A Review of Applied Probability Theory

Continuous Random Variables

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Continuous Random Variables

- In this set of slides, we introduce a couple of continuous random variables and discuss their properties. Before this, we review some general properties of continuous R.Vs and related functions.
- Properties of the pdf of a continuous R.V:

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Continuous Random Variables cont.

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

$$f(x) = \frac{dF(x)}{dx}$$

$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} f(x)dx = F(\beta) - F(\alpha)$$

$$P(X = \alpha) = \lim_{\varepsilon \to 0} \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} f(x)dx = \lim_{\varepsilon \to 0} [F(\alpha + \varepsilon) - F(\alpha - \varepsilon)] = 0$$

1- Uniform Distribution

$$f(x) = k \qquad a \le x \le b$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{a}^{b} kdx = k(b-a) = 1 \quad \rightarrow \quad k = \frac{1}{b-a}$$

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

$$E[X] = \int_{a}^{b} x \frac{1}{b-a} dx = \frac{a+b}{2} \qquad , \qquad Var[X] = E[X^2] - E^2[X] = \frac{(b-a)^2}{12}$$

$$m_X(t) = E[e^{tX}] = \int_{a}^{b} e^{tx} \frac{1}{b-a} dx = \frac{e^{tb} - e^{ta}}{b-a}$$

$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} \frac{1}{b-a} dx = \frac{\beta - \alpha}{b-a} \qquad (a < \alpha < \beta < b)$$

2- Exponential Distribution

$$f(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

$$E[X] = \frac{1}{\lambda} \qquad , \qquad Var[X] = \frac{1}{\lambda^2}$$

$$m_X(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t} \quad t < \lambda$$

The Exponential distribution is used to model the lifetime of electronic components.

2- Exponential Distribution cont.

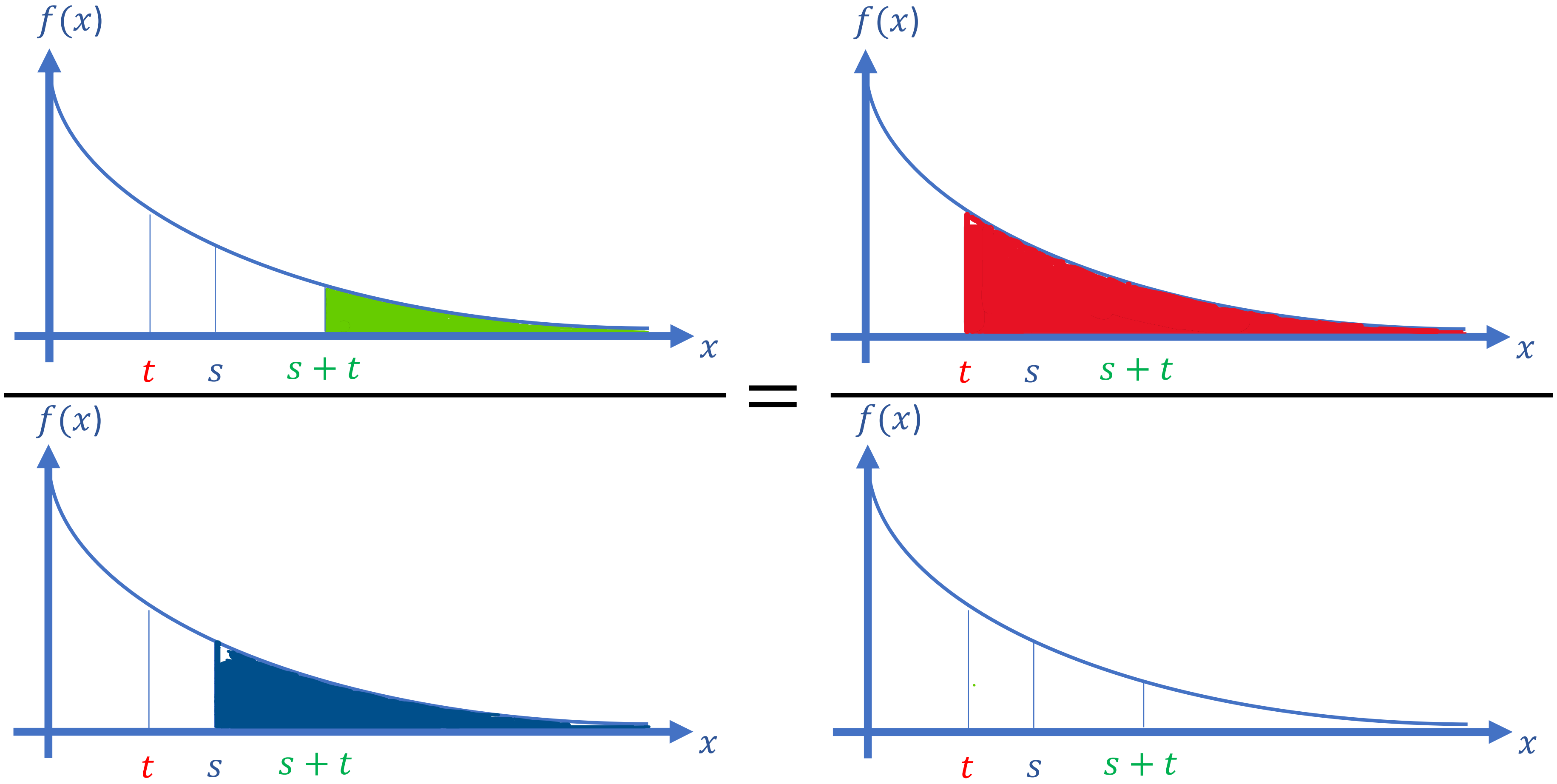
- **Theorem 1 –** Exponential distribution is memoryless.

