CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #7: Intensity

Fergal Shevlin, Ph.D.

School of Computer Science and Statistics,
Trinity College Dublin

November 19, 2021

Intensity

Forces exerted by electric and magnetic fields can move or heat matter and move charges. This means electromagnetic waves have *energy*. (Actually they *transport* energy.)

The energy required to accelerate an object over $1\,\text{m}$ distance with $1\,\text{N}$ force is,

$$1\,\text{J} = 1\,\text{N}\,\text{m} = 1\,\text{kg}\,\text{m}\,\text{s}^{-2} \cdot 1\,\text{m} = 1\,\text{kg}\,\text{m}^2\,\text{s}^{-2}.$$

Electromagnetic wave intensity at a point in space at time t is $\propto$ to the product of field amplitudes,

$$^1/_\mu \|\mathbf{E}(t)\|\,\|\mathbf{B}(t)\|.$$

Average intensity over a wave time period is a more useful quantity,
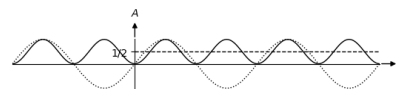
$$I = \frac{1}{2}\frac{1}{\mu}EB = \frac{E^2}{2\mu c} = \frac{cB^2}{2\mu}$$

where E and B are max. amplitudes.

Power is energy per unit time, $1\,\text{W} = 1\,\text{J}\,\text{s}^{-1}$. Power per unit area, $\text{W}\,\text{m}^{-2}$, is energy flux or *intensity*.

Intensity of electromagnetic waves is what what our eyes see; and what is measured by the photosensitive elements in cameras.

Note that the average of the product of two equal sinusoids is $\propto$ average of $\cos^2$ or $\sin^2$ which is $^1/_2$.

For simplicity, scaling factors can be ignored and field amplitudes denoted by A to consider avg. intensity $I \propto A^2$.

Intensity PDF

$f_A(a)$ for amplitude A of a random phasor sum was found to follow a Rayleigh distribution. The *derived* PDF for intensity $I = A^2$ is,

$$f_I(i) = f_A(\sqrt{i}) \left| \frac{\mathrm{d}\sqrt{i}}{\mathrm{d}i} \right| = f_A(\sqrt{i}) \frac{1}{2\sqrt{i}}$$

since $\dfrac{\mathrm{d} i^{\frac{1}{2}}}{\mathrm{d}i} = \dfrac{1}{2} i^{\frac{-1}{2}} = \dfrac{1}{2} \dfrac{1}{\sqrt{i}} = \dfrac{1}{2\sqrt{i}}.$

$$f_A(a) = \exp\left\{ -\frac{a^2}{2\sigma_A^2} \right\} \frac{a}{\sigma_A^2}.$$

$$f_I(i) = \exp\left\{ -\frac{i}{2\sigma_A^2} \right\} \frac{\sqrt{i}}{\sigma_A^2} \frac{1}{2\sqrt{i}}$$

$$= \exp\left\{ -\frac{i}{2\sigma_A^2} \right\} \frac{1}{2\sigma_A^2}.$$

So intensity PDF follows an *exponential* distribution, i.e. $f_X(x) = \lambda\, e^{-\lambda x}$.
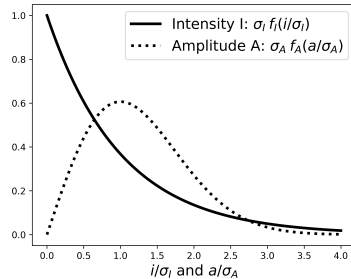

Legend:
— Intensity I: $\sigma_I f_I(i/\sigma_I)$
⋯ Amplitude A: $\sigma_A f_A(a/\sigma_A)$

x-axis: i/σ_I and a/σ_A

Mean intensity $\overline{I}$ can be found as $2\sigma_A^2$ so the PDF can be written,

$$f_I(i) = \exp\left\{ -\frac{i}{\overline{I}} \right\} \frac{1}{\overline{I}}.$$

$$\text{Variance } \sigma_I^2 = \overline{I}^2, \quad \text{Std. dev. } \sigma_I = \overline{I},$$

