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CS7GV2: Mathematics of Light and Sound

Lecture #4: Simulation

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October 30, 2020

Analytical versus numerical methods

- For a quadratic polynomial $f(x) = a\,x^2 + b\,x + c$, the roots (zero-crossings) are found with the well-known formula,

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}.$$

- In science and engineering it's more-often-than-not the case that problems do not have neat *closed-form* or *analytical* solutions except in very specific circumstances.
- What can we do about it? Approximation, iteration. For example, the "method of bisection" for root finding: guess where a root might be; keep halving the length of an interval around it such that $f(x)$ has different signs at the start and the end.
- Such solutions often described as *numerical methods* because they use numbers (and computers) versus *analytical methods* which use symbols (and thinking.)

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Analytic or Closed-form solution

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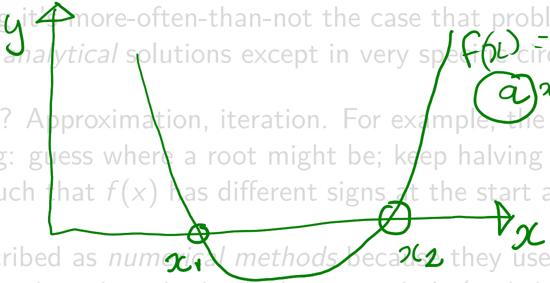

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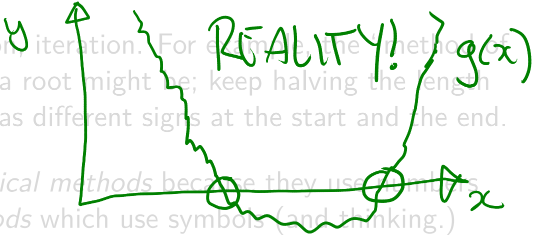

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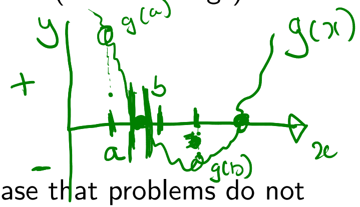

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HIDDEN FIGURES!

Wave Motion

- We've seen that wave motion is described by the second order PDE known as the wave equation,

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2}} = \underline{c^2} \boxed{\frac{\partial^2 u(x,t)}{\partial x^2}}.$$

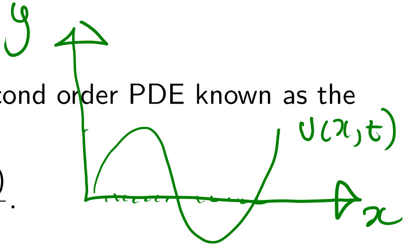
- We've seen a *closed-form* solution for wave propagation,

$$u(x,t) = R\cos(kx - \omega t) + (1 - R)\cos(kx + \omega t).$$

- This is perfect when there are no constraints. For example, light in a homogeneous medium, a wave on an infinitely long string (no end points,) or a sound in a huge volume of air.

- But the closed-form solution doesn't tell us, for example, how a string plucked in a particular way is going to move: `https://tinyurl.com/y4ncymx7`.

Wave Motion



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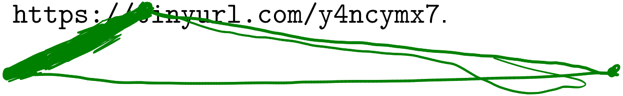

Wave Simulation

- When there are specific constraints (also known as conditions,) there is usually no alternative but to *simulate* wave motion in an *iterative* way.
- Iterative means doing more-or-less the same sequence of calculations again and again.
- Usually the current iteration's calculations use results calculated in the previous iteration(s.)
- An iterative simulation can never be perfect. Error is inevitable, for example, because descretization is required.
- Error is typically cumulative so the results become less correct at each iteration.
- There are lots of nice interactive simulations of wave motions available, for example: `https://tinyurl.com/2xrsrz` and `https://tinyurl.com/mtwczmj`.

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EM Wave simulation

- Solve Maxwell's equations to find local wave characteristics at many discrete volumes of space at successive steps in time.
- The results for one discrete volume are used in the calculation of the characteristics of its neighbors.
- One of the most used techniques (e.g. in MEEP) is called *finite difference time domain* (FDTD.)
- Approaches like this in general are called *finite element methods* for the approximate solution of *boundary value problems* with *partial differential equations.*
- Advantages: can deal with complex geometries and different materials.
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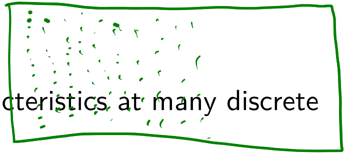

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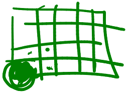
Cf. String Simulation



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google it!

