Math 308 Cheat Sheet

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Contents

1 Systems of Linear Equations

A matrix is in **row echelon form** if it meets the following conditions:

- All rows consisting of only zeroes are at the bottom
- The leading coefficient (also called the **pivot**) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

A matrix is in **reduced row echelon form** if:

- It is in row echelon form
- The leading entry in each nonzero row is a 1 (called a leading 1).
- Each column containing a leading 1 has zeros in all its other entries.

All variables which are not leading (pivot) variables are called **free variables**.

Gaussian elimination converts a matrix to row echelon form, using the following steps:

- Blah blah blah

2 Linear Independence

A set of vectors is **linearly dependent** if and only if one of the vectors is in the span of the others.

3 Matrices

3.1 Linear Transformations

A **linear transformation** is a function

$$T : \mathbb{R}^m \to \mathbb{R}^n$$

such that for any two vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^m$:

- $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$

Note that the zero vector is always mapped to itself under a linear transformation.

Any linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is given by an n by m matrix (n rows and m columns). Conversely, any matrix defines a linear transformation.

3.1.1 Domain, Codomain, Range

If $T : \mathbb{R}^m \to \mathbb{R}^n$, then:

- $\mathbb{R}^m$ is the **domain** of T.
- $\mathbb{R}^n$ is the **codomain** of T.
- The vector $T(\mathbf{x})$ in $\mathbb{R}^n$ is the **image** of the vector $\mathbf{x}$ in $\mathbb{R}^m$.
- The set of all $T(\mathbf{x})$ in $\mathbb{R}^n$ as $\mathbf{x}$ takes all possible values in $\mathbb{R}^m$ is called the **range** of T.

Let A be an n by m matrix and $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then:

- The range of T is the **span** of the **columns** of the matrix A.
- A vector $\mathbf{w}$ is in the range of T if and only if $T(\mathbf{x}) = \mathbf{w}$ has a solution with $\mathbf{x}$ in $\mathbb{R}^m$.

3.1.2 One-to-One, Onto

If $T : \mathbb{R}^m \to \mathbb{R}^n$, then:

- The linear transformation T is **one-to-one** if no two vectors $\mathbf{x} \neq \mathbf{y}$ in $\mathbb{R}^m$ get mapped to the same vector in $\mathbb{R}^n$. In other words, $T(\mathbf{x}) = T(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$.
- The linear transformation T is **onto** if the range is all of the codomain $\mathbb{R}^n$.

Let A be the n by m matrix such that $T(x) = A\mathbf{x}$, and let B be a matrix in echelon form which is equivalent to A. Then:

(a) T is onto if and only if the columns span $\mathbb{R}^n$, which happens when B has a pivot in every row. This is impossible if $m < n$ because the matrix will have m columns (and thus m vectors) and m vectors cannot span $\mathbb{R}^n$ when $m < n$.

(b) T is one-to-one if and only if the only solution to $T(x) = 0$ is $\mathbf{x} = \mathbf{0}$. This is the same as checking if the columns are linearly independent vectors or if each column has a pivot. This is impossible if $m > n$ because we have m linearly independent vectors in $\mathbb{R}^n$ with $m > n$.

3.1.3 Geometry of Linear Transformations

The columns of a matrix are the images of $e_1, e_2, ..., e_n$. If we know where those vectors are mapped to, we know where everything else is mapped to as well.

3.2 Matrix Algebra

Given a function, we can:
- Multiply a function by a number
- Add two functions together
- Compose two functions (note that this is not commutative)

Applying this to linear transformations, we can:
- Multiply a matrix by a scalar (equivalent to multiplying each entry of the matrix by that scalar)
- Add two transformations together (they must have the same domain and same codomain; same as adding corresponding entries)
- Combine the two transformations (matrix multiplication - For $S(T(v))$ to be possible, the image $T(v)$ of v must be in the domain of S, meaning that the codomain of T must be the same space as the domain of S).

If the matrix of T is A and the matrix of S is B, the matrix of $S(T(x))$ is BA.
The multiplication is only possible if the number of rows of A is the same as the number of columns of B.

