CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Mathematics

Abstract object

Lecture #5: $\boxed{\text{Phasors}}$

C.S. object

Some data
some methods

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data structure + methods

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e.g. Array list

Complex Numbers

A kind of mathematical object!

A complex number is expressed as the sum of a *real* part and an *imaginary* part,

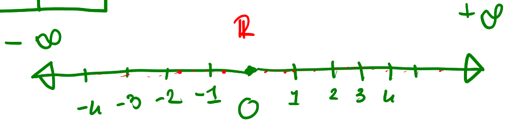
$$a + bj \in \mathbb{C} \quad \text{for } a, b \in \mathbb{R}.$$

Imaginary unit j is defined as,

$$j^2 = -1 \quad \text{so} \quad j = \pm\sqrt{-1}.$$

(In engineering, $j = -i$ is used to denote imaginary to avoid confusion with electrical current I.)

a b



i

$j = -i$

$\sqrt{-1}$ doesn't exist!

Cardano denoted it by symbol j and considered it "imaginary".

$a + bj$

$a + bi$

$x + yj$

$x + yi$

$a + jb$

$a + ib$

Complex Numbers

$(xy)^2 = x^2 y^2$

They allow expressions that wouldn't be possible otherwise, e.g. roots of $\boxed{(x+1)^2 = -9}$ are at $\boxed{x = \boxed{-1 \pm 3\,\mathrm{j}}}$.

$$(\cancel{-1} \pm 3\,\mathrm{j} \cancel{+1})^2 =$$
$$(\pm 3\,\mathrm{j})^2 = (\pm 3)^2\,\mathrm{j}^2 =$$
$$(+3)^2\,\mathrm{j}^2 \text{ and } (-3)^2\,\mathrm{j}^2 =$$
$$\boxed{(9)}(-1) = -9.$$

9 9

LHS = RHS

$\boxed{-9 = -9}$

original use

$(x+1)^2 = -9$

$\sqrt{(x+1)^2} = \pm\sqrt{-9}$

$\boxed{(x+1) = \pm\sqrt{-9}}$

$x = -1 + 3j$

and

$x = -1 - 3j$

Complex Numbers

used as a data structure to keep relevant data together

$\mathrm{Re}\{z\} = x$

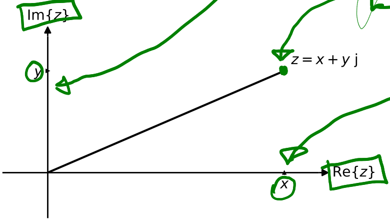
$\mathrm{Im}\{z\} = y$

$z = x + yj$

They can be used to associate numbers that go together, such as point vector coordinates $\boxed{(x, y)}$.

2-D

Y



X

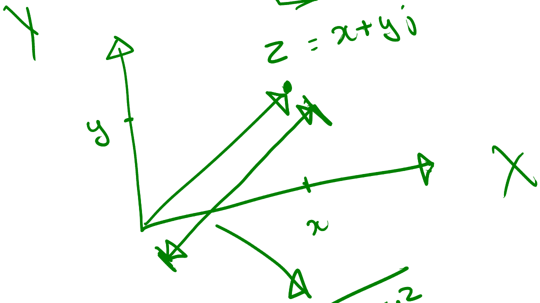
(x, y)

$z = x + yj$

Complex Numbers

2-D Point Vector

$z = x + yj$



$|z| = \sqrt{x^2+y^2}$

But some consideration required, e.g compensation for $\boxed{j^2 = -1}$ to express vector magnitude,

$$
\begin{aligned}
\text{for } \; z &= x + y\,j\,, \\
\text{complex conjugate } \; \bar{z} &= x - y\,j\,, \\
\text{magnitude squared } \; |z|^2 &= z\bar{z} = \boxed{x^2 + y^2,} \\
\text{magnitude } \; |z| &= \sqrt{z\bar{z}}.
\end{aligned}
$$

Complex Numbers

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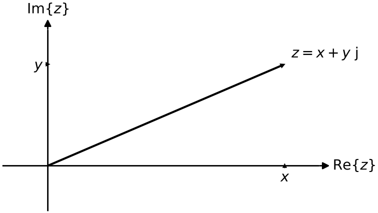
(In engineering, $\mathrm{j} = -\mathrm{i}$ is used to denote imaginary to avoid confusion with electrical current I.)

