CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #9: Diffraction

Fergal Shevlin, Ph.D.

School of Computer Science and Statistics,
Trinity College Dublin

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Amplitude away from source

Speed of light* in $\mathrm{m\,s^{-1}}$ is c

Wavelength in m is λ

Wave period in s is $T = \lambda/c$

Wave frequency in Hz is $\nu = 1/T$

Angular freq. in $\mathrm{rad\,s^{-1}}$ is $\omega = 2\pi/T$

Wave number in $\mathrm{rad\,m^{-1}}$ is $k = 2\pi/\lambda$

Amplitude away from source

A phasor encodes max. amplitude $A(\boldsymbol{p})$ and phase $\phi(\boldsymbol{p})$ at position $\boldsymbol{p}$,

$$U(\boldsymbol{p}) = A(\boldsymbol{p}) \exp\{\mathrm{j}\ \phi(\boldsymbol{p})\}.$$

The scalar value of an EM wave vector component at time t can be found as,

$$\begin{aligned} u(\boldsymbol{p}, t) &= \mathrm{Re}\{U(\boldsymbol{p})\ \exp\{-\mathrm{j}\,\omega t\}\} \\ &= A(\boldsymbol{p}) \cos(\omega\ t - \phi(\boldsymbol{p})). \end{aligned}$$

Amplitude away from source

Let $\boldsymbol{p}_1$ be the point source of a wave and let $\boldsymbol{p}_0$ be somewhere else. Let t_{01} be the time it takes for the wave to travel.

$$u(\boldsymbol{p}_0, t_{01}) = \mathrm{Re}\left\{ U(\boldsymbol{p}_1) \, \frac{\exp\{-\mathrm{j}\,\omega t_{01}\}}{r_{01}} \right\}$$

where $r_{01} = \|\boldsymbol{p}_0 - \boldsymbol{p}_1\| = c\, t_{01}$ is the Euclidean distance between the points.

Amplitude away from source

$$\omega\ t_{01} = \frac{2\pi}{T}\ t_{01} = \frac{2\pi}{\lambda/c}\ t_{01}$$
$$= \frac{2\pi\ c}{\lambda}\ t_{01} = k\ r_{01}$$

Since there is no explicit time term, this can be used to express the phasor at $\boldsymbol{p}_0$,

$$U(\boldsymbol{p}_0) = U(\boldsymbol{p}_1)\ \frac{\exp\{-\mathrm{j}\ \ k\ r_{01}\}}{r_{01}}$$

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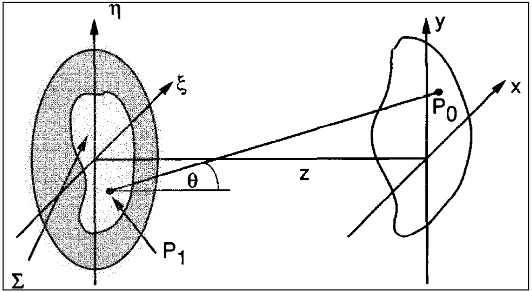
Amplitude after an aperture



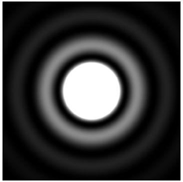
Amplitude after an aperture

Amplitude at $\boldsymbol{p}_0$ is the integral of the contributions from all possible points $\boldsymbol{p}_1$ in the aperture,

$$U(\boldsymbol{p}_0) = \frac{1}{\mathrm{j}\,\lambda} \iint\limits_{\Sigma} U(\boldsymbol{p}_1)\, \frac{\exp\{\mathrm{j}\, k\, r_{01}\}}{r_{01}} \cos\theta \,\mathrm{d}s.$$

This expresses the Huygens-Fresnel principle of wave summation.

Amplitude after an aperture



An aperture affects wave summation such that unusual constructive and destructive interference arises. Spot dia. $2.44\,\lambda\,f/D$.

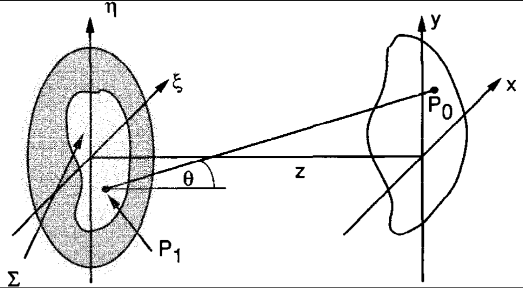
Amplitude after an aperture

This is termed *diffraction* and it happens to all physical waves:

- light
- sound
- vibration (e.g. of water)
- gravitational waves

Diffuse reflection from a rough surface can also be understood as diffraction.

