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CS7GV1 Computer vision

PCA

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Introduction

We have seen that images can be reconstructed efficiently with wavelets.

We introduce Principal Component Analysis (PCA) as a way to learn a basis of functions (~ learnt DCT wavelets!).

PCA is also associated with the whitening operation

https://en.wikipedia.org/wiki/Whitening_transformation

and "batch normalisation" in CNN

Mean and Covariance of a set of vectors I

Consider that we have a set of vectors $\{\mathbf{x}_i\}_{i=1\cdots N}$ in $\mathbb{R}^d$. We can define

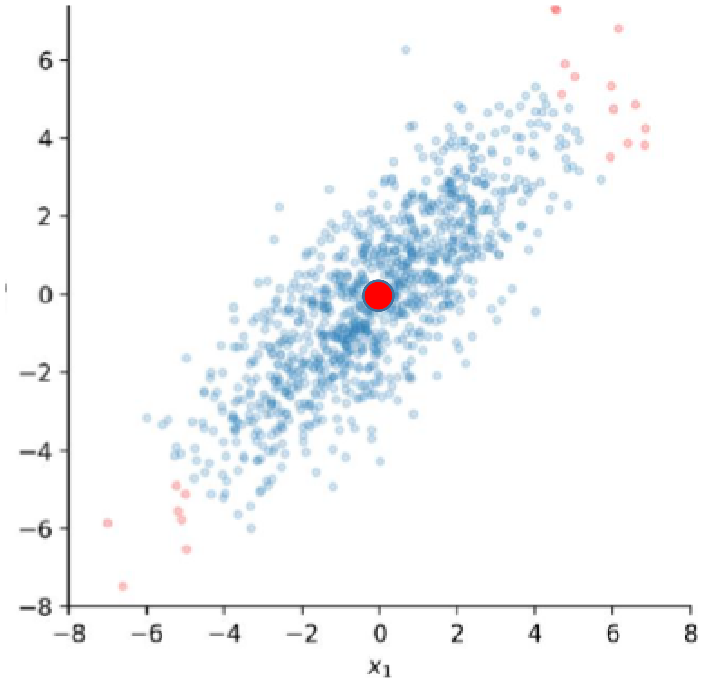
- the mean $\overline{\mathbf{x}}$ such that

$$\overline{\mathbf{x}} = \frac{\sum_{i=1}^{N} \mathbf{x}_i}{N}$$

  spatially the mean can be understood as the center of gravity of the clouds of points $\{\mathbf{x}_i\}_{i=1\cdots N}$.

- the covariance matrix:

$$\mathrm{C} = \frac{1}{N} \sum_{i=1}^{N} (\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^T$$



https://en.wikipedia.org/wiki/Multivariate_normal_distribution

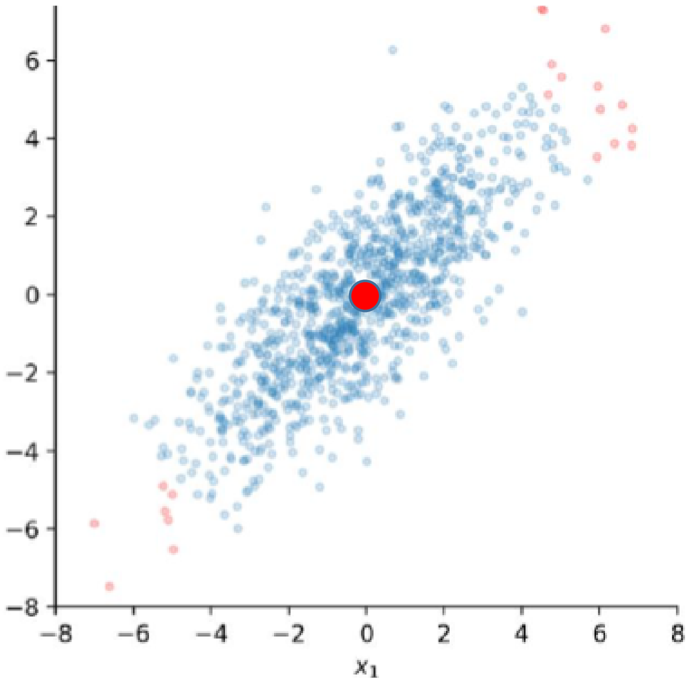
Mean and Covariance of a set of vectors II

with defining $\tilde{\mathbf{x}}_i = \mathbf{x} - \overline{\mathbf{x}}, \; \forall i$ then

$$\mathrm{C} = \frac{1}{N} \sum_{i=1}^{N} \tilde{\mathbf{x}}_i \tilde{\mathbf{x}}_i^T$$

or

$$\mathrm{C} = \frac{1}{N} \begin{bmatrix} \sum_{i=1}^{N} \tilde{x}_{1i}^2 & \sum_{i=1}^{N} \tilde{x}_{1i}\tilde{x}_{2i} & \cdots & \sum_{i=1}^{N} \tilde{x}_{1i}\tilde{x}_{di} \\ \sum_{i=1}^{N} \tilde{x}_{1i}\tilde{x}_{2i} & \sum_{i=1}^{N} \tilde{x}_{2i}^2 & & \\ & & \ddots & \\ & & \ddots & \\ & & & \sum_{i=1}^{N} \tilde{x}_{di}^2 \end{bmatrix}$$



The Lagrangian I

Definition

We consider the optimization problem:

$$\begin{array}{lll} \text{minimize} & f_0(\mathbf{x}) & \\ \text{subject to} & f_i(\mathbf{x}) = 0 & i = 1, \cdots, m \\ & h_j(\mathbf{x}) \leq 0 & j = 1, \cdots, p \end{array}$$

with $\mathbf{x} \in \mathbf{R}^d$.

