

# CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

## Lecture#8: Intensity sums

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## Descriptive statistics of a sum of two terms

The means of random intensities  $I_X$  and  $I_Y$  are,

$$\overline{I_X} = \mathbf{E}[I_X] \quad \text{and} \quad \overline{I_Y} = \mathbf{E}[I_Y].$$

Linearity of expectation allows the mean of their sum to be expressed as,

$$\mathbf{E}[I_X + I_Y] = \mathbf{E}[I_X] + \mathbf{E}[I_Y]$$

$$\overline{I_Z} = \overline{I_X + I_Y} = \overline{I_X} + \overline{I_Y}.$$

## Descriptive statistics of a sum of two terms

When  $I_X$  and  $I_Y$  are independent,

$$\mathbf{E}[I_X I_Y] = \mathbf{E}[I_X] \mathbf{E}[I_Y]$$

$$\overline{I_X I_Y} = \overline{I_X} \overline{I_Y} .$$

Hence,

$$\begin{aligned} \overline{I_Z^2} &= \overline{(I_X + I_Y)^2} \\ &= \overline{I_X^2 + 2 I_X I_Y + I_Y^2} \\ &= \overline{I_X^2} + 2 \overline{I_X} \overline{I_Y} + \overline{I_Y^2} , \end{aligned}$$

## Descriptive statistics of a sum of two terms

By integration of exponential PDF,

$$\overline{l_X^2} = 2\overline{l_X}^2 \quad \text{and} \quad \overline{l_Y^2} = 2\overline{l_Y}^2.$$

So the variance can be expressed as,

$$\begin{aligned}\sigma_Z^2 &= \mathbf{E}[l_Z^2] - \mathbf{E}[l_Z]^2 = \overline{l_Z^2} - \overline{l_Z}^2 \\ &= 2\overline{l_X}^2 + 2\overline{l_X} \overline{l_Y} + 2\overline{l_Y}^2 \\ &\quad - \overline{l_X}^2 - 2\overline{l_X} \overline{l_Y} - \overline{l_Y}^2 \\ &= \overline{l_X}^2 + \overline{l_Y}^2.\end{aligned}$$

## Descriptive statistics of a sum of two terms

$$C = \frac{\sigma_Z}{\overline{l_Z}} = \frac{\sqrt{\overline{l_X}^2 + \overline{l_Y}^2}}{\overline{l_X} + \overline{l_Y}} = \frac{\sqrt{1 + r^2}}{1 + r}$$

where  $r = \overline{l_Y} / \overline{l_X}$  is the ratio of one mean to the other. Note,

$$\frac{\sqrt{a^2 + b^2}}{\sqrt{a^2}} = \sqrt{\frac{a^2}{a^2} + \frac{b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

When  $r = 1$  the contrast is minimized,

$$C = \sqrt{2}/2 = 2/2\sqrt{2} = 1/\sqrt{2}.$$

## Descriptive statistics of a sum of two terms

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Hence,

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$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\overline{I_X}^2 + \overline{I_Y}^2}}{\overline{I_X} + \overline{I_Y}} = \frac{\sqrt{1 + r^2}}{1 + r}$$

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When  $r = 1$  the contrast is minimized,

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## Descriptive statistics of a sum of $N$ terms

For a sum of random intensities  $I_Z = I_1 + I_2 + \dots + I_N$ ,  
the mean  $\overline{I_Z} = \sum_{n=1}^N \overline{I_n}$  by LOE.

$$\begin{aligned}\overline{I_Z^2} &= \sum_{n=1}^N \sum_{m=1}^N \overline{I_n I_m} \\ &= \sum_{n=1}^N \overline{I_n^2} + \sum_{n=1}^N \sum_{\substack{m=1, \\ m \neq n}}^N \overline{I_n I_m}\end{aligned}$$

## Descriptive statistics of a sum of $N$ terms

Note that  $(a + b + c)^2$

$$= (a + b + c)(a + b + c)$$

$$= a^2 + b^2 + c^2$$

$$+ 2(ab + ac + bc)$$

$$= \sum_{\substack{n = \\ \{a, b, c\}}} n^2 + \sum_{\substack{n = \\ \{a, b, c\}}} \sum_{\substack{m = \\ \{a, b, c\}, \\ m \neq n}} nm.$$



