CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture#8: Intensity sums

Fergal Shevlin, Ph.D.

School of Computer Science and Statistics,
Trinity College Dublin

November 26, 2021

Descriptive statistics of a sum of two terms

The means of random intensities I_X and I_Y are,

$$\overline{I_X} = \mathbf{E}[I_X] \quad \text{and} \quad \overline{I_Y} = \mathbf{E}[I_Y].$$

Linearity of expectation allows the mean of their sum to be expressed as,

$$\mathbf{E}[I_X + I_Y] = \mathbf{E}[I_X] + \mathbf{E}[I_Y]$$

$$\overline{I_Z} = \overline{I_X + I_Y} = \overline{I_X} + \overline{I_Y}.$$

Descriptive statistics of a sum of two terms

When I_X and I_Y are independent,

$$\mathbf{E}[I_X I_Y] = \mathbf{E}[I_X]\,\mathbf{E}[I_Y]$$
$$\overline{I_X I_Y} = \overline{I_X}\,\overline{I_Y}\ .$$

Hence,

$$\begin{aligned}\overline{I_Z^2} &= \overline{(I_X + I_Y)^2} \\ &= \overline{I_X^2 + 2\, I_X I_Y + I_Y^2} \\ &= \overline{I_X^2} + 2\overline{I_X}\,\overline{I_Y} + \overline{I_Y^2}\ ,\end{aligned}$$

Descriptive statistics of a sum of two terms

By integration of exponential PDF,

$$\overline{I_X^2} = 2\overline{I_X}^2 \quad \text{and} \quad \overline{I_Y^2} = 2\overline{I_Y}^2.$$

So the variance can be expressed as,

$$\begin{aligned}
\sigma_Z^2 = \mathbf{E}[I_Z^2] - \mathbf{E}[I_Z]^2 = \overline{I_Z^2} - \overline{I_Z}^2 \\
= 2\overline{I_X}^2 + 2\overline{I_X}\,\overline{I_Y} + 2\overline{I_Y}^2 \\
- \overline{I_X}^2 - 2\overline{I_X}\,\overline{I_Y} - \overline{I_Y}^2 \\
= \overline{I_X}^2 + \overline{I_Y}^2.
\end{aligned}$$

Descriptive statistics of a sum of two terms

$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\overline{I_X}^2 + \overline{I_Y}^2}}{\overline{I_X} + \overline{I_Y}} = \frac{\sqrt{1+r^2}}{1+r}$$

where $r = \overline{I_Y} / \overline{I_X}$ is the ratio of one mean to the other. Note,

$$\frac{\sqrt{a^2+b^2}}{\sqrt{a^2}} = \sqrt{\frac{a^2}{a^2} + \frac{b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

When $r = 1$ the contrast is minimized,

$$C = \sqrt{2}/2 = 2/2\sqrt{2} = 1/\sqrt{2}.$$

Descriptive statistics of a sum of two terms

The means of random intensities I_X and I_Y are,

$$\overline{I_X} = \mathbf{E}[I_X] \quad \text{and} \quad \overline{I_Y} = \mathbf{E}[I_Y].$$

Linearity of expectation allows the mean of their sum to be expressed as,

$$\mathbf{E}[I_X + I_Y] = \mathbf{E}[I_X] + \mathbf{E}[I_Y]$$

$$\overline{I_Z} = \overline{I_X + I_Y} = \overline{I_X} + \overline{I_Y}.$$

When I_X and I_Y are independent,

$$\mathbf{E}[I_X I_Y] = \mathbf{E}[I_X]\,\mathbf{E}[I_Y]$$

$$\overline{I_X I_Y} = \overline{I_X}\,\overline{I_Y}.$$

Hence,

$$\begin{aligned} \overline{I_Z^2} &= \overline{(I_X + I_Y)^2} \\ &= \overline{I_X^2 + 2\,I_X I_Y + I_Y^2} \\ &= \overline{I_X^2} + 2\overline{I_X}\,\overline{I_Y} + \overline{I_Y^2}, \end{aligned}$$

By integration of exponential PDF,

$$\overline{I_X^2} = 2\overline{I_X}^2 \quad \text{and} \quad \overline{I_Y^2} = 2\overline{I_Y}^2.$$

So the variance can be expressed as,

$$\begin{aligned} \sigma_Z^2 = \mathbf{E}[I_Z^2] - \mathbf{E}[I_Z]^2 &= \overline{I_Z^2} - \overline{I_Z}^2 \\ &= 2\overline{I_X}^2 + 2\overline{I_X}\,\overline{I_Y} + 2\overline{I_Y}^2 \\ &\quad - \overline{I_X}^2 - 2\overline{I_X}\,\overline{I_Y} - \overline{I_Y}^2 \\ &= \overline{I_X}^2 + \overline{I_Y}^2. \end{aligned}$$