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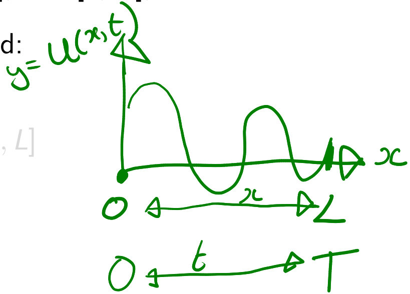
Simulated weather < 3 days

Initial and Boundary Conditions

- To simulate a specific solution for $\boxed{u(x,t)}$ described by the wave equation,

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}} \quad x \in [0, L], \ \ t \in [0, T],$$

  for a string of length L over a time period T, we need:



- two *initial conditions* at time $t = 0$,

$$u(x,0) = I(x), \quad x \in [0, L]$$
$$\frac{\partial}{\partial t} u(x,0) = 0, \quad x \in [0, L]$$

  where $I(x)$ specifies the initial shape of the string,

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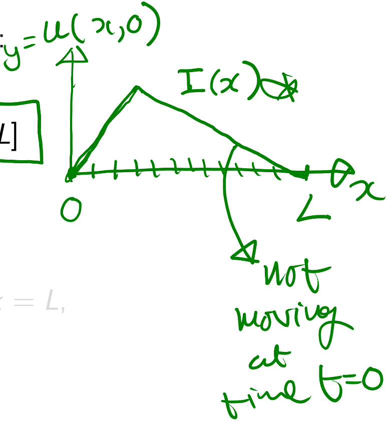

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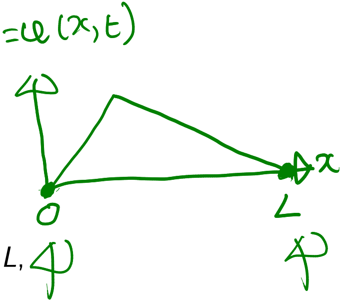

Discretization of domain

- Computer operations take a finite amount of time to complete so there can't be infinitely many time steps in the simulation.

  The time period $[0, T]$ has to be descretized, e.g. into intervals of equal duration Δt,

  $$t_i = i\,\Delta t, \quad i = 0, \ldots N_t \text{ (where } N_t = T/\Delta t.\text{)}$$

- Computer memory is finite so there can't be infinitely many distances in the simulation.

  The length $[0, L]$ have to be descretized, e.g. into intervals of equal distance Δx,

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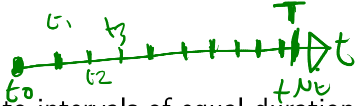
T = 5s



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✱ to keep calculation + memory requirements finite

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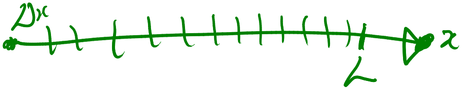

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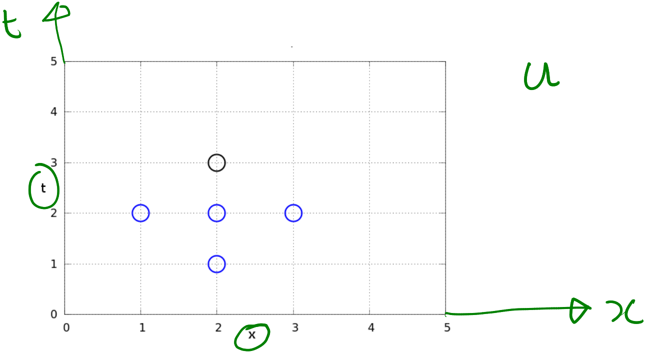
Solution mesh

- The discrete points in space and time can be visualized as a two-dimensional *mesh* (or net.)



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- . . . except for certain exterior mesh points whose values have been specified through the initial conditions, i.e. $I(x)$.