Amplitude after an aperture



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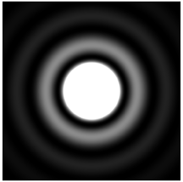

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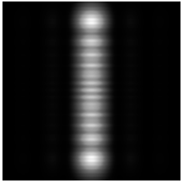
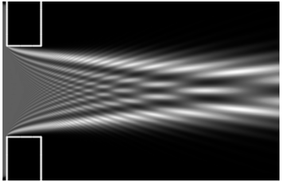


Intensity in $x-y$ plane after a narrow rectangular aperture.



Intensity in $x-z$ plane through and after a narrow rectangular aperture.

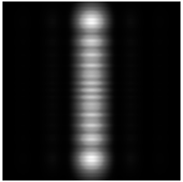
Significant computation is required for *numerical solutions* that simulate diffraction effects through summation of wave amplitudes.

Many different techniques can be used, e.g. *finite element methods*, to find wave amplitudes at discrete volumes in space at successive steps in time.

Note that intensity at distance r_{01} is distributed over a sphere whose surface area is $4\pi r_{01}^2$. So intensity scales $\propto 1/r_{01}^2$.

Since amplitude is the square root of intensity, it scales $\propto \sqrt{1/r_{01}^2} = 1/r_{01}$.

In two dimensional wave propagation (e.g. on water) the amplitude scales $\propto 1/\sqrt{r_{01}}$.



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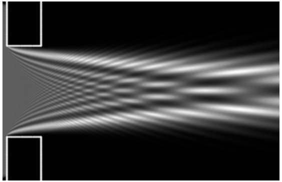

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Fresnel Approximation

Since $\cos\theta = z/r_{01}$, wave summation can be rewritten in more explicit rectangular coordinates as,

$$U(x, y, z) = \frac{z}{\mathrm{j}\,\lambda} \iint_{\Sigma} U(\xi, \eta)\, \frac{\exp\{\mathrm{j}\; k\; r_{01}\}}{r_{01}^2}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

with distance calculated as,

$$r_{01} = \sqrt{z^2 + (x - \xi)^2 + (y - \eta)^2}$$

Fresnel Approximation

To facilitate *analytical solutions*, an approximation for distance r_{01} uses a binomial expansion to replace the square root,

$$\begin{aligned}\sqrt{1+b} &= (1+b)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \dots\end{aligned}$$

when $|b| < 1$.

Fresnel Approximation

$$r_{01} = \sqrt{z^2}\sqrt{1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2}$$

$$\approx z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]$$

using only the first two terms of the expansion with $b = \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2$.

(cf. parabolic approx. of spherical wavefront.)

Fresnel Approximation

The same approximation for r_{01} doesn't have to be used for all occurances.

Using the first term only, the denominator $r_{01}^2 \approx z^2$. This can be factored out of the integral into the scaling term,

$$\frac{z}{\mathrm{j}\,\lambda\,z^2} = \frac{1}{\mathrm{j}\,\lambda\,z}.$$

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when $|b| < 1$.

$$\begin{aligned} r_{01} &= \sqrt{z^2}\sqrt{1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2} \\ &\approx z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]\end{aligned}$$

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Convolution

Using the first two terms of the approximation,

$$\exp\{\mathrm{j}\ k\ r_{01}\} \approx \exp\Big\{\mathrm{j}\ k\ z\Big[1 + \frac{1}{2}\Big(\frac{x-\xi}{z}\Big)^2 + \frac{1}{2}\Big(\frac{y-\eta}{z}\Big)^2\Big]\Big\}$$

$$= \exp\{\mathrm{j}\ k\ z\}\ \exp\Big\{\mathrm{j}\ k\ z\Big[\frac{1}{2}\Big(\frac{x-\xi}{z}\Big)^2 + \frac{1}{2}\Big(\frac{y-\eta}{z}\Big)^2\Big]\Big\}$$

$$= \exp\{\mathrm{j}\ k\ z\}\ \exp\Big\{\mathrm{j}\ \frac{k}{2z}[(x-\xi)^2 + (y-\eta)^2]\Big\}$$

Convolution

$$U(x, y, z) \approx \frac{\exp\{\mathrm{j}\; k\; z\}}{\mathrm{j}\; \lambda z} \iint_{-\infty}^{+\infty} U(\xi, \eta) \times$$
$$\exp\left\{\mathrm{j}\; \frac{k}{2z} [(x-\xi)^2 + (y-\eta)^2]\right\}\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

Accurate only for the "near field" close to the aperture because of distance approximation.