$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\overline{I_X}^2 + \overline{I_Y}^2}}{\overline{I_X} + \overline{I_Y}} = \frac{\sqrt{1 + r^2}}{1 + r}$$

where $r = \overline{I_Y}/\overline{I_X}$ is the ratio of one mean to the other. Note,

$$\frac{\sqrt{a^2 + b^2}}{\sqrt{a^2}} = \sqrt{\frac{a^2}{a^2} + \frac{b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

When $r = 1$ the contrast is minimized,

$$C = \sqrt{2}/2 = 2/2\sqrt{2} = 1/\sqrt{2}.$$

Descriptive statistics of a sum of N terms

For a sum of random intensities $I_Z = I_1 + I_2 + \ldots I_N$, the mean $\overline{I_Z} = \sum_{n=1}^{N} \overline{I_n}$ by LOE.

$$\overline{I_Z^2} = \sum_{n=1}^{N} \sum_{m=1}^{N} \overline{I_n I_m}$$

$$= \sum_{n=1}^{N} \overline{I_n^2} + \sum_{n=1}^{N} \sum_{\substack{m=1, \\ m \neq n}}^{N} \overline{I_n}\,\overline{I_m}$$

Descriptive statistics of a sum of N terms

Note that $(a+b+c)^2$

$$
\begin{aligned}
&= (a+b+c)(a+b+c) \\
&= a^2 + b^2 + c^2 \\
&\quad + 2(ab + ac + bc) \\
&= \sum_{\substack{n= \\ \{a,b,c\}}} n^2 + \sum_{\substack{n= \\ \{a,b,c\}}} \sum_{\substack{m= \\ \{a,b,c\}, \\ m\neq n}} nm.
\end{aligned}
$$

Descriptive statistics of a sum of N terms

Using $\overline{I_n^2} = 2\overline{I_n}^2$,

$$\overline{I_Z^2} = 2\sum_{n=1}^{N} \overline{I_n}^2 + \sum_{n=1}^{N} \sum_{\substack{m=1,\\ m\neq n}}^{N} \overline{I_n}\,\overline{I_m}$$

$$= \sum_{n=1}^{N} \overline{I_n}^2 + \Big(\sum_{n=1}^{N} \overline{I_n}\Big)^2$$

$$= \sum_{n=1}^{N} \overline{I_n}^2 + \overline{I_Z}^2.$$

Descriptive statistics of a sum of N terms

$$\sigma_Z^2 = \overline{I_Z^2} - \overline{I_Z}^{\,2} = \sum_{n=1}^{N} \overline{I_n}^{\,2}$$

$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\sum_{n=1}^{N} \overline{I_n}^{\,2}}}{\sum_{n=1}^{N} \overline{I_n}}$$

When mean intensities $\overline{I_n}$ are equal,

$$C = \frac{\sqrt{N\overline{I}^{\,2}}}{N\overline{I}} = \frac{\sqrt{N}\sqrt{\overline{I}^{\,2}}}{N\overline{I}} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}.$$

Descriptive statistics of a sum of N terms

For a sum of random intensities $I_Z = I_1 + I_2 + \ldots I_N$, the mean $\overline{I_Z} = \sum_{n=1}^{N} \overline{I_n}$ by LOE.

$$\overline{I_Z^2} = \sum_{n=1}^{N} \sum_{m=1}^{N} \overline{I_n I_m}$$

$$= \sum_{n=1}^{N} \overline{I_n^2} + \sum_{n=1}^{N} \sum_{\substack{m=1,\\ m \neq n}}^{N} \overline{I_n}\,\overline{I_m}$$

Note that $(a+b+c)^2$

$$= (a+b+c)(a+b+c)$$

$$= a^2 + b^2 + c^2$$

$$+ 2(ab + ac + bc)$$

$$= \sum_{\substack{n = \\ \{a,b,c\}}} n^2 + \sum_{\substack{n = \\ \{a,b,c\}}} \sum_{\substack{m = \\ \{a,b,c\}, \\ m \neq n}} nm.$$

