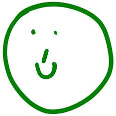
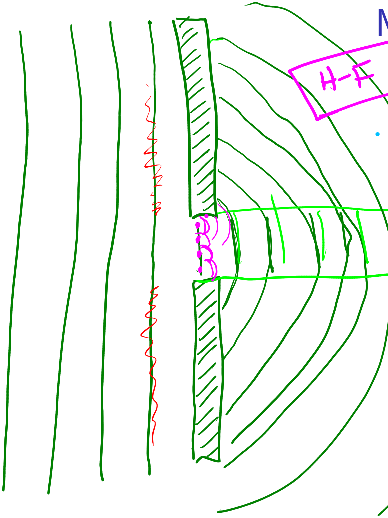
Aperture ≈ a hole!



CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.



H-F

does happen!

Lecture #9: Diffraction

Sound
Water
Light } any physical wave

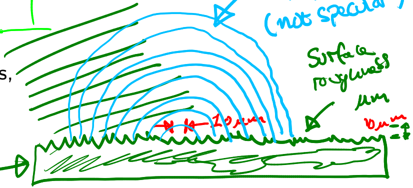
doesn't happen!

diffuse reflection (not specular)

Fergal Shevlin, Ph.D.

School of Computer Science and Statistics,
Trinity College Dublin

December 3, 2021



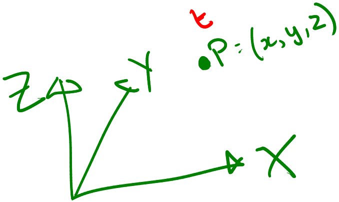
magnified cross-section of table

Amplitude away from source

$$\text{Speed of light}^* \text{ in } \mathrm{m\,s^{-1}} \text{ is } c$$
$$\text{Wavelength in m is } \lambda$$
$$\text{Wave period in s is } T = \lambda/c$$
$$\text{Wave frequency in Hz is } \nu = 1/T$$
$$\text{Angular freq. in } \mathrm{rad\,s^{-1}} \text{ is } \omega = 2\pi/T$$
$$\text{Wave number in } \mathrm{rad\,m^{-1}} \text{ is } k = 2\pi/\lambda$$

Green $\lambda = 532$ nm $= 532 \times 10^{-9}$ m

Amplitude away from source



A phasor encodes max. amplitude $A(\boldsymbol{p})$ and phase $\phi(\boldsymbol{p})$ at position $\boldsymbol{p}$, (x, y, z) =

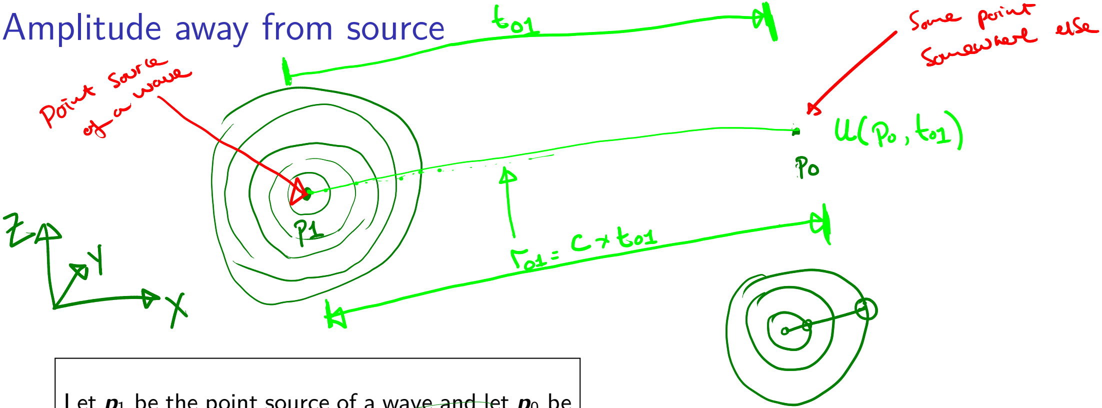
$$U(\boldsymbol{p}) = A(\boldsymbol{p}) \exp\{\mathrm{j}\ \phi(\boldsymbol{p})\}.$$

