# Math 327 Cheat Sheet

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# 1 Tools for Analysis

### Field Axioms

A field is a set $F$ with two operations $+$ and $\cdot$ where $a + b \in F$, $a \cdot b \in F$ $\forall a, b \in F$. The following apply for both addition and multiplication:

- Commutativity
- Associativity
- Identity
- Inverse

In addition to those, there are two more properties which combine addition and multiplication:

- Distributivity
- Non-triviality - the additive identity and multiplicative identity must not equal each other

Some results of the above:

- $0 \cdot A = 0$
- 0 (the additive identity) is unique
- $a \cdot b = 0$ implies $a = 0$ or $b = 0$
- Additive and multiplicative **inverses** are unique ($-a$, $a^{-1}$)
- $-a = (-1) \cdot a$
- $-(-a) = a$, $(a^{-1})^{-1} = a$
- The the additive inverse of the multiplicative identity is its own multiplicative inverse (i.e. $(-1)^{-1} = -1$)

### Bounds

A field $F$ with a subset $P$ satisfying the positivity axioms is called an **ordered field**.
A subset $S$ of an ordered field is said to be **bounded above** provided that there exists $c \in F$ such that $s \leq c$ for any $s \in S$. Then $c$ is called an upper bound of $S$. The smallest upper bound of $S$ is called the **least upper bound** of $S$, or the **supremum** of $S$, denoted as

$$\alpha = \sup S \tag{1}$$

Formally, this is defined as follows:

- $\alpha$ is an upper bound of $S$
- For any $\beta$, if $\beta < \alpha$, $\beta$ is not an upper bound of $S$

Least upper bounds may or may not be in the set $S$.
The supremum equals the maxmimum if and only if it is in the set.
We can define lower bounds and the greatest lower bound (also called the **infimum**, or $\inf S$) in a similar fashion.

### The Reals

$\mathbb{R}$ is a field with the following properties:

- $\mathbb{R}$ contains $\mathbb{Q}$
- $\mathbb{R}$ is ordered
- $\mathbb{R}$ is complete

Completeness axiom:
Any $S \subseteq \mathbb{R}$ which is bounded above has a least upper bound.
As a result of that we have the following theorem:
Any $S \subseteq \mathbb{R}$ which is bounded below has a greatest lower bound.

**Density** A set $S$ is said to be dense in $\mathbb{R}$ if for any interval $(x, y)$ in $\mathbb{R}$ you can find $s \in S$.

# 2 Convergent Sequences

## 2.1 The Convergence of Sequences

A **sequence** is a real-valued function whose domain is the set of natural numbers.
A sequence $\{a_n\}$ is said to **converge** to the number $a$ provided that for very positive number $\epsilon$ there exists and index $N$ such that $|a_n - a| < \epsilon$ for all indices $n \geq N$.

**The Comparison Lemma** Let the sequence $\{a_n\}$ converge to the number $d$. Then the sequence $\{b_n\}$ converges to the number $b$ if there is a nonnegative number $C$ and an index $N_1$ such that for all indices $n \geq N$,

$$|b_n - b| \leq C|a_n - a|$$

## 2.2 Sequences and Sets

### Boundedness of Convergent Sequences

A set $S$ of numbers is said to be bounded provided that there is a number $M$ such that $|x| < M$ for all points $x$ in $S$.
A sequence $\{a_n\}$ is said to be bounded provided that there is a number $M$ such that $|a_n| < M$ for every index $n$.

### Properties of Convergent Sequences

- All convergent sequences are bounded
- A convergent sequence has exactly one limit
- If $\{a_n\}$ converges to $a$ and $c$ is a constant, then $\{c \cdot a_n\}$ converges to $c \cdot a$.
- Suppose that $\{a_n\}$ converges to $a$ and $\{b_n\}$ converges to $b$. Then $\{a_n + b_n\}$ converges to $a + b$ and $\{a_n \cdot b_n\}$ converges to $ab$.
- If $\{a_n\}$ is a sequence of non-zero numbers and $\lim_{n\to\infty} a_n = a \neq 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

### Sequences and Density

A set $S$ is dense in $\mathbb{R}$ if and only if every number $x$ is the limit of a sequence in $S$.