The Lagrangian $\mathscr{L} : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^p$ associated with the problem is defined as:

$$\mathscr{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x})$$

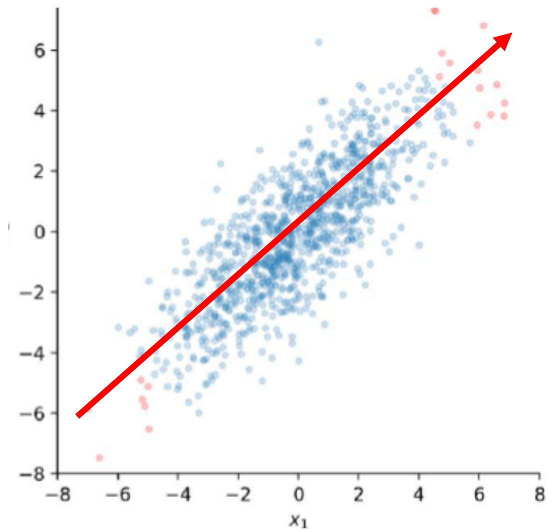
The vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are called the Lagrange multiplier vectors.

The Lagrangian III

We are mainly interested in minimizing functions with equality constraints.
Differentiating the Lagrangian $\mathscr{L}(\mathbf{x}, \boldsymbol{\lambda})$ in $\mathbf{x}$ gives d equations and differentiating the Lagrangian $\mathscr{L}(\mathbf{x}, \boldsymbol{\lambda})$ in $\boldsymbol{\lambda}$ gives m equations.
Solving the optimization problem then become solving

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = 0 \\ \frac{\partial \mathscr{L}}{\partial \boldsymbol{\lambda}} = 0 \end{cases}$$

Determining Principal components I



Consider that we have a set of vectors $\{\mathbf{x}_i\}_{i=1 \cdots N}$ in $\mathbb{R}^d$ arranged in a matrix X:

$$\mathrm{X} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,N} \\ \vdots & & & \vdots \\ x_{d,1} & x_{d,2} & \cdots & x_{d,N} \end{bmatrix}$$

we are looking in a direction or vector $\mathbf{v} \in \mathbb{R}^d$ such that the projections of $\{\mathbf{x}_i\}_{i=1 \cdots N}$ on $\mathbf{v}$ leads to the scatter of N points with the highest dispersion.

Determining Principal components II

v: unit vector

- The projection of $\mathbf{x}_i$ on to $\mathbf{v}$ is $\mathbf{v}\mathbf{v}^T\mathbf{x}_i$.

- The distance between two projections is

$$\begin{aligned}\|\mathbf{v}\mathbf{v}^T\mathbf{x}_i - \mathbf{v}\mathbf{v}^T\mathbf{x}_j\|^2 &= (\mathbf{v}\mathbf{v}^T\mathbf{x}_i - \mathbf{v}\mathbf{v}^T\mathbf{x}_j)^T(\mathbf{v}\mathbf{v}^T\mathbf{x}_i - \mathbf{v}\mathbf{v}^T\mathbf{x}_j) \\ &= \mathbf{v}^T(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T\mathbf{v}\end{aligned}$$

- Considering all the vectors $\{\mathbf{x}_i\}_{i=1\cdots N}$, the criterion to be maximized is:

$$\begin{aligned}\mathscr{J}(\mathbf{v}) &= \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{v}^T(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T\mathbf{v} \\ &= \mathbf{v}^T\mathrm{C}\mathbf{v}\end{aligned}$$

Determining Principal components III

so the problem can be summarized as finding $\mathbf{v}$ such that:

$$\begin{cases} \max_{\mathbf{v}} \mathscr{J}(\mathbf{v}) \\ \text{subject to } \ \mathbf{v}^T\mathbf{v} = 1 \end{cases}$$

Using the Lagrange multipliers, the equivalent problem is:

$$\max \mathscr{J}(\mathbf{v}) - \lambda(\mathbf{v}^T\mathbf{v} - 1)$$

Determining Principal components IV

The solution is found by solving $\frac{\partial \mathscr{J}(\mathbf{v})}{\partial \mathbf{v}} = 0$ and $\frac{\partial \mathscr{J}(\mathbf{v})}{\partial \lambda} = 0$:

$$\begin{cases} \mathrm{C}\mathbf{v} - \lambda \mathbf{v} = 0 \\ \mathbf{v}^T \mathbf{v} = 1 \end{cases}$$

So $\mathbf{v}$ is an eigenvector of C, and $\mathscr{J}(\mathbf{v}) = \lambda$. Then to get the biggest dispersion we should choose the eigenvector associated with the highest eigenvalue. Hence the name of this method principal component analysis.

This result can be generalized: the k principal directions are the k eigendirections of the highest eigenvalues.

Determining Principal components V

Theorem (Principal Component Analysis)

From a set of vectors $\{\mathbf{x}_i\}$

1. *Compute the mean* $\overline{\mathbf{x}}$
2. *Center each observations* $\tilde{\mathbf{x}}_i = \mathbf{x}_i - \overline{\mathbf{x}}$
3. *Compute the covariance matrix*

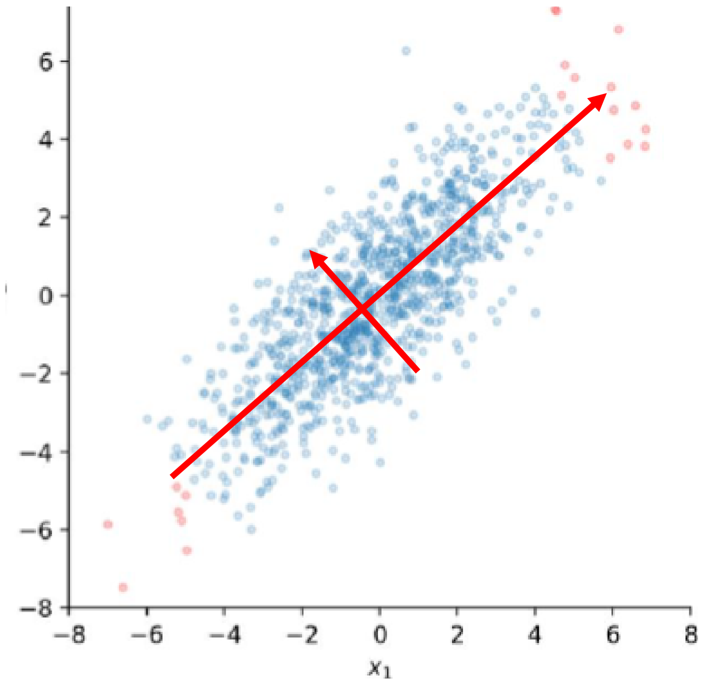
$$\mathrm{C} = \frac{1}{N}\sum_{i=1}^{N} \tilde{\mathbf{x}}_i \tilde{\mathbf{x}}_i^T$$