$$P(X > s + t \mid X > s) = P(X > t)$$

$$P(X > s + t \mid X > s) = \frac{\boldsymbol{P(X > s + t)}}{\boldsymbol{P(X > s)}} = \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x} dx}{\int_{s}^{\infty} \lambda e^{-\lambda x} dx}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \int_{t}^{\infty} \lambda e^{-\lambda x} dx = P(X > t) = \frac{\boldsymbol{P(X > t)}}{\boldsymbol{1}}$$

2- Exponential Distribution cont.



2- Exponential Distribution cont.

- **Theorem 2 –** Exponential distribution is the only continuous distribution that is memoryless.

$$P(X > s + t \mid X > s) = P(X > t) \quad \rightarrow \quad \frac{P(X > s + t)}{P(X > s)} = P(X > t) \quad \rightarrow \quad P(X > s + t) = P(X > s)P(X > t)$$

$$[1 - F(s + t)] = [1 - F(s)][1 - F(t)] \quad \rightarrow \quad G(s + t) = G(s)G(t) \quad knowing\ that\ \ 0 \leq G(.) \leq 1$$

$$s \rightarrow t \quad \rightarrow \quad G(2t) = G^2(t), and\ similarly\ \ G(3t) = G^3(t), \dots, G(kt) = G^k(t) \quad \rightarrow \quad G(t) = G^{\frac{1}{k}}(kt)$$

$$then\ easily \quad \rightarrow \quad G\left(\frac{m}{n}t\right) = G^{\frac{m}{n}}(t) \quad and\ therefore \quad \rightarrow \quad G(xt) = G^x(t)$$

$$Now\ let\ t = 1 \quad \rightarrow G(x) = G^x(1) \quad therefore \quad G(x) = e^{x\ lnG(1)}$$

$$1 - F(x) = e^{x\ lnG(1)} \quad \rightarrow \quad F(x) = 1 - e^{x\ lnG(1)}$$

$$\rightarrow \quad f(x) = -lnG(1)e^{x\ lnG(1)}\ \ that\ is\ in\ form\ of\ Exponential\ distribution.$$

2- Exponential Distribution cont.

- **Theorem 3 (link between Poisson and exponential distributions).** If the number of events is a random variable Poisson distribution wit rate r, the time between two consecutive events has exponential distribution.
- Assume

$$X\text{: } number\ of\ Poisson\ events$$
$$T\text{: } Time\ between\ two\ consecutive\ events$$

$$P(T > t_0) = P\left(X_{t_0} = 0\right) = \frac{e^{-rt_0}\,(rt_0)^0}{0!} = e^{-rt_0}$$

$$on\ the\ other\ hand\quad P(T > t_0) = 1 - P(T \le t_0) = 1 - F_T(t_0)$$

$$we\ therefore\ have\quad 1 - F_T(t_0) = e^{-rt_0}$$

$$and\ therefore\quad f_T(t_0) = re^{-rt_0}\quad or\quad f_T(t) = re^{-r}$$

$$that\ indicates\ exponential\ distribution\ with\ \lambda = r.$$

3- Erlang Distribution

If $X_1, X_2, \ldots, X_n$ are independent, identical Exponential RVs, then

$$X \; = \; X_1 + X_2 + \cdots + X_n$$

has Erlang distribution with

$$f(x) = (\lambda x)^{n-1} \lambda e^{-\lambda} \qquad x \geq 0$$

and

$$E[X] = \frac{n}{\lambda} \qquad , \qquad Var[X] = \frac{n}{\lambda^2}$$

$$m_X(t) = E[e^{tX}] = \left(\frac{\lambda}{\lambda - t}\right)^n \quad t < \lambda$$

4- Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < +\infty$$

$$E[X] = \mu, \qquad \mathrm{Var[X]} = \sigma^2 \ , \qquad m_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