$$\text{Contrast } C = \sigma_I/\overline{I} = 1.0,$$

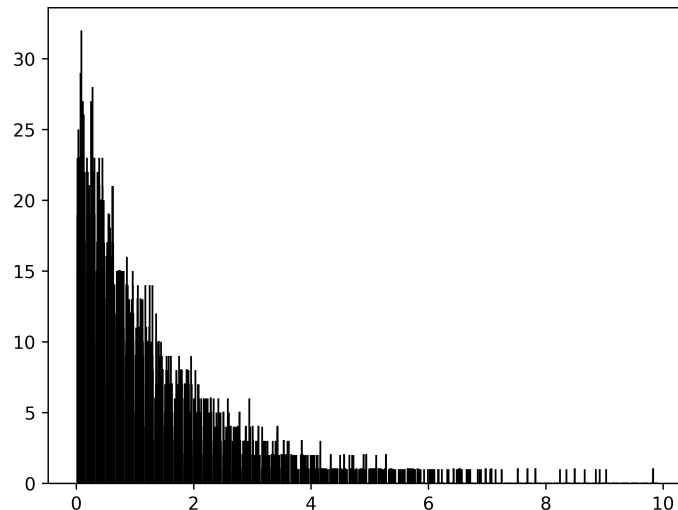$$\text{S/N ratio } = 1/C = \overline{I}/\sigma_I = 1.0.$$

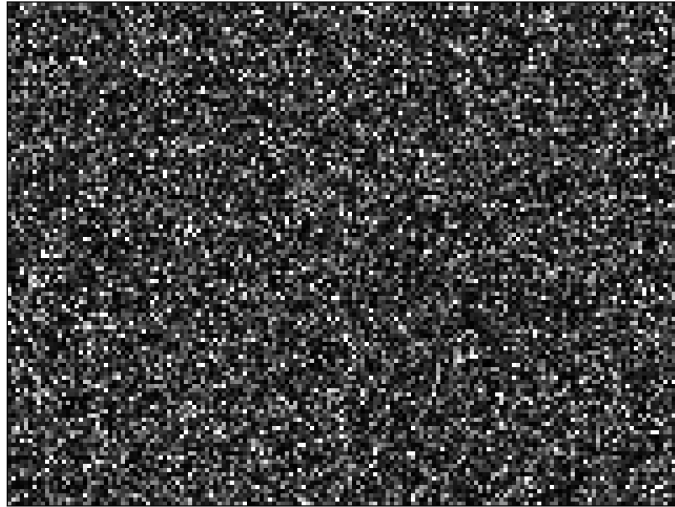# Histogram of simulated intensities

19,200 intensity values chosen randomly in accordance with an exponential PDF with $\overline{I} = 1.0$ and $\sigma_I = 1.0$. These could be many observations over time at a single point in space or many observations over space at a single point in time.
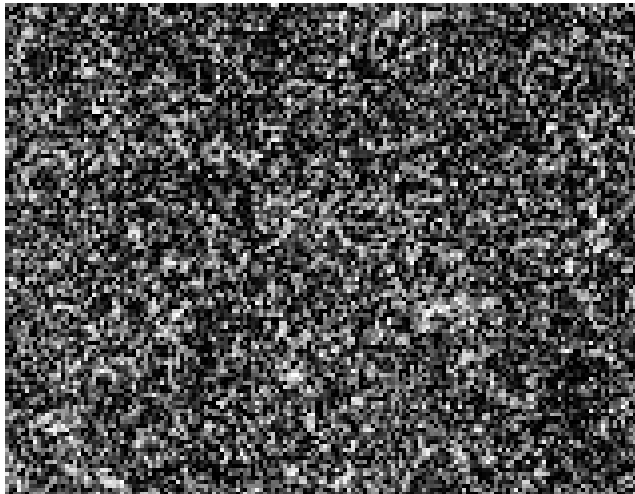
Notes

Image of simulated intensities

The same* intensity values i arranged as a 160×120 pixel image.
Contrast $C = \sigma_I / \overline{I} = 1.0$.

