CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Adding together

encodes Amplitude + phase of a sinusoid

Lecture #6: Random Phasor Sums

written as an exponent with a complex number

Amplitudes and Phases of different waves are independent of one another

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November 12, 2021

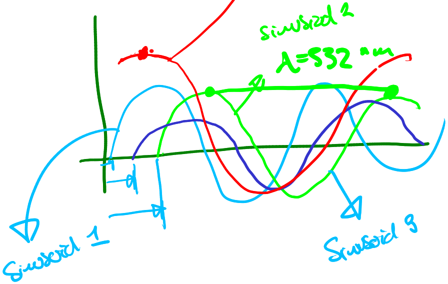
Random walks

Multiple wave phasors can be summed like vectors to find the *resultant* wave phasor at a point in space and time.

When wave amplitudes are independent and wave phases are independent* their summation is called a "random walk."

$\underbrace{3+4+5}_{\text{To sum = to add}} = 12$

Resultant



How do we sum / add waves with the same wavelength λ?

λ wavelength

green 532 nm

Random walks

Amplitude A

phase ϕ

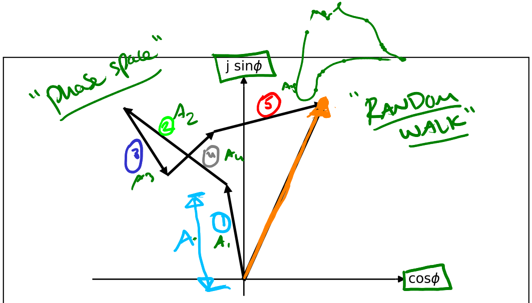
$$\boxed{A e^{j\phi}} \rightarrow A\left[\underline{\cos\phi} + j\,\underline{\sin\phi}\right]$$

Sinusoids

→ Given 5 different ① ② ③ ④ ⑤

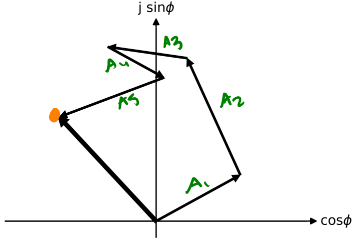
→ write as phasors

→ Add corresponding terms of each phasor



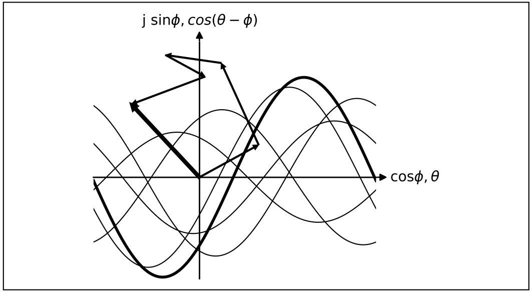
Multiple phasors and their resultant sum.

Random walks



Same amplitudes, different phases, smaller resultant.

Random walks



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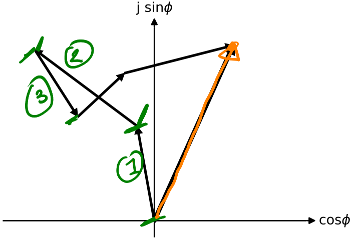
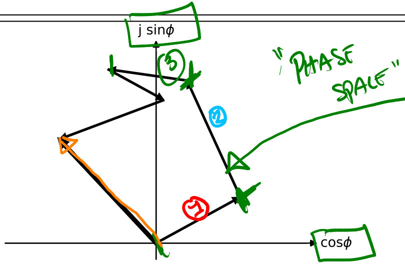


Multiple phasors and their resultant sum.



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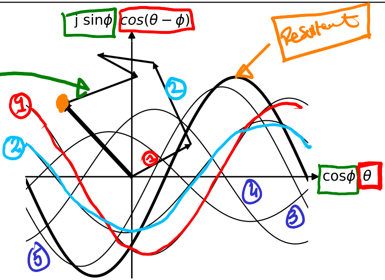


Independence and randomness

Independence means that one event or value of a random quantity X (e.g. a wave's amplitude) has no effect on another, Y (e.g. its phase,)

$$\mathbf{P}(X|Y) = \mathbf{P}(X \cap Y)/\mathbf{P}(Y)$$

Conditional Probability definition

$$\mathbf{P}(X \cap Y) = \mathbf{P}(X|Y)\mathbf{P}(Y)$$

$$\mathbf{P}(X \cap Y) = \mathbf{P}(X)\mathbf{P}(Y) \text{ i.f.f.}$$

$$\mathbf{P}(X) = \mathbf{P}(X|Y) \text{ and } \mathbf{P}(Y) = \mathbf{P}(Y|X).$$

Independence and randomness

Probability distributions

A random quantity is one whose value depends on the outcome of a random phenomenon.