Other points about matrix multiplication:
(a) You can have $AB = 0$ with neither A nor B being 0.
(b) Matrix multiplication is not commutative: $AB \neq BA$
(c) Matrix multiplication is associative: $(AB)C = A(BC)$
(d) Matrix multiplication distributes over addition
(e) Location of scalars doesn't matter
(f) $AI = IA = A$ where I is the identity matrix

3.2.1 Special Matrices

Zero Matrix 0 in every entry. Can have any number of rows/columns.

Identity Matrix Always square. 0 in every entry except the diagonal, which is all 1's.

Diagonal Matrix Like identity matrix, but values on the diagonal can be anything. Powers are easy to compute (raise entries on diagonal to the power). Another property of diagonal matrices is that multiplication of diagonal matrices is commutative ($AB = BA$). Note that this is not true for matrices in general.

Triangular Matrix Like diagonal matrix but either upper half or lower half has non-zero values. Remain triangular when taking powers. Upper triangular has 0's below diagonal, lower triangular has 0's above.

Rotation Matrix The following matrix rotates $\mathbb{R}^2$ by θ radians counterclockwise.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3.2.2 Transpose of a Matrix

Turn rows into columns/vice versa. Can be done to any matrix.
Properties:
- $(A+B)^T = A^T + B^T$
- $\alpha(A^T) = (\alpha A)^T$
- $(AB)^T = B^T A^T$

3.3 Inverses of Matrices

$T : \mathbb{R}^m \to \mathbb{R}^n$ has an inverse $T^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ if and only if
- $m = n$
- T is one-to-one
- T is onto

If A the matrix of T, then $AA^{-1} = A^{-1}A = I_n$
If A and B are invertible, $(AB)^{-1} = B^{-1}A^{-1}$ (this holds for any number of matrices - just flip the order and invert all of them)

Process:
1. Augment the matrix with the identity matrix
2. Convert left side of the matrix to identity matrix
3. If possible, right side of matrix will be the inverse. If impossible (zero rows), inverse does not exist.

2×2 **Special Case** The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if $ad - bc \neq 0$. If it exists,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4 Subspaces

4.1 Subspaces

All subspaces are spans of a set of vectors, and conversely, all spans of sets of vectors are subspaces.

Three criteria that need to be met to be a subspace:
1. Contains the zero vector
2. Closed under scalar multiplication
3. Closed under vector addition

4.1.1 Kernel, Null Space

The **kernel** of T is the set of all vectors in the domain $\mathbb{R}^m$ which get mapped to $\mathbf{0}$ in the codomain $\mathbb{R}^n$.
The **null space** is the set of all vectors in the domain $\mathbb{R}^m$ which satisfy $A\mathbf{x}=\mathbf{0}$.
Note: If the transformation T is represented by the matrix A then they are the same.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then:
- Ker T is a subspace of $\mathbb{R}^m$. (This can be shown by checking the three properties of a subspace)
- Ran T is a subspace of $\mathbb{R}^n$. (Recall that the range is the span of the columns of T and all spans are subspaces)

4.2 Basis and Dimension

A set of vectors B is a **basis** for a subspace S of $\mathbb{R}^n$ if it both spans S and is linearly independent.
Because B spans S, any vector $\mathbf{v}$ in S can be written as a linear combination of the vectors in B in a **unique** way.

4.2.1 Finding the Basis by Removing Vectors

Given a set of vectors and their span S, we want to find the basis for S. There are two methods:

Method 1: Take the vectors and make them the ROWS (not the columns!) of a matrix. Reduce to row echelon or reduced row echelon form. The non-zero vectors will form a basis for S (They will be different from the original vectors). Advantage: vectors in basis are "simple" because they have "lots of zeros"

Method 2: Take the vectors and make them the COLUMNS of a matrix. Reduce to echelon or row echelon form. Use this to determine which vectors to discard (keep pivots, throw out the rest). Advantage: basis is subset of original spanning set

4.2.2 Dimension

All bases for the same subspace S have the same number of elements. This number is called the **dimension** of S.

