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Notes

CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #2: Sound and Vibration

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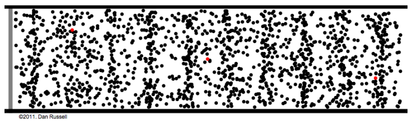
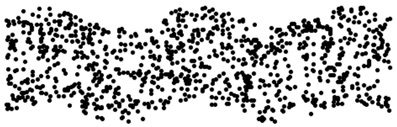
Notes

Acoustic waves

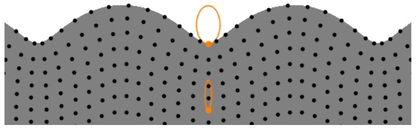
- Propagation of energy through matter by oscillation of pressure or of displacement. Speed $c_s = \sqrt{K/\rho}\,\mathrm{m\,s^{-1}}$.
- No heat or mass is transferred.
- Can be reflected, refracted, diffracted, and/or attenuated by the medium.
- For lots of interesting simulations, see `https://tinyurl.com/yyv5sajz`



Longitudinal waves have variations around equilibrium pressure due to compression and rarefaction of the medium in the direction of propagation.



Transverse waves have surface deformations perpendicular to the direction of wave propagation.



In solids, combinations of wave types can cause particles to move in elliptical trajectories with depth-dependent direction. In liquids, particles move in anti-clockwise circular trajectories.

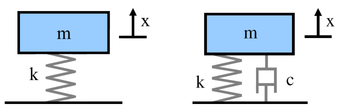
Notes

Vibration

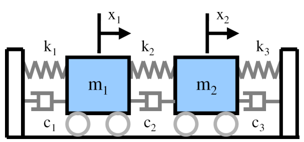
- Oscillations about equlibrium of a material, a structure, or a mechanical system.
- Can be periodic (e.g. a pendulum) or random (e.g. wheel on gravel road.)
- *Free vibration* is when there is an initial disturbance only (e.g. a tuning fork.)
- Objects have natural *frequencies* and *modes* of free vibration.

- Modes of a cantilevered I-beam. `https://tinyurl.com/yyvosd69`
- Modes of a membrane under tension. `https://tinyurl.com/yad48x9r`
- Mode-like basis functions used in linear combination to describe other shapes. `https://tinyurl.com/y4prwl9q`

Analysis of vibration of an object with one degree-of-freedom to move (out of six in three-dimensional space) usually starts with a simple mass-spring or mass-spring-damper model.



Analysis of an object with multiple degrees-of-freedom usually starts with an assembly of simple models, one for each degree.

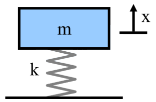


Notes

Simple Harmonic Oscillator

To develop a mathematical expression for vibration, consider some physical laws that a simple model should obey,

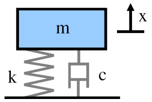
- Hooke's law: Spring force scales linearly with distance of displacement from equilibrium. $F = -k\,x$.
- Newton's second law of motion: Force equals mass by acceleration. $F = m\,a$.



Equating the two expressions for F yields a single differential equation that relates all parameters,

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k\,x = 0.$$

cf. "free-body diagrams."



Friction can be modelled as a damper with force $F = -c\,\frac{\mathrm{d}x}{\mathrm{d}t}$ that can be added into the equation,

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + c\,\frac{\mathrm{d}x}{\mathrm{d}t} + k\,x = 0.$$

A solution for the undamped system is,

$$x(t) = x_0 \cos(\omega\,t + \phi)$$

where $\omega = \sqrt{k/m}$ is angular freq. $f = \omega/2\pi$ is temporal freq. x_0 is the amplitude of initial displacement and ϕ is phase which is this case is 0.