Using $\overline{I_n^2} = 2\overline{I_n}^2$,

$$\overline{I_Z^2} = 2\sum_{n=1}^{N} \overline{I_n}^2 + \sum_{n=1}^{N} \sum_{\substack{m=1,\\ m \neq n}}^{N} \overline{I_n}\,\overline{I_m}$$

$$= \sum_{n=1}^{N} \overline{I_n}^2 + \left(\sum_{n=1}^{N} \overline{I_n}\right)^2$$

$$= \sum_{n=1}^{N} \overline{I_n}^2 + \overline{I_Z}^2.$$

$$\sigma_Z^2 = \overline{I_Z^2} - \overline{I_Z}^2 = \sum_{n=1}^{N} \overline{I_n}^2$$

$$C = \frac{\sigma_Z}{\overline{I_Z}} = \frac{\sqrt{\sum_{n=1}^{N} \overline{I_n}^2}}{\sum_{n=1}^{N} \overline{I_n}}$$

When mean intensities $\overline{I_n}$ are equal,

$$C = \frac{\sqrt{N\overline{I}^2}}{N\overline{I}} = \frac{\sqrt{N}\sqrt{\overline{I}^2}}{N\overline{I}} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}.$$

PDF of a sum of two terms

For random intensities I_X and I_Y decribed as exponential PDFs,

$$f_{I_X}(i_X) = \frac{1}{\overline{I_X}} \exp\left\{ \frac{-i_X}{\overline{I_X}} \right\}$$

$$f_{I_Y}(i_Y) = \frac{1}{\overline{I_Y}} \exp\left\{ \frac{-i_Y}{\overline{I_Y}} \right\},$$

their sum $I_Z = I_X + I_Y$ is described by another PDF, $f_{I_Z}(i_Z)$.

PDF of a sum of two terms

For *discrete* random quantities $X, Y, Z \in \mathbb{N}_0$, let $Z = X + Y$. A sum of 3 could result from $0 + 3$ or $1 + 2$ or $2 + 1$ or $3 + 0$ so,

$$\begin{aligned}
\mathbf{P}(Z = 3) = \mathbf{P}(X = 0)\,\mathbf{P}(Y = 3) \\
+ \mathbf{P}(X = 1)\,\mathbf{P}(Y = 2) \\
+ \mathbf{P}(X = 2)\,\mathbf{P}(Y = 1) \\
+ \mathbf{P}(X = 3)\,\mathbf{P}(Y = 0)
\end{aligned}$$

PDF of a sum of two terms

$$= \sum_{x=0}^{3} \mathbf{P}(X = x)\, \mathbf{P}(Y = 3 - x).$$

Letting $p_Z(z)$ etc. be discrete equivalents of a PDF to denote probabilities,

$$p_Z(z) = \sum_{x=0}^{z} p_X(x)\, p_Y(z - x).$$

Such a sum (or integral) of products of functions is called a *convolution*.

PDF of a sum of two terms

For *continuous* random quantities $X, Y, Z \in \mathbb{R}_{\geq 0}$, let $Z = X + Y$. The PDF for Z has an integral instead of a sum,

$$f_Z(z) = \int_0^z f_X(x)\, f_Y(z - x)\, \mathrm{d}x.$$

PDF of a sum of two terms

For random intensities I_X and I_Y decribed as exponential PDFs,

$$f_{I_X}(i_X) = \frac{1}{\overline{I_X}} \exp\left\{\frac{-i_X}{\overline{I_X}}\right\}$$

$$f_{I_Y}(i_Y) = \frac{1}{\overline{I_Y}} \exp\left\{\frac{-i_Y}{\overline{I_Y}}\right\},$$

their sum $I_Z = I_X + I_Y$ is described by another PDF, $f_{I_Z}(i_Z)$.

For *discrete* random quantities $X, Y, Z \in \mathbb{N}_0$, let $Z = X + Y$. A sum of 3 could result from $0+3$ or $1+2$ or $2+1$ or $3+0$ so,

$$\begin{aligned} \mathbf{P}(Z=3) &= \mathbf{P}(X=0)\,\mathbf{P}(Y=3) \\ &\quad + \mathbf{P}(X=1)\,\mathbf{P}(Y=2) \\ &\quad + \mathbf{P}(X=2)\,\mathbf{P}(Y=1) \\ &\quad + \mathbf{P}(X=3)\,\mathbf{P}(Y=0) \end{aligned}$$

$$= \sum_{x=0}^{3} \mathbf{P}(X=x)\,\mathbf{P}(Y=3-x).$$

Letting $p_Z(z)$ etc. be discrete equivalents of a PDF to denote probabilities,

$$p_Z(z) = \sum_{x=0}^{z} p_X(x)\,p_Y(z-x).$$

Such a sum (or integral) of products of functions is called a *convolution*.