4.2.3 Reaching the Basis by Adding Vectors

Method: Add vectors until we are sure that the set spans S, then reduce using one of the previous methods. Usually this is done by adding the set $\{e_1, e_2, ..., e_n\}$.

4.2.4 Misc. Notes

- One basis for the range of T is given by the vectors corresponding to pivot columns in the row echelon form.
- One basis for the kernel/null space of T is given by the vectors in the general solution to A, where A is the matrix corresponding to T. The number of vectors will equal the number of free variables in the row echelon form of A.
- The column space of a matrix will always be in its codomain
- The null space and row space of a matrix will always be in its domain

4.3 Row and Column Spaces

The **column space** is the span of the columns of a matrix A. If A is the matrix for transformation T, **the column space is a subspace of the codomain of** T.
The **row space** is the span of the rows of a matrix A. If A is the matrix for transformation T, **the row space is a subspace of the domain of** T.
$\dim(\text{Row}(A))=\dim(\text{Col}(A))$. This number is referred to as the **rank** of the matrix A, or $\text{Rank}(A)$.

4.3.1 Rank-Nullity Theorem

The following are all true and say pretty much the same thing:

$$\dim(\text{Ker}(T)) + \dim(\text{Ran}(T)) = m$$
$$\dim(\text{Null}(A)) + \dim(\text{Col}(A)) = m$$
$$\text{Nullity}(A) + \text{Rank}(A) = m$$

4.3.2 Solving for Subspaces

To find **rank and column spaces**, reduce the matrix to row echelon form. The rows and columns containing the pivots are the rows and columns to pull for the basis of $\text{Row}(A)$ and $\text{Col}(A)$.
The **range** is simply the column space.
The **null space/kernel** can be found by solving $A\mathbf{x} = 0$ for $\mathbf{x}$ in terms of vectors multiplied by the free variables, then using those vectors as the basis for the null space.

4.3.3 Misc. Notes

If A and B are equivalent matrices, the subspace spanned by the rows of A is the same as the subspace spanned by the rows of B.

4.4 Change of Basis

If $B = \{u_1, u_2, ..., u_n\}$ is a basis for $\mathbb{R}^n$ and

$$\mathbf{x} = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$$

then we write

$$\mathbf{x}_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_B$$

Generally speaking, if U is the matrix with columns $u_1, u_2, ..., u_n$, for any vector $\mathbf{x}$ in $\mathbb{R}^n$ we have:

$$\mathbf{x} = U\mathbf{x}_B$$

The matrix U is called the **change of basis matrix**.
This means that to convert a vector into the standard basis, one only needs to multiply it by U. To find the change of basis matrix from the standard basis to the basis defined by U, simply find U^{-1}. To convert a vector $\mathbf{x}$ from the standard basis to the basis defined by U, multiply $\mathbf{x}$ by U^{-1}.

5 Determinants

5.1 The Determinant Function

The **determinant** is a number assigned to a square matrix, or equivalently, assigned to a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. The determinant measures how much the unit space in $\mathbb{R}^n$ is being expanded or shrunk.

For $n = 1$, det $A = a$.
For $n = 2$, det $A = ad - bc$. T is invertible if $ad - bc \neq 0$.
For $n \geq 3$ we use **cofactor expansion**:

1. Pick any row or column (you'll want to pick one with a lot of zeros to reduce the amount of calculation necessary)

2. Every entry has a +/- sign attached to it like a checkerboard (top left is +). Note that this is independent of the +/- signs of the values in the entries.

3. The **minor matrix** of each entry is created by ignoring the row and column of that entry, and the **cofactor** of each entry is the determinant of its minor matrix. Multiply each entry in the row/column you chose by its sign and cofactor and add them together.

5.1.1 3×3 Special Case

The cross product of two 3-dimensional vectors $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (d, e, f)$ is simply the determinant of the matrix:

$$\mathbf{x}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{bmatrix}$$

5.1.2 Theorem

Let A be an n by n square matrix. Then:
1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is invertible if and only if $\det(A) \neq 0$.
2. If T is invertible, $\det(A^{-1}) = \dfrac{1}{\det(A)}$
3. $\det(A) = \det\left(A^T\right)$
4. If the matrix has a row or column of zeros, $\det(A) = 0$.
5. If the matrix contains two identical rows or columns, $\det(A)$.