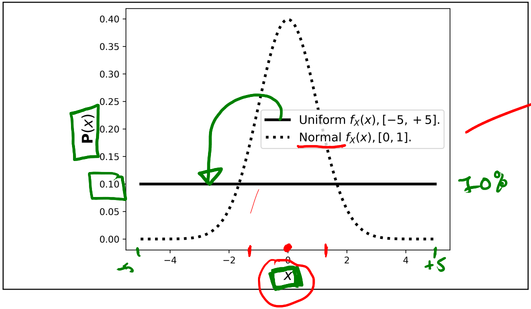
Its occurance may be known* to follow a particular probability density function f_X, or probability mass function p_X, with discriptive parameters μ, σ, *etc*.

Example PDFs are *Uniform*, *Normal*, *Exponential*, *Poisson*, *Rayleigh*.

Independence and randomness

$\frac{0.1}{1.0} = 10\%$ $\frac{0.4}{1.0} = 40\%$

PDF

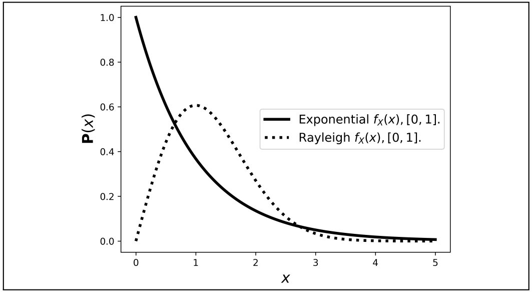


10%

Normal or Gaussian

Independence and randomness

PDF



Independence and randomness

Probability density functions

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$$\mathbf{P}(X|Y) = \mathbf{P}(X \cap Y)/\mathbf{P}(Y)$$
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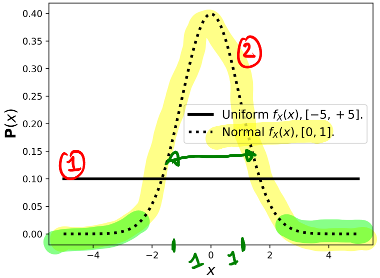
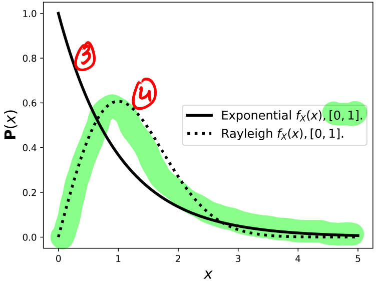
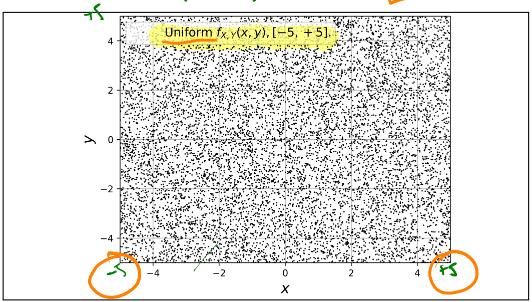
Example PDFs are *Uniform* 1, *Normal* 2, *Exponential* 3, *Poisson*, *Rayleigh*. 4





10,000 values (x, y) chosen randomly



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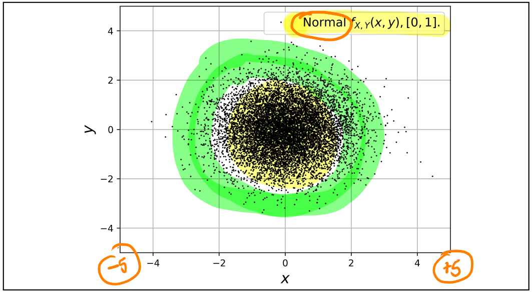

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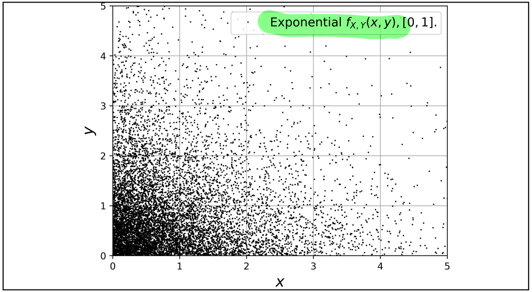

