Math 327 Cheat Sheet

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Contents

1 Tools for Analysis

Field Axioms

A field is a set F with two operations $+$ and $\cdot$ where $a + b \in F$, $a \cdot b \in F$ $\forall a, b \in F$. The following apply for both addition and multiplication:

- Commutativity
- Associativity
- Identity
- Inverse

In addition to those, there are two more properties which combine addition and multiplication:

- Distributivity
- Non-triviality - the additive identity and multiplicative identity must not equal each other

Some results of the above:

- $0 \cdot A = 0$
- 0 (the additive identity) is unique
- $a \cdot b = 0$ implies $a = 0$ or $b = 0$
- Additive and multiplicative **inverses** are unique ($-a$, a^{-1})
- $-a = (-1) \cdot a$
- $-(-a) = a$, $(a^{-1})^{-1} = a$
- The the additive inverse of the multiplicative identity is its own multiplicative inverse (i.e. $(-1)^{-1} = -1$)

Bounds

A field F with a subset P satisfying the positivity axioms is called an **ordered field**.
A subset S of an ordered field is said to be **bounded above** provided that there exists $c \in F$ such that $s \leq c$ for any $s \in S$. Then c is called an upper bound of S. The smallest upper bound of S is called the **least upper bound** of S, or the **supremum** of S, denoted as

$$\alpha = \sup S \tag{1}$$

Formally, this is defined as follows:

- α is an upper bound of S
- For any β, if $\beta < \alpha$, β is not an upper bound of S

Least upper bounds may or may not be in the set S.
The supremum equals the maxmimum if and only if it is in the set.
We can define lower bounds and the greatest lower bound (also called the **infimum**, or $\inf S$) in a similar fashion.

The Reals

$\mathbb{R}$ is a field with the following properties:

- $\mathbb{R}$ contains $\mathbb{Q}$
- $\mathbb{R}$ is ordered
- $\mathbb{R}$ is complete

Completeness axiom:
Any $S \subseteq \mathbb{R}$ which is bounded above has a least upper bound.
As a result of that we have the following theorem:
Any $S \subseteq \mathbb{R}$ which is bounded below has a greatest lower bound.

Density A set S is said to be dense in $\mathbb{R}$ if for any interval (x, y) in $\mathbb{R}$ you can find $s \in S$.

2 Convergent Sequences

2.1 The Convergence of Sequences

A **sequence** is a real-valued function whose domain is the set of natural numbers.
A sequence $\{a_n\}$ is said to **converge** to the number a provided that for very positive number ϵ there exists and index N such that $|a_n - a| < \epsilon$ for all indices $n \geq N$.

The Comparison Lemma Let the sequence $\{a_n\}$ converge to the number d. Then the sequence $\{b_n\}$ converges to the number b if there is a nonnegative number C and an index N_1 such that for all indices $n \geq N$,

$$|b_n - b| \leq C\,|a_n - a|$$

2.2 Sequences and Sets

Boundedness of Convergent Sequences

A set S of numbers is said to be bounded provided that there is a number M such that $|x| < M$ for all points x in S.
A sequence $\{a_n\}$ is said to be bounded provided that there is a number M such that $|a_n| < M$ for every index n.

Properties of Convergent Sequences

- All convergent sequences are bounded
- A convergent sequence has exactly one limit
- If $\{a_n\}$ converges to a and c is a constant, then $\{c \cdot a_n\}$ converges to $c \cdot a$.
- Suppose that $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. Then $\{a_n + b_n\}$ converges to $a + b$ and $\{a_n \cdot b_n\}$ converges to ab.
- If $\{a_n\}$ is a sequence of non-zero numbers and $\lim\limits_{n\to\infty} a_n = a \neq 0$, then $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

Sequences and Density

A set S is dense in $\mathbb{R}$ if and only if every number x is the limit of a sequence in S.

Closed Sets

Let $\{c_n\}$ be a sequence in the interval $[a, b]$. If $\{c_n\}$ converges to the number c, then c is also in the interval $[a, b]$.
A subset S of $\mathbb{R}$ is said to be **closed** if every convergent sequence in S converges to an element of S.