4. *Compute the eigenvectors of* C *and sort them from the one associated with the highest eigenvalue , to the one associated with the lowest eigenvalue.*

Using PCA

Each vector in the set can be written as a linear combination of the mean and the eigenvectors:

$$\mathbf{x} = \overline{\mathbf{x}} + \sum_j \alpha_j \mathbf{v}_j$$

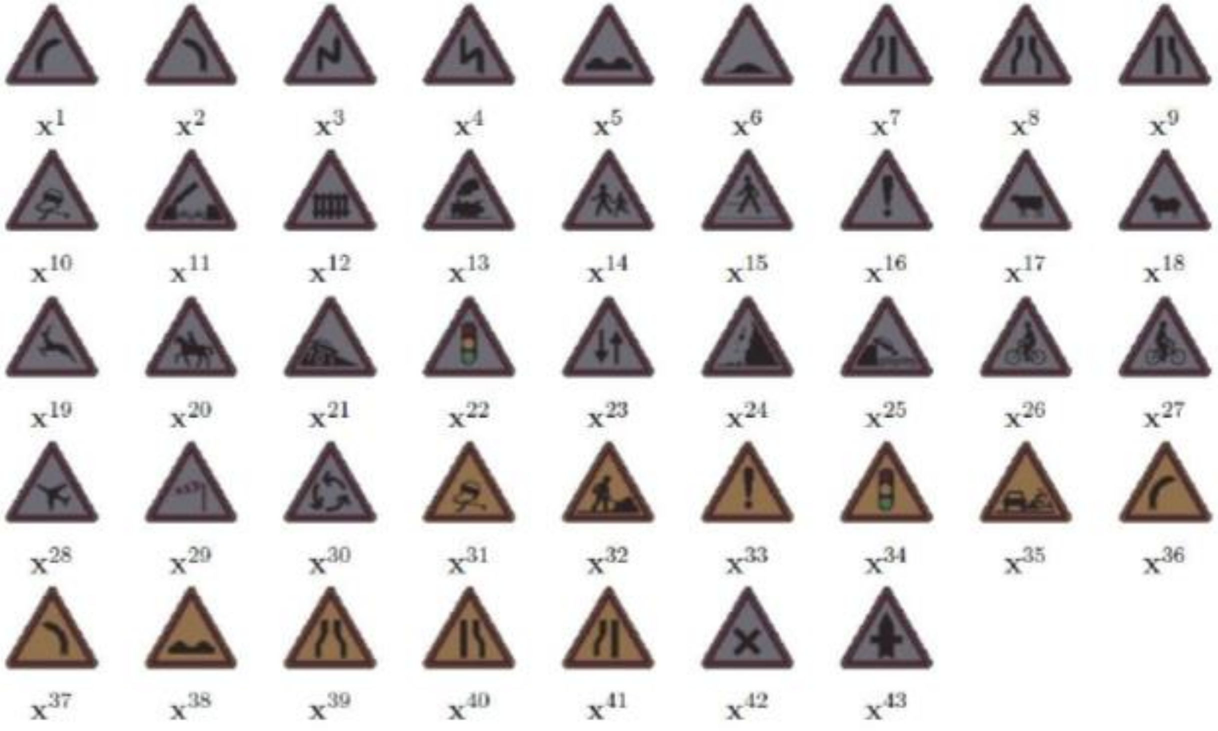


How many eigenvectors really needed?

Applications:

- Compression / Dimensionality reduction
- Visualisation of data distribution
- etc.

Training Dataset



vector embedding of image data

Pixels $p_1\ p_2 \ldots p_N$

$$\mathbf{x} = \begin{pmatrix} r_1 \\ g_1 \\ b_1 \\ \vdots \\ r_N \\ g_N \\ b_N \end{pmatrix}$$

Robust Visual Recognition of Colour Images
R. Dahyot at al, IEEE conference on Computer Vision and Pattern Recognition (CVPR'00), DOI:10.1109/CVPR.2000.855886

