A Review of Applied Probability Theory

Discrete Random Variables

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Autumn 2022

Discrete Random Variables

- In this set of slides, we introduce a couple of discrete random variables and discuss their properties.

1- Bernoulli Distribution

- **Bernoulli Trials.** An experiment with 2 complement outcomes, which are usually called ***Success*** and ***Failure***, with probabilities $P(S) = p\,, P(F) = q \quad (p + q = 1)$.

- X denotes the number of success in 1 Bernoulli trial.

$$f(x) = P(X = x) = \begin{cases} p & x = 1 \\ q & x = 0 \end{cases} = p^x q^{1-x} \quad x = 0,1$$

$$E[X] = \sum_x x.f(x) = 0 \times q + 1 \times p = p$$

$$E[X^2] = \sum_x x^2.f(x) = 0^2 \times q + 1^2 \times p = p$$

$$Var[X] = E[X^2] - E^2[X] = p - p^2 = p(1 - p) = pq$$

$$m_X(t) = E[e^{tX}] = \sum_x e^{tx} f(x) = e^{t\times 0} \times q + e^{t\times 1} \times p = q + pe^t$$

2- Binomial Distribution

X denotes the number of successes in n identical-independent Bernoulli trial.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} \quad x = 0,1, \dots , n$$

A Binomial random variable is the sum of n identical-independent Bernoulli random variables

$$X = X_1 + X_2 + \cdots + X_n$$

Therefore

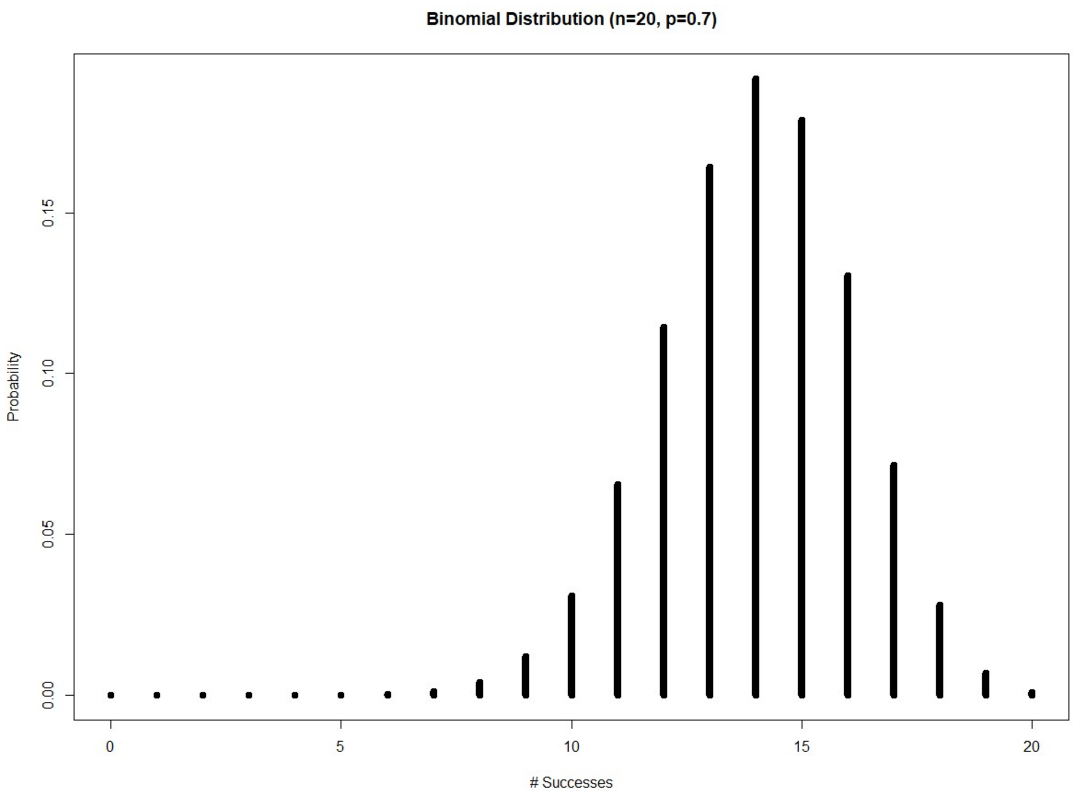
$$E[X] = E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = np$$

