac{\partial^{i+j}f(0,0)}{\partial x^i\partial y^j}x^iy^j+o(\rho^n)=\sum_{i+j=0}^{n}A_{ij}x^ix^j+o(\rho^n),$$

所以 $A_{ij}=0$（$i,\ j$ 为非负数，$i+j=0,1,\cdots,n$），因而 Taylor 公式是唯一的.

$$f(x,y)=\ln(1+x+y)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}(x+y)^k+o(\rho^n)$$

$$=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\sum_{i=1}^{k}C_k^i x^{k-i}y^i+\frac{(-1)^n}{n+1}\frac{(x+y)^{n+1}}{[1+\theta(x+y)]^{n+1}}，\ 0<\theta<1.$$

6．通过对 $f(x,y)=\sin x\cos y$ 用中值定理，证明存在 $\theta\in(0,1)$，使

$$\frac{3}{4}=\frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}.$$

证明　$f(0,0)=0$，$f_x(x,y)=\cos x\cos y$，$f_y(x,y)=-\sin x\sin y$，

$$f(x,y)=\sin x\cos y=f(0,0)+f_x(\theta x,\theta y)x+f_y(\theta x,\theta y)y$$

$$=x\cos\theta x\cos\theta y-y\sin\theta x\sin\theta y\ （0<\theta<1），$$

因而当 $x=\frac{\pi}{3}$，$y=\frac{\pi}{6}$ 时，有 $f(\frac{\pi}{3},\frac{\pi}{6})=\sin\frac{\pi}{3}\cos\frac{\pi}{6}=\frac{3}{4}$，从而存在 $\theta\in(0,1)$，使

$$\frac{3}{4}=\frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}.$$

7. 设 $f(x,y)$ 在区域 D 内有偏导数存在，且 $f_x(x,y)=f_y(x,y)\equiv 0$. 证明 $f(x,y)$ 在 D 内为常数.

证明　对 $f(x,y)$ 用中值定理有（$(x_0,y_0)\in D$ 是固定的一点），

$$f(x,y)=f(x_0,y_0)+f_x(x_0+\theta(x-x_0),y_0+\theta(y-y_0))(x-x_0)$$

$$+f_y(x_0+\theta(x-x_0),y_0+\theta(y-y_0))(y-y_0)，\ 0<\theta<1$$

对一切 $(x_0,y_0)\in D$ 成立，显然对不同的 $(x,y)\in D$，θ 与 (x,y) 有关，所以，

$$f(x,y)=f(x_0,y_0)，\ \forall(x,y)\in D，$$

因而 $f(x,y)$ 在 D 内为常数.

8．若 $|x|$，$|y|$ 是很小的量，导出下列函数准确到二次项的近似公式：

（1）$\dfrac{\cos x}{\cos y}$；　　（2）$\arctan\dfrac{1+x+y}{1-xy}$.

解（1）$f(x,y)=\dfrac{\cos x}{\cos y}$，$f(0,0)=1$，$f_x(x,y)=-\dfrac{\sin x}{\cos y}$，$f_x(0,0)=0$，

$f_y(x,y)=-\dfrac{\cos x\sin y}{\cos^2 y}$，$f_y(0,0)=0$，$f_{x^2}(x,y)=-\dfrac{\cos x}{\cos y}$，$f_{x^2}(0,0)=-1$，

$f_{xy}(x,y)=-\dfrac{\sin x\sin y}{\cos^2 y}$，$f_{xy}(0,0)=0$，$f_{y^2}(x,y)=-\dfrac{\cos x(1+\sin^2 y)}{\cos^3 y}$，$f_{y^2}(0,0)=1$，

所以当$|x|$，$|y|$很小时，有

$$f(x,y)=\frac{\cos x}{\cos y}$$

$$\approx f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}[f_{x^2}(0,0)x^2+2f_{xy}(0,0)xy+f_{y^2}(0,0)y^2]$$

$$=1+\frac{1}{2}(-x^2+y^2).$$

（2）令 $f(x,y)=\arctan\dfrac{1+x+y}{1-xy}$，则 $f(0,0)=\dfrac{\pi}{4}$，

$f_x(x,y)=\dfrac{1+y+y^2}{2+2x+2y+x^2+y^2+x^2y^2}$，$f_x(0,0)=\dfrac{1}{2}$，

$f_y(x,y)=\dfrac{1+x+x^2}{2+2x+2y+x^2+y^2+x^2y^2}$，$f_y(0,0)=\dfrac{1}{2}$，

$f_{x^2}(x,y)=-\dfrac{2(1+2y+y^2)(1+x+xy^2)}{2+2x+2y+x^2+y^2+x^2y^2}$，$f_{x^2}(0,0)=-\dfrac{1}{2}$，

$f_{xy}(x,y)=\dfrac{(1+2y)(2+2x+2y+x^2+y^2+x^2y^2)-(1+y+y^2)(2+2y+2x^2y)}{(2+2x+2y+x^2+y^2+x^2y^2)^2}$，

$f_{xy}(0,0)=0$，$f_{y^2}(x,y)=-\dfrac{2(1+x+x^2)(1+yx^2y)}{(2+2x+2y+x^2+y^2+x^2y^2)^2}$，$f_{y^2}(0,0)=-\dfrac{1}{2}$.

