CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

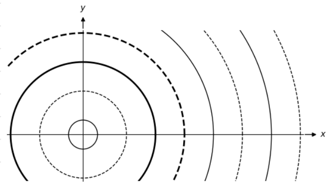
Lecture #1: Waves

Fergal Shevlin, Ph.D.

School of Computer Science and Statistics,
Trinity College Dublin

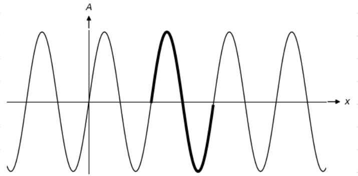
September 16, 2022

Physical waves



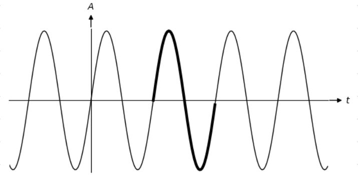
Emission from a point source in space.
Amplitude $\propto r^{-1}$ for 2-D and $\propto r^{-2}$ for 3-D.

Physical waves



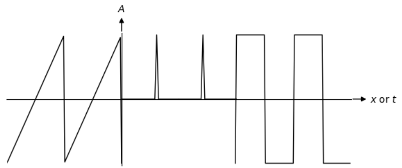
Approx. amplitude along a radial line through space. Spatial wavelength $\lambda := p$ m. Spatial freq. $\xi := (1\,\mathrm{m}/p\,\mathrm{m})\ \mathrm{c/m} = 1\,\mathrm{m}/\lambda\ \mathrm{c/m}$.

Physical waves



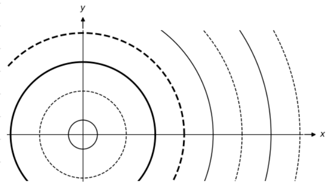
Approx. amplitude at any point x in space wrt time t. Temporal period $T := q\,\mathrm{s}$. Temporal freq. $\nu := (1\,\mathrm{s}/q\,\mathrm{s})\,\mathrm{c/s} = 1/q\,\mathrm{s} = 1\,\mathrm{s}/T\,\mathrm{c/s}$.

Physical waves



Non-smooth (instantaneous change) or non-changing wrt space and time is non-physical.

Physical waves



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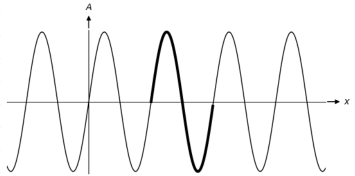

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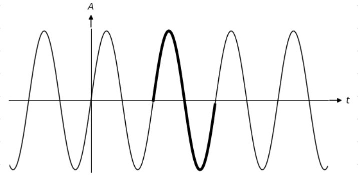

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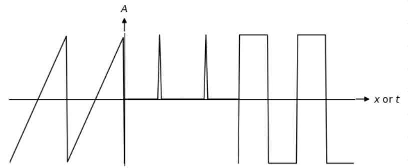
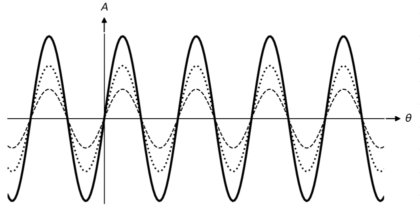


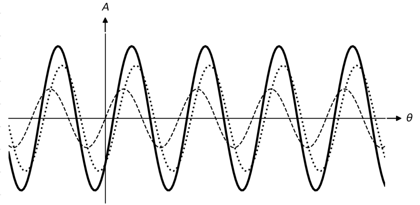
Non-smooth (instantaneous change) or non-changing wrt space and time is non-physical.

Wave summation, $A_{1+2} := A_1 + A_2$



When $\lambda_1 = \lambda_2$ and $\phi_1 = \phi_2$,
$\lambda_{1+2} = \lambda_1 = \lambda_2$ and $\phi_{1+2} = \phi_1 = \phi_2$.