### Closed Sets

Let $\{c_n\}$ be a sequence in the interval $[a, b]$. If $\{c_n\}$ converges to the number $c$, then $c$ is also in the interval $[a, b]$.
A subset $S$ of $\mathbb{R}$ is said to be **closed** if every convergent sequence in $S$ converges to an element of $S$.

**Common Sequences**

- $\lim_{n\to\infty} \frac{1}{n} = 0$
- If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
- If $p > 0$ then $\lim_{n\to\infty} p^{1/n} = 0$
- $\lim_{n\to\infty} n^{1/n} = 1$
- If $p > 0$ and $\alpha$ is real then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
- If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$

## 2.3 The Monotone Convergence Theorem

A monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone sequence $\{a_n\}$ converges to
- $\sup\{a_n \mid n \in N\}$ if it is monotonically increasing, and to
- $\inf\{a_n \mid n \in N\}$ if it is monotonically decreasing.

**The Nested Interval Theorem**

**The Squeeze Theorem**

Given sequences $\{a_n\}$ $\{b_n\}$ and $\{c_n\}$ with $\{a_n\}$ and $\{c_n\}$ convergent and $a_n < b_n < c_n$, if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$, then $\{b_n\}$ converges to $l$.

## 2.4 The Sequential Compactness Theorem

**Subsequences**

Define a strictly increasing sequence of natural numbers $\{n_k\}$. Then $\{b_n\} = \{a_{n_k}\}$ is called a subsequence.
Now, if $\{a_n\}$ converges to $a$, $\{b_n\}$ also converges to $a$.
Every sequence has a monotonic subsequence.

## 2.5 The Cauchy Convergence Criterion for Sequences

A sequence $\{a_n\}$ is called Cauchy provided that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \epsilon$.

A sequence converges if and only if it is Cauchy.

## 2.6 Rudin

### Lim Sup and Lim Inf

Let $\{s_n\}$ be a sequence of real numbers. Let $E$ be the set of all real numbers $x$ such that $s_{n_k} \to x$ for some subsequence $s_{n_k}$. This set includes all possible subsequence limits, along with (possibly) $+\infty$ and $-\infty$. Define $s^* = \sup E$ and $s_* = \inf E$. $s^*$ and $s_*$ are called the upper and lower limits of $\{s_n\}$.

Now $s^*$ has the following properties:

- $s^* \in E$
- If $x > s^*$, there is some integer $N$ such that $n \geq N$ implies $s_n < x$

Furthermore, $s^*$ is the only number with these two properties.

If $s_n \leq t_n$ for $n \geq N$, where $N$ is fixed, then we have the following:

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

# 3 Continuous Functions

## 3.1 Continuity

### Definition of Continuity

Let $f : D \to \mathbb{R}$. We call a function $f$ **s-continuous** at $x \in D$ provided that $\lim_{n\to\infty} f(x_n) = f(x)$ for any sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x$.

### Properties of Continuity

If $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ are both continuous at $x \in D$, then:

- $f + g$ is continuous at $x \in D$
- $fg$ is continuous at $x \in D$
- $\frac{f}{g}$ is continuous at $x \in D$ if $g(y) \neq 0$ for any $y \in D$

All polynomials are continuous functions. As a result, if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are polynomials, then $f/g : D \to \mathbb{R}$ is continuous, where $D = \{x \in \mathbb{R} \mid g(x) \neq 0\}$

### Compositions of Continuous Functions

Let $f : D \to \mathbb{R}$, $g : U \to \mathbb{R}$ such that $f(D) \subseteq U$. If $f$ is continuous at $x_0 \in D$ and $g$ is continuous at $f(x_0) \in U$, then $g \circ f : D \to \mathbb{R}$ is continuous at $x_0$.