Common Sequences

- $\lim_{n \to \infty} \frac{1}{n} = 0$
- If $p > 0$ then $\lim_{n \to \infty} \frac{1}{n^p} = 0$
- If $p > 0$ then $\lim_{n \to \infty} p^{1/n} = 0$
- $\lim_{n \to \infty} n^{1/n} = 1$
- If $p > 0$ and α is real then $\lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$
- If $|x| < 1$ then $\lim_{n \to \infty} x^n = 0$

2.3 The Monotone Convergence Theorem

A monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone sequence $\{a_n\}$ converges to

- $\sup \{a_n \mid n \in N\}$ if it is monotonically increasing, and to
- $\inf \{a_n \mid n \in N\}$ if it is monotonically decreasing.

The Nested Interval Theorem

For each natural number n, let a_n and b_n be numbers such that $a_n < b_n$ and consider the interval $I_n = [a_n, b_n]$. Assume that $I_{n+1} \subseteq I_n$ for every index n. Also assume that $\lim_{n \to \infty} [b_n - a_n] = 0$. Then there is exactly one point x that belongs to the interval I_n for all n, and both $\{a_n\}$ and $\{b_n\}$ converge to this point.

The Squeeze Theorem

Given sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ with $\{a_n\}$ and $\{c_n\}$ convergent and $a_n < b_n < c_n$, if $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = l$, then $\{b_n\}$ converges to l.

2.4 The Sequential Compactness Theorem

Subsequences

Define a strictly increasing sequence of natural numbers $\{n_k\}$. Then $\{b_n\} = \{a_{n_k}\}$ is called a subsequence.

Now, if $\{a_n\}$ converges to a, $\{b_n\}$ also converges to a.

Every sequence has a monotonic subsequence.

2.5 The Cauchy Convergence Criterion for Sequences

A sequence $\{a_n\}$ is called Cauchy provided that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \epsilon$.

A sequence converges if and only if it is Cauchy.

2.6 Rudin

Lim Sup and Lim Inf

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all real numbers x such that $s_{n_k} \to x$ for some subsequence s_{n_k}. This set includes all possible subsequence limits, along with (possibly) $+\infty$ and $-\infty$. Define $s^* = \sup E$ and $s_* = \inf E$. s^* and s_* are called the upper and lower limits of $\{s_n\}$.

Now s^* has the following properties:

- $s^* \in E$
- If $x > s^*$, there is some integer N such that $n \geq N$ implies $s_n < x$

Furthermore, s^* is the only number with these two properties.

If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then we have the following:

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

3 Continuous Functions

3.1 Continuity

Definition of Continuity

Let $f : D \to \mathbb{R}$. We call a function f **s-continuous** at $x \in D$ provided that $\lim_{n\to\infty} f(x_n) = f(x)$ for any sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x$.

Properties of Continuity

If $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ are both continuous at $x \in D$, then:

- $f + g$ is continuous at $x \in D$
- fg is continuous at $x \in D$
- $\frac{f}{g}$ is continuous at $x \in D$ if $g(y) \neq 0$ for any $y \in D$

All polynomials are continuous functions. As a result, if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are polynomials, then $f/g : D \to \mathbb{R}$ is continuous, where $D = \{x \in \mathbb{R} \mid g(x) \neq 0\}$

Compositions of Continuous Functions

Let $f : D \to \mathbb{R}$, $g : U \to \mathbb{R}$ such that $f(D) \subseteq U$. If f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$, then $g \circ f : D \to \mathbb{R}$ is continuous at x_0.

3.2 The Extreme Value Theorem

For a function $f : D \to \mathbb{R}$ we define $f(D) \equiv \{y \mid y = f(x)$ for some x in D and call the set $f(D)$ the **image** of the function $f : D \to \mathbb{R}$. The function $f : D \to \mathbb{R}$ has a maximum value if its image $f(D)$ has a maximum, i.e. $f(x) \leq f(x_0)$ for all x in D. Then x_0 is the **maximizer** of $f : D \to \mathbb{R}$. We can define a minimum value and thus **minimizer** in the same way.

In general, a nonempty set has a maximum provided that the set is bounded above and contains its supremum.

3.3 The Intermediate Value Theorem

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Let $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = y$.

A subset D of $\mathbb{R}$ is said to be convex provided that whenever the points u and v are in D and $u < v$, then the whole interval $[u, v]$ is contained in D.

Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then its image $f(I)$ is also an interval.

3.4 Uniform Continuity

A function $f : D \to \mathbb{R}$ is said to be uniformly continuous provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D such that $\lim_{n\to\infty} [u_n - v_n] = 0$, then $\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0$.