Training set with Data Augmentation

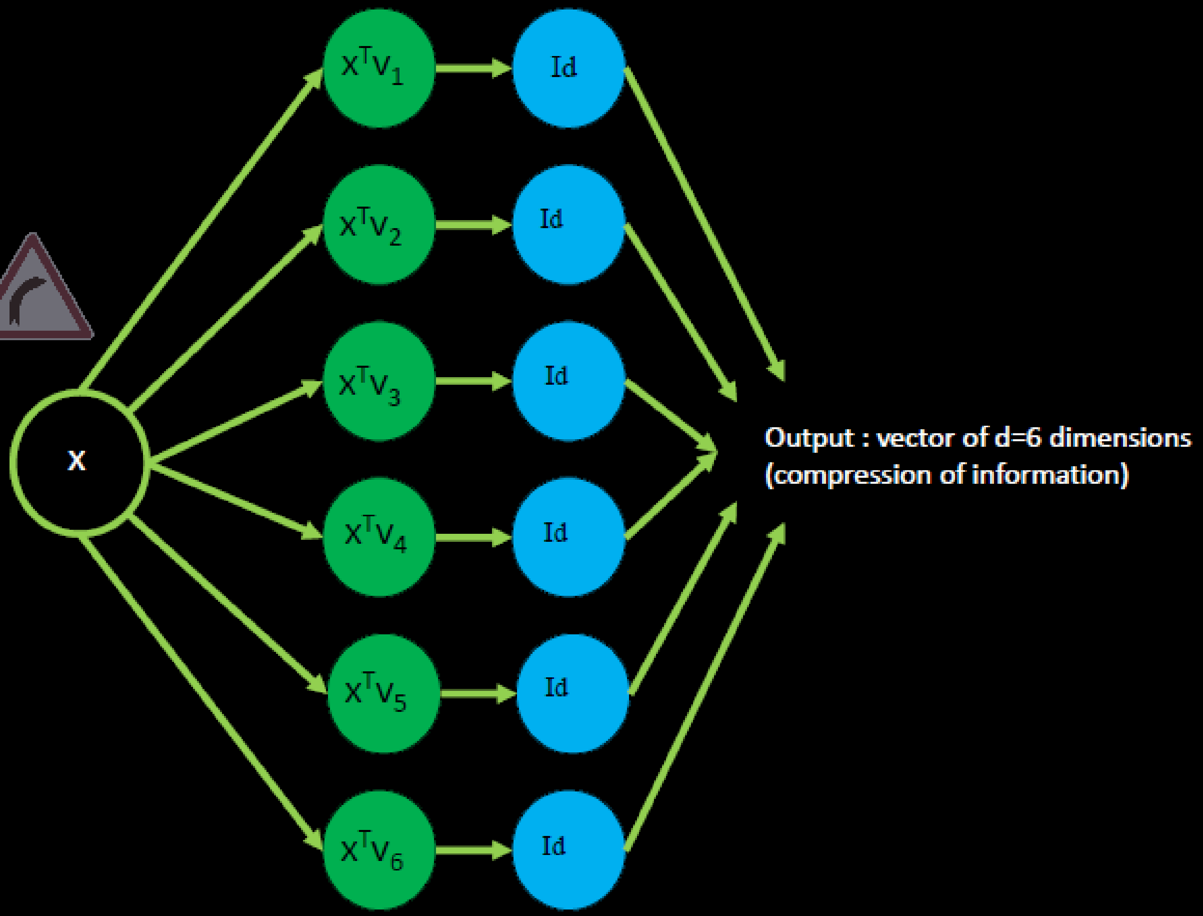
Example data augmentation with rotation

[Robust Visual Recognition of Colour Images](#)
R. Dahyot at al, IEEE conference on Computer Vision and Pattern Recognition (CVPR'00), DOI:10.1109/CVPR.2000.855886

Training dataset can be augmented by generated imaged, e.g. using geometric transformation

https://www.tensorflow.org/tutorials/images/data_augmentation
https://en.wikipedia.org/wiki/Data_augmentation

Learning the Principal Components

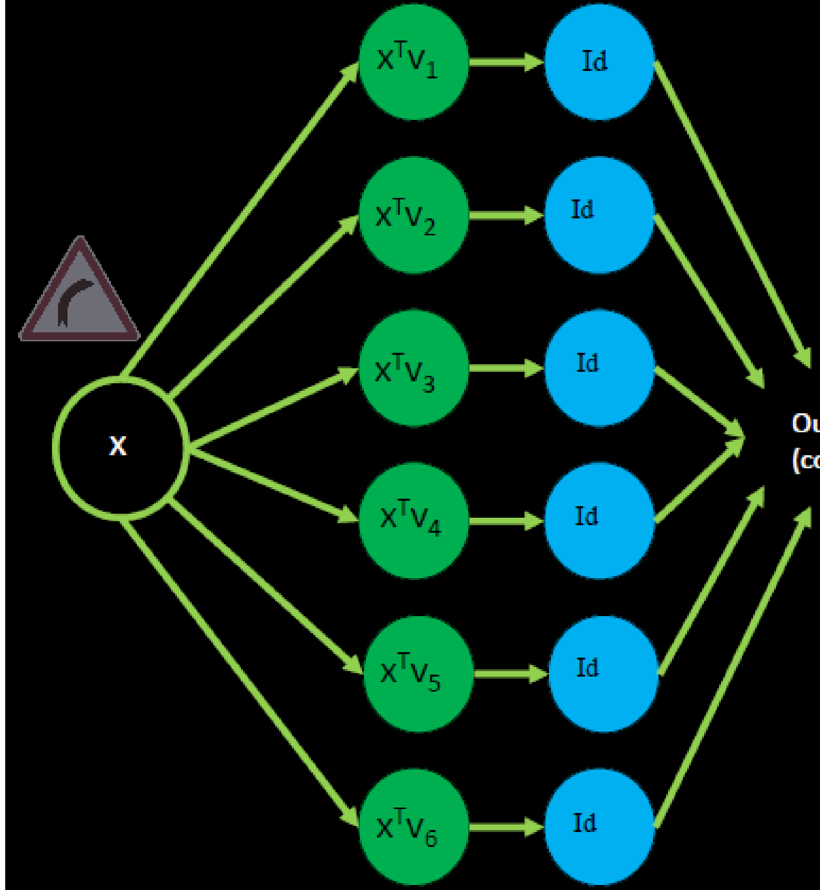


$$\begin{aligned}\mathscr{J}(\mathbf{v}) &= \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{v}^T(\mathbf{x}_i-\mathbf{x}_j)(\mathbf{x}_i-\mathbf{x}_j)^T\mathbf{v}\\ &= \mathbf{v}^T C\mathbf{v}\end{aligned}$$

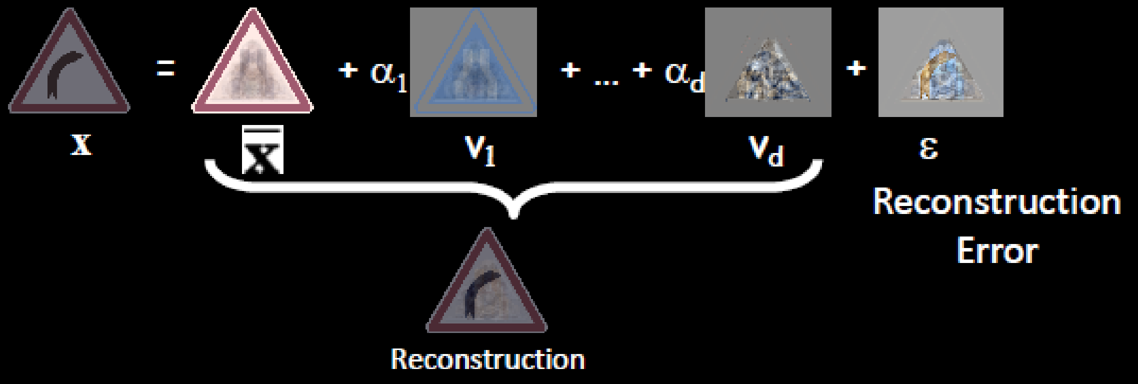
$$\begin{cases} \max_{\mathbf{v}} \mathscr{J}(\mathbf{v}) \\ \text{subject to} \quad \mathbf{v}^T\mathbf{v} = 1 \end{cases}$$

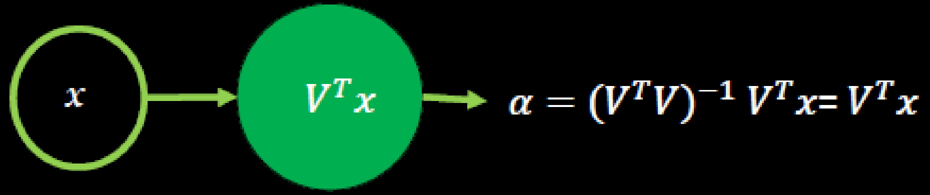
- **Projection** with Basis of learnt vectors $\{v_j\}$. Learning is performed as a solution for minimising a cost function *J* defined the training dataset (with some constraints)
- **Activation** function is chosen as the identity function.