Convolution

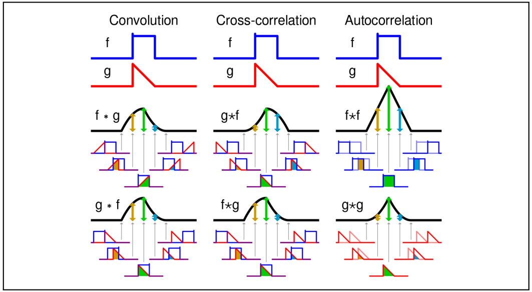
To facilitate analysis, it can be written as a *convolution* of the aperture with a function h.

$$U(x, y, z) \approx \iint_{-\infty}^{+\infty} U(\xi, \eta) \times h(x-\xi, y-\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

Convolution *kernel* $h(v, w) =$

$$\frac{\exp\{\mathrm{j}\ k\ z\}}{\mathrm{j}\ \lambda z} \exp\left\{\mathrm{j}\ \frac{k}{2z}(v^2 + w^2)\right\}.$$

Convolution



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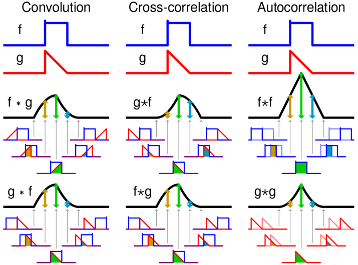

Fourier transform

$$(x-\xi)^2 = x^2 - 2x\xi + \xi^2,$$
$$(y-\eta)^2 = y^2 - 2y\eta + \eta^2.$$

Hence further factorization outside the integral is possible since only those terms in ξ and η need to remain inside.

$$\exp\left\{\mathrm{j}\,\frac{k}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} =$$

Fourier transform

$$\exp\left\{\mathrm{j}\,\frac{k}{2z}(x^2+y^2)\right\}\times$$
$$\exp\left\{\mathrm{j}\,\frac{k}{2z}(\xi^2+\eta^2)\right\}\times$$
$$\exp\left\{-2\,\mathrm{j}\,\frac{k}{2z}(x\xi+y\eta)\right\}.$$

Note that $k=2\pi/\lambda$,

$$\frac{k}{2z}=\frac{2\pi}{\lambda}\frac{1}{2z}=\frac{2\pi}{\lambda 2z}=\frac{\pi}{\lambda z}.$$

Fourier transform

$$\boxed{\begin{gathered} U(x,y,z) \approx \\ \frac{\exp\{\mathrm{j}\;k\;z\}}{\mathrm{j}\;\lambda z}\exp\left\{\mathrm{j}\;\frac{k}{2z}(x^2+y^2)\right\}\times \\ \iint_{-\infty}^{+\infty} U(\xi,\eta)\exp\left\{\mathrm{j}\;\frac{k}{2z}(\xi^2+\eta^2)\right\}\times \\ \exp\left\{-\mathrm{j}\;\frac{2\pi}{\lambda z}(x\xi+y\eta)\right\}\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{gathered}}$$

Fourier transform

This integral can be recognised as the (scaled) *Fourier transform* of the (scaled) aperture evaluated at spatial frequencies,

$$f_X = \frac{x}{\lambda z} \qquad f_Y = \frac{y}{\lambda z}.$$

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Fraunhofer Approximation

When z in $\frac{k}{2z}(\xi^2 + \eta^2)$ is very big,* this expression is close to 0 so its exponent is close to 1. So it is not essential to use it a scaling factor,

$$U(x, y, z) = \frac{\exp\{\mathrm{j}\, kz\} \exp\left\{\mathrm{j}\, \frac{k}{2z}(x^2 + y^2)\right\}}{\mathrm{j}\, \lambda z}$$

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Hence wave summation can be expressed as the (scaled) Fourier transform of the (unscaled) aperture evaluated at frequencies,

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Accurate only for the "far field" distant from the aperture because of distance aassumption.

Fraunhofer Approximation

For intensity $I(x, y, z) = |U(x, y, z)|^2$, the numerator and denominator of the scaling term simplify as follows.

$$\begin{aligned} |\exp\{\mathrm{j}\,kz\}|^2 &= \exp\{+\mathrm{j}\,kz\}\times \\ &\qquad \exp\{-\mathrm{j}\,kz\} \\ &= \exp\{+\mathrm{j}\,kz - \mathrm{j}\,kz\} \\ &= \exp\{0\} = 1. \end{aligned}$$

Fraunhofer Approximation

$$\left|\mathrm{j}\,\lambda z\right|^2 = (+\mathrm{j}\,\lambda z)(-\mathrm{j}\,\lambda z)$$
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For a nice alternative derivation of the material in this lecture, see `https://www.youtube.com/watch?v=JKxDa5D3GnQ`.

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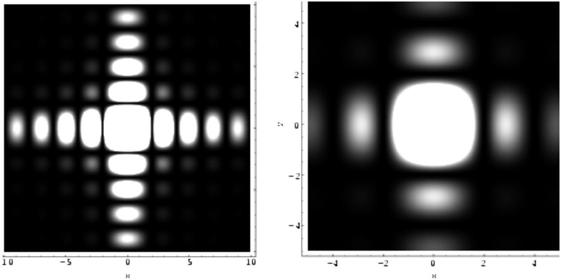
More examples



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