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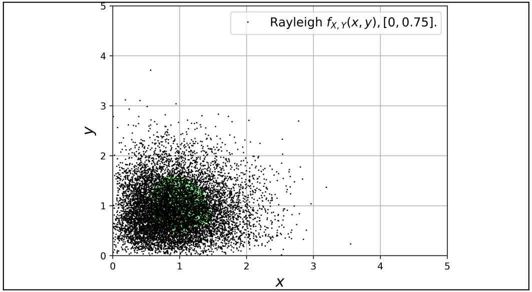
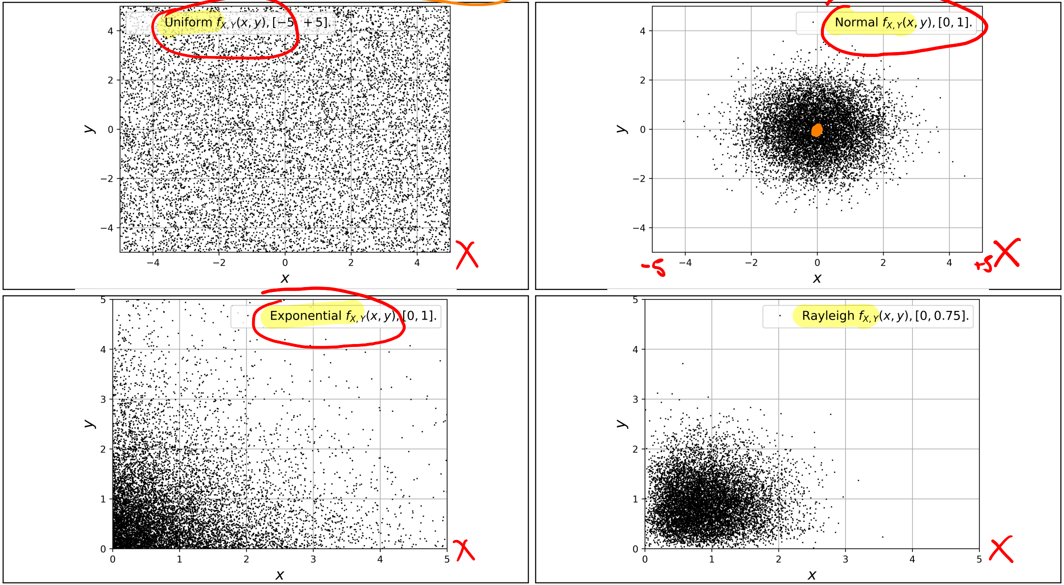


10,000 values (x, y) chosen randomly

$N = 10,000$



Descriptive statistics

"Average"

Expected value or *mean* of a continuous random quantity X with probability density function f_X,

$$\mathbb{E} \qquad \boxed{\mathbf{E}[X]} = \int_{-\infty}^{+\infty} x \, f_X(x) \, \mathrm{d}x.$$

+5

-5

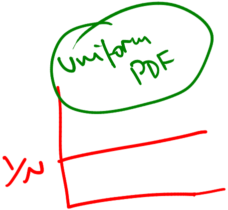
And for a discrete, finite, random quantity X with probability mass function p_X,

10,000

$$\mathbf{E}[X] = \sum_{i=1}^{N} x_i \boxed{p_X(x_i)}$$

PDF

Descriptive statistics



AVERAGE

This is the *arithmetic mean* when probability mass function p_X is uniformly $^1/_N$.

$$
\begin{aligned}
\mathbf{E}[X] &= \textstyle\sum_{i=1}^{N} x_i \boxed{^1/_N} \\
&= {^1/_N} \textstyle\sum_{i=1}^{N} x_i \\
&= (x_1 + x_2 + \ldots x_N)/N
\end{aligned}
$$

Descriptive statistics

Linearity of expectation,

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y].$$

If $Y = aX + b$ for $a, b \in \mathbb{R}$,

$$\mathbf{E}[Y] = a\mathbf{E}[X] + b.$$

If X, Y independent,

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y].$$

Descriptive statistics

Variance is mean distance squared to the mean (when uniform,)

$$\begin{aligned}\sigma_X^2 &= \mathbf{E}[(X - \mathbf{E}[X])^2] \\ &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\ &= \mathbf{E}[X^2] \text{ when } \mathbf{E}[X] = 0.\end{aligned}$$

Standard deviation $\sigma_X = \sqrt{\sigma_X^2}$.

[cf. SCFT STD vs MAD.]