They allow expressions that wouldn't be possible otherwise, e.g. roots of $(x+1)^2 = -9$ are at $x = -1 \pm 3\,\mathrm{j}$.

$$(-1 \pm 3\,\mathrm{j} + 1)^2 =$$
$$(\pm 3\,\mathrm{j})^2 = (\pm 3)^2\,\mathrm{j}^2 =$$
$$(+3)^2\,\mathrm{j}^2 \text{ and } (-3)^2\,\mathrm{j}^2 =$$
$$(9)(-1) = -9.$$

They can be used to associate numbers that go together, such as point vector coordinates (x, y).



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$$\text{magnitude squared } |z|^2 = z\bar{z} = x^2 + y^2,$$
$$\text{magnitude } |z| = \sqrt{z\bar{z}}.$$

Euler's Formula

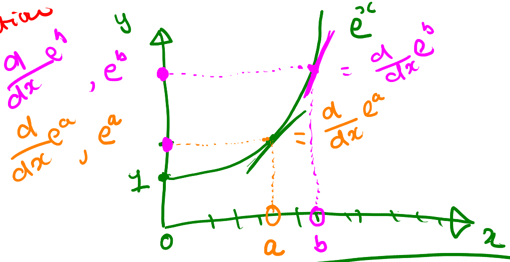
Slope function

Euler's constant $e \approx 2.71828$. $\boxed{\frac{d}{dx}e^x = e^x.}$

e^x is called the natural exponential function and can be written $\exp x$ or $\exp(x)$.

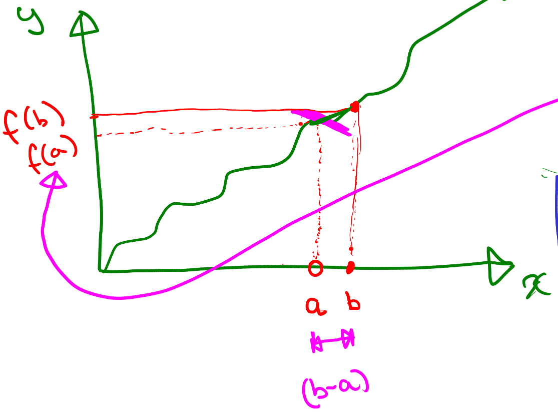
Euler's formula expresses sinusoidal functions through the natural exponential function,

$$\boxed{e^{j\theta}} = \boxed{\cos\theta + j\sin\theta.}$$



$$\boxed{e^x \equiv \exp x \equiv \exp(x)}$$

Euler's Formula



$2! = 2\times 1$
$3! = 3\times 2\times 1$
$n! = n\times(n-1)\times(n-2)\times\cdots\times 1$

The value of any smooth function f at point b in the neighbourhood of point a can be expressed through the Taylor Series,

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a)^1 + \frac{f''(a)}{2!}(b-a)^2 + \ldots$$

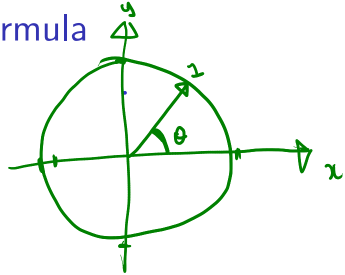
At $x{=}a{=}0$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}\mathrm{e}^x{=}\mathrm{e}^0{=}1$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^x{=}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}$ $\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^x{=}\mathrm{e}^0 = 1$, etc.,

so, $\exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots$

$$f'(a) = \frac{d}{dx} f(a)$$

$$f''(a) = \frac{d^2}{dx^2} f(a)$$

Euler's Formula



$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

+

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

$$= \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

Euler's Formula

instead of b in the Taylor series expansion, write $j\theta$

$$
\begin{aligned}
e^{j\,\theta} = \exp(j\,\theta) &= 1 + \frac{(j\,\theta)^1}{1!} + \frac{(j\,\theta)^2}{2!} + \frac{(j\,\theta)^3}{3!} \ldots \\
&= \boxed{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots} \quad \text{Real terms} \\
&\quad + j\,\boxed{(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)} \quad \text{imaginary terms} \\
&= \cos\theta + j\,\sin\theta.
\end{aligned}
$$

(Hence the expression $e^{i\,\pi} - 1 = 0$.)