5.1.3 Effect of row operations on Determinant

Swapping rows: Swapping rows will change the sign of the determinant.

Scaling rows: Multiplying a row by a scalar will multiply the determinant by that factor.

Adding rows: Adding a multiple of one row to another row will not change the determinant.

Note that $\det(A) = \det\left(A^T\right)$, and as a result these rules also work for column operations. However, never combine row and column operations in the same step.

5.1.4 Misc. Notes

- The determinant of a triangular matrix is equal to the product of the values on the diagonal running from upper left to bottom right.
- $\det(AB) = \det(A)\det(B)$

6 Eigenvalues and Eigenvectors

An **eigenvector** is one which does not change direction (though it can change magnitude, including going negative) under the transformation defined by the matrix A. Its corresponding **eigenvalue** λ is the scalar by which the vector is stretched, with a negative sign if it flips around.
In matrix terms, this means that $A\mathbf{v} = \lambda\mathbf{v}$

6.1 Finding Eigenvalues and Eigenvectors

For any eigenvalue λ of A, $\det(A - I\lambda) = 0$, where I is the identity matrix. Thus, finding eigenvalues involves solving for λ by setting the determinant equal to 0 and solving the resultant **characteristic polynomial**.

6.1.1 Eigenspaces

The **eigenspace** of an eigenvalue is the span of all eigenvectors associated with that eigenvalue. **By definition, it is the null space of $A - I\lambda$.** Some consequences of this:

- We can find the eigenspace for λ by reducing $A - I\lambda$ and finding the vectors corresponding to the free variables.
- The basis for the nullity of a square matrix (if it exists) is equal to the basis of the eigenspace for the eigenvalue 0. This also allows us to find $\text{rank}(A - I\lambda)$ for any eigenvalue.
- If 0 is an eigenvalue of A, A is not invertible (because $A = A - 0\lambda$ has a non-zero null space).

6.2 Diagonalization

Diagonalizing a matrix A means finding matrices P and D such that $A = PDP^{-1}$ and D is a diagonal matrix. **This is only possible if the eigenvectors of A span $\mathbb{R}^n$, or equivalently, if the sum of the dimensions of the eigenspaces of A equals n.**

Steps: First, we find the eigenvalues and eigenvectors for the A. Now create a diagonal matrix D using the eigenvalues. This is A but in the basis defined by the eigenvectors. Then use the eigenvectors (in the same order as the eigenvalues in D) to create the **change of basis matrix** U. Then $UDU^{-1} = A$. **This process also gives the steps for finding A given its eigenvalues and eigenvectors.**

A Unifying Theorem

Let $\boldsymbol{S = \{u_1, u_2, ..., u_n\}}$ be a set of vectors in $\mathbb{R}^n$. Let A be the matrix whose columns are the vectors $\boldsymbol{u_1, u_2, ..., u_n}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation given by $T(x) = A\mathbf{x}$.

Then the following statements are equivalent (i.e. either all are true or all are false):

(a) The set $\mathbf{S}$ spans $\mathbb{R}^n$.

(b) The transformation T is onto.

(c) $\mathrm{Col}(A) = \mathrm{Range}(T) = \mathbb{R}^n$.

(d) $\dim(\mathrm{Ran}(T)) = \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)) = \mathrm{Rank}(A) = n$.

(e) The set $\mathbf{S}$ is linearly independent.

(f) The transformation T is one-to-one.

(g) The kernel of T is $\{\mathbf{0}\}$.

(h) $\mathrm{Null}(A) = \mathrm{Ker}(T) = \{\mathbf{0}\}$.

(i) The system Ax=b has a unique solution for every b in $\mathbb{R}^n$.

(j) The matrix A is invertible.

(k) $\det(A) \neq 0$.

(l) $\lambda = 0$ is not an eigenvalue of A.

(m) The set S is a basis for $\mathbb{R}^n$.

Lines (a)-(d) all say that T is onto.
Lines (e)-(h) all say that T is one-to-one.
Lines (i)-(l) both say that there exists an inverse.