## Descriptive statistics of a sum of $N$ terms

Using  $\overline{l_n^2} = 2\overline{l_n}^2$ ,

$$\begin{aligned}\overline{l_Z^2} &= 2 \sum_{n=1}^N \overline{l_n}^2 + \sum_{n=1}^N \sum_{\substack{m=1, \\ m \neq n}}^N \overline{l_n} \overline{l_m} \\ &= \sum_{n=1}^N \overline{l_n}^2 + \left( \sum_{n=1}^N \overline{l_n} \right)^2 \\ &= \sum_{n=1}^N \overline{l_n}^2 + \overline{l_Z}^2.\end{aligned}$$

## Descriptive statistics of a sum of $N$ terms

$$\sigma_Z^2 = \overline{I_Z^2} - \overline{I_Z}^2 = \sum_{n=1}^N \overline{I_n}^2$$

$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\sum_{n=1}^N \overline{I_n}^2}}{\sum_{n=1}^N \overline{I_n}}$$

When mean intensities  $\overline{I_n}$  are equal,

$$C = \frac{\sqrt{N\overline{I}^2}}{N\overline{I}} = \frac{\sqrt{N}\sqrt{\overline{I}^2}}{N\overline{I}} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}.$$

## Descriptive statistics of a sum of $N$ terms

For a sum of random intensities  $I_Z = I_1 + I_2 + \dots + I_N$ , the mean  $\overline{I_Z} = \sum_{n=1}^N \overline{I_n}$  by LOE.

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Note that  $(a + b + c)^2$

$$\begin{aligned}&= (a + b + c)(a + b + c) \\ &= a^2 + b^2 + c^2 \\ &\quad + 2(ab + ac + bc) \\ &= \sum_{\substack{n= \\ \{a,b,c\}}} n^2 + \sum_{\substack{n= \\ \{a,b,c\}}} \sum_{\substack{m= \\ \{a,b,c\}, \\ m \neq n}} nm.\end{aligned}$$

Using  $\overline{I_n^2} = 2\overline{I_n}^2$ ,

$$\begin{aligned}\overline{I_Z^2} &= 2 \sum_{n=1}^N \overline{I_n}^2 + \sum_{n=1}^N \sum_{\substack{m=1, \\ m \neq n}}^N \overline{I_n I_m} \\ &= \sum_{n=1}^N \overline{I_n}^2 + \left( \sum_{n=1}^N \overline{I_n} \right)^2 \\ &= \sum_{n=1}^N \overline{I_n}^2 + \overline{I_Z}^2.\end{aligned}$$

$$\sigma_Z^2 = \overline{I_Z^2} - \overline{I_Z}^2 = \sum_{n=1}^N \overline{I_n}^2$$

$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\sum_{n=1}^N \overline{I_n}^2}}{\sum_{n=1}^N \overline{I_n}}$$

When mean intensities  $\overline{I_n}$  are equal,

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## PDF of a sum of two terms

For random intensities  $I_X$  and  $I_Y$  described as exponential PDFs,

$$f_{I_X}(i_X) = \frac{1}{I_X} \exp\left\{-\frac{i_X}{I_X}\right\}$$

$$f_{I_Y}(i_Y) = \frac{1}{I_Y} \exp\left\{-\frac{i_Y}{I_Y}\right\},$$

their sum  $I_Z = I_X + I_Y$  is described by another PDF,  $f_{I_Z}(i_Z)$ .