Descriptive statistics

Expected value or *mean* of a continuous random quantity X with probability density function f_X,

$$\mathbf{E}[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, \mathrm{d}x.$$

And for a discrete, finite, random quantity X with probability mass function p_X,

$$\mathbf{E}[X] = \sum_{i=1}^{N} x_i\, p_X(x_i).$$

This is the *arithmetic mean* when probability mass function p_X is uniformly $1/N$.

$$\begin{aligned}\mathbf{E}[X] &= \textstyle\sum_{i=1}^{N} x_i\, 1/N \\ &= 1/N \textstyle\sum_{i=1}^{N} x_i \\ &= (x_1 + x_2 + \ldots x_N)/N\end{aligned}$$

Linearity of expectation,

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y].$$

If $Y = aX + b$ for $a, b \in \mathbb{R}$,

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If X, Y independent,

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y].$$

Useful identities which allow simplification of analysis

Variance is mean distance squared to the mean (when uniform,) → How spread out is the PDF

$$\begin{aligned}\sigma_X^2 &= \mathbf{E}[(X - \mathbf{E}[X])^2] \\ &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\ &= \mathbf{E}[X^2] \text{ when } \mathbf{E}[X] = 0.\end{aligned}$$

Standard deviation $\sigma_X = \sqrt{\sigma_X^2}$.

[cf. SCFT STD vs MAD.]

Random phasor sum

wave amplitudes + phases are independent of one-another

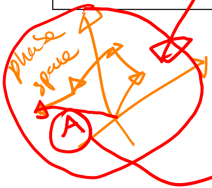
Defined as a weighted sum of random phasors:

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n e^{j\phi_n} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} \boldsymbol{a}_n = A e^{j\theta}$$

$$= \boldsymbol{A} \quad \text{(the "resultant.")}$$

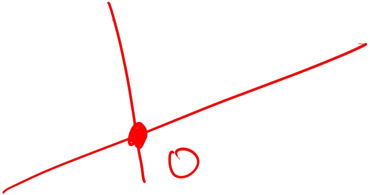
phase of resultant

N



Amplitude of resultant

Random phasor sum



$$\mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}] = \mathbf{E}[{}^1\!/\!\sqrt{N} \sum_{n=1}^{N} a_n \cos\phi_n]$$

$$= {}^1\!/\!\sqrt{N} \sum \mathbf{E}[a_n \cos\phi_n]$$

$$= {}^1\!/\!\sqrt{N} \sum \mathbf{E}[a_n]\mathbf{E}[\cos\phi_n]$$

$$= 0.$$

Similarly, $\mathbf{E}[\mathrm{Im}\{\boldsymbol{A}\}] = 0$.

Random phasor sum

$$\begin{aligned}
\sigma^2_{\mathrm{Re}\{\boldsymbol{A}\}} &= \mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}^2]. \\
\mathrm{Re}\{\boldsymbol{A}\}^2 &= {}^1\!/\!\sqrt{N}(a_1 \cos\phi_1 + a_2 \cos\phi_2 + \ldots)\times \\
&\qquad {}^1\!/\!\sqrt{N}(a_1 \cos\phi_1 + a_2 \cos\phi_2 + \ldots) \\
&= {}^1\!/\!N \textstyle\sum_n \sum_m a_n a_m \cos\phi_n \cos\phi_m. \\
\mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}^2] &= {}^1\!/\!N \textstyle\sum_n \sum_m \mathbf{E}[a_n a_m]\times \\
&\qquad\qquad \mathbf{E}[\cos\phi_n \cos\phi_m] \\
&= {}^1\!/\!N \textstyle\sum_n \mathbf{E}[a_n^2]\mathbf{E}[\cos^2\phi_n]
\end{aligned}$$

Random phasor sum

$$\text{(since for } n \neq m, \mathbf{E}[\cos\phi_n \cos\phi_m]$$
$$= \mathbf{E}[\cos\phi_n]\mathbf{E}[\cos\phi_m] = 0)$$

$$= {}^1\!/_N \sum_n \mathbf{E}[a_n^2] \times \mathbf{E}[{}^1\!/_2 + {}^1\!/_2 \cos 2\phi_n]$$
$$\text{(since } \cos^2\phi = (1 + \cos 2\phi)/2)$$

$$= {}^1\!/_N \sum_n \mathbf{E}[a_n^2]/2.$$

Similarly, $\sigma^2_{\mathrm{Im}\{\boldsymbol{A}\}} = {}^1\!/_N \sum_n \mathbf{E}[a_n^2]/2.$

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&= {}^1\!/\!\sqrt{N}\sum \mathbf{E}[a_n \cos\phi_n] \\
&= {}^1\!/\!\sqrt{N}\sum \mathbf{E}[a_n]\mathbf{E}[\cos\phi_n] \\
&= 0.
\end{aligned}$$