For *continuous* random quantities $X, Y, Z \in \mathbb{R}_{\geq 0}$, let $Z = X + Y$. The PDF for Z has an integral instead of a sum,

$$f_Z(z) = \int_0^z f_X(x)\,f_Y(z-x)\,\mathrm{d}x.$$

Let $f_X(x) = \lambda_X \, \mathrm{e}^{-\lambda_X \, x}$ and $f_Y(y) = \lambda_Y \, \mathrm{e}^{-\lambda_Y \, y}$ be exponential PDFs with rate parameters λ_X and λ_Y. The PDF of their sum is,

$$\begin{aligned} f_Z(z) &= \int_0^z \lambda_X \, \mathrm{e}^{-\lambda_X \, x} \lambda_Y \, \mathrm{e}^{-\lambda_Y \, (z-x)} \, \mathrm{d}x \\ &= \lambda_X \, \lambda_Y \, \mathrm{e}^{-\lambda_Y \, z} \int_0^z \mathrm{e}^{(\lambda_Y - \lambda_X) \, x} \, \mathrm{d}x. \end{aligned}$$

By FTC the definite integral is the difference between the antiderivative evaluated at the limits,

$$\frac{1}{\lambda_Y - \lambda_X} e^{(\lambda_Y - \lambda_X)\,x} \Big|_0^z$$

$$= \frac{1}{\lambda_Y - \lambda_X} e^{(\lambda_Y - \lambda_X)\,z} - \frac{1}{\lambda_Y - \lambda_X}.$$

Or $\int_0^z 1 \, dx = z$ when $\lambda_X = \lambda_Y$.

$$f_Z(z) = \begin{cases} \frac{\lambda_X\,\lambda_Y}{\lambda_Y-\lambda_X}(\mathrm{e}^{-\lambda_X\,z} - \mathrm{e}^{-\lambda_Y\,z}) \\ \qquad \text{when } \lambda_Y \neq \lambda_X. \\ \\ z\,\lambda^2\,\mathrm{e}^{-\lambda\,z} \\ \qquad \text{when } \lambda = \lambda_X = \lambda_Y. \end{cases}$$

$\lambda = {}^{1}\!/\overline{I}$ for $\overline{I} = \overline{I_X} = \overline{I_Y}$, so the PDF of the sum is a scaled* version of the originals,

$$f_{I_Z}(i_Z) = \frac{i_Z}{\overline{I}} \frac{1}{\overline{I}} \exp\left\{ \frac{-i_Z}{\overline{I}} \right\}.$$

Let $f_X(x) = \lambda_X\, e^{-\lambda_X\, x}$ and $f_Y(y) = \lambda_Y\, e^{-\lambda_Y\, y}$ be exponential PDFs with rate parameters λ_X and λ_Y. The PDF of their sum is,

$$f_Z(z) = \int_0^z \lambda_X\, e^{-\lambda_X\, x} \lambda_Y\, e^{-\lambda_Y\,(z-x)}\, dx$$
$$= \lambda_X\, \lambda_Y\, e^{-\lambda_Y\, z} \int_0^z e^{(\lambda_Y - \lambda_X)\, x}\, dx.$$

By FTC the definite integral is the difference between the antiderivative evaluated at the limits,

$$\frac{1}{\lambda_Y - \lambda_X} e^{(\lambda_Y - \lambda_X)\, x}\Big|_0^z$$
$$= \frac{1}{\lambda_Y - \lambda_X} e^{(\lambda_Y - \lambda_X)\, z} - \frac{1}{\lambda_Y - \lambda_X}.$$

Or $\int_0^z 1\, dx = z$ when $\lambda_X = \lambda_Y$.

$$f_Z(z) = \begin{cases} \frac{\lambda_X\, \lambda_Y}{\lambda_Y - \lambda_X}\left(e^{-\lambda_X\, z} - e^{-\lambda_Y\, z}\right) \\ \qquad \text{when } \lambda_Y \neq \lambda_X. \\ \\ z\, \lambda^2\, e^{-\lambda\, z} \\ \qquad \text{when } \lambda = \lambda_X = \lambda_Y. \end{cases}$$

$\lambda = 1/\overline{I}$ for $\overline{I} = \overline{I_X} = \overline{I_Y}$, so the PDF of the sum is a scaled* version of the originals,

$$f_{I_Z}(i_Z) = \frac{i_Z}{\overline{I}} \frac{1}{\overline{I}} \exp\left\{\frac{-i_Z}{\overline{I}}\right\}.$$

PDF of a sum of N terms

A bit complicated. If interested, see Goodman,

PDF of a sum of N terms

Speckle Phenomena in Optics,

PDF of a sum of N terms

page

PDF of a sum of N terms

42.

PDF of a sum of N terms

A bit complicated. If interested, see Goodman,	*Speckle Phenomena in Optics,*
page	42.