Note for Measure Theory

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The axiom of choice is obviously true; the well-ordering principle is obviously false; and who can tell about Zorn's lemma?

—Jerry Bona

Theorem 1 (Axiom of choice). *Suppose that $\{X_j\}_{j\in J}$ is a nonempty collection of nonempty sets, then $\prod_{j\in J} X_j \neq \emptyset$.*

Here we prove theorem 1 by admitting theorem 6. In the following context we will prove theorem 2 by admitting theorem 1, theorem 4 by admitting theorem 2, and theorem 5 by admitting theorem 4, theorem 6 by admitting 5. In this way we show that theorem 1, 2, 4, 5, and 6 are essentially equivalent.

Proof. Pick a well order on $X = \cup_{j\in J} X_j$ (where the existence is guaranteed by theorem 6), define $f : J \to X, j \mapsto \min X_j$. Thus $f \in \prod_{j\in J} X_j$. $\square$

Theorem 2. *Suppose that for any $x \in X$, there exists $y \in Y$ such that property $P(x, y)$ holds, then there exists $f : X \to Y$ satisfies $\forall x \in X, P(x, f(x))$ holds.*

Proof. For any $x \in X$, let $Y_x = \{y \in Y : P(x, y)\}$. Through theorem 1, there exists $f \in \prod_{x\in X} Y_x$ satisfies that $\forall x \in X, P(x, f(x))$ holds. $\square$

0.1 Order

We refer that $\preceq$ is a partially order iff it satisfies

- Reflexivity: $x \preceq x$
- Antisymmetry: $x \preceq y, y \preceq x \Rightarrow x = y$
- Transmissibility: $x \preceq y, y \preceq z \Rightarrow x \preceq z$

We define that $(X, \preceq)$ is a partially ordered set iff $\preceq$ is a partially order on X, $(X, \preceq)$ is a totally ordered set iff for any $x, y \in X$, either $x \preceq y$ or $y \preceq x$ holds, and X is a well ordered set iff for any $\emptyset \neq A \subset X$, A has a minimum.

Theorem 3 (Strong induction principle). *Suppose that X is a well ordered set and property $P(n)$ satisfies $\forall n > n' \in X, P(n')$ holds implies $P(n)$ holds, then $\forall n \in X, P(n)$ holds.*

Proof. Assume there exists $n \in X$ such that $\neg P(n)$ holds, then $Y = \{n \in X : \exists n \geq m \in X, \neg P(m)\} \neq \emptyset$. Denote $n' = \min Y$. It is clear that $\forall n' > n \in X, P(n)$ holds, thus $\neg P(n')$ holds. Meanwhile we can derive $P(n')$ holds, which forms a contradiction. $\square$

Theorem 4. *Every partially ordered set X has a well ordered subset with no strict upper bound.*

Proof. The case for emptyset is too trivial to be specified. Suppose that $x_0 \in X$. Assume that every well ordered subset has a strict upper bound. Through theorem 2, there exists mapping f satisfies that for any well ordered subset Y, $f(Y)$ is a strict upper bound of Y. In the following context we constrct a contradiction to find that such a mapping cannot exist. We refer $\mathcal{G}$ as a collection of well ordered subsets of X where $Y \in \mathcal{G}$ iff $\min Y = x_0$ and $\forall x_0 \neq x \in Y, x = f(\{y \in Y : y < x\})$. Firstly we verify that

$$\forall n \in Y \cap Y', \{y \in Y : y < n\} = \{y \in Y' : y < n\} = \{y \in Y \cap Y' : y < n\} \quad (1)$$

Since $Y \cap Y'$ is well ordered, through theorem 3, to varify (1), we can use the condition

$$\forall n > n' \in Y \cap Y', \{y \in Y : y < n'\} = \{y \in Y' : y < n'\} = \{y \in Y \cap Y' : y < n'\}$$

which implies

$$\begin{aligned}\bigcup_{n > n' \in Y \cap Y'} \{y \in Y : y \leq n'\} &= \bigcup_{n > n' \in Y \cap Y'} \{y \in Y' : y \leq n'\} \\ &= \bigcup_{n > n' \in Y \cap Y'} \{y \in Y \cap Y' : y \leq n'\}\end{aligned}$$

or, equivalently

$$\begin{aligned}&\{y \in Y : \exists n > n' \in Y \cap Y' : y \in Y : y \leq n'\} \\ &= \{y \in Y' : \exists n > n' \in Y \cap Y' : y \in Y : y \leq n'\} \\ &= \{y \in Y \cap Y' : \exists n > n' \in Y \cap Y' : y \in Y : y \leq n'\}\end{aligned}$$

Thus, we only need to prove

$$Z = \{y \in Y \backslash Y' : \forall n > n' \in Y \cap Y', n' < y < n\} = \emptyset$$

$$Z' = \{y \in Y' \backslash Y : \forall n > n' \in Y \cap Y', n' < y < n\} = \emptyset$$

Assume $Z \neq \emptyset, Z' = \emptyset$ and denote $z = \min Z$. We have $\{y \in Y : y < z\} = \{y \in Y \cap Y' : y < n\} = \{y \in Y' : y < n\}$, which implies $z = n$ and forms a contradition. By symmetry, $Z = \emptyset, Z' \neq \emptyset$ also impossibly holds. Assume $Z, Z' \neq \emptyset$ and denote $z' = \min Z'$. We have $\{y \in Y : y < z\} = \{y \in Y \cap Y' : y < n\} = \{y \in Y' : y < z'\}$, which implies $z = z'$ and forms a contradition. Thus there must be $Z = Z' = \emptyset$, which indicates that (1) holds.

