

# CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

## Lecture #9: Diffraction

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## Amplitude away from source

Speed of light\* in  $\text{m s}^{-1}$  is  $c$

Wavelength in m is  $\lambda$

Wave period in s is  $T = \lambda/c$

Wave frequency in Hz is  $\nu = 1/T$

Angular freq. in  $\text{rad s}^{-1}$  is  $\omega = 2\pi/T$

Wave number in  $\text{rad m}^{-1}$  is  $k = 2\pi/\lambda$

## Amplitude away from source

A phasor encodes max. amplitude  $A(\mathbf{p})$  and phase  $\phi(\mathbf{p})$  at position  $\mathbf{p}$ ,

$$U(\mathbf{p}) = A(\mathbf{p}) \exp\{j \phi(\mathbf{p})\}.$$

The scalar value of an EM wave vector component at time  $t$  can be found as,

$$\begin{aligned} u(\mathbf{p}, t) &= \text{Re}\{U(\mathbf{p}) \exp\{-j \omega t\}\} \\ &= A(\mathbf{p}) \cos(\omega t - \phi(\mathbf{p})). \end{aligned}$$

## Amplitude away from source

Let  $\mathbf{p}_1$  be the point source of a wave and let  $\mathbf{p}_0$  be somewhere else. Let  $t_{01}$  be the time it takes for the wave to travel.

$$u(\mathbf{p}_0, t_{01}) = \operatorname{Re} \left\{ U(\mathbf{p}_1) \frac{\exp\{-j \omega t_{01}\}}{r_{01}} \right\}$$

where  $r_{01} = \|\mathbf{p}_0 - \mathbf{p}_1\| = c t_{01}$  is the Euclidean distance between the points.

## Amplitude away from source

$$\begin{aligned}\omega t_{01} &= \frac{2\pi}{T} t_{01} = \frac{2\pi}{\lambda/c} t_{01} \\ &= \frac{2\pi c}{\lambda} t_{01} = k r_{01}\end{aligned}$$

Since there is no explicit time term, this can be used to express the phasor at  $\mathbf{p}_0$ ,

$$U(\mathbf{p}_0) = U(\mathbf{p}_1) \frac{\exp\{-j k r_{01}\}}{r_{01}}$$

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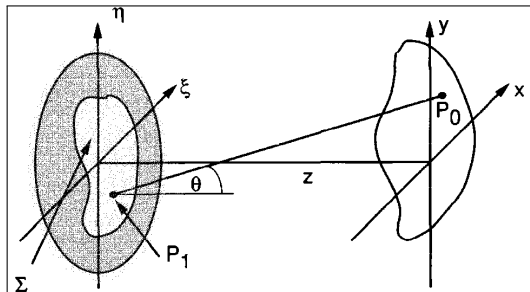
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## Amplitude after an aperture

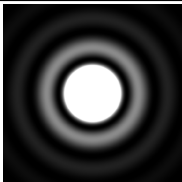
Amplitude at  $\mathbf{p}_0$  is the integral of the contributions from all possible points  $\mathbf{p}_1$  in the aperture,

$$U(\mathbf{p}_0) = \frac{1}{j\lambda} \iint_{\Sigma} U(\mathbf{p}_1) \frac{\exp\{j k r_{01}\}}{r_{01}} \cos \theta \, ds.$$

This expresses the Huygens-Fresnel principle of wave summation.



## Amplitude after an aperture



An aperture affects wave summation such that unusual constructive and destructive interference arises. Spot dia.  $2.44 \lambda f/D$ .

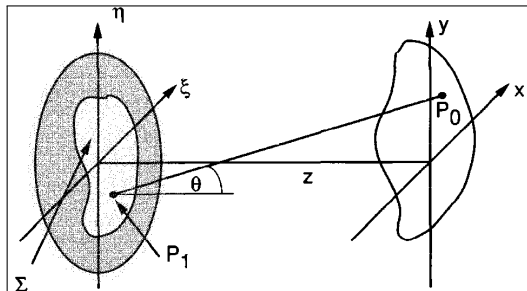
# Amplitude after an aperture

This is termed *diffraction* and it happens to all physical waves:

- ▶ light
- ▶ sound
- ▶ vibration (e.g. of water)
- ▶ gravitational waves

Diffuse reflection from a rough surface can also be understood as diffraction.

