CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

2 real numbers

Mathematical object

Integers
Reals

wave amplitude
wave phase

Object

e.g.

Array of numbers

1-D

2-D

Lecture #5: Phasors

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data structures

string	string	int
Dog	Sauce	2
Cat	Furry	1

Complex Numbers

Cardano 1600s

A complex number is expressed as the sum of a *real* part and an *imaginary* part,

$$a + bj \in \mathbb{C} \quad \text{for } a, b \in \mathbb{R}.$$

Imaginary unit j is defined as,

$$j^2 = -1 \quad \text{so } j = \pm\sqrt{-1}.$$

(In engineering, j $= -$i is used to denote imaginary to avoid confusion with electrical current *I*.)

$a+bj$

a	b
7.1	6.3

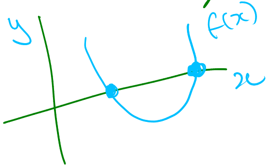
lateral ?

Complex Numbers

$(x+1)^2 =$

$(x+1)(x+1)$

$$\begin{array}{r} x^2 + x + 1 \\ + x \\ \hline x^2 + 2x + 1 \end{array} = -9$$



They allow expressions that wouldn't be possible otherwise, e.g. roots of $(x+1)^2 = -9$ are at $x = -1 \pm 3\,\mathrm{j}$.

$$(-1 \pm 3\,\mathrm{j} + 1)^2 =$$
$$(\pm 3\,\mathrm{j})^2 = (\pm 3)^2\,\mathrm{j}^2 =$$
$$(+3)^2\,\mathrm{j}^2 \text{ and } (-3)^2\,\mathrm{j}^2 =$$
$$(9)(-1) = -9.$$

$x = -1 + 3j$

$x = -1 - 3j$ ✓

Complex Numbers

data structure
program object

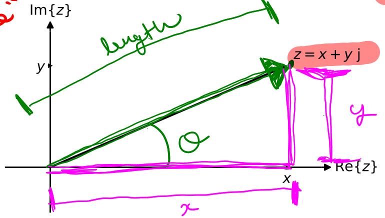
A 2-D point coordinate (x, y) is an example of keeping 2 numbers together for practical reasons.

$c^2 = a^2 + b^2$

$\text{length}^2 = x^2 + y^2$ $\quad$ $\text{length} = \sqrt{x^2 + y^2}$

They can be used to associate numbers that go together, such as point vector coordinates (x, y).

"Complex plane"



$\boxed{(x, y)}$ an ordered pair

$(7, 16)$

$z = \boxed{x + y j}$

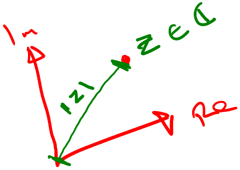
$z \in \mathbb{C}$

$\mathrm{Re}\{z\} \to x \in \mathbb{R}$

$\mathrm{Im}\{z\} \to y \in \mathbb{R}$

Also vectors length + direction

Complex Numbers



$z \in \mathbb{C}$

$\text{length}^2 = x^2 + y^2$

$|z|^2 = x^2 + y^2$

$= z\bar{z}$

$|z| = \sqrt{z\bar{z}}$

But some consideration required, e.g compensation for $\mathrm{j}^2 = -1$ to express vector magnitude,

$$\text{for } z = x + y\,\mathrm{j}\,,$$
$$\text{complex conjugate } \bar{z} = x - y\,\mathrm{j}\,,$$
$$\text{magnitude squared } |z|^2 = z\bar{z} = x^2 + y^2,$$
$$\text{magnitude } |z| = \sqrt{z\bar{z}}.$$

Complex Numbers

$\mathrm{Re}(z)$ $\mathrm{Im}(z)$

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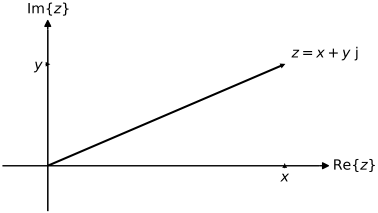

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Euler's Formula

Swiss

Euler's constant e ≈ 2.71828. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$.

e^x is called the natural exponential function and can be written $\exp x$ or $\exp(x)$.