Compression with PCA



Output : vector of d=6 dimensions $\alpha_1, \ldots, \alpha_d$
(compression of information)





<u>No cost function to optimise</u> at <u>inference time</u>

Eigenvalues of principal components

Fig. 10. The first eight eigenfaces.

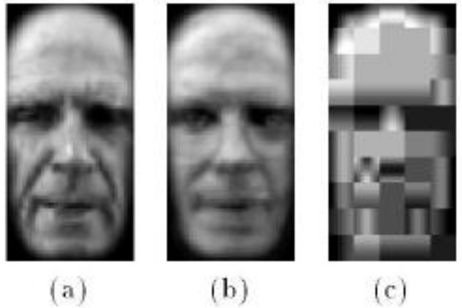
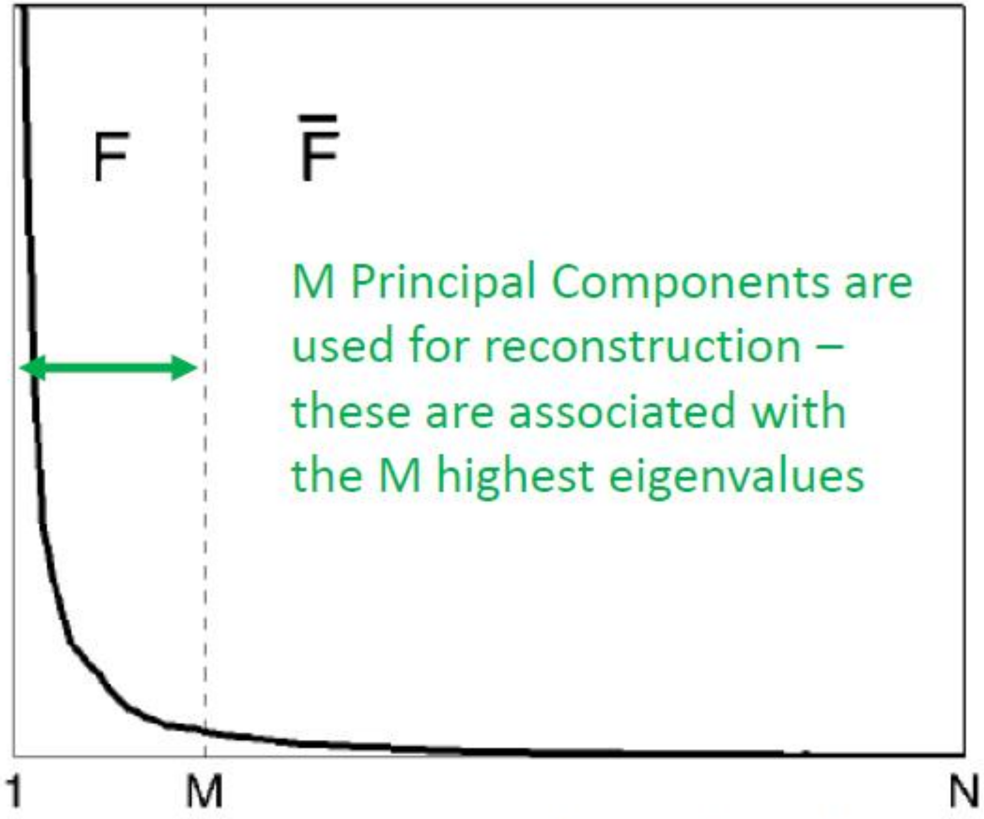


Figure 9: *(a) aligned face, (b) eigenspace reconstruction (85 bytes) (c) JPEG reconstruction (530 bytes).*

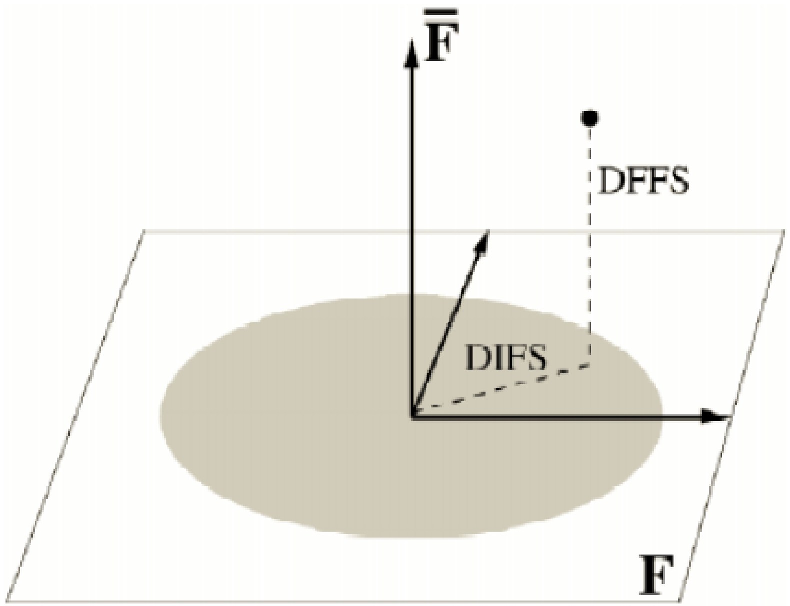
Eigenvalue spectrum obtained by PCA



Notation: N=dim(x)

Probabilistic visual learning for object representation, IEEE Trans. Pattern Analysis and Machine Intelligence (1997), DOI: 10.1109/34.598227

