Math 424 Cheat Sheet

Chenkai Luo

Contents

1 Lecture 1: Limits

1.1 Limits of Sequences

Given a sequence $\{x_n\} \subset \mathbb{R}$, we say that x_n converges to a number $l \in \mathbb{R}$ if and only if for all $\epsilon > 0$, there exists N such that for all $n > N$, $|x_n - l| < \epsilon$.

Given $D \subset \mathbb{R}$, we say that $x \in \mathbb{R}$ is a limit point of D if there exists $\{x_n\} \subset D$ such that $x_n \to x$.
Example: the limit points of $[a, b]$ are (a, b).

1.2 Limits of Functions

Let $D \subset \mathbb{R}$, $f\colon D \to \mathbb{R}$, and let x be a limit point in D. Then we say that f has limit l at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in D$ and $|x - x_0| \leq \delta$, then $|f(x) - l| \leq \epsilon$.

If f has a limit at x_0, then this limit is unique.

Given $f\colon D \to \mathbb{R}$, and a limit point x_0 of D, with $x_0 \in \mathbb{R}$, then $f(x) \to l$ as $x \to x_0$ if and only if for all $\{x_n\} \to x_0$, $f(x_n) \to l$.

1.3 Triangle Inequality

The following two inequalities are true for any real a and b:

$|a + b| \leq |a| + |b|$
$|a - b| \geq ||a| - |b||$

2 Lecture 2: Continuity

$\epsilon - \delta$ **Definition of Continuity** Let $D \subset \mathbb{R}$, $f \colon D \to \mathbb{R}$, $x_0 \in D$ We say that a function is continuous at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $|x - x_0| \leq \delta$, then $|f(x) - f(x_0)| \leq \epsilon$.

Let $D \subset \mathbb{R}$, $f \colon D \to \mathbb{R}$, $x_0 \in D$. Then the following are equivalent:

- f is continuous at x_0
- $\lim\limits_{x \to x_0} f(x) = f(x_0)$
- For all $\{x_n\} \to x_0$, $\lim\limits_{n \to \infty} f(x_n) = f(x_0)$

Let $D \subset \mathbb{R}$, and let $f, g \colon D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then the following are all continuous at x_0:

- $f + g$
- $f - g$
- fg

If $g(x_0) \neq 0$, then $\dfrac{f}{g}$ is also continuous at x_0.

Let $D, U \subset \mathbb{R}$, $f \colon D \to \mathbb{R}$, $g \colon U \to \mathbb{R}$, and $x_0 \in D$.

Assume that $f(D) \subset U$. Then $g \circ f(x) = g\left(f(x)\right)$ is defined on D. Furthermore, if f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

3 Lecture 3: Properties of Continuous Functions

Let $f\colon D \to \mathbb{R}$, $D \subset \mathbb{R}$. We say that f is continuous on D if f is continuous at any $x_0 \in D$.

Let $f, g\colon D \to \mathbb{R}$ be continuous functions on $D \subset \mathbb{R}$. Then

- $\max(f, g)$ is continuous on D
- $\min(f, g)$ is continuous on D

Intermediate Value Theorem Let $f\colon\ [a,\, b] \to \mathbb{R}$ be a continuous function and choose c between $f(a)$ and $f(b)$. Then there exists $x_0 \in [a,\, b]$ such that $f(x_0) = c$.

Extreme Value Theorem Let $f\colon\ [a,\, b] \to \mathbb{R}$ be a continuous function. Then f is bounded and reaches both its maximum and minimum.
Note: All assumptions are essential. f must be continuous and bounded

Sequential Compactness Theorem Every bounded sequence contains a convergent subsequence. This is also known as the Bolzano-Weierstrass Theorem.

4 Lecture 4: Uniform Continuity

A function $f\colon D \to \mathbb{R}$ is uniformly continuous if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in D$, $|x - y| \le \delta$ implies that $|f(x) - f(y)| \le \epsilon$.
Note: $<$ and $\le$ are equivalent when working with infinitesimal values, such as ϵ.

Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function (note the closed interval). Then f is uniformly continuous. In other words, a continuous function on a closed bounded interval is uniformly continuous.

Note Lack of uniform continuity comes from open intervals.