Euler's Formula

Euler's constant $\mathrm{e} \approx 2.71828$. $\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x.}$

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$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a)^1 + \frac{f''(a)}{2!}(b-a)^2 + \dots$$

At $x{=}a{=}0$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}\mathrm{e}^x{=}\mathrm{e}^0{=}1$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^x{=}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}$ $\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^x{=}\mathrm{e}^0 = 1$, etc.,

$$\text{so,}\quad \exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = \boxed{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots}$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

$$= \boxed{\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots}$$

$$\mathrm{e}^{\mathrm{j}\,\theta} = \exp(\mathrm{j}\,\theta) = \boxed{1 + \frac{(\mathrm{j}\,\theta)^1}{1!} + \frac{(\mathrm{j}\,\theta)^2}{2!} + \frac{(\mathrm{j}\,\theta)^3}{3!} \dots}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$+\,\mathrm{j}\left(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$= \cos\theta + \mathrm{j}\,\sin\theta.$$

(Hence the expression $\boxed{\mathrm{e}^{\mathrm{j}\pi} - 1 = 0.}$)

a neat equation including all the most important

Phasors → Another Mathematical object used to describe a concept → in this case, a wave

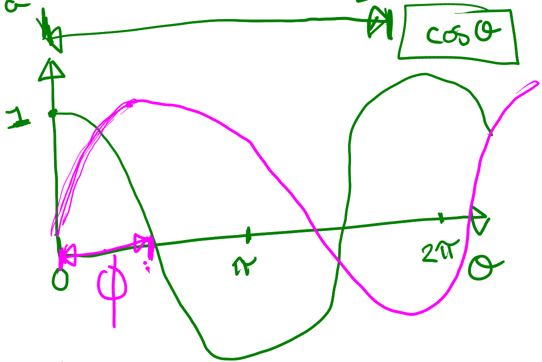
Any

A sinusoid can be described as $A \cos(\omega t + \phi)$ in terms that are *variable*:

- time t, $-\infty < t < +\infty$
- angular frequency $\omega = 2\pi\nu$
- temporal frequency ν

and *constant*:

- amplitude A
- phase ϕ



ν = nu
number of wave periods per unit time

ω = omega
number of radians per unit time

Phasors

phasor

A phasor is used to encode the constants,

$$A e^{j\phi} = A(\cos\phi + j\sin\phi).$$

Multiply by exponential function encoding the variables $e^{j\omega t}$ to get a sinusoid,

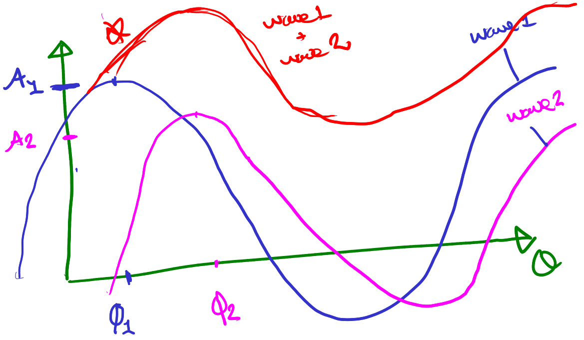
$$A e^{j\phi} e^{j\omega t} = A e^{j(\phi+\omega t)}.$$

$$A\ \mathrm{Re}\left\{e^{j(\phi+\omega t)}\right\} = A\cos(\omega t + \phi).$$

sinusoid

Phasors

Allow us to sum (add) waves in a (relatively) easy way



Amplitude

Phase

Amp

Phase

Sum of two sinusoids with the same ang. freq. ω,

$$A_1\,\cos(\omega t+\phi_1)+A_2\,\cos(\omega t+\phi_2)=$$

$$\mathrm{Re}\left\{A_1\,e^{j\,(\omega t+\phi_1)}+A_2\,e^{j\,(\omega t+\phi_2)}\right\}=$$

$$\mathrm{Re}\left\{\left(A_1\,e^{j\,\phi_1}+A_2\,e^{j\,\phi_2}\right)e^{j\,\omega t}\right\}.$$

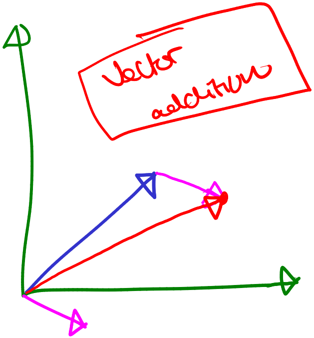
Which is the sum of phasors encoding the constants multiplied by an exponential function encoding the variables.