Modelling P(x) using PCA

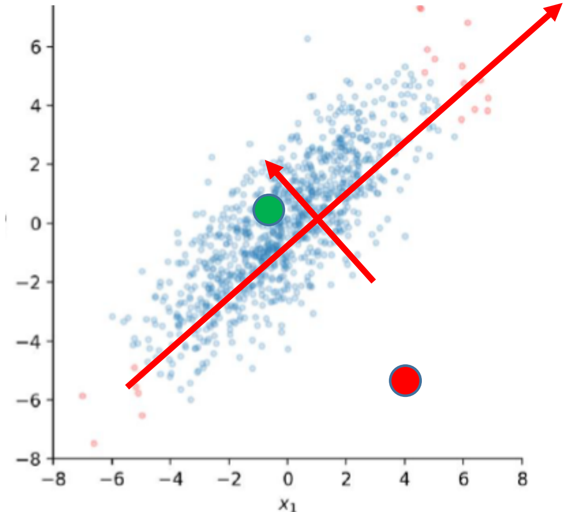


(a)

<u>Notation</u>: $y_i = x^T v_i$
projection on principal component v_i
associated with eigenvalue λ_i

$$\hat{P}(\mathbf{x}\,|\,\Omega) = \left[\frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{M}\frac{y_i^2}{\lambda_i}\right)}{(2\pi)^{M/2}\prod_{i=1}^{M}\lambda_i^{1/2}}\right]\cdot\left[\frac{\exp\left(-\frac{\epsilon^2(\mathbf{x})}{2\rho}\right)}{(2\pi\rho)^{(N-M)/2}}\right]$$

$$= P_F(\mathbf{x}\,|\,\Omega)\,\hat{P}_{\bar{F}}(\mathbf{x}\,|\,\Omega)$$



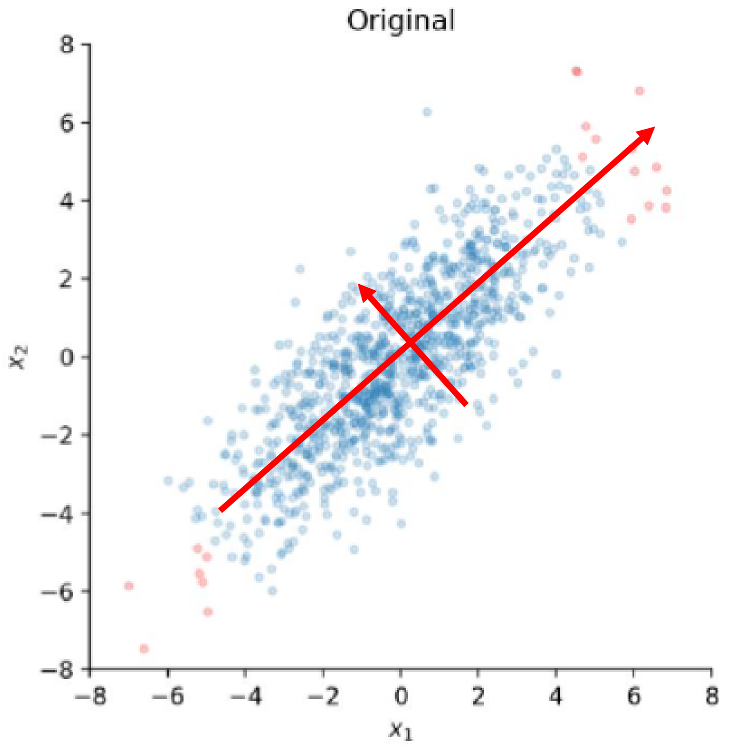
[Probabilistic visual learning for object representation](), IEEE Trans. Pattern Analysis and Machine Intelligence (1997), DOI: 10.1109/34.598227

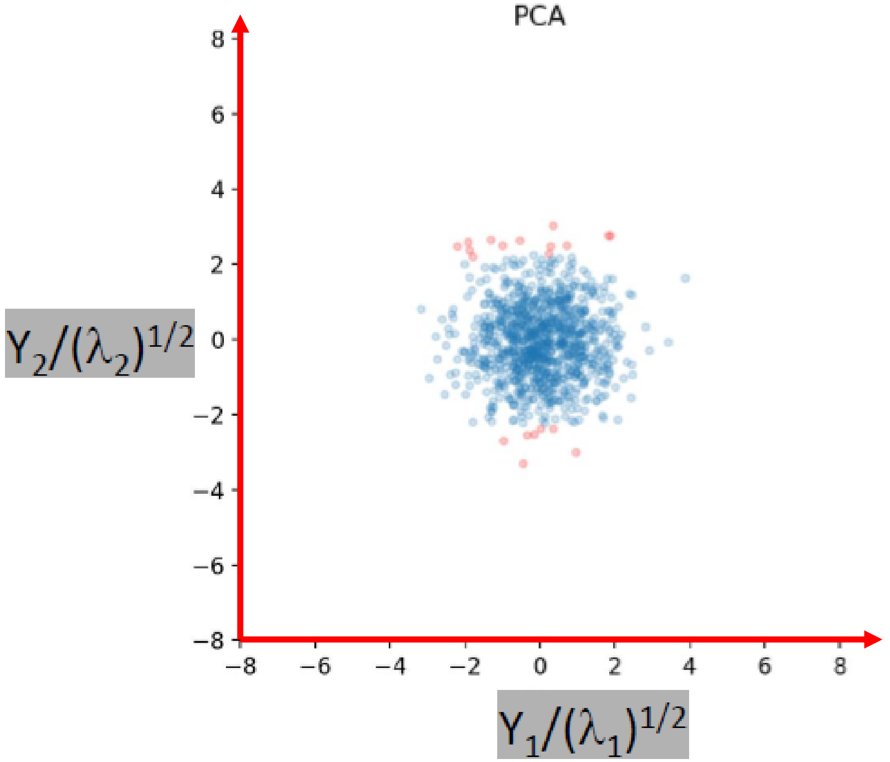
Whitening

https://cbrnr.github.io/2018/12/17/whitening-pca-zca/
https://en.wikipedia.org/wiki/Whitening_transformation

Notation: $y_i = x^T v_i$
projection on principal component v_i
associated with eigenvalue λ_i