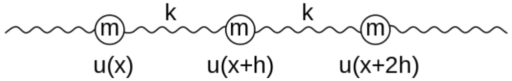
A solution for damped with $\zeta = \frac{c}{2\sqrt{km}}$ is,

$$x(t) = x_0\,\mathrm{e}^{-\zeta\omega t} \cos\left(\sqrt{1-\zeta^2}\,\omega\,t + \phi\right).$$

Notes

Wave equation*

One way to model a continuous (non-discrete) system such as a string is to consider it as a series of masses m connected by springs of lengths h and spring constants k,



where $u(x)$ is the distance from its equilibrium position of a mass at position x.

The force acting on mass m at position $x + h$ at time t can be described independently:

$$\text{Newton's, } F(x+h,t) = m\,\frac{\partial^2}{\partial t^2}\,u(x+h,t).$$

$$\begin{aligned}\text{Hooke's, } F(x+h,t) &= F(x+2h,t) - F(x,t)\\ &= k\,[u(x+2h,t) - u(x+h,t)]\\ &\quad -k\,[u(x+h,t) - u(x,t)].\end{aligned}$$

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m}\,[u(x+2h,t) - u(x+h,t) - u(x+h,t) + u(x,t)].$$

For N masses evenly spaced over total length $L = Nh$ and total mass $M = Nm$ and a total spring constant $K = k/N$, the rhs becomes,

$$\frac{KL^2}{M}\,\frac{[u(x+2h,t) - 2u(x+h,t) + u(x,t)]}{h^2}.$$

Let $c^2 = \frac{KL^2}{M}$ and consider the continuous system situation where $N \to \infty$ (which means taking the limit as $h \to 0$,)

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\,\frac{\partial^2 u(x,t)}{\partial x^2}.}$$

*See `https://tinyurl.com/y6acqr4j`.

Notes

Taking the limit

$$\lim_{h\to 0} \frac{u(x+2h,t) - 2u(x+h,t) + u(x,t)}{h^2}$$

This expression is *indeterminite* because for $h = 0$ it becomes $\frac{0}{0}$ which has multiple interpretations:

0? ∞? 1?

Luckily, we have l'Hospital's Rule which says that, under certain conditions,

$$\lim_{x\to y} \frac{f(x)}{g(x)} = \lim_{x\to y} \frac{f'(x)}{g'(x)}.$$

So the quotient terms can be replaced by their derivates, $\frac{\mathrm{d}}{\mathrm{d}h} h^2 = 2h$ and

$$\frac{\partial}{\partial h}[u(x+2h,t) - 2u(x+h,t) + u(x,t)] =$$
$$2\,u'(x+2h,t) - 2\,u'(x+h,t).$$

$$\lim_{h\to 0} \frac{u(x+2h,t) - 2\,u(x+h,t) + u(x,t)}{h^2} =$$
$$\lim_{h\to 0} \frac{2\,u'(x+2h,t) - 2\,u'(x+h,t)}{2h} =$$
$$\lim_{h\to 0} \frac{4\,u''(x+2h,t) - 2\,u''(x+h,t)}{2} =$$

$$\lim_{h\to 0} [2\,u''(x+2h,t) - u''(x+h,t)] =$$
$$2 \lim_{h\to 0} u''(x+2h,t) - \lim_{h\to 0} u''(x+h,t) =$$
$$2\,u''(x,t) - u''(x,t) =$$
$$u''(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

Notes

Wave equation solution

A solution for $u(x,t)$ is:

$$R\cos(kx - \omega t) + (1-R)\cos(kx + \omega t)$$

ω is angular frequency $2\pi\,\nu$ in rad s^{-1}
k is the wave number $2\pi/\lambda$ in rad m^{-1}
$|R| \leq 1$ specifies direction of travel.

Which is the superposition of two sinusoidal waves travelling in opposite directions. Non-sinusoidal solutions are possible too.

Notes

d'Alembert's approach

The wave equation is a hyperbolic linear second order partial differential equation (PDE,)

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

A solution is the function $u(x,t)$ which satisfies it. One approach to finding a solution is to transform the differential equation into a form for which a solution is already known.