constants of any wave

The scalar value of an EM wave vector component at time t can be found as,

$$\begin{aligned} u(\boldsymbol{p}, t) &= \mathrm{Re}\{U(\boldsymbol{p})\ \boxed{\exp\{-\mathrm{j}\,\omega t\}}\} \\ &= \boxed{A(\boldsymbol{p}) \cos(\omega\ t - \phi(\boldsymbol{p}))}. \end{aligned}$$

Amplitude away from source



Let $\boldsymbol{p}_1$ be the point source of a wave and let $\boldsymbol{p}_0$ be somewhere else. Let t_{01} be the time it takes for the wave to travel.

phasor @ at pos p1

$$\boxed{u(\boldsymbol{p}_0, t_{01})} = \mathrm{Re}\left\{ U(\boldsymbol{p}_1) \frac{\exp\{-\mathrm{j}\,\omega t_{01}\}}{r_{01}} \right\}$$

where $r_{01} = \|\boldsymbol{p}_0 - \boldsymbol{p}_1\| = c\, t_{01}$ is the Euclidean distance between the points.

Amplitude decreases with distance from point source.

Amplitude away from source

$$\omega\ t_{01} = \frac{2\pi}{T}\ t_{01} = \frac{2\pi}{\lambda/c}\ t_{01}$$

$$= \frac{2\pi\ c}{\lambda}\ t_{01} = k\ r_{01}$$

Since there is no explicit time term, this can be used to express the phasor at $\boldsymbol{p}_0$,

phasor!

$$\boxed{U(\boldsymbol{p}_0)} = \boxed{U(\boldsymbol{p}_1)}\ \boxed{\frac{\exp\{-\mathrm{j}\ \ k\ r_{01}\}}{r_{01}}}$$

Amplitude away from source

Speed of light* in $\mathrm{m\,s^{-1}}$ is c

Wavelength in m is λ

Wave period in s is $T = \lambda / c$

Wave frequency in Hz is $\nu = 1/T$

Angular freq. in $\mathrm{rad\,s^{-1}}$ is $\omega = 2\pi / T$

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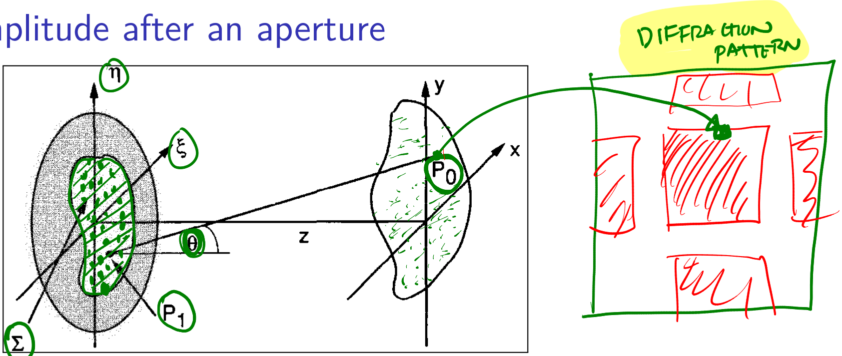
Amplitude after an aperture



Sigma Aperture

Amplitude after an aperture

Phasor at source point p_1

wave propagation

Amplitude at $\boldsymbol{p}_0$ is the integral of the contributions from all possible points $\boldsymbol{p}_1$ in the aperture,

Scale with angle

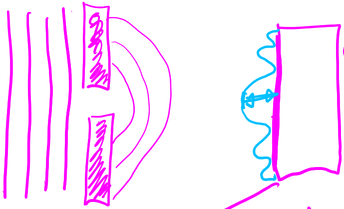
$$U(\boldsymbol{p}_0) = \frac{1}{\mathrm{j}\,\lambda} \iint_{\Sigma} \boxed{U(\boldsymbol{p}_1)} \frac{\boxed{\exp\{\mathrm{j}\,k\,r_{01}\}}}{r_{01}} \cos\theta \,\mathrm{d}s.$$

Scale Amplitude with distance

This expresses the Huygens-Fresnel principle of wave summation.