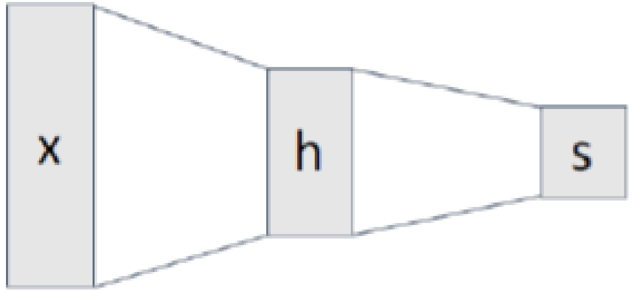
Whitening with PCA



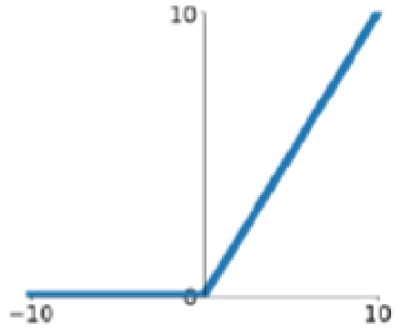


Components of a Convolutional Network

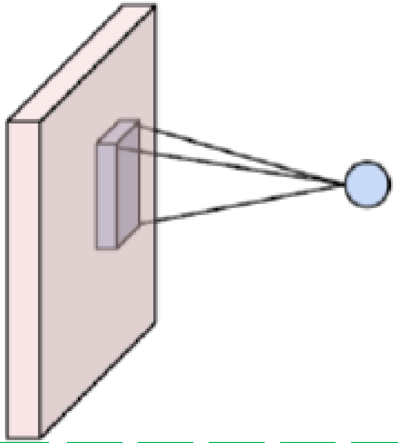
Fully-Connected Layers



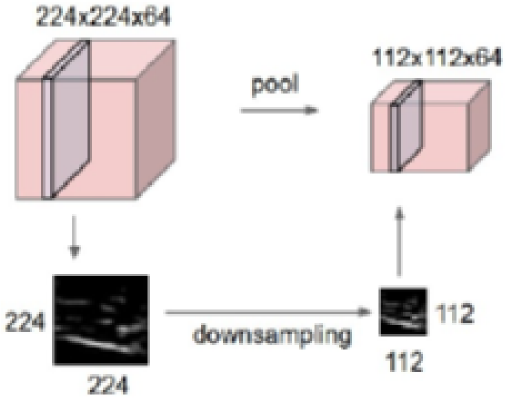
Activation Function



Convolution Layers



Pooling Layers



Normalization

$$\hat{x}_{i,j} = \frac{x_{i,j} - \mu_j}{\sqrt{\sigma_j^2 + \varepsilon}}$$

Batch Normalization

Idea: "Normalize" the outputs of a layer so they have zero mean and unit variance

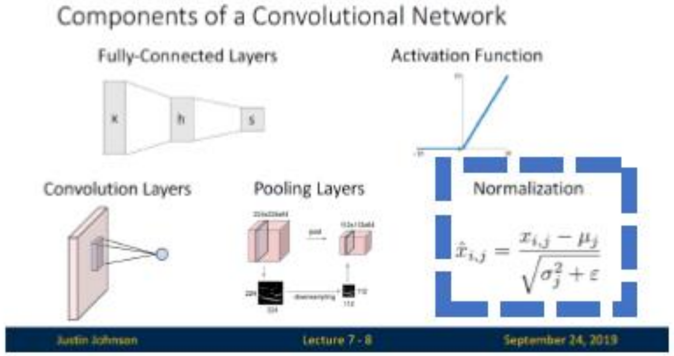
Why? Helps reduce "internal covariate shift", improves optimization

We can normalize a batch of activations like this:

$$\widehat{x}^{(k)} = \frac{x^{(k)} - \mathrm{E}[x^{(k)}]}{\sqrt{\mathrm{Var}[x^{(k)}]}}$$

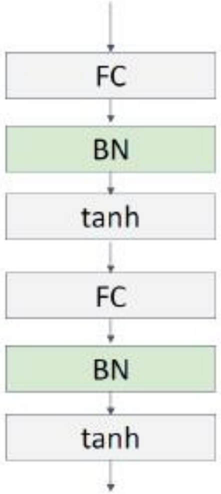
This is a **differentiable function**, so we can use it as an operator in our networks and backprop through it!

Ioffe and Szegedy, "Batch normalization: Accelerating deep network training by reducing internal covariate shift", ICML 2015



Lecture 7: Convolutional Networks (Michigan Online 2020)
https://youtu.be/ANyxBVxmdZ0

Batch Normalization



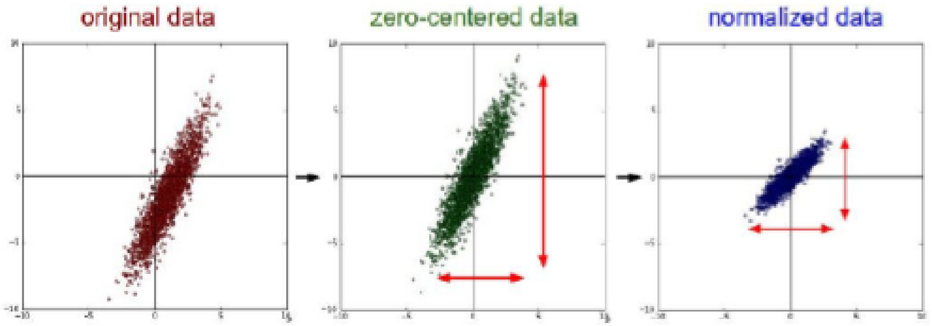
- Makes deep networks **much** easier to train!
- Allows higher learning rates, faster convergence
- Networks become more robust to initialization
- Acts as regularization during training
- Zero overhead at test-time: can be fused with conv!
- Not well-understood theoretically (yet)
- Behaves differently during training and testing: this is a very common source of bugs!