The first step is to change the variables, e.g. let $\xi \equiv x - c_s t$ and $\eta \equiv x + c_s t$,

$$x = \frac{1}{2}(\xi - \eta) \text{ and } t = \frac{1}{2c_s}(\xi + \eta).$$

Notes

By the Chain Rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -c_s\frac{\partial u}{\partial \xi} + c_s\frac{\partial u}{\partial \eta}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\\
&= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-c_s\frac{\partial u}{\partial \xi} + c_s\frac{\partial u}{\partial \eta}\right)\\
&= \left(-c_s\frac{\partial}{\partial \xi} + c_s\frac{\partial}{\partial \eta}\right)\left(-c_s\frac{\partial u}{\partial \xi} + c_s\frac{\partial u}{\partial \eta}\right) = c_s^2\frac{\partial^2 u}{\partial \xi^2} - 2c_s^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c_s^2\frac{\partial^2 u}{\partial \eta^2}.
\end{aligned}$$

Notes

So the wave equation $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 u}{\partial t^2} = 0$ becomes

$$(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}) - \frac{1}{c_s^2}(c_s^2\frac{\partial^2 u}{\partial \xi^2} - 2c_s^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c_s^2\frac{\partial^2 u}{\partial \eta^2}) = 0$$

$$\text{which is } \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi} = 0.$$

for which any solution is known to have the form:

$$p(\xi, \eta) = f(\eta) + g(\xi) = f(x + c_s t) + g(x - c_s t)$$

Notes

Specific solutions

To find functions f and g which describe a specific wave p we need to have some:

Boundary conditions e.g. $p(0, t) = 0$ and $p(L, t) = 0$.

Initial conditions e.g. $p(x, 0) = f(x)$ and $\frac{\partial u}{\partial t} = g(x)$.

We might revisit this topic in more detail later.

Notes

Chain Rule(s) of differentiation

- Rules that specify derivatives for compositions of functions, e.g. for $f(g(x))$, the derivative is $f'(g(x))\,g'(x)$.
- Leibnitz notation often used: let $y \equiv f(g(x))$ and $u \equiv g(x)$,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = f'(g(x))\,g'(x).$$
- Let $z \equiv f(x,y)$ and $x \equiv g(t)$ and $y \equiv h(t)$ then
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$
- Let $z \equiv f(x,y)$ and $x \equiv g(s,t)$ and $y \equiv h(s,t)$ then
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}x} + \frac{\partial z}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}x}$$

Notes

An easier approach

- To solve PDEs, you can use a mathematics-oriented symbolic programming system like Mathematica (e.g. through the free Wolfram Alpha web interface.)

```
DSolve[D[u[x, t], {x, 2}] - (1/c^2) D[u[x, t], {t, 2}]
       == 0, u[,], {t, x}]

{{u[x, t]->C[1][-sqrt(c^2) t + x] +
           C[2][sqrt(c^2) t + x]}}
```

For arbitrary functions `C[1]` and `C[2]` which can be found once initial values and/or boundary conditions are known.

- Note that this looks different to previous solution whose parameters are expressed in angular distance (radians) not linear distance (metres.)

Notes

Assignment # 2: Acoustic wave simulation

- Write a SciPy program to make at least one plot similar to those shown the acoustic wave slide.
- Demonstrate wave propagation over time through either a set of pictures or an animation.
- Make it into a self-contained project repository in your personal account on gitlab.scss.tcd.ie.
- For this and every other assignment, feel free to collaborate with your classmates about the the non-mathematical parts like plotting.

Notes

Assignment # 3: Simple harmonic oscillator simulation

- Write a SciPy program to make an animation of simple harmonic motion.
- In addition to an object moving up and down, you could show a sinusoidal plot of its amplitude.
- Make it into a self-contained project repository in your personal account on gitlab.scss.tcd.ie.
- For this and every other assignment, feel free to collaborate with your classmates about the the non-mathematical parts like plotting.

Notes