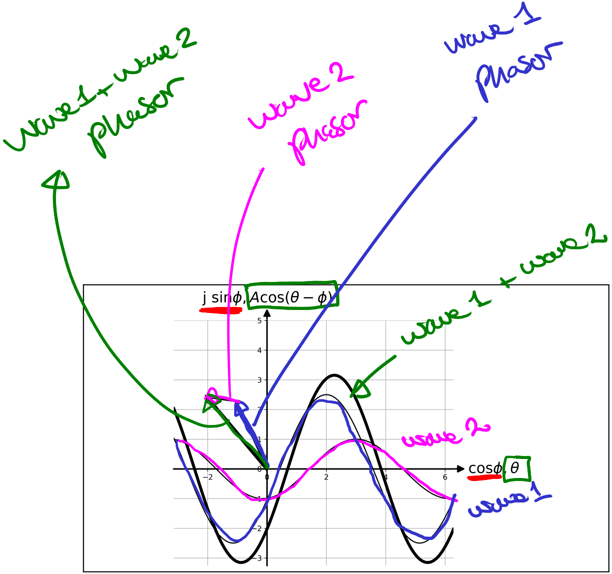
Sum of Sinusoids

Sum of sinusoids expressed using Euler's formula

Sum of phasors

Phasors


Vector addition


wave 1 + wave 2 phasor
wave 2 phasor
wave 1 phasor
j sinϕ, Acos(θ − ϕ)
wave 1 + wave 2
wave 2
cosϕ, θ
wave 1

Phasors

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and *constant*:

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Multiply by exponential function encoding the variables $\mathrm{e}^{\mathrm{j}\,\omega t}$ to get a sinusoid,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t} = A\,\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}.$$

$$A\ \mathrm{Re}\left\{\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}\right\} = A\ \cos(\omega t+\phi).$$

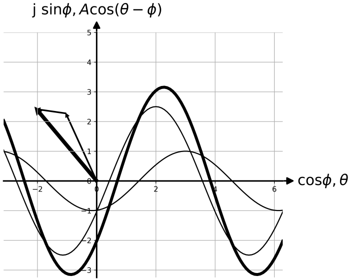
Sum of two sinusoids with the same ang. freq. ω,

$$A_1\ \cos(\omega t+\phi_1) + A_2\ \cos(\omega t+\phi_2) =$$

$$\mathrm{Re}\left\{A_1\ \mathrm{e}^{\mathrm{j}\,(\omega t+\phi_1)} + A_2\ \mathrm{e}^{\mathrm{j}\,(\omega t+\phi_2)}\right\} =$$

$$\mathrm{Re}\left\{(A_1\ \mathrm{e}^{\mathrm{j}\,\phi_1} + A_2\ \mathrm{e}^{\mathrm{j}\,\phi_2})\ \mathrm{e}^{\mathrm{j}\,\omega t}\right\}.$$

Which is the sum of phasors encoding the constants multiplied by an exponential function encoding the variables.



Phasor calculus

The derivative with respect to time of the exponential function of the variables can be expressed as a phasor,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t} = \mathrm{e}^{t} \text{ so } \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\omega t} = \omega \mathrm{e}^{\omega t} \text{ and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{j}\,\omega t} = \mathrm{j}\,\omega\,\mathrm{e}^{\mathrm{j}\,\omega t} = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}\,\mathrm{e}^{\mathrm{j}\,\omega t}$$

since $\mathrm{j} = 0 + \mathrm{j}\,1 = \cos\frac{\pi}{2} + \mathrm{j}\,\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{j}\,\pi/2}$.

Phasor calculus

To differentiate: multiply by $\mathrm{j}\,\omega = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}$.

To integrate: multiply by $\frac{1}{\mathrm{j}\,\omega} = \frac{1}{\omega}\mathrm{e}^{-\mathrm{j}\,\pi/2}$.

Phasor calculus

For example, find the derivative with respect to time of a sinusoid expressed using a phasor,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(A e^{j\phi} e^{j\omega t}\right) &= A e^{j\phi}(j\omega) e^{j\omega t} \\ &= A e^{j\phi} e^{j\pi/2} \omega e^{j\omega t} \\ &= \omega A e^{j(\phi+\pi/2)} e^{j\omega t}.\end{aligned}$$

Phasor calculus

$$\begin{aligned}\mathrm{Re}\left\{\omega A e^{\mathrm{j}\,(\phi+\pi/2)} e^{\mathrm{j}\,\omega t}\right\} &= \omega A \cos(\omega t + \phi + \pi/2) \\ &= \omega A \sin(\omega t + \phi).\end{aligned}$$

Phasor calculus

← we will use some of these results in later analysis!

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For example, find the derivative with respect to time of a sinusoid expressed using a phasor,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t}\right) &= A\mathrm{e}^{\mathrm{j}\,\phi}(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\pi/2}\omega\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= \omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}.\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\left\{\omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}\right\} &= \omega A\cos(\omega t + \phi + \pi/2)\\ &= \omega A\sin(\omega t + \phi).\end{aligned}$$