Wave summation, $A_{1+2} := A_1 + A_2$



When $\lambda_1 = \lambda_2$ and $\phi_1 > \phi_2$ and $A_1 > A_2$,
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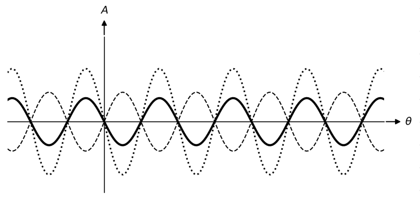

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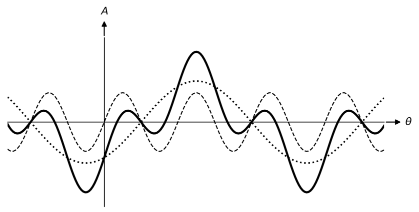

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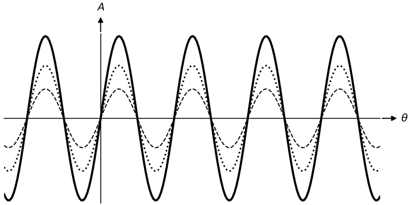

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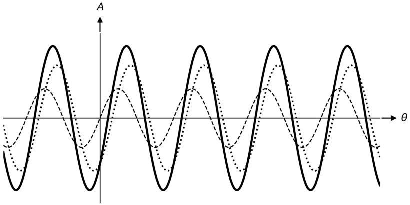

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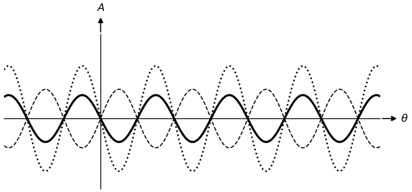
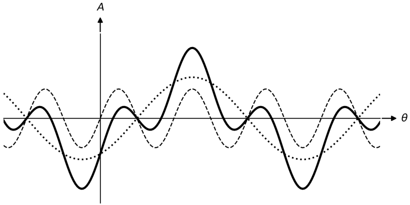
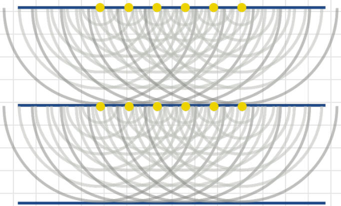


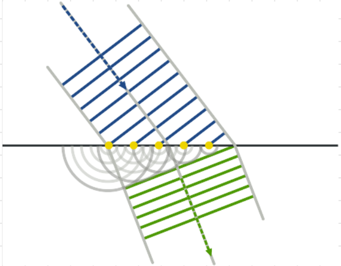


Wavefront propagation



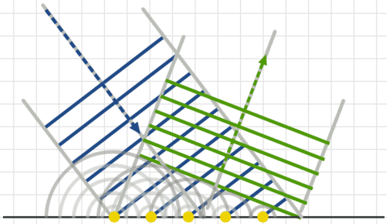
Huygens-Fresnel Construction.

Wavefront propagation



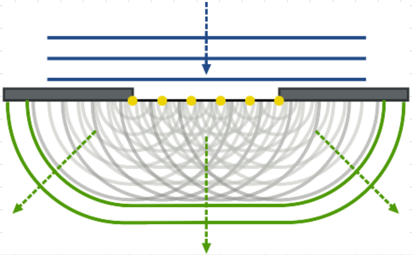
Refraction.

Wavefront propagation



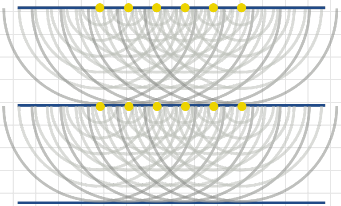
Reflection.

Wavefront propagation



Diffraction.

Wavefront propagation



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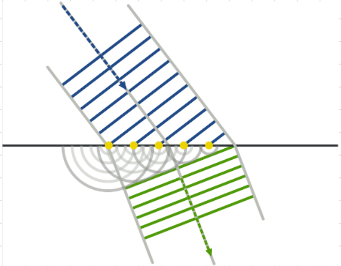

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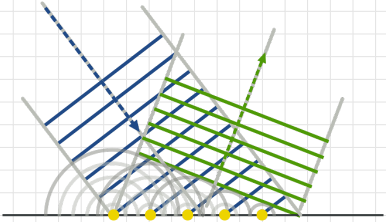