Now assume $Y \backslash Y', Y' \backslash Y \neq \emptyset$ and denote $x = \min Y \backslash Y', x' = \min Y' \backslash Y$. Because of (1) we have $\forall y \in Y \cap Y', x > y, x' > y$, then

$$\{y \in Y : y < x\} = \{y \in Y' : y < x'\} = Y \cap Y'$$

which leads to $x = x'$ and forms a contradiction. Thus either $Y \backslash Y' = \emptyset$ or $Y' \backslash Y = \emptyset$ and hence either $Y \subset Y'$ or $Y' \subset Y$ is true.

Next we prove $Y_\infty = \cup_{Y \in \mathcal{G}} Y$ is well ordered. For any $\emptyset \neq A \subset Y_\infty$, there exists $x \in A, Y \in \mathcal{G}$ such that $x \in Y$, thus $\emptyset \neq A \cap Y \subset Y$. Denote $a = \min(A \cap Y)$. Since for any $Y' \in \mathcal{G}$, every element in $Y' \backslash Y$ are greater than a, we can derive $a = \min(A \cap Y_\infty) = \min A$.

It is clear that $Y_\infty \in \mathcal{G}$ and furthermore $Y_\infty \cup \{f(Y_\infty)\} \in \mathcal{G}$, thus $Y_\infty \cup \{f(Y_\infty)\} \subset Y_\infty \Rightarrow f(Y_\infty) \in Y_\infty$, which forms a contradiction. □

Theorem 5 (Zorn's lemma)**.** *Suppose X is an nonempty partially ordered set where every totally ordered subset has an upper bound, then X has a maximal element.*

Proof. Assume that X has no maximal element, then every totally subset of X has a strict upper bound, which contrdicts theorem 4. □

Theorem 6 (Well ordering principle)**.** *Every nonempty set X can be well ordered.*

Proof. Suppose that $\mathcal{P}$ is the set of partially order on the subsets of X, $\leq_1, \leq_2$ are well orders on X_1, X_2, which are both subsets of X. We refer that $\leq_1 \preceq \leq_2$ iff $X_1 \subset X_2$ and $\leq_2 |_{X_1} = \leq_1$. It is clear that $(\mathcal{P}, \preceq)$ is a partially ordered set. For any totally ordered subset $\mathcal{T} \subset \mathcal{P}$, we define $a \leq_\infty b$ iff there exists $\leq \in \mathcal{T}$ satisfies that $a \leq b$. It can be examined that $\leq_\infty$ is an upper bound of $\mathcal{T}$, thus through theorem 5, there exists a maximal element $\leq_M \in \mathcal{P}$. Here $\leq_M$ must be defined on X, or by denoting the domain by M and $x \in X \backslash M$, we can define $a \leq_N b$ iff $a, b \in M$ and $a \leq_M b$ and $a \leq_N x$ for any $a \in M$, where $\leq_M \preceq \leq_N$. □

0.2 Cardinality

We refer that $\mathrm{card}(X) \leq \mathrm{card}(Y), \mathrm{card}(X) = \mathrm{card}(Y), \mathrm{card}(X) \geq \mathrm{card}(Y)$ iff there exists injective, bijective, and surjective from X to Y, respectively. In the following context we verify that $\leq$ is a totally order.

Theorem 7 (Schröder-Bernstein)**.** *Suppose that* $\mathrm{card}(X) \leq \mathrm{card}(Y)$ *and* $\mathrm{card}(X) \geq \mathrm{card}(Y)$ *both hold, then* $\mathrm{card}(X) = \mathrm{card}(Y)$*.*

Proof. Let $f : X \to Y, g : Y \to X$ be injections, $\square$

Theorem 8. *For any sets* X *and* Y*, either* $\mathrm{card}(X) \leq \mathrm{card}(Y)$ *or* $\mathrm{card}(X) \geq \mathrm{card}(Y)$ *holds.*

Proof. Consider the set of injections from subset of X to Y. We refer that $f \preceq g$ iff $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $g|_{\mathrm{dom}(f)} = f$, which can be verified to satisfy the condition of theorem 5. Thus there exists a maximal injection h, where either $\mathrm{dom}(h) = X$ or $\mathrm{image}(h) = Y$ holds. $\square$