Euler's formula expresses sinusoidal functions through the natural exponential function,

$$\mathrm{e}^{\mathrm{j}\,\theta} = \cos\theta + \mathrm{j}\,\sin\theta.$$

LHS RHS

$f(x) = e^x$

$\frac{df(x)}{dx} = e^x$

$f'(x) = e^x$

$\boxed{x + yj}$

$\boxed{x + jy}$

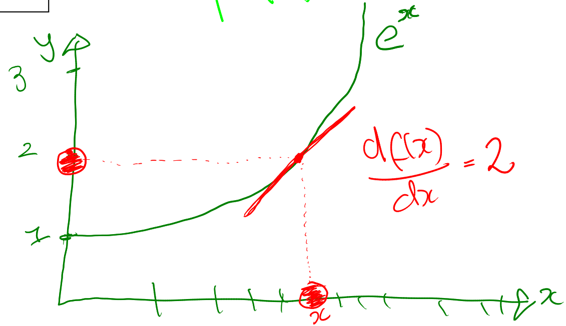
$0 + \boxed{jy} = jy$

$\theta \in \mathbb{R}$

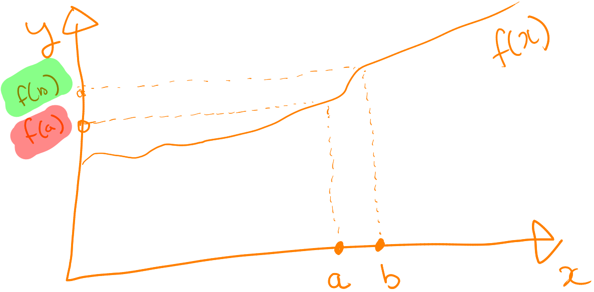
e^x

e^{jx}

θ



Euler's Formula



The value of any smooth function f at point b in the neighbourhood of point a can be expressed through the Taylor Series,

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a)^1 + \frac{f''(a)}{2!}(b-a)^2 + \dots$$

At $x = a = 0$, $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x = e^0 = 1$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}e^x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}e^x = \frac{\mathrm{d}}{\mathrm{d}y}e^x = e^0 = 1$, etc.,

$$\text{so, } \exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$$

$e^b = e^0 +$

Euler's Formula

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

$$= \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

Euler's Formula

$$\begin{aligned} e^{j\,\theta} = \exp(j\,\theta) &= 1 + \frac{(j\,\theta)^1}{1!} + \frac{(j\,\theta)^2}{2!} + \frac{(j\,\theta)^3}{3!} \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \\ &\quad + j\,(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots) \\ &= \cos\theta + j\,\sin\theta. \end{aligned}$$

(Hence the expression $e^{i\,\pi} - 1 = 0$.)

Euler's Formula

$e^{j\theta}$

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At $x{=}a{=}0$, $\frac{d}{dx}\mathrm{e}^x{=}\mathrm{e}^x{=}\mathrm{e}^0{=}1$, $\frac{d^2}{dx^2}\mathrm{e}^x{=}\frac{d}{dx}\frac{d}{dx}\mathrm{e}^x{=}\frac{d}{dy}\mathrm{e}^x{=}\mathrm{e}^0 = 1$, etc.,

so, $\exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = \boxed{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots}$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

$$= \boxed{\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots}$$

$$\boxed{\mathrm{e}^{\mathrm{j}\,\theta}} = \exp(\mathrm{j}\,\theta) = 1 + \frac{(\mathrm{j}\,\theta)^1}{1!} + \frac{(\mathrm{j}\,\theta)^2}{2!} + \frac{(\mathrm{j}\,\theta)^3}{3!}\dots$$

$$= \boxed{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots}$$

$$+ \mathrm{j}\left(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$= \boxed{\cos\theta + \mathrm{j}\,\sin\theta.}$$

(Hence the expression $\mathrm{e}^{\mathrm{i}\,\pi} - 1 = 0$.)