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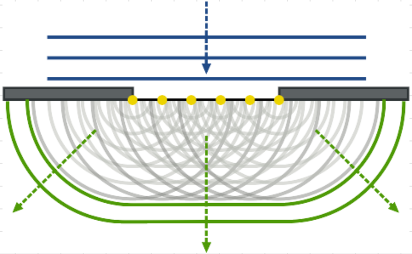
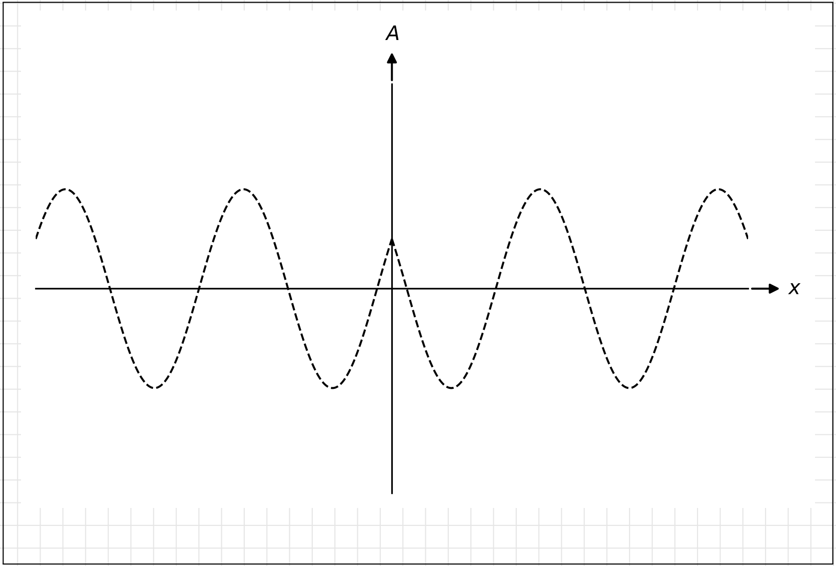
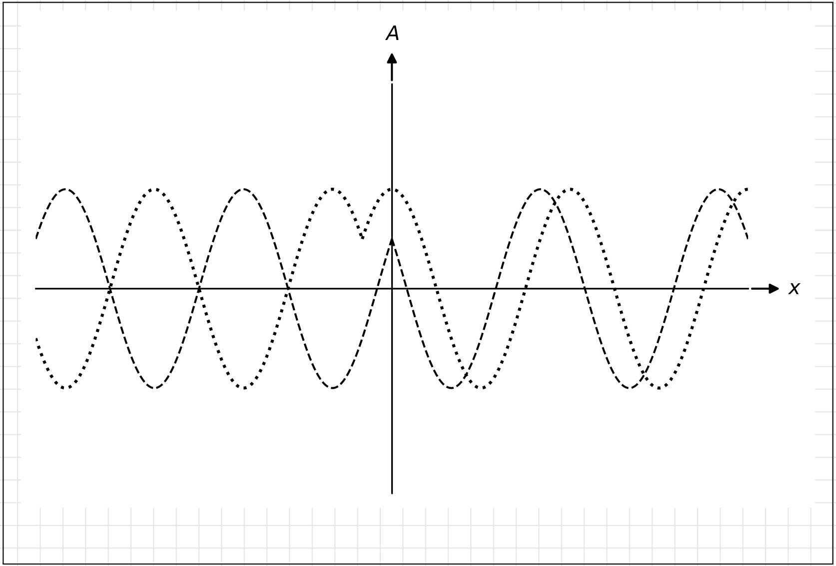


Diffraction.

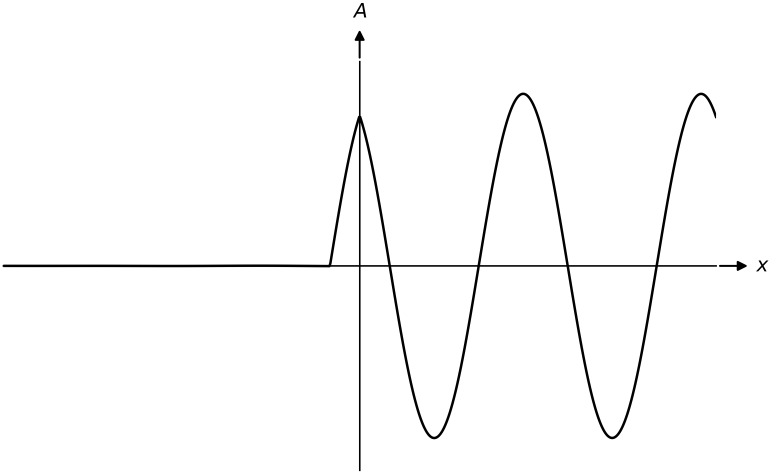
Propagation from a point where amplitude is increasing



. . . and from another point where amplitude is decreasing

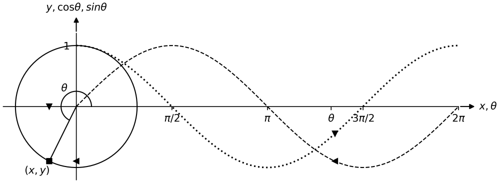


Radial propagation



Huygens-Miller construction.