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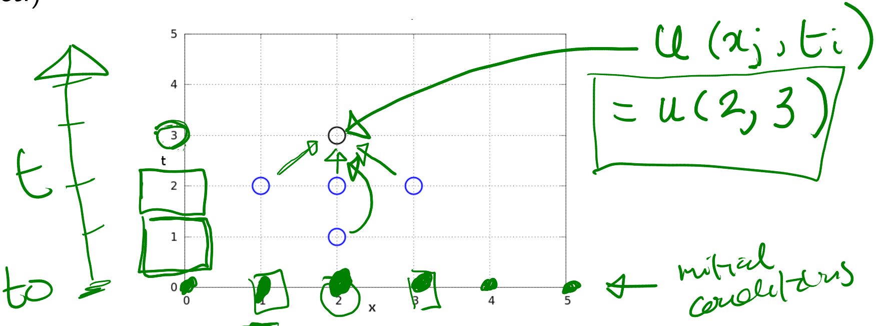

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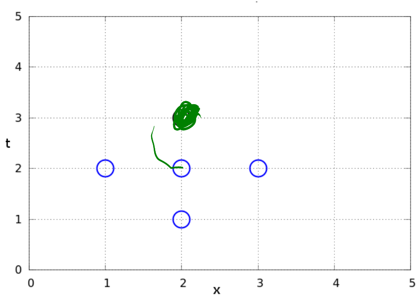

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Discretization of equations

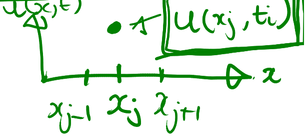
t_i

- *Wave equation.* Use the symmetric second difference approximation of the second derivative,

$$\frac{\partial^2 u(x,t)}{\partial t^2}$$

=

$$\frac{u(x_j,t_{i+1}) - 2\,u(x_j,t_i) + u(x_j,t_{i-1})}{\Delta t^2} \approx c^2\frac{u(x_{j+1},t_i) - 2\,u(x_j,t_i) + u(x_{j-1},t_i)}{\Delta x^2}$$



$$\frac{\partial^2 u(x,t)}{\partial^2 x}$$

t_i

Alternative notation can be used to make the parameters more obvious,

$$\frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{\Delta t^2} \approx c^2\frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{\Delta x^2}, \tag{1}$$

- *Initial condition.* Use the centered first difference approximation of the first derivative,

$$\frac{\partial}{\partial t}u(x_j,t_i) \approx \frac{u_j^{i+1} - u_j^{i-1}}{2\Delta t} \tag{2}$$

Note division by $2\Delta t$ because the difference is between values of $u(x,t)$ separated by two time intervals.

Discretization of equations

- *Wave equation.* Use the symmetric second difference approximation of the second derivative,

$$\frac{u(x_j,t_{i+1}) - 2\,u(x_j,t_i) + u(x_j,t_{i-1})}{\Delta t^2} \approx c^2\,\frac{u(x_{j+1},t_i) - 2\,u(x_j,t_i) + u(x_{j-1},t_i)}{\Delta x^2}.$$

$u(x_j, t_{i+1})$

$= u_j^{i+1}$

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Initial Conditions

- Using approximation (2), initial condition $\frac{\partial}{\partial t}u(x_j, 0) = 0$ means,

$$u_j^{i-1} = u_j^{i+1}, \quad j = 0, \ldots, N_x. \quad i = 0.$$

- The intial condition of shape is simply,

$$u_j^0 = I(x_j), \quad j = 0, \ldots, N_x.$$

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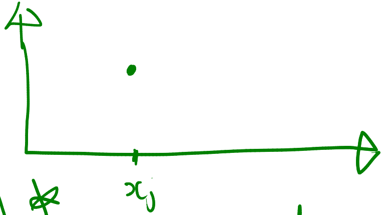
Formulae

$u(x_j, t_{i+1})$

LHS = RHS

- $u_j^{i+1} = -u_j^{i-1} + 2u_j^i + C^2\left(u_{j+1}^i - 2u_j^i + u_{j-1}^i\right)$
- $C = c\frac{\Delta t}{\Delta x}$.
- $u_j^1 = u_j^0 - \frac{1}{2}C^2\left(u_{j+1}^i - 2u_j^i + u_{j-1}^i\right)$



where did this come from?

Rearranged wave equation

t_{i-1}

t_i

t_{i+1}

RHS terms are blue

look at solution mesh

Iterative Simulation Algorithm

1. Initialize $u_j^0 = I(x_j)$ for $j = 0, \dots N_x$.
2. Compute u_j^1 and set $u_j^1 = 0$ for the boundary points $i = 0$ and $i = N_x$, for $i = 1, \dots N - 1$
3. For each time level $i = 1, \dots N_t - 1$
   - 3.1 find u_j^{i+1} for $j = 1, \dots N_x - 1$.
   - 3.2 set $u_j^{i+1} = 0$ for the boundary points $j = 0, j = N_x$.

Write a program in SciPy to simulate String motion using this Algorithm!