## 3.2 The Extreme Value Theorem

For a function $f : D \to \mathbb{R}$ we define $f(D) \equiv \{y \mid y = f(x)$ for some $x$ in $D$ and call the set $f(D)$ the **image** of the function $f : D \to \mathbb{R}$. The function $f : D \to \mathbb{R}$ has a maximum value if its image $f(D)$ has a maximum, i.e. $f(x) \leq f(x_0)$ for all $x$ in $D$. Then $x_0$ is the **maximizer** of $f : D \to \mathbb{R}$. We can define a minimum value and thus **minimizer** in the same way.

In general, a nonempty set has a maximum provided that the set is bounded above and contains its supremum.

## 3.3 The Intermediate Value Theorem

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Let $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = y$.

A subset $D$ of $\mathbb{R}$ is said to be convex provided that whenever the points $u$ and $v$ are in $D$ and $u < v$, then the whole interval $[u, v]$ is contained in $D$.

Let $I$ be an interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then its image $f(I)$ is also an interval.

## 3.4 Uniform Continuity

A function $f : D \to \mathbb{R}$ is said to be uniformly continuous provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in $D$ such that $\lim_{n\to\infty} [u_n - v_n] = 0$, then $\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0$.

If a function $f : D \to \mathbb{R}$ is uniformly continuous, it is continuous.

A continuous function on a closed bounded interval $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

## 3.5 The $\epsilon - \delta$ Criterion for Continuity

We say that $f$ is **$\epsilon\delta$-continuous** provided that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.

#### The $\epsilon - \delta$ Criterion at a Point

The function $f : D \to \mathbb{R}$ is said to satisfy the $\epsilon - \delta$ criterion for continuity at a point $x_0$ in the domain $D$ provided that for each positive number $\epsilon$ there is a positive number $\delta$ such that for $x$ in $D$, $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$.

s-continuity and $\epsilon\delta$-continuity are equivalent.

#### Misc. Notes

If $f : D \to \mathbb{R}$ and $D \in \mathbb{Z}$ then $f$ is uniformly continuous.

## 3.6 Images, Inverses, Monotone Functions

Let $f : D \to \mathbb{R}$ be a monotone function. If its image $f(D)$ is an interval, then $f$ is continuous.

A function $f : D \to \mathbb{R}$ is said to be one-to-one provided that for each point $y$ on its image $f(D)$, there is exactly one point $x$ in its domain $D$ such that $f(x) = y$.

Let $I$ be an interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

Notes:

- One-to-one functions have inverses.
- Strictly monotone functions are one-to-one.
- The inverse of a strictly monotone function is strictly monotone.
- Define the function $f(x) = x^r$ for $x \geq 0$ for any rational number $r$. Then $f : [0, \infty) \to \mathbb{R}$ is continuous.

# 4 Convergence Tests for Series

Given a sequence $\{a_n\}$, we define $s_n = \sum\limits_{k=1}^{n} a_k$. Then $\sum\limits_{k=1}^{\infty} a_k = \lim\limits_{n\to\infty} s_n$. If $\{s_n\}$ converges we say the series converges, otherwise we say the series diverges.
If $\sum\limits_{k=1}^{\infty} a_k$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.

### Common Series

- $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ converges.
- $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.
- $\sum\limits_{n=1}^{\infty} k^n$ converges if and only if $r < 1$, in which case the sum equals $\frac{1}{1-r}$.
- If $p \leq 1$ then $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

### Comparison Test

Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $0 \leq \{a_n\} \leq \{b_n\}$ for all $n \in \mathbb{R}$. Then:

- If $\sum\limits_{n=1}^{\infty} b_n$ converges, $\sum\limits_{n=1}^{\infty} a_n$ converges.
- If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges.

Note that because $0 \leq \{a_n\} \leq \{b_n\}$, $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are monotonically increasing, and thus they converge if and only if the partial sums are bounded.

### Cauchy Condensation Test

Suppose that $\{a_n\}$ is monotonically decreasing and all terms are nonnegative. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{k=0}^{\infty} 2^k a_{2^k}$ does.

# 5 Misc. Notes

Need to add triangle inequality
$||a| - |b|| \leq |a - b|$ (Need a proof)