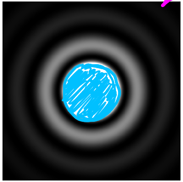
Too much Calculation required!

Amplitude after an aperture



CAMERA SENSOR

PICTURE FROM CAMERA!



AIRY disc

limit of resolution of any telescope or microscope, etc.

An aperture affects wave summation such that unusual constructive and destructive interference arises. Spot dia. $2.44\,\lambda\, f/D$.

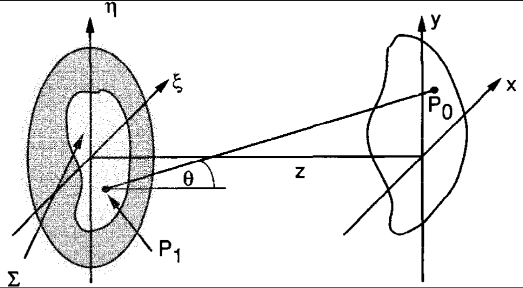
Amplitude after an aperture

This is termed *diffraction* and it happens to all physical waves:

- light
- sound
- vibration (e.g. of water)
- gravitational waves

Diffuse reflection from a rough surface can also be understood as diffraction.

Amplitude after an aperture



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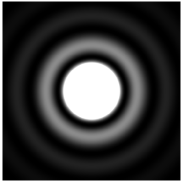

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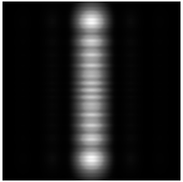
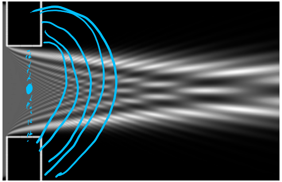


Intensity in $x-y$ plane after a narrow rectangular aperture.



Intensity in $x-z$ plane through and after a narrow rectangular aperture.

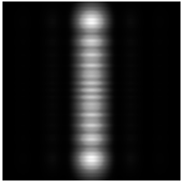
Significant computation is required for *numerical solutions* that simulate diffraction effects through summation of wave amplitudes.

Many different techniques can be used, e.g. *finite element methods*, to find wave amplitudes at discrete volumes in space at successive steps in time.

Note that intensity at distance r_{01} is distributed over a sphere whose surface area is $4\pi r_{01}^2$. So intensity scales $\propto 1/r_{01}^2$.

Since amplitude is the square root of intensity, it scales $\propto \sqrt{1/r_{01}^2} = 1/r_{01}$.

In two dimensional wave propagation (e.g. on water) the amplitude scales $\propto 1/\sqrt{r_{01}}$.



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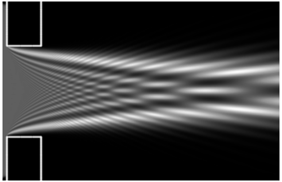

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Fresnel Approximation

Since $\cos\theta = z/r_{01}$, wave summation can be rewritten in more explicit rectangular coordinates as,

$$U(x, y, z) = \frac{z}{\mathrm{j}\,\lambda} \iint_{\Sigma} U(\xi, \eta)\, \frac{\exp\{\mathrm{j}\; k\; r_{01}\}}{r_{01}^2}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

with distance calculated as,

$$r_{01} = \sqrt{z^2 + (x - \xi)^2 + (y - \eta)^2}$$

Fresnel Approximation

To facilitate *analytical solutions*, an approximation for distance r_{01} uses a binomial expansion to replace the square root,

$$\sqrt{1+b} = (1+b)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \ldots$$

when $|b| < 1$.