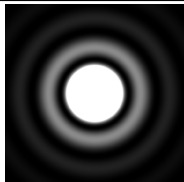
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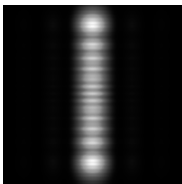


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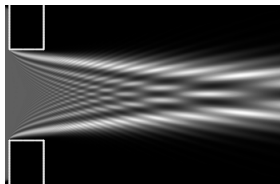
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Intensity in  $x-y$  plane after a narrow rectangular aperture.



Intensity in  $x$ - $z$  plane through and after a narrow rectangular aperture.

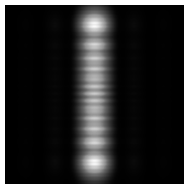
Significant computation is required for *numerical solutions* that simulate diffraction effects through summation of wave amplitudes.

Many different techniques can be used, e.g. *finite element methods*, to find wave amplitudes at discrete volumes in space at successive steps in time.

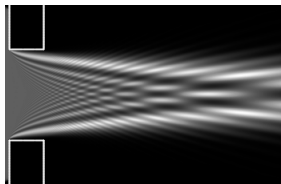
Note that intensity at distance  $r_{01}$  is distributed over a sphere whose surface area is  $4\pi r_{01}^2$ . So intensity scales  $\propto 1/r_{01}^2$ .

Since amplitude is the square root of intensity, it scales  $\propto \sqrt{1/r_{01}^2} = 1/r_{01}$ .

In two dimensional wave propagation (e.g. on water) the amplitude scales  $\propto 1/\sqrt{r_{01}}$ .



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# Fresnel Approximation

Since  $\cos \theta = z/r_{01}$ , wave summation can be rewritten in more explicit rectangular coordinates as,

$$U(x, y, z) = \frac{z}{j\lambda} \iint_{\Sigma} U(\xi, \eta) \frac{\exp\{j k r_{01}\}}{r_{01}^2} d\xi d\eta$$

with distance calculated as,

$$r_{01} = \sqrt{z^2 + (x - \xi)^2 + (y - \eta)^2}$$

# Fresnel Approximation

To facilitate *analytical solutions*, an approximation for distance  $r_{01}$  uses a binomial expansion to replace the square root,

$$\begin{aligned}\sqrt{1+b} &= (1+b)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \dots \\ &\text{when } |b| < 1.\end{aligned}$$

# Fresnel Approximation

$$r_{01} = \sqrt{z^2} \sqrt{1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2}$$
$$\approx z \left[ 1 + \frac{1}{2} \left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2} \left(\frac{y-\eta}{z}\right)^2 \right]$$

using only the first two terms of the expansion  
with  $b = \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2$ .

(cf. parabolic approx. of spherical wavefront.)

# Fresnel Approximation

The same approximation for  $r_{01}$  doesn't have to be used for all occurrences.

Using the first term only, the denominator  $r_{01}^2 \approx z^2$ . This can be factored out of the integral into the scaling term,

$$\frac{z}{j \lambda z^2} = \frac{1}{j \lambda z}.$$

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$$\begin{aligned}r_{01} &= \sqrt{z^2} \sqrt{1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2} \\ &\approx z \left[ 1 + \frac{1}{2} \left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2} \left(\frac{y-\eta}{z}\right)^2 \right]\end{aligned}$$

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# Convolution

Using the first two terms of the approximation,

$$\begin{aligned}\exp\{j k r_{01}\} &\approx \exp\left\{j k z \left[1 + \frac{1}{2} \left(\frac{x - \xi}{z}\right)^2 + \frac{1}{2} \left(\frac{y - \eta}{z}\right)^2\right]\right\} \\&= \exp\{j k z\} \exp\left\{j k z \left[\frac{1}{2} \left(\frac{x - \xi}{z}\right)^2 + \frac{1}{2} \left(\frac{y - \eta}{z}\right)^2\right]\right\} \\&= \exp\{j k z\} \exp\left\{j \frac{k}{2z} [(x - \xi)^2 + (y - \eta)^2]\right\}\end{aligned}$$

# Convolution

$$U(x, y, z) \approx \frac{\exp\{j k z\}}{j \lambda z} \iint_{-\infty}^{+\infty} U(\xi, \eta) \times \\ \exp\left\{j \frac{k}{2z} [(x - \xi)^2 + (y - \eta)^2]\right\} d\xi d\eta.$$

Accurate only for the “near field” close to the aperture because of distance approximation.

