CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #5: Phasors

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Notes

Phasor is a math object which can be used to store some info
math object easily does some calculation with.

Complex Numbers

Cardano 1600s

A complex number is expressed as the sum of a *real* part and an *imaginary* part,

$$a + b\,\mathrm{j} \in \mathbb{C} \quad \text{for } a, b \in \mathbb{R}.$$

Imaginary unit j is defined as,

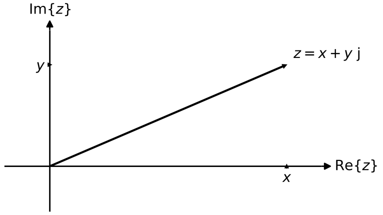
$$\mathrm{j}^2 = -1 \quad \text{so } \mathrm{j} = \pm\sqrt{-1}.$$

(In engineering, j $= -\mathrm{i}$ is used to denote imaginary to avoid confusion with electrical current I.)

They allow expressions that wouldn't be possible otherwise, e.g. roots of $(x+1)^2 = -9$ are at $x = -1 \pm 3\,\mathrm{j}$.

$$\begin{aligned} (-1 \pm 3\,\mathrm{j} + 1)^2 &= \\ (\pm 3\,\mathrm{j})^2 = (\pm 3)^2\,\mathrm{j}^2 &= \\ (+3)^2\,\mathrm{j}^2 \text{ and } (-3)^2\,\mathrm{j}^2 &= \\ (9)(-1) &= -9. \end{aligned}$$

They can be used to associate numbers that go together, such as point vector coordinates (x, y).



But some consideration required, e.g compensation for $\mathrm{j}^2 = -1$ to express vector magnitude,

$$\begin{aligned} \text{for } z &= x + y\,\mathrm{j}, \\ \text{complex conjugate } \bar{z} &= x - y\,\mathrm{j}, \\ \text{magnitude squared } |z|^2 &= z\bar{z} = x^2 + y^2, \\ \text{magnitude } |z| &= \sqrt{z\bar{z}}. \end{aligned}$$

Notes

Just use j instead of i, i => imaginary unit

Euler's Formula

Euler's constant $\mathrm{e} \approx 2.71828$. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$.

e^x is called the natural exponential function and can be written $\exp x$ or $\exp(x)$.

Euler's formula expresses sinusoidal functions through the natural exponential function,

$$\mathrm{e}^{\mathrm{j}\,\theta} = \cos\theta + \mathrm{j}\,\sin\theta.$$

The value of any smooth function f at point b in the neighbourhood of point a can be expressed through the Taylor Series,

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a)^1 + \frac{f''(a)}{2!}(b-a)^2 + \ldots$$

At $x{=}a{=}0$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}\mathrm{e}^x{=}\mathrm{e}^0{=}1$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^x{=}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}$ $\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^x{=}\mathrm{e}^0 = 1$, etc.,

$$\text{so,}\;\; \exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

$$= \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$\begin{aligned}\mathrm{e}^{\mathrm{j}\,\theta} = \exp(\mathrm{j}\,\theta) &= 1 + \frac{(\mathrm{j}\,\theta)^1}{1!} + \frac{(\mathrm{j}\,\theta)^2}{2!} + \frac{(\mathrm{j}\,\theta)^3}{3!}\ldots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \\ &\quad + \mathrm{j}\,(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots) \\ &= \cos\theta + \mathrm{j}\,\sin\theta.\end{aligned}$$

(Hence the expression $\mathrm{e}^{\mathrm{i}\,\pi} - 1 = 0$.)

Notes

Phasors

A sinusoid can be described as $A\,\cos(\omega t + \phi)$ in terms that are *variable*:

- time t, $-\infty < t < +\infty$
- angular frequency $\omega = 2\pi\,\nu$
- temporal frequency ν

and *constant*:

- amplitude A
- phase ϕ

A phasor is used to encode the constants,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi} = A\,(\cos\phi + \mathrm{j}\,\sin\phi).$$

Multiply by exponential function encoding the variables $\mathrm{e}^{\mathrm{j}\,\omega t}$ to get a sinusoid,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t} = A\,\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}.$$

$$A\;\mathrm{Re}\left\{\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}\right\} = A\;\cos(\omega t + \phi).$$

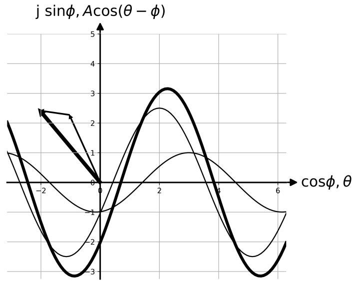
Sum of two sinusoids with the same ang. freq. ω,

$$A_1\;\cos(\omega t + \phi_1) + A_2\;\cos(\omega t + \phi_2) =$$

$$\mathrm{Re}\left\{A_1\;\mathrm{e}^{\mathrm{j}\,(\omega t+\phi_1)} + A_2\;\mathrm{e}^{\mathrm{j}\,(\omega t+\phi_2)}\right\} =$$

$$\mathrm{Re}\left\{(A_1\;\mathrm{e}^{\mathrm{j}\,\phi_1} + A_2\;\mathrm{e}^{\mathrm{j}\,\phi_2})\;\mathrm{e}^{\mathrm{j}\,\omega t}\right\}.$$

Which is the sum of phasors encoding the constants multiplied by an exponential function encoding the variables.



Notes

Phasor calculus

The derivative with respect to time of the exponential function of the variables can be expressed as a phasor,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t} = \mathrm{e}^{t} \text{ so } \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\omega t} = \omega \mathrm{e}^{\omega t} \text{ and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{j}\,\omega t} = \mathrm{j}\,\omega\,\mathrm{e}^{\mathrm{j}\,\omega t} = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}\,\mathrm{e}^{\mathrm{j}\,\omega t}$$

since $\mathrm{j} = 0 + \mathrm{j}\,1 = \cos\frac{\pi}{2} + \mathrm{j}\,\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{j}\,\pi/2}$.

To differentiate: multiply by $\mathrm{j}\,\omega = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}$.

To integrate: multiply by $\frac{1}{\mathrm{j}\,\omega} = \frac{1}{\omega}\mathrm{e}^{-\mathrm{j}\,\pi/2}$.

For example, find the derivative with respect to time of a sinusoid expressed using a phasor,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t}\right) &= A\mathrm{e}^{\mathrm{j}\,\phi}(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\pi/2}\omega \mathrm{e}^{\mathrm{j}\,\omega t}\\ &= \omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}.\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\left\{\omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}\right\} &= \omega A\cos(\omega t + \phi + \pi/2)\\ &= \omega A\sin(\omega t + \phi).\end{aligned}$$

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