所以当$|x|$，$|y|$很小时，有

$$f(x,y)=\arctan\frac{1+x+y}{1-xy}$$

$$\approx f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2!}[f_{x^2}(0,0)x^2+2f_{xy}(0,0)xy+f_{y^2}(0,0)y^2]$$

$$=\frac{\pi}{4}+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}(-\frac{1}{2}x^2-\frac{1}{2}y^2)=\frac{\pi}{4}+\frac{x+y}{2}-\frac{x^2+y^2}{2}.$$

9．设函数 $f(x,y)$ 有直到 n 阶连续偏导数，试证 $u(t)=f(a+ht,b+kt)$ 的 n 阶导数

$$u^{(n)}(t)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^n f(a+ht,b+kt)\,.$$

证明 $u'(t)=\frac{\partial f}{\partial x}(x+ht,y+kt)h+\frac{\partial f}{\partial y}(x+ht,y+kt)k$

$$=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^1 f(x+ht,y+kt)\,,$$

假设$u^{(n)}(t)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^n f(x+ht,y+kt)$，则

$$u^{(n+1)}(t)=[u^{(n)}(t)]'=[(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^n f(x+ht,y+kt)]'$$

$$=\left[\sum_{m=0}^{n}C_n^m\frac{\partial^n f(a+ht,b+kt)}{\partial x^{n-m}\partial y^m}h^{n-m}k^m\right]'$$

$$=\sum_{m=0}^{n}C_n^m[\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n-m+1}\partial y^m}h+\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n-m}\partial y^{m+1}}k]h^{n-m}k^m$$

$$=\sum_{m=0}^{n}C_n^m\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n-m+1}\partial y^m}h^{n-m+1}k^m+\sum_{m=0}^{n}\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n-m}\partial y^{m+1}}h^{n-m}k^{m+1}$$

$$=C_n^0\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n+1}}h^{n+1}+\sum_{m=1}^{n}(C_n^m+C_n^{m-1})\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n+1-m}\partial y^m}h^{n+1-m}k^m$$

$$+C_n^n\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial y^{n+1}}k^{n+1}$$

$$=C_{n+1}^0\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n+1}}h^{n+1}+\sum_{m=1}^{n}C_{n+1}^m\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n+1-m}\partial y^m}h^{n+1-m}k^m$$

$$+C_{n+1}^{n+1}\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial y^{n+1}}k^{n+1}$$

$$=\sum_{m=0}^{n+1}C_{n+1}^m\frac{\partial^{n+1} f(a+ht,b+kt)}{\partial x^{n+1-m}\partial y^m}h^{n+1-m}k^m=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1} f(a+ht,b+kt)\,,$$

由归纳法原理，知所证等式成立.

10．设 $f(x,y)$ 为 n 次齐次函数，证明

$$(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^m f=n(n-1)\cdots(n-m+1)f\,.$$

证明 设 $f(x,y)$ 为 n 次齐次函数，则 $\forall t \in R^{+}$，有 $f(tx,ty)=t^{n}f(x,y)$．两边求对 t 求导，有

$$\frac{\partial}{\partial x}f(tx,ty)x+\frac{\partial}{\partial y}f(tx,ty)y=nt^{n-1}f(x,y),$$

上式两边再对 t 求导，有

$$(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^{2}f(tx,ty)=n(n-1)t^{n-2}f(x,y).$$

假设 $(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^{k}f(tx,ty)=n(n-1)\cdots(n-k+1)t^{n-k}f(x,y)$，则该式两边再对 t 求导，有

$$(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^{k+1}f(tx,ty)=n(n-1)\cdots(n-k+1)(n-k)t^{n-k}f(x,y),$$

令 $t=1$，对一切 $m=0,1,2,\cdots,n$，有

$$(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^{m}f=n(n-1)\cdots(n-m+1)f.$$

11．设 $f(x,y)=\psi(ax+by)$，其中 a,b 为常数，在包含原点的某邻域内，ψ 有 q 阶连续导数．求证：在 $(0,0)$ 点邻域的 Taylor 公式是

$$f(x,y)=\sum_{k=0}^{q-1}\frac{\psi^{(k)}(0)}{k!}\sum_{j=0}^{k}C_{k}^{j}(ax)^{j}(by)^{k-j}+R_{q}(x,y).$$

证明 $\frac{\partial^{k}f}{\partial x^{j}\partial y^{k-j}}=\psi^{(k)}(ax+by)a^{j}b^{k-j}$， $j=0,1,2,\cdots,k$， $k=0,1,2,\cdots,q-1$，

所以 $\frac{\partial^{k}f(0,0)}{\partial x^{i}\partial y^{k-j}}=\psi^{(k)}(0)a^{j}b^{k-j}$， $j=0,1,2,\cdots,k$， $k=0,1,2,\cdots,q-1$．

因而 $f(x,y)=\psi(ax+by)$ 在 $(0,0)$ 点邻域的 Taylor 公式是

$$f(x,y)=\sum_{k=0}^{q-1}\frac{1}{k!}(\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y)^{k}f(0,0)+R_{q}(x,y)$$

$$=\sum_{k=0}^{q-1}\frac{1}{k!}\sum_{j=0}^{k}C_{k}^{j}\frac{\partial^{k}f(0,0)}{\partial x^{j}y^{k-j}}x^{j}y^{k-j}+R_{q}(x,y)$$

$$=\sum_{k=0}^{q-1}\frac{1}{k!}\sum_{j=0}^{k}C_k^j\psi^{(k)}(0)a^jb^{k-j}x^jy^{k-j}+R_q(x,y)$$

$$=\sum_{k=0}^{q-1}\frac{\psi^{(k)}(0)}{k!}\sum_{j=0}^{k}C_k^j(ax)^j(by)^{k-j}+R_q(x,y)\,.$$