Fresnel Approximation

$$r_{01} = \sqrt{z^2}\sqrt{1 + \Big(\frac{x-\xi}{z}\Big)^2 + \Big(\frac{y-\eta}{z}\Big)^2}$$
$$\approx z\Big[1 + \frac{1}{2}\Big(\frac{x-\xi}{z}\Big)^2 + \frac{1}{2}\Big(\frac{y-\eta}{z}\Big)^2\Big]$$

using only the first two terms of the expansion with $b = \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2$.

(cf. parabolic approx. of spherical wavefront.)

Fresnel Approximation

The same approximation for r_{01} doesn't have to be used for all occurances.

Using the first term only, the denominator $r_{01}^2 \approx z^2$. This can be factored out of the integral into the scaling term,

$$\frac{z}{\mathrm{j}\,\lambda\,z^2} = \frac{1}{\mathrm{j}\,\lambda\,z}.$$

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with distance calculated as, Euclidean distance

$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2}$$

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Convolution

Using the first two terms of the approximation,

$$\exp\{j\,k\,r_{01}\} \approx \exp\left\{j\,k\,z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]\right\}$$

$$= \exp\{j\,k\,z\}\ \exp\left\{j\,k\,z\left[\frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]\right\}$$

$$= \exp\{j\,k\,z\}\ \exp\left\{j\,\frac{k}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\}$$

Convolution

approximation

$$U(x,y,z) \approx \frac{\exp\{\mathrm{j}\,k\,z\}}{\mathrm{j}\,\lambda z} \iint_{-\infty}^{+\infty} U(\xi,\eta) \times$$

$$\exp\left\{\mathrm{j}\,\frac{k}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Accurate only for the "near field" close to the aperture because of distance approximation.

Convolution

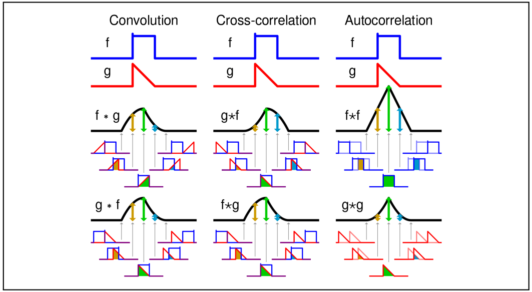
To facilitate analysis, it can be written as a *convolution* of the aperture with a function h.

$$U(x,y,z) \approx \iint_{-\infty}^{+\infty} U(\xi,\eta) \times h(x-\xi, y-\eta)\ \mathrm{d}\xi\,\mathrm{d}\eta.$$

Convolution *kernel* $h(v,w) =$

$$\frac{\exp\{\mathrm{j}\ k\ z\}}{\mathrm{j}\ \lambda z} \exp\left\{\mathrm{j}\ \frac{k}{2z}(v^2 + w^2)\right\}.$$

Convolution



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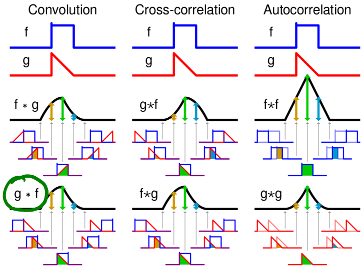

Fourier transform

$$(x-\xi)^2 = x^2 - 2x\xi + \xi^2,$$
$$(y-\eta)^2 = y^2 - 2y\eta + \eta^2.$$

Hence further factorization outside the integral is possible since only those terms in ξ and η need to remain inside.

$$\exp\left\{\mathrm{j}\,\frac{k}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} =$$

Fourier transform

$$\exp\left\{\mathrm{j}\,\frac{k}{2z}(x^2+y^2)\right\}\times$$
$$\exp\left\{\mathrm{j}\,\frac{k}{2z}(\xi^2+\eta^2)\right\}\times$$
$$\exp\left\{-2\,\mathrm{j}\,\frac{k}{2z}(x\xi+y\eta)\right\}.$$