Let $f\colon [a, b) \to \mathbb{R}$ be uniformly continuous. Then $f(x)$ has a finite limit as $x \to b$.

4.1 Cauchy Sequences

We say that $\{x_n\}$ is a Cauchy sequence if and only if for any positive number ϵ there is a positive number N such that for all natural numbers $m, n > N$, $|x_m - x_n| < \epsilon$

Cauchy sequences always converge. This is useful when you want to prove convergence but do not have ways to identify the limit.

4.2 Sequential Characterization of a Limit

The following two statements are equivalent:

- $l = \lim_{x \to b} f(x)$
- For all sequences $\{x_n\}$ where $x_n \to b$ as $n \to \infty$, $f(x_n) \to l$ as $n \to \infty$.

5 Lecture 5: Monotonicity and Continuity

We call a function $f \colon I \to \mathbb{R}$ increasing if and only if for all $x, x' \in I$, if $x < x'$ then $f(x) < f(x')$.

Let I be an open interval and $f \colon I \to \mathbb{R}$ be an increasing function. Then the following are equivalent:

i. f is continuous

ii. $f(I)$ is an interval

This is true for any interval, whether open/closed/otherwise

Let $I \subset \mathbb{R}$. We say that $f \colon I \to J$ admits an inverse $g \colon J \to I$ if

> For every x in I, $g \circ f(x) = x$ and
>
> For every y in J, $f \circ g(y) = y$

Let the function $f \colon I \to f(I)$ be both increasing and continuous. Then its inverse f^{-1} is also increasing and continuous.

6 Lecture 6: Differentiability

Let $I \subset \mathbb{R}$ be an open interval. Let $f\colon I \to \mathbb{R}$, and $x_0 \in I$. We say that f has a derivative at x_0 (f is differentiable at x_0) if and only if:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Then we call this value the derivative of f at x_0 (or $f'(x_0)$).

If f is differentiable at every $x_0 \in I$, we say that f is differentiable on I, and we have $f'\colon I \to \mathbb{R}$.

Let $f\colon I \to \mathbb{R}$ and let I be an open interval with $x_0 \in I$. If f is differentiable at x_0, then it is continuous at x_0.

6.1 Operations and Differentiability

Let $f, g\colon I \to \mathbb{R}$, $x_0 \in I$, and let f be differentiable at x_0. Then we have the following:

i. $f + g$ is differentiable at x_0 and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0)$$

ii. fg is differentiable at x_0 and

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

iii. If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Let f be differentiable at x_0 with $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $f(x_0)$ and

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$

equivalently,

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

where $y_0 = f(x_0)$.

7 Lecture 7: Properties of Differentiable Functions

7.1 Chain Rule

Let $f\colon I \to \mathbb{R}$ and $g\colon J \to \mathbb{R}$, with $f(I) \subseteq J$. Assume that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = f'(x_0) \cdot g'(f(x_0))$$

7.2 Properties of Differentiable Functions

Let I be an open interval and let $f\colon I \to \mathbb{R}$ be differentiable at $x_0 \in I$ such that f reaches a minimum or a maximum at x_0. Then

$$f'(x_0) = 0$$

7.3 Rolle's Theorem

Let $f\colon [a,\, b] \to \mathbb{R}$ be a continuous function which is differentiable on $[a, b)$. Then if $f(a) = f(b)$, there exists $c \in (a,\, b)$ such that $f'(c) = 0$.

7.4 Mean Value Theorem

Let $f\colon [a,\, b] \to \mathbb{R}$ be a continuous function and differentiable on $(a,\, b)$. Then there exists $c \in (a,\, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

8 Lecture 8: Derivatives and Variations

Let I be an open interval and let $f\colon I \to \mathbb{R}$ be differentiable. Then

i. If $f'(x) > 0$ for all $x \in I$, f is increasing

ii. If $f'(x) < 0$ for all $x \in I$, f is decreasing

Let $f\colon I \to \mathbb{R}$ be a differentiable function and let $f'\colon I \to \mathbb{R}$ also be differentiable. Then we say that f has two derivatives: f' and f'', and write $f^{(2)} = f''$. Likewise we say that f has n derivatives if $f', f'', \ldots, f^{(n)}$ exist.