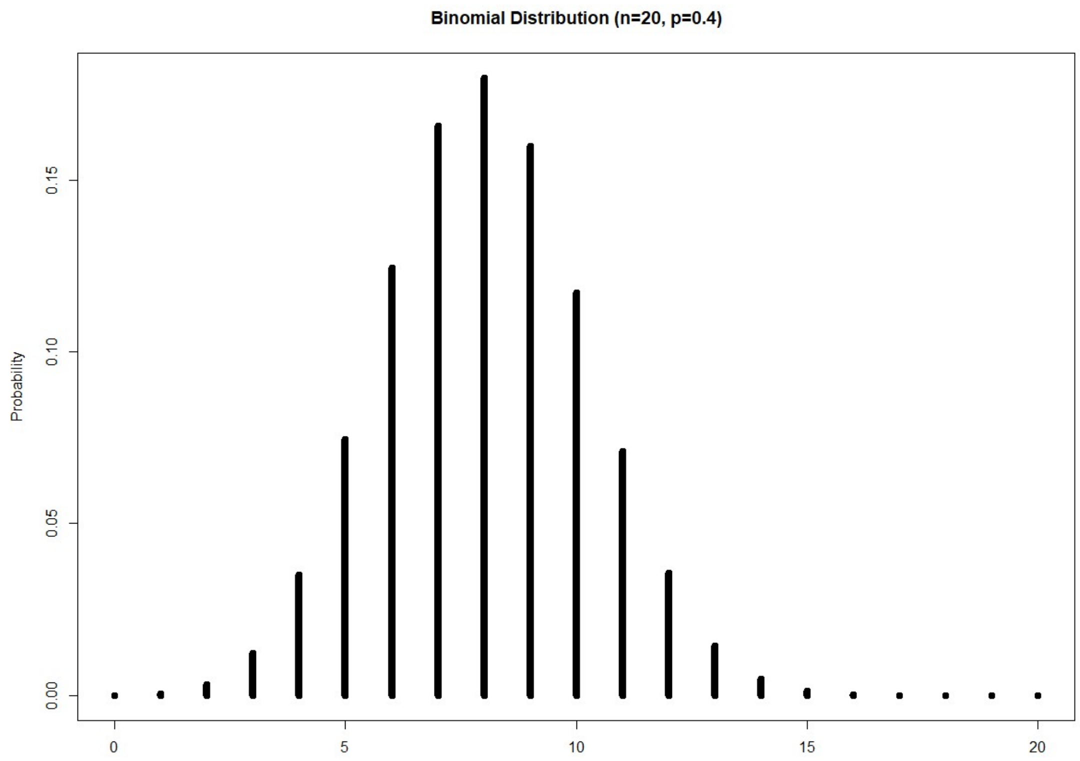
$$Var[X] = Var[X_1 + X_2 + \cdots + X_n] =^{*} Var[X_1] + Var[X_2] + \cdots + Var[X_n] = npq$$

$$m_X(t) =^{**} \prod_i m_{X_i}(t) = (q + pe^t) \dots (q + pe^t) = (q + pe^t)^n$$

*&** We'll prove these later.

2- Binomial Distribution





For any values of n and p, the function $f(x)$ is increasing until “somewhere” and then decreasing. This “somewhere” is actually the “Mode” of the distribution.

2- Binomial Distribution

If $\hat{x}$ denotes the Mode, we'll have:

number of successes in n identical-independent Bernoulli trial.

$$\begin{cases} f(\hat{x}) \geq f(\hat{x}-1) \\ \& \\ f(\hat{x}) \geq f(\hat{x}+1) \end{cases} \rightarrow \begin{cases} \binom{n}{\hat{x}} p^{\hat{x}} q^{n-\hat{x}} \geq \binom{n}{\hat{x}-1} p^{\hat{x}-1} q^{n-(\hat{x}-1)} \\ \& \\ \binom{n}{\hat{x}} p^{\hat{x}} q^{n-\hat{x}} \geq \binom{n}{\hat{x}+1} p^{\hat{x}+1} q^{n-(\hat{x}+1)} \end{cases}$$