*These aren't exactly the same as those in the histogram. Values > 4.0 were replaced with 4.0 to avoid too much compression of dynamic range in display.
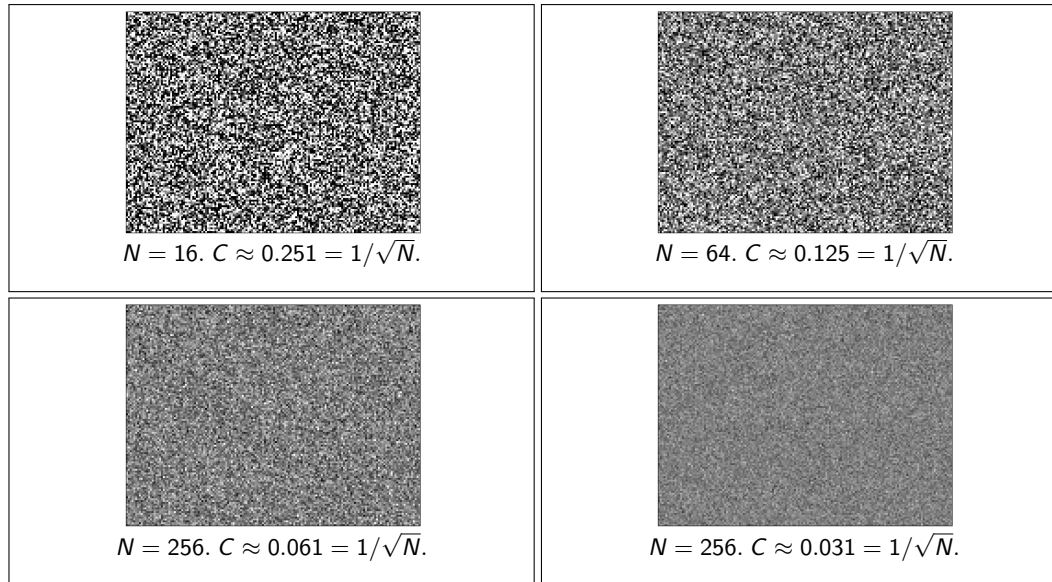
Image of actual intensities

Material with diffuse reflectance characteristics illuminated evenly with *monochromatic* light with *no phase or amplitude change* during the observation time period. $C \approx 0.83$.
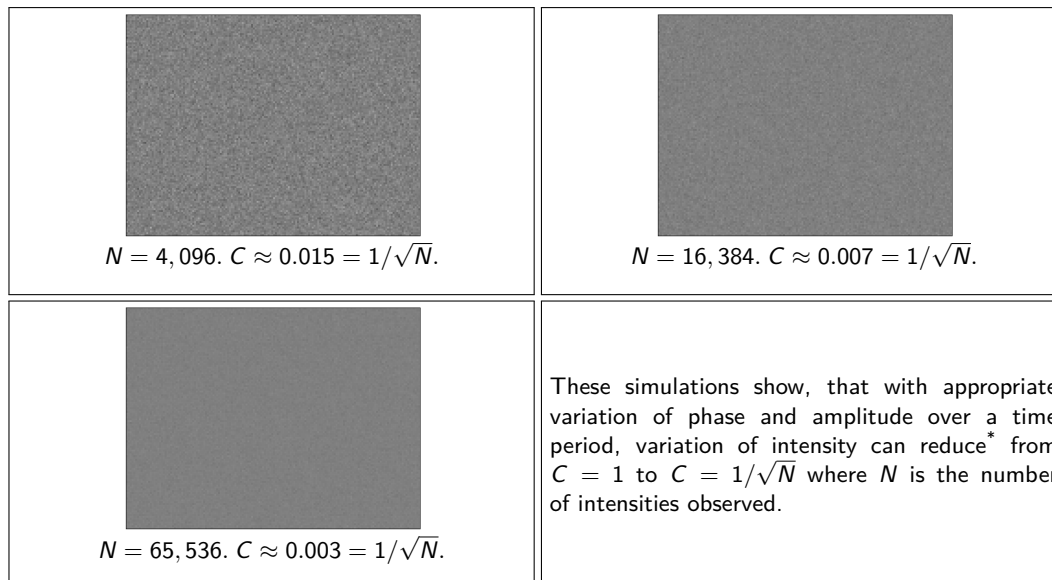
Notes

Averages of simulated intensity images

$N = 16. \; C \approx 0.251 = 1/\sqrt{N}.$

$N = 64. \; C \approx 0.125 = 1/\sqrt{N}.$

$N = 256. \; C \approx 0.061 = 1/\sqrt{N}.$

$N = 256. \; C \approx 0.031 = 1/\sqrt{N}.$

Averages of simulated intensity images

$N = 4,096. \; C \approx 0.015 = 1/\sqrt{N}.$

$N = 16,384. \; C \approx 0.007 = 1/\sqrt{N}.$

$N = 65,536. \; C \approx 0.003 = 1/\sqrt{N}.$

These simulations show, that with appropriate variation of phase and amplitude over a time period, variation of intensity can reduce[*] from $C = 1$ to $C = 1/\sqrt{N}$ where N is the number of intensities observed.