8.1 Taylor's Theorem

LEt $f\colon [a, b] \to \mathbb{R}$ be a function with n derivatives. Assume that $f^{(n-1)}$ is continuous on $[a, b]$ and define

$$P(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Then there exists $c \in (a, b)$ such that

$$\frac{f^{(n)}(c)}{n!} = \frac{f(b) - P(b)}{(b-a)^n}$$

Note: Taylor's Theorem helps us understand how f varies near x_0, if $f'(x_0)$ vanishes.

8.2 Local Extrema

Let $f\colon I \to \mathbb{R}$, $x_0 \in I$. We say that f has a local minimum at x_0 if there exists $\delta > 0$ such that if $|x - x_0| \leq \delta$, then $f(x) \geq f(x_0)$ (for a local maximum, $f(x) \leq f(x_0)$).

Let $f\colon I \to \mathbb{R}$ have two derivatives, and let f'' be continuous. If there exists $x_0 \in I$ such that $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local minimum at x_0.

9 Lecture 9: Continuity of Derivatives

Let $f\colon [a, b] \to \mathbb{R}$ with n continuous derivatives. If

$$P(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Then there exists c in (a, b) with

$$\frac{f^(n)(c)}{n!} = \frac{f(b) - P(b)}{(b-a)^n}$$

9.1 Continuity of Derivatives

Derivatives do not need to be continuous. However, they must satisfy the Intermediate Value Theorem.

10 Lecture 10: Additional Topics About Derivatives

10.1 Convex Functions

Let $I \subset \mathbb{R}$ be an open interval and define a function $f\colon I \to \mathbb{R}$. f is convex if and only if for every $x, y \in I$, and all $t \in (0, 1)$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

The definition of strictly convex is the same, but replace the $\leq$ with a $<$.

Let $f\colon I \to \mathbb{R}$ be a differentiable function with a continuous derivative. If f is convex, then the graph of f is always above its tangent. If f is strictly convex, then its graph is always strictly above its tangent.

Let $f\colon I \to \mathbb{R}$ be a differentiable function with two continuous derivatives. Then f is continuous if and only if f' is non-decreasing, or in other words, if and only if $f'' \geq 0$

10.2 Counting Local Minima

Let $f\colon I \to \mathbb{R}$ be a differentiable function with two continuous derivatives. We say that f has a non-degenerate local minimum at x_0 if and only if

$$f'(x_0) = 0 \text{ and } f''(x_0) > 0$$

The set of non-degenerate local minima of f is at most countable. The proof for this is based on the it being a countable union of finite sets.

11 Lecture 11: Lower and Upper Sums

A **partition** of $[a, b]$ is a collection $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

11.1 Upper and Lower Sums

Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Define

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$$
$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$$

The lower sum for f with respect to P is

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

The upper sum for f with respect to P is

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and define

$$m = \inf\{f(x)\colon x \in [a, b]\}$$
$$M = \sup\{f(x)\colon x \in [a, b]\}$$

Then $m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$ for any partition P of $[a, b]$.

Refinement of Partitions

Let P, P^* be two partitions of $[a, b]$. We say that P^* is finer than P if $P \subset P^*$.

Let P, P^* be two partitions of $[a, b]$ such that $P*$ is a refinement of P. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Then

$$L(f, P) \leq L(f, P^*) \leq U(f, P*) \leq U(f, P)$$

12 Lecture 12: Integrable Functions

Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Let P be a partition of $[a, b]$.

We define the **lower Riemann integral** of f (over $[a, b]$) as

$$\underline{\int_a^b} f = \sup \{L(f, P)\}$$

and the **upper Riemann integral** as

$$\overline{\int_a^b} f = \inf \{U(f, P)\}$$

Notes:

a. By definition, the lower integral is an upper bound for the collection of lower Darboux sums, and the upper integral is a lower bound for the collection of upper Darboux sums.

b. The lower integral can be thought of as the maximal area that fits under the graph of f, while the upper interval is the minimal area that covers the graph of f.

We call a bounded function $f\colon [a, b] \to \mathbb{R}$ **Riemann-integrable** if and only if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

Then we write $\int_a^b f$ for the common value.