Ioffe and Szegedy, "Batch normalization: Accelerating deep network training by reducing internal covariate shift", ICML 2015

Whitening http://cs231n.stanford.edu/slides/2018/cs231n_2018_lecture07.pdf

Decorrelated Batch Normalization

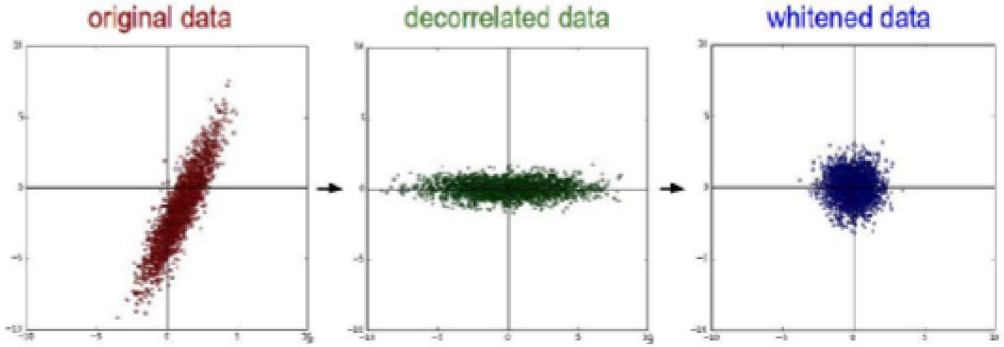
Batch Normalization



$$\hat{x}_{i,j} = \frac{x_{i,j} - \mu_j}{\sqrt{\sigma_j^2 + \varepsilon}}$$

BatchNorm normalizes the data, but cannot correct for correlations among the input features

Decorrelated Batch Normalization



$$\hat{x}_i = \Sigma^{-\frac{1}{2}}(x_i - \mu)$$

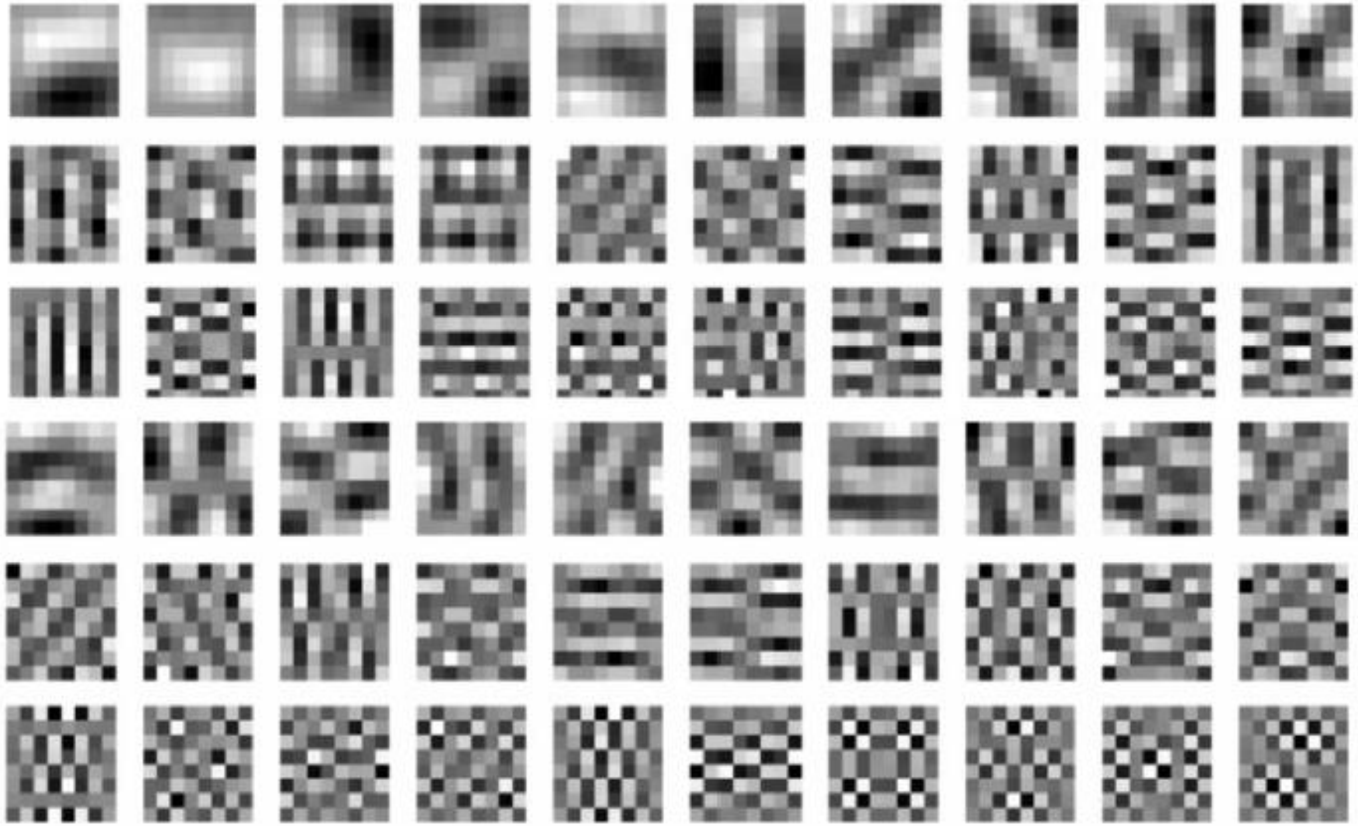
DBN **whitens** the data using the full covariance matrix of the minibatch; this corrects for correlations

Huang et al, "Decorrelated Batch Normalization", arXiv 2018 (Appeared 4/23/2018)

PCA & DCT Wavelets

https://www.cs.cmu.edu/~mgormley/courses/10601-s17/slides/lecture18-pca.pdf

60 most important eigenvectors



Looks like the discrete cosine bases of JPG!...

DCT in Neural networks:

[Harmonic Networks for Image Classification](#)
M. Ulicny, V. Krylov and R. Dahyot, British Machine Vision Conference (BMVC) 2019.

Harmonic Networks with Limited Training Samples
M. Ulicny, V. Krylov and R. Dahyot, European Signal Processing Conference (Eusipco) 2019.

Summary

- PCA allows to have a learnt orthogonal basis of vectors to represent a set of vectors (images).
- PCA basis is learnt from a training dataset – In comparison Wavelet and Fourier basis are not learnt from a dataset!
- Hence PCA basis is a fine tuned basis of vectors tailored to the training dataset
- Similarly CNN will be learning its own filters/kernel for convolutions from a training dataset
- The whitening transformation used with PCA is also a type of operation used in CNN/NN (== “batch normalisation”)