Similarly, $\mathbf{E}[\mathrm{Im}\{\boldsymbol{A}\}] = 0$.

$$\begin{aligned}
\sigma^2_{\mathrm{Re}\{\boldsymbol{A}\}} &= \mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}^2]. \\
\mathrm{Re}\{\boldsymbol{A}\}^2 &= {}^1\!/\!\sqrt{N}(a_1\cos\phi_1 + a_2\cos\phi_2 + \ldots)\times \\
&\qquad {}^1\!/\!\sqrt{N}(a_1\cos\phi_1 + a_2\cos\phi_2 + \ldots) \\
&= {}^1\!/\!N\sum_n\sum_m a_n a_m \cos\phi_n\cos\phi_m. \\
\mathbf{E}[\mathrm{Re}\{\boldsymbol{A}\}^2] &= {}^1\!/\!N\sum_n\sum_m \mathbf{E}[a_n a_m]\times \\
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&= {}^1\!/\!N\sum_n \mathbf{E}[a_n^2]\mathbf{E}[\cos^2\phi_n]
\end{aligned}$$

(since for $n \neq m$, $\mathbf{E}[\cos\phi_n\cos\phi_m]$
$= \mathbf{E}[\cos\phi_n]\mathbf{E}[\cos\phi_m] = 0$)

$$= {}^1\!/\!N\sum_n \mathbf{E}[a_n^2]\times\mathbf{E}[{}^1\!/\!2 + {}^1\!/\!2\cos 2\phi_n]$$

(since $\cos^2\phi = (1+\cos 2\phi)/2$)

$$= {}^1\!/\!N\sum_n \mathbf{E}[a_n^2]/2.$$

Similarly, $\sigma^2_{\mathrm{Im}\{\boldsymbol{A}\}} = {}^1\!/\!N\sum_n \mathbf{E}[a_n^2]/2$.

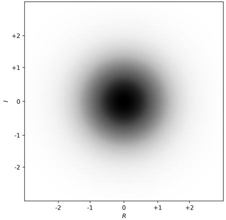
Large numbers

Central Limit Theorem says that the probability density of the *sum* of N independent, identically-distributed, random quantities approaches Normal as $N \to \infty$, ,

$$f_{R,I}(r,i) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{r^2 + i^2}{2\sigma^2} \right\}$$

Where $R = \mathrm{Re}\{\boldsymbol{A}\}$ and $I = \mathrm{Im}\{\boldsymbol{A}\}$ and $\sigma^2 = \sigma_R^2 = \sigma_I^2$. [cf. SCFT p. 125.]

Large numbers



Normal $f_{R,I}(r, i)$, $[0, 1]$.

Large numbers

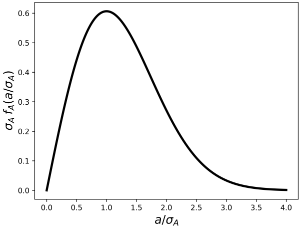
Through transformation of variables,* marginal statistics for A and θ are found as Rayleigh and Uniform respectively,

$$f_A(a) = {}^{a}\!/_{\sigma^2} \exp\left\{ \frac{a^2}{2\sigma^2} \right\}$$

$$f_\theta(\phi) = {}^{1}\!/_{2\pi}$$

$$\mathbf{E}[A] = \sqrt{{}^{\pi}\!/_{2}}\ \sigma, \quad \sigma_A = (2 - {}^{\pi}\!/_{2})\sigma^2$$

Large numbers



Rayleigh $f_A(a)$, $[0, 1]$ (scaled by σ_A.)

Large numbers

RANDOM PHASOR SUMS FOLLOW A NORMAL distribution

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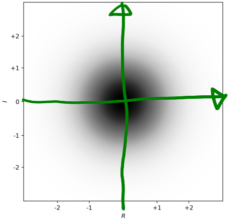
Where $R = \mathrm{Re}\{\boldsymbol{A}\}$ and $I = \mathrm{Im}\{\boldsymbol{A}\}$ and $\sigma^2 = \sigma_R^2 = \sigma_I^2$. [cf. SCFT p. 125.]

2-D



1-D

Normal $f_{R,I}(r,i)$, [0, 1].

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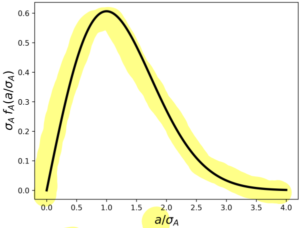

Rayleigh $f_A(a)$, [0, 1] (scaled by σ_A.)