

Math 327 Cheat Sheet

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1 Tools for Analysis

Field Axioms

A field is a set F with two operations $+$ and \cdot where $a + b \in F$, $a \cdot b \in F \forall a, b \in F$. The following apply for both addition and multiplication:

- Commutativity
- Associativity
- Identity
- Inverse

In addition to those, there are two more properties which combine addition and multiplication:

- Distributivity
- Non-triviality - the additive identity and multiplicative identity must not equal each other

Some results of the above:

- $0 \cdot A = 0$
- 0 (the additive identity) is unique
- $a \cdot b = 0$ implies $a = 0$ or $b = 0$
- Additive and multiplicative **inverses** are unique $(-a, a^{-1})$
- $-a = (-1) \cdot a$
- $-(-a) = a$, $(a^{-1})^{-1} = a$
- The additive inverse of the multiplicative identity is its own multiplicative inverse (i.e. $(-1)^{-1} = -1$)

Bounds

A field F with a subset P satisfying the positivity axioms is called an **ordered field**.

A subset S of an ordered field is said to be **bounded above** provided that there exists $c \in F$ such that $s \leq c$ for any $s \in S$. Then c is called an upper bound of S . The smallest upper bound of S is called the **least upper bound** of S , or the **supremum** of S , denoted as

$$\alpha = \sup S \tag{1}$$

Formally, this is defined as follows:

- α is an upper bound of S
- For any β , if $\beta < \alpha$, β is not an upper bound of S

Least upper bounds may or may not be in the set S .

The supremum equals the maximum if and only if it is in the set.

We can define lower bounds and the greatest lower bound (also called the **infimum**, or $\inf S$) in a similar fashion.

The Reals

\mathbb{R} is a field with the following properties:

- \mathbb{R} contains \mathbb{Q}
- \mathbb{R} is ordered
- \mathbb{R} is complete

Completeness axiom:

Any $S \subseteq \mathbb{R}$ which is bounded above has a least upper bound.

As a result of that we have the following theorem:

Any $S \subseteq \mathbb{R}$ which is bounded below has a greatest lower bound.

Density A set S is said to be dense in \mathbb{R} if for any interval (x, y) in \mathbb{R} you can find $s \in S$.

2 Convergent Sequences

2.1 The Convergence of Sequences

A **sequence** is a real-valued function whose domain is the set of natural numbers.

A sequence $\{a_n\}$ is said to **converge** to the number a provided that for every positive number ϵ there exists an index N such that $|a_n - a| < \epsilon$ for all indices $n \geq N$.

The Comparison Lemma Let the sequence $\{a_n\}$ converge to the number d . Then the sequence $\{b_n\}$ converges to the number b if there is a nonnegative number C and an index N_1 such that for all indices $n \geq N$,

$$|b_n - b| \leq C|a_n - a|$$

2.2 Sequences and Sets

Boundedness of Convergent Sequences

A set S of numbers is said to be bounded provided that there is a number M such that $|x| < M$ for all points x in S .

A sequence $\{a_n\}$ is said to be bounded provided that there is a number M such that $|a_n| < M$ for every index n .

Properties of Convergent Sequences

- All convergent sequences are bounded
- A convergent sequence has exactly one limit
- If $\{a_n\}$ converges to a and c is a constant, then $\{c \cdot a_n\}$ converges to $c \cdot a$.
- Suppose that $\{a_n\}$ converges to a and $\{b_n\}$ converges to b . Then $\{a_n + b_n\}$ converges to $a + b$ and $\{a_n \cdot b_n\}$ converges to ab .
- If $\{a_n\}$ is a sequence of non-zero numbers and $\lim_{n \rightarrow \infty} a_n = a \neq 0$, then $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}$

Sequences and Density

A set S is dense in \mathbb{R} if and only if every number x is the limit of a sequence in S .

Closed Sets

Let $\{c_n\}$ be a sequence in the interval $[a, b]$. If $\{c_n\}$ converges to the number c , then c is also in the interval $[a, b]$.

A subset S of \mathbb{R} is said to be **closed** if every convergent sequence in S converges to an element of S .