If a function $f : D \to \mathbb{R}$ is uniformly continuous, it is continuous.

A continuous function on a closed bounded interval $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

3.5 The $\epsilon - \delta$ Criterion for Continuity

We say that f is **$\epsilon\delta$-continuous** provided that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$.

The $\epsilon - \delta$ Criterion at a Point

The function $f : D \to \mathbb{R}$ is said to satisfy the $\epsilon - \delta$ criterion for continuity at a point x_0 in the domain D provided that for each positive number ϵ there is a positive number δ such that for x in D, $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$.

s-continuity and $\epsilon\delta$-continuity are equivalent.

Misc. Notes

If $f : D \to \mathbb{R}$ and $D \in \mathbb{Z}$ then f is uniformly continuous.

3.6 Images, Inverses, Monotone Functions

Let $f : D \to \mathbb{R}$ be a monotone function. If its image $f(D)$ is an interval, then f is continuous.

A function $f : D \to \mathbb{R}$ is said to be one-to-one provided that for each point y on its image $f(D)$, there is exactly one point x in its domain D such that $f(x) = y$.

Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

Notes:

- One-to-one functions have inverses.
- Strictly monotone functions are one-to-one.
- The inverse of a strictly monotone function is strictly monotone.
- Define the function $f(x) = x^r$ for $x \geq 0$ for any rational number r. Then $f : [0, \infty) \to \mathbb{R}$ is continuous.

4 Convergence Tests for Series

4.1 Common Series and Tests

Given a sequence $\{a_n\}$, we define $s_n = \sum_{k=1}^{n} a_k$. Then $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$. If $\{s_n\}$ converges we say the series converges, otherwise we say the series diverges.

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

Let $\{a_k\}$ be a sequence of nonnegative numbers. The series $\sum_{n=1}^{\infty} a_k$ converges if and only if the sequence of partial sums is bounded, i.e. there is a positive number M such that $a_1 + a_2 + \cdots + a_n \leq M$ for every index n.

Common Series

- $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- $\sum_{n=1}^{\infty} k^r$ converges if and only if $r < 1$, in which case the sum equals $\frac{1}{1-r}$.
- If $p \leq 1$ then $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ diverges. It converges if $p > 1$.
- If $p > 1$, then $\sum_{n=2}^{\infty} \dfrac{1}{n(\log n)^p}$ converges. It diverges otherwise.

Comparison Test

Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $0 \leq \{a_n\} \leq \{b_n\}$ for all $n \in \mathbb{R}$. Then:

- If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note that because $0 \leq \{a_n\} \leq \{b_n\}$, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are monotonically increasing, and thus they converge if and only if the partial sums are bounded.

Cauchy Condensation Test

Suppose that $\{a_n\}$ is monotonically decreasing and all terms are nonnegative. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ does.

4.2 The Root and Ratio Tests

The Root Test

Given $\sum a_n$, let $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$. Then:

- If $\alpha < 1$, $\sum a_n$ converges.
- If $\alpha > 1$, $\sum a_n$ diverges.
- If $\alpha = 1$, the test gives no information.

The Ratio Test

The series $\sum a_n$

- converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
- diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$ where n_0 is some fixed integer.

Notes

The root test has a wider scope than the ratio test. Whenever the ratio test shows convergence, the root test does as well, and whenever the root test is inconclusive, the ratio test is also inconclusive. Furthermore, both the root and ratio tests check absolute convergence, i.e. if one of the tests tells you that a series converges, it converges absolutely.

4.3 Alternating Sequences

Alternating Series Test

Suppose $\{a_n\}$ is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

Rearrangements

Convergence does not imply that a sum converges to a single value. It is possible for a sum's value to depend on the order of summation, in which which case the sum is said to be convergent but not absolutely convergent.

The alternating harmonic series can converge to any value or even diverge, depending on the order in which you sum the terms.

Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely provided that $\sum_{n=1}^{\infty} |a_n|$ converges.

If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, it converges, and for any rearrangement $\{a_{n'}\}$ of the series, $\sum_{n=1}^{\infty} a_{n'} = \sum_{n=1}^{\infty} a_n$.

5 Misc. Notes

Need to add triangle inequality
$||a| - |b|| \leq |a - b|$ (Need a proof)