Phasors

Any

A sinusoid can be described as $A\cos(\omega t + \phi)$ in terms that are *variable*:

0 - 2π

- time t, $-\infty < t < \pm\infty$
- angular frequency $\omega = 2\pi\nu$
- temporal frequency ν

2π x periods per unit time

Hz

ω omega

number of wave periods per unit time

nu

and *constant*:

- amplitude A
- phase ϕ

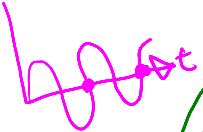
ϕ phi

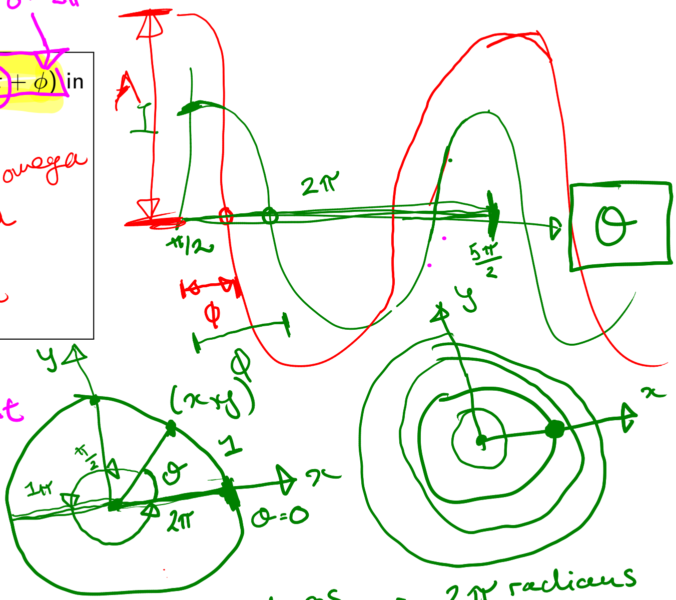
How can physical waves be different from one another?

* Amplitude

* frequency

* phase





360 degrees or 2π radians

Phasors

A phasor is used to encode the constants, A, ϕ

$$\boxed{A\,e^{j\,\phi} = A\,(\cos\phi + j\,\sin\phi).}$$

Multiply by exponential function encoding the variables $\boxed{e^{j\,\omega t}}$ to get a sinusoid,

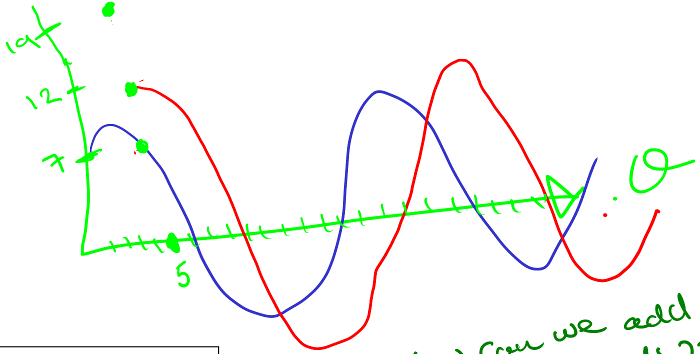
$$A\,e^{j\,\phi}e^{j\,\omega t} = \boxed{A\,e^{j\,(\phi+\omega t)}}.$$

$$A\,\mathrm{Re}\left\{e^{j\,(\phi+\omega t)}\right\} = \boxed{A\,\cos(\omega t + \phi).}$$

$a^x \times a^y$

$= a^{(x+y)}$

Phasors



How can we add two sinusoids???

→ Numerically?