Note that $k = 2\pi/\lambda$,

$$\frac{k}{2z} = \frac{2\pi}{\lambda}\frac{1}{2z} = \frac{2\pi}{\lambda 2z} = \frac{\pi}{\lambda z}.$$

Fourier transform

$$U(x,y,z) \approx$$
$$\frac{\exp\{\mathrm{j}\; k\; z\}}{\mathrm{j}\;\lambda z} \exp\left\{\mathrm{j}\;\frac{k}{2z}(x^2+y^2)\right\} \times$$
$$\iint_{-\infty}^{+\infty} U(\xi,\eta) \exp\left\{\mathrm{j}\;\frac{k}{2z}(\xi^2+\eta^2)\right\} \times$$
$$\exp\left\{-\mathrm{j}\;\frac{2\pi}{\lambda z}(x\xi+y\eta)\right\}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Fourier transform

This integral can be recognised as the (scaled) *Fourier transform* of the (scaled) aperture evaluated at spatial frequencies,

$$f_X = \frac{x}{\lambda z} \qquad f_Y = \frac{y}{\lambda z}.$$

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Fraunhofer Approximation

When z in $\frac{k}{2z}(\xi^2 + \eta^2)$ is very big,* this expression is close to 0 so its exponent is close to 1. So it is not essential to use it a scaling factor,

$$U(x, y, z) = \frac{\exp\{\mathrm{j}\, kz\} \exp\left\{\mathrm{j}\, \frac{k}{2z}(x^2 + y^2)\right\}}{\mathrm{j}\, \lambda z}$$

$$\iint_{-\infty}^{+\infty} U(\xi, \eta) \exp\left\{-\mathrm{j}\, \frac{2\pi}{\lambda z}(x\xi + y\eta)\right\} \mathrm{d}\xi\, \mathrm{d}\eta$$

Fraunhofer Approximation

Hence wave summation can be expressed as the (scaled) Fourier transform of the (unscaled) aperture evaluated at frequencies,

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Accurate only for the “far field” distant from the aperture because of distance aassumption.

Fraunhofer Approximation

For intensity $I(x, y, z) = |U(x, y, z)|^2$, the numerator and denominator of the scaling term simplify as follows.

$$\begin{aligned} |\exp\{\mathrm{j}\, kz\}|^2 &= \exp\{+\mathrm{j}\, kz\}\times \\ &\qquad \exp\{-\mathrm{j}\, kz\} \\ &= \exp\{+\mathrm{j}\, kz - \mathrm{j}\, kz\} \\ &= \exp\{0\} = 1. \end{aligned}$$

Fraunhofer Approximation

$$
\begin{aligned}
|\mathrm{j}\,\lambda z|^2 &= (+\mathrm{j}\,\lambda z)(-\mathrm{j}\,\lambda z) \\
&= +1\,\lambda^2 z^2 = \lambda^2 z^2.
\end{aligned}
$$

For a nice alternative derivation of the material in this lecture, see `https://www.youtube.com/watch?v=JKxDa5D3GnQ`.

Fraunhofer Approximation

scipy has FT methods!

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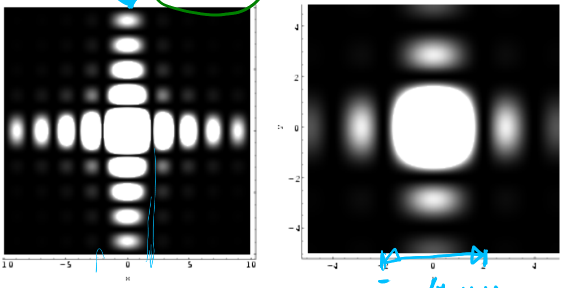
More examples

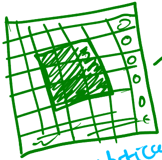
Q6!!!

Small square aperture

Larger square aperture



(a)



Vertical slit aperture

Horizontal slit aperture