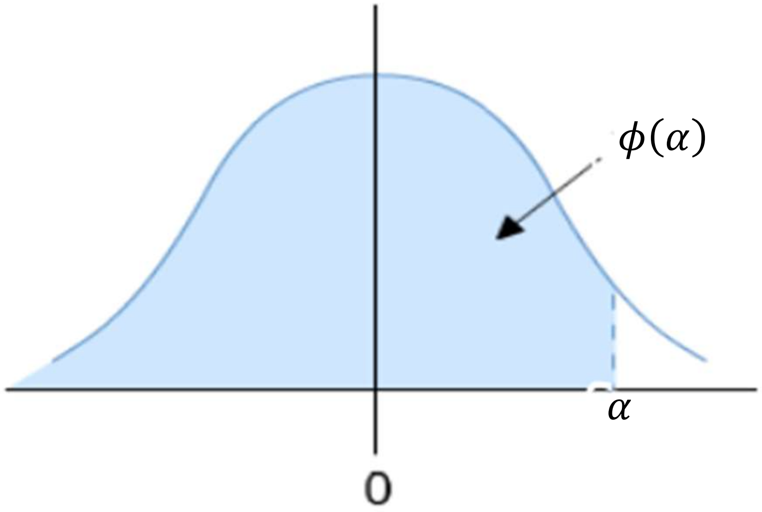
$$Z = \frac{X-\mu}{\sigma}$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad -\infty < z < +\infty$$

$$E[Z] = 0, \qquad \mathrm{Var[Z]} = 1$$

4- Normal Distribution Cont.

$$\phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$



$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

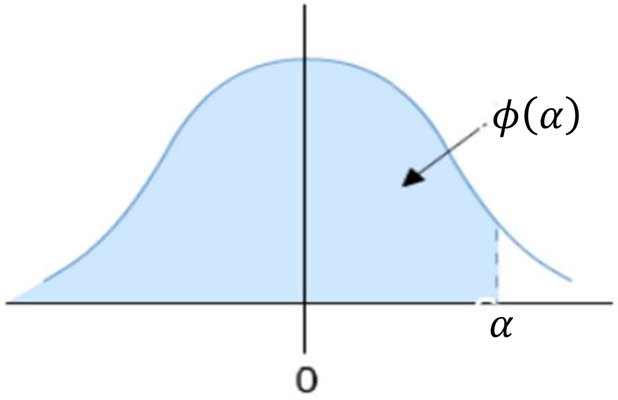
$$= P\left(Z < \frac{b-\mu}{\sigma}\right) - P\left(Z < \frac{a-\mu}{\sigma}\right) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

4- Normal Distribution Cont.

α	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

$$\phi(0.53) = 0.7019$$
$$\phi(1.05) = 0.8531$$

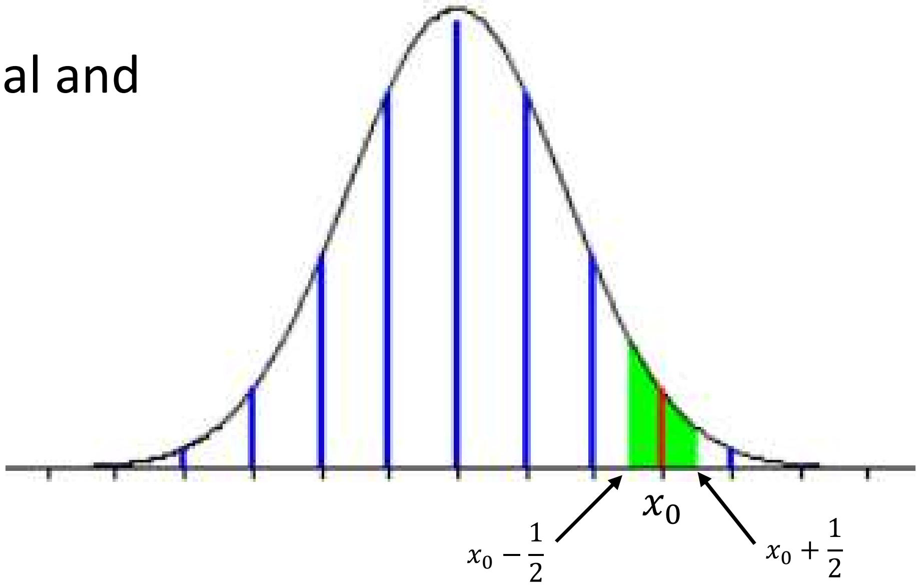
$$\phi^{-1}(0.9292) = 1.47$$



Normal Approximation for Binomial Probabilities

We can show under some assumption as below, Normal distribution can be used to find approximate probabilities for both Binomial and Poisson distributions.

$X \sim B(n,p), \qquad large\ n, and\ small\ p$



$$P(X = x_0) = \binom{n}{x_0} p^{x_0} q^{n-x_0} \approx \phi\left(\frac{x_0 + \frac{1}{2} - \mu}{\sigma}\right) - \phi\left(\frac{x_0 - \frac{1}{2} - \mu}{\sigma}\right) \text{ where we put } \begin{cases} \mu = np \\ \sigma = \sqrt{npq} \end{cases}$$

Normal Approximation for Poisson Probabilities

- Using a visual illustration similar to the previous slide, we can show

$X \sim P(\lambda)$, $\quad$ $large\ \lambda$

$$P(X = x_0) = \frac{e^{-\lambda}\lambda^{x_0}}{x_0!} \approx \phi\left(\frac{x_0 + \frac{1}{2} - \mu}{\sigma}\right) - \phi\left(\frac{x_0 - \frac{1}{2} - \mu}{\sigma}\right) \text{ where we put } \begin{cases} \mu = \lambda \\ \sigma = \sqrt{\lambda} \end{cases}$$