θ

Sum of two sinusoids with the same ang. freq. ω,

$$A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) =$$

$$\mathrm{Re}\left\{A_1\, e^{j\,(\omega t + \phi_1)} + A_2\, e^{j\,(\omega t + \phi_2)}\right\} =$$

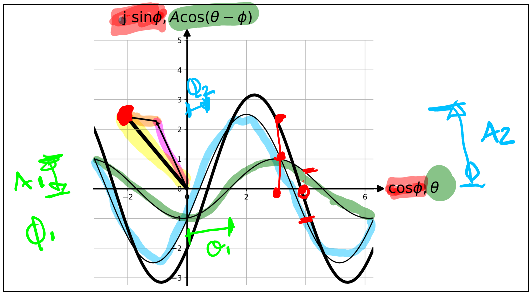
$$\mathrm{Re}\left\{(A_1\, e^{j\,\phi_1} + A_2\, e^{j\,\phi_2})\, e^{j\,\omega t}\right\}.$$

Which is the sum of phasors encoding the constants multiplied by an exponential function encoding the variables.

Phasors

A$_2$ ϕ_2

A$_1$ ϕ_1



Phasors

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and *constant*:

- amplitude A
- phase ϕ

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Multiply by exponential function encoding the variables $\mathrm{e}^{\mathrm{j}\,\omega t}$ to get a sinusoid,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t} = A\,\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}.$$

$$A\,\mathrm{Re}\left\{\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}\right\} = A\cos(\omega t+\phi).$$

Sum of two sinusoids with the same ang. freq. ω,

$$A_1\cos(\omega t+\phi_1) + A_2\cos(\omega t+\phi_2) =$$

$$\mathrm{Re}\left\{A_1\,\mathrm{e}^{\mathrm{j}\,(\omega t+\phi_1)} + A_2\,\mathrm{e}^{\mathrm{j}\,(\omega t+\phi_2)}\right\} =$$

$$\mathrm{Re}\left\{(A_1\,\mathrm{e}^{\mathrm{j}\,\phi_1} + A_2\,\mathrm{e}^{\mathrm{j}\,\phi_2})\,\mathrm{e}^{\mathrm{j}\,\omega t}\right\}.$$

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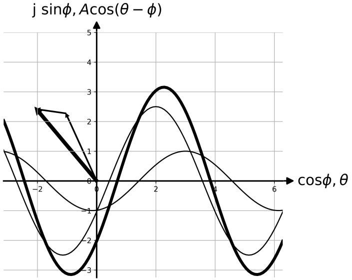

Phasor calculus

The derivative with respect to time of the exponential function of the variables can be expressed as a phasor,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t} = \mathrm{e}^{t} \text{ so } \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\omega t} = \omega\mathrm{e}^{\omega t} \text{ and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{j}\,\omega t} = \mathrm{j}\,\omega\,\mathrm{e}^{\mathrm{j}\,\omega t} = \omega\mathrm{e}^{\mathrm{j}\,\pi/2}\,\mathrm{e}^{\mathrm{j}\,\omega t}$$

since $\mathrm{j} = 0 + \mathrm{j}\,1 = \cos\frac{\pi}{2} + \mathrm{j}\,\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{j}\,\pi/2}$.

Phasor calculus

To differentiate: multiply by $\mathrm{j}\,\omega = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}$.

To integrate: multiply by $\frac{1}{\mathrm{j}\,\omega} = \frac{1}{\omega}\mathrm{e}^{-\mathrm{j}\,\pi/2}$.

Phasor calculus

For example, find the derivative with respect to time of a sinusoid expressed using a phasor,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t}\right) &= A\mathrm{e}^{\mathrm{j}\,\phi}(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\pi/2}\omega\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= \omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}.\end{aligned}$$

Phasor calculus

$$\mathrm{Re}\left\{\omega A e^{\mathrm{j}\,(\phi+\pi/2)} e^{\mathrm{j}\,\omega t}\right\} = \omega A \cos(\omega t + \phi + \pi/2)$$
$$= \omega A \sin(\omega t + \phi).$$

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$$\begin{aligned}\mathrm{Re}\left\{\omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}\right\} &= \omega A\cos(\omega t + \phi + \pi/2)\\ &= \omega A\sin(\omega t + \phi).\end{aligned}$$