By solving the above we will get

$$(n+1)p - 1 \leq \hat{x} \leq (n+1)p$$

3- Poisson Distribution

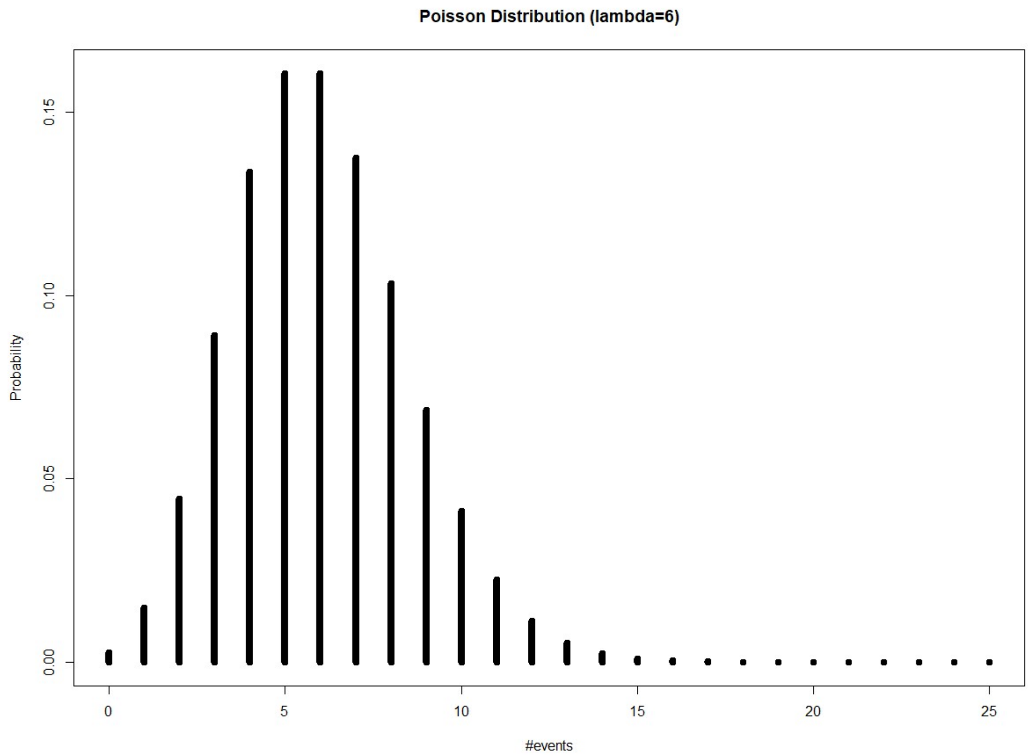
X denotes the number of independent events.

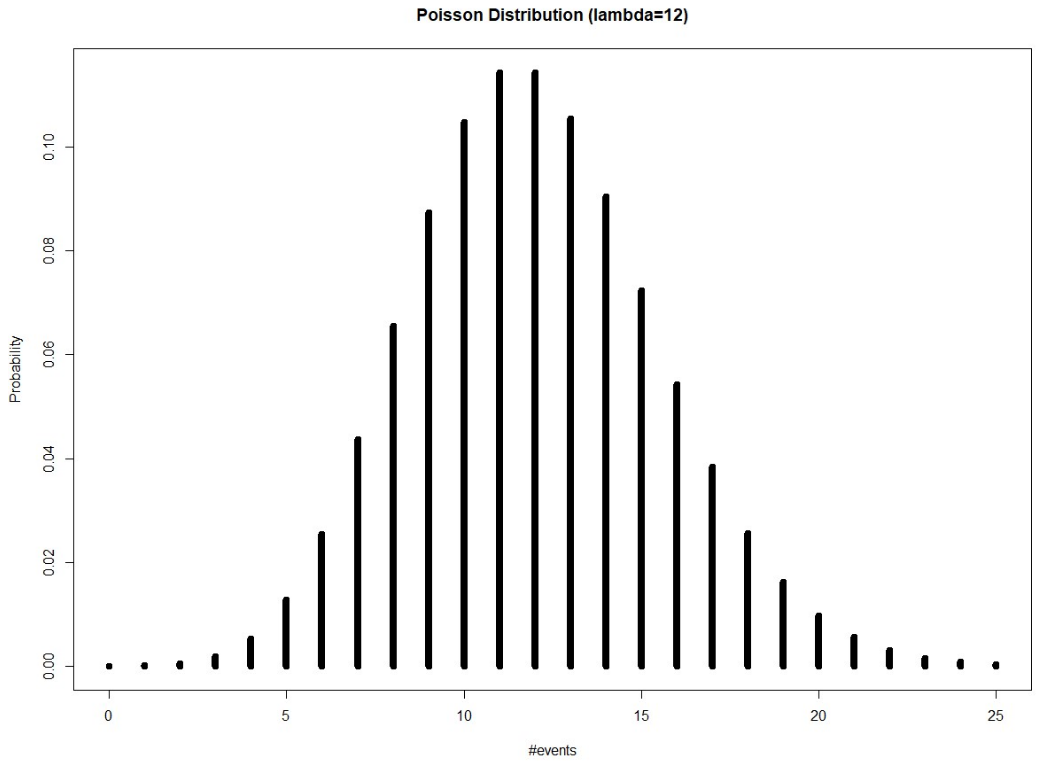
$$f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0,1,2,\ldots$$

$$E[X] = \lambda \qquad Var[X] = \lambda \qquad m_X(t) = e^{\lambda(e^t - 1)}$$

As number of events is meaningful when the time interval is known, λ is usually shown as rt, where r is the rate of events and t is the interval that the events are being studied.

3- Poisson Distribution





For any values of λ, the function $f(x)$ is increasing until "somewhere" and then decreasing. This "somewhere" is actually the "Mode" of the distribution.