## PDF of a sum of two terms

For *discrete* random quantities  $X, Y, Z \in \mathbb{N}_0$ , let  $Z = X + Y$ . A sum of 3 could result from  $0 + 3$  or  $1 + 2$  or  $2 + 1$  or  $3 + 0$  so,

$$\begin{aligned}\mathbf{P}(Z = 3) &= \mathbf{P}(X = 0) \mathbf{P}(Y = 3) \\ &\quad + \mathbf{P}(X = 1) \mathbf{P}(Y = 2) \\ &\quad + \mathbf{P}(X = 2) \mathbf{P}(Y = 1) \\ &\quad + \mathbf{P}(X = 3) \mathbf{P}(Y = 0)\end{aligned}$$

## PDF of a sum of two terms

$$= \sum_{x=0}^3 \mathbf{P}(X = x) \mathbf{P}(Y = 3 - x).$$

Letting  $p_Z(z)$  etc. be discrete equivalents of a PDF to denote probabilities,

$$p_Z(z) = \sum_{x=0}^z p_X(x) p_Y(z - x).$$

Such a sum (or integral) of products of functions is called a *convolution*.

## PDF of a sum of two terms

For *continuous* random quantities  $X, Y, Z \in \mathbb{R}_{\geq 0}$ , let  $Z = X + Y$ . The PDF for  $Z$  has an integral instead of a sum,

$$f_Z(z) = \int_0^z f_X(x) f_Y(z - x) dx.$$

## PDF of a sum of two terms

For random intensities  $I_X$  and  $I_Y$  described as exponential PDFs,

$$f_{I_X}(i_X) = \frac{1}{I_X} \exp\left\{\frac{-i_X}{I_X}\right\}$$
$$f_{I_Y}(i_Y) = \frac{1}{I_Y} \exp\left\{\frac{-i_Y}{I_Y}\right\},$$

their sum  $I_Z = I_X + I_Y$  is described by another PDF,  $f_{I_Z}(i_Z)$ .

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For *discrete* random quantities  $X, Y, Z \in \mathbb{N}_0$ , let  $Z = X + Y$ . A sum of 3 could result from  $0 + 3$  or  $1 + 2$  or  $2 + 1$  or  $3 + 0$  so,

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For *continuous* random quantities  $X, Y, Z \in \mathbb{R}_{\geq 0}$ , let  $Z = X + Y$ . The PDF for  $Z$  has an integral instead of a sum,

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Let  $f_X(x) = \lambda_X e^{-\lambda_X x}$  and  $f_Y(y) = \lambda_Y e^{-\lambda_Y y}$  be exponential PDFs with rate parameters  $\lambda_X$  and  $\lambda_Y$ . The PDF of their sum is,

$$\begin{aligned} f_Z(z) &= \int_0^z \lambda_X e^{-\lambda_X x} \lambda_Y e^{-\lambda_Y (z-x)} dx \\ &= \lambda_X \lambda_Y e^{-\lambda_Y z} \int_0^z e^{(\lambda_Y - \lambda_X) x} dx. \end{aligned}$$

By FTC the definite integral is the difference between the antiderivative evaluated at the limits,

$$\left. \frac{1}{\lambda_Y - \lambda_X} e^{(\lambda_Y - \lambda_X)x} \right|_0^z \\ = \frac{1}{\lambda_Y - \lambda_X} e^{(\lambda_Y - \lambda_X)z} - \frac{1}{\lambda_Y - \lambda_X}.$$

Or  $\int_0^z 1 \, dx = z$  when  $\lambda_X = \lambda_Y$ .

$$f_Z(z) = \begin{cases} \frac{\lambda_X \lambda_Y}{\lambda_Y - \lambda_X} (e^{-\lambda_X z} - e^{-\lambda_Y z}) & \text{when } \lambda_Y \neq \lambda_X. \\ z \lambda^2 e^{-\lambda z} & \text{when } \lambda = \lambda_X = \lambda_Y. \end{cases}$$

$\lambda = 1/\bar{l}$  for  $\bar{l} = \overline{l_X} = \overline{l_Y}$ , so the PDF of the sum is a scaled\* version of the originals,

$$f_{l_Z}(iz) = \frac{iz}{\bar{l}} \frac{1}{\bar{l}} \exp\left\{\frac{-iz}{\bar{l}}\right\}.$$

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## PDF of a sum of $N$ terms

A bit complicated. If interested, see Goodman,

PDF of a sum of  $N$  terms

*Speckle Phenomena in Optics,*

# PDF of a sum of $N$ terms

page



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42.

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