Common Sequences

- $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$
- If $p > 0$ then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$
- If $p > 0$ then $\lim_{n \rightarrow \infty} p^{1/n} = 0$
- $\lim_{n \rightarrow \infty} n^{1/n} = 1$
- If $p > 0$ and α is real then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$
- If $|x| < 1$ then $\lim_{n \rightarrow \infty} x^n = 0$

2.3 The Monotone Convergence Theorem

A monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone sequence $\{a_n\}$ converges to

- $\sup \{a_n \mid n \in \mathbb{N}\}$ if it is monotonically increasing, and to
- $\inf \{a_n \mid n \in \mathbb{N}\}$ if it is monotonically decreasing.

The Nested Interval Theorem

The Squeeze Theorem

Given sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with $\{a_n\}$ and $\{c_n\}$ convergent and $a_n < b_n < c_n$, if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = l$, then $\{b_n\}$ converges to l .

2.4 The Sequential Compactness Theorem

Subsequences

Define a strictly increasing sequence of natural numbers $\{n_k\}$. Then $\{b_n\} = \{a_{n_k}\}$ is called a subsequence.

Now, if $\{a_n\}$ converges to a , $\{b_n\}$ also converges to a .

Every sequence has a monotonic subsequence.

2.5 The Cauchy Convergence Criterion for Sequences

A sequence $\{a_n\}$ is called Cauchy provided that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \epsilon$.

A sequence converges if and only if it is Cauchy.

2.6 Rudin

Lim Sup and Lim Inf

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all real numbers x such that $s_{n_k} \rightarrow x$ for some subsequence s_{n_k} . This set includes all possible subsequence limits, along with (possibly) $+\infty$ and $-\infty$. Define $s^* = \sup E$ and $s_* = \inf E$. s^* and s_* are called the upper and lower limits of $\{s_n\}$.

Now s^* has the following properties:

- $s^* \in E$
- If $x > s^*$, there is some integer N such that $n \geq N$ implies $s_n < x$

Furthermore, s^* is the only number with these two properties.

If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then we have the following:

$$\begin{aligned}\liminf_{n \rightarrow \infty} s_n &\leq \liminf_{n \rightarrow \infty} t_n \\ \limsup_{n \rightarrow \infty} s_n &\leq \limsup_{n \rightarrow \infty} t_n\end{aligned}$$

3 Continuous Functions

3.1 Continuity

Definitions of Continuity

Let $f : D \rightarrow \mathbb{R}$. We call a function f **s-continuous** at $x \in D$ provided that $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ for any sequence $\{x_n\}$ with $\lim_{n \rightarrow \infty} x_n = x$.

We say that f is **$\epsilon\delta$ -continuous** provided that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.

s-continuity and $\epsilon\delta$ -continuity are equivalent.

Properties of Continuity

If $f : D \rightarrow \mathbb{R}$, $g : D \rightarrow \mathbb{R}$ are both continuous at $x \in D$, then:

- $f + g$ is continuous at $x \in D$
- fg is continuous at $x \in D$
- $\frac{f}{g}$ is continuous at $x \in D$ if $g(y) \neq 0$ for any $y \in D$

Let $f : D \rightarrow \mathbb{R}$, $g : U \rightarrow \mathbb{R}$ such that $f(D) \subseteq U$. If f is continuous at $x^* \in D$ and g is continuous at $f(x^*) \in U$, then $g \circ f$ is continuous at x^* .

3.2 The Extreme Value Theorem

For a function $f : D \rightarrow \mathbb{R}$ we define $f(D) \equiv \{y \mid y = f(x) \text{ for some } x \text{ in } D\}$ and call the set $f(D)$ the **image** of the function $f : D \rightarrow \mathbb{R}$. The function $f : D \rightarrow \mathbb{R}$ has a maximum value if its image $f(D)$ has a maximum, i.e. $f(x) \leq f(x_0)$ for all x in D . Then x_0 is the **maximizer** of $f : D \rightarrow \mathbb{R}$. We can define a minimum value and thus **minimizer** in the same way.

In general, a nonempty set has a maximum provided that the set is bounded above and contains its supremum.

4 Misc. Notes

Need to add triangle inequality

$$||a| - |b|| \leq |a - b| \text{ (Need a proof)}$$