2- Poisson Distribution

If $\hat{x}$ denotes the Mode, we'll have:

number of successes in n identical-independent Bernoulli trial.

$$\begin{cases} f(\hat{x}) \geq f(\hat{x}-1) \\ f(\hat{x}) \geq f(\hat{x}+1) \end{cases} \rightarrow \begin{cases} \dfrac{e^{-\lambda}\lambda^{\hat{x}}}{\hat{x}!} \geq \dfrac{e^{-\lambda}\lambda^{\hat{x}-1}}{(\hat{x}-1)!} \\ \dfrac{e^{-\lambda}\lambda^{\hat{x}}}{\hat{x}!} \geq \dfrac{e^{-\lambda}\lambda^{\hat{x}+1}}{(\hat{x}+1)!} \end{cases}$$

By solving the above we will get

$$\lambda - 1 \leq \hat{x} \leq \lambda$$

3- Poisson Distribution

- **Theorem 1 (sum of independent Poisson rvs).** If X and Y are independent Poisson random variables with parameters λ_1 and λ_2, the random variable $Z = X + Y$ has a Poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2$.

- **Theorem 2 (Poisson approximation for Binomial rv).** For a Binomial rv X with parameters n and p, if $n \rightarrow \infty$ and $p \rightarrow 0$, the Binomial probabilities can be approximated using a Poisson distribution with $\lambda = np$, in other words:

$$P(X = x_0) = \binom{n}{x_0} p^{x_0} q^{n-x_0} \approx \frac{e^{-np}(np)^{x_0}}{x_0!}$$

4- Geometric Distribution

X denotes the number of failures before the first success when repeating identical-independent Bernoulli trials.

$$f(x) = P(X = x) = q^x p \qquad\qquad x = 0,1,2,\ldots$$

$$E[X] = \frac{q}{p} \qquad Var[X] = \frac{q}{p^2} \qquad m_X(t) = \frac{p}{1 - qe^t}$$

4- Geometric Distribution

- **Theorem 1 (memoryless property of Geometric rv).**

$$P(X = m + n \mid X \geq m) = P(X = n)$$

- **Theorem 2.**

Geometric distribution is the **only** memoryless discrete distribution!

5- Negative-Binomial Distribution

X denotes the number of failures before the k^{th} success when repeating identical-independent Bernoulli trials.

$$f(x) = P(X = x) = \binom{x+k-1}{k-1} p^k q^x \qquad x = 0,1,\ldots$$

A Negative-Binomial random variable is the sum of n identical-independent Geometric random variables

$$X = X_1 + X_2 + \cdots + X_k$$

Therefore

$$E[X] = E[X_1 + X_2 + \cdots + X_k] = E[X_1] + E[X_2] + \cdots + E[X_k] = \frac{kq}{p}$$

$$Var[X] = Var[X_1 + X_2 + \cdots + X_k] =^{*} Var[X_1] + Var[X_2] + \cdots + Var[X_k] = \frac{kq}{p^2}$$

$$m_X(t) =^{**} \prod_i m_{X_i}(t) = \left(\frac{p}{1-qe^t}\right) \ldots \left(\frac{p}{1-qe^t}\right) = \left(\frac{p}{1-qe^t}\right)^k$$

*&** We'll prove these later.

6- Hyper-Geometric Distribution

X denotes the number of chips from favorite color, among n chips, taken randomly and without replacement from a box that contains k chips from the favorite color among N total number of chips.

$$f(x) = P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad , \qquad x = Max\big(0, n - (N-k)\big), \dots, min(k, n)$$

$$E[X] = n\frac{k}{N} \qquad\qquad Var[X] = n\frac{k}{N}\frac{N-k}{N} \times \frac{N-n}{N-1}$$

6- Hyper-Geometric Distribution

- **Theorem (Binomial approximation for Hyper-Geometric rv).**
- For a Hyper-Geometric rv X with parameters N, n, k, if $\frac{n}{N} \leq 0.1$, the Hyper-Geometric probabilities can be approximated using a Binomial distribution with $p = \frac{k}{N}$. In other words: $\left(\frac{k}{N}\right)$

$$f(x) = P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \approx \binom{n}{x}\left(\frac{k}{N}\right)^{x}\left(\frac{N-k}{N}\right)^{n-x}$$

7- Uniform Distribution

X denotes the observed number on a card, taken randomly from a set of numbered cards from 1 to n.

$$f(x) = P(X = x) = \frac{1}{n} \quad , \qquad x = 1, 2, \dots, n$$

$$E[X] = \frac{n+1}{2} \quad , \qquad Var[X] = \frac{n^2 - 1}{12} \quad , \qquad m_X(t) = \frac{1 - e^{(n+1)t}}{n(1 - e^t)}$$