Sinusoids



For any point (x, y) on unit circle,
$\cos\theta := x$ and $\sin\theta := y$.

Sinusoids

Point coordinates corresponding to angle θ are,

$$(\cos\theta, \sin\theta).$$

Angle θ corresponding to point coords (x, y) is,

$$\arccos x \text{ and } \arcsin y.$$

Sinusoids

Geometric contruction is impractical and mathematical expression is complicated:

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

so calculators with programmed buttons or printed tables are used.

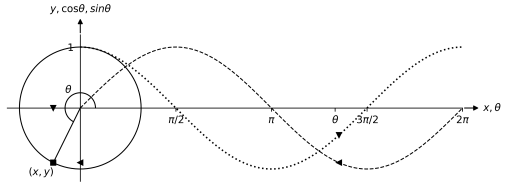
Sinusoids

Sinusoids with same λ but arbitrary ϕ and A sum to a sinusoid with same λ.

This is how physical waves behave.

Sinusoids are *only* periodic functions with this property.

Sinusoids



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Wave equation

How can waves be described so their behaviour can be analysed mathematically?

“They look like sinusoids” isn’t rigorous enough.

Wave equation

We will soon derive this constraint equation from Hooke's and Newton's Laws:

$$\frac{\partial^2 A}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0$$

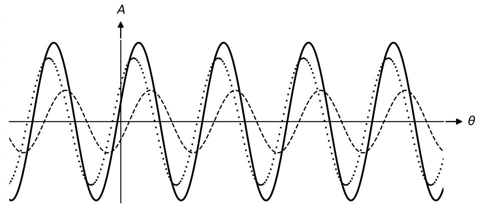
Wave equation

One of many solutions can be found algebraically as,

$$\begin{aligned} A(x,t) =& R\cos(k\,x - \omega\,t) \\ &+ (1-R)\cos(k\,x + \omega\,t) \end{aligned}$$

where k and ω are constants related to (angular) wavelength and frequency and $|R| \leq 1$.

Wave equation



The sum of two weighted sinusoids travelling in opposite directions. Here for $R = 0.33$.

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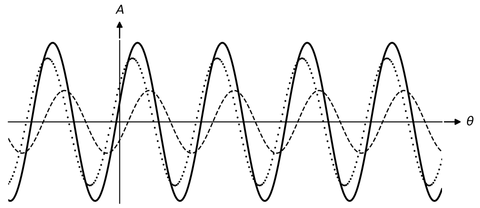

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Assignment # 1: Huygens-Fresnel construction

- Write a SciPy program to make at least one plot similar to those shown the wavefront propagation slide.
- Use Huygens-Fresnel construction to determine where the wavefront should be at different times.
- Make it into a self-contained project repository in your personal account on gitlab.scss.tcd.ie.
- Add `fshevlin@tcd.ie` as a member with "reporter" privileges.

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Greek letters* often used as symbols in mathematics

α	alpha	θ	theta	o	omicron	τ	tau
β	beta	ϑ	caligr. theta	π	pi	υ	upsilon
γ	gamma	ι	iota	ϖ	caligr. pi	ϕ	phi
δ	delta	κ	kappa	ρ	rho	φ	caligr. phi
ϵ	epsilon	λ	lambda	ϱ	caligr. rho	χ	chi
ε	caligr. epsilon	μ	mu	σ	sigma	ψ	psi
ζ	zeta	ν	nu	ς	caligr. sigma	ω	omega
η	eta	ξ	xi				

Γ	big gamma	Λ	big lambda	Σ	big sigma	Ψ	big psi
Δ	big delta	Ξ	big xi	Υ	big upsilon	Ω	big omega
Θ	big theta	Π	big pi	Φ	big phi		

*With their anglophone pronunciations.