# Convolution

To facilitate analysis, it can be written as a *convolution* of the aperture with a function  $h$ .

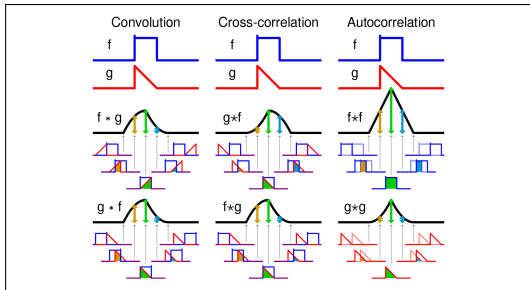
$$U(x, y, z) \approx \iint_{-\infty}^{+\infty} U(\xi, \eta) \times \\ h(x - \xi, y - \eta) \, d\xi \, d\eta.$$

Convolution *kernel*  $h(v, w) =$

$$\frac{\exp\{j k z\}}{j \lambda z} \exp\left\{j \frac{k}{2z} (v^2 + w^2)\right\}.$$



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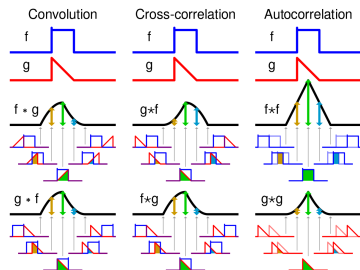
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## Fourier transform

$$(x - \xi)^2 = x^2 - 2x\xi + \xi^2,$$

$$(y - \eta)^2 = y^2 - 2y\eta + \eta^2.$$

Hence further factorization outside the integral is possible since only those terms in  $\xi$  and  $\eta$  need to remain inside.

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## Fourier transform

$$\begin{aligned} & \exp\left\{j \frac{k}{2z}(x^2 + y^2)\right\} \times \\ & \exp\left\{j \frac{k}{2z}(\xi^2 + \eta^2)\right\} \times \\ & \exp\left\{-2j \frac{k}{2z}(x\xi + y\eta)\right\}. \end{aligned}$$

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# Fourier transform

This integral can be recognised as the (scaled) *Fourier transform* of the (scaled) aperture evaluated at spatial frequencies,

$$f_X = \frac{x}{\lambda z} \quad f_Y = \frac{y}{\lambda z}.$$

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# Fraunhofer Approximation

When  $z$  in  $\frac{k}{2z}(\xi^2 + \eta^2)$  is very big,\* this expression is close to 0 so its exponent is close to 1. So it is not essential to use it a scaling factor,

$$U(x, y, z) = \frac{\exp\{j k z\} \exp\left\{j \frac{k}{2z}(x^2 + y^2)\right\}}{j \lambda z}$$

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# Fraunhofer Approximation

Hence wave summation can be expressed as the (scaled) Fourier transform of the (unscaled) aperture evaluated at frequencies,

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Accurate only for the “far field” distant from the aperture because of distance assumption.

# Fraunhofer Approximation

For intensity  $I(x, y, z) = |U(x, y, z)|^2$ , the numerator and denominator of the scaling term simplify as follows.

$$\begin{aligned} |\exp\{j\,kz\}|^2 &= \exp\{+j\,kz\} \times \\ &\quad \exp\{-j\,kz\} \\ &= \exp\{+j\,kz - j\,kz\} \\ &= \exp\{0\} = 1. \end{aligned}$$

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$$\begin{aligned} |j \lambda z|^2 &= (+j \lambda z)(-j \lambda z) \\ &= +1 \lambda^2 z^2 = \lambda^2 z^2. \end{aligned}$$

For a nice alternative derivation of the material in this lecture, see <https://www.youtube.com/watch?v=JKxDa5D3GnQ>.

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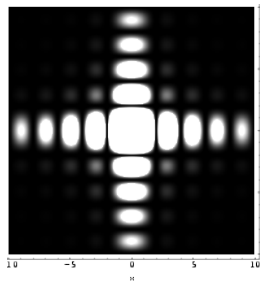
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## More examples



(a)