12.1 Archimedes-Riemann Theorem

A bounded function $f\colon [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if there exists a sequence of partitions P_n such that

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$$

Such a sequence of partitions is called an **Archimedean sequence of partitions**.
Furthermore, for all such sequences of partitions, we have

$$\int_a^b f = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$$

This theorem can be reworded as follows: A bounded function $f\colon [a, b] \to \mathbb{R}$ is integrable if and only if there is an Archimedean sequence of partitions for f on $[a, b]$ and for any such sequence of partitions, the corresponding sequences of upper and lower Darboux sums converge to the integral of f on $[a, b]$.

Note: Integration using upper/lower sums allows you to integrate more functions.

12.2 Useful Inequalities About Darboux Sums and Upper/Lower Integrals

a.

$$L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P)$$

b.

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P) - L(f,P)$$

c.

$$0 \le U(f,P) - \overline{\int_a^b} f \le U(f,P) - L(f,P)$$

d.

$$0 \le \underline{\int_a^b} - L(f,P) \le U(f,P) - L(f,P)$$

12.3 Regular Partitions

For a natural number n, the partition $P = \{x_0, x_1, \dots, x_n\}$ of the interval $[a,\, b]$ defined by $x_i = a + i\frac{b-a}{n}$ for $0 \le i \le n$ is called the **regular partition** of $[a,\, b]$ into n partition intervals.

12.4 The Gap of a Partition

The gap of a partition P, denoted by gap P, is the length of the longest partition interval of P.

Given an integrable function f,

$$\int_a^b f(x)dx = \lim_{\text{gap } P \to 0} L(f,P) = \lim_{\text{gap } P \to 0} U(f,P)$$

13 Lecture 13: Examples of Integrable Functions

Suppose that a function f is integrable. Let $\{P_n\}$ be a sequence of partitions such that

$$\lim_{n\to\infty} \text{gap } P_n = 0$$

Then $\{P_n\}$ is an Archimedean sequence of partitions.

Step functions are integrable.

If f is continuous on abc, then f is uniformly continuous and integrable.

14 Lecture 14: Properties of Integrable Functions

Let $f\colon [a, b] \to \mathbb{R}$ be an integrable function. Then for all $c \in (a, b)$,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Let $f, g\colon [a, b] \to \mathbb{R}$ be bounded and integrable. Then

i. if $f \le g$, $\int_a^b f \le \int_a^b g$

ii. if $c \in \mathbb{R}$, $c\int_a^b f = \int_a^b cf$

iii. $\int_a^b f + g = \int_a^b f + \int_a^b g$

When f and $|f|$ are both integrable,

$$\left|\int_a^b f\right| \le \int_a^b |f|$$

This can be thought of as a continuous version of the triangle inequality.

If f is integrable, then $|f|$ is also integrable. Note that the converse is not true, i.e. $|f|$ being integrable does not imply that f is integrable!

15 Lecture 15: Applications

Let $f\colon [a, b] \to \mathbb{R}$ be integrable and let $S \subset [a, b]$ be a finite set of x values. Then $\int_a^b f$ does not depend on the value of f on S. In other words, changing f at finitely many points has no effect on the integral.

If $f, g\colon [a, b] \to \mathbb{R}$ are bounded and integrable, then fg is also integrable.

Let $f\colon\ [a,\, b] \to \mathbb{R}$ be bounded and integrable such that

$$m \leq f(x) \leq M, x \in [a,\, b]$$

Let $g\colon [m, M] \to \mathbb{R}$ be a continuous function. Then $g \circ f$ is integrable.

16 Uncategorized

A sequence $\{a_n\}$ has a limit if and only if every subsequence shares the same limit.
Every function $f : \mathbb{N} \to \mathbb{R}$ is continuous
f is continuous if for any open interval D, $f^{-1}(D)$ is also open.
x is an **accumulation point** of D if there exists δ for every $y \in D$
If x is not an accumulation point of D then every function on D is continuous at x.
If D is discrete (has no accumulation points) any function on it will be continuous.
A function $f\colon D \to \mathbb{R}$ is said to be a Lipschitz function provided that there is a nonnegative number C such that $|f(u) - f(v)| \leq C\,|u - v|$ for all u and v in D. All Lipschitz functions are continuous.
Uniform continuity: for any u, v in D, if $|u - v| \leq \delta$, $|f(u) - f(v)| \leq \epsilon$
The only continuous functions from $\mathbb{R}$ to $\mathbb{N}$ are constant functions.