CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

Lecture #5: Phasors

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Complex Numbers

A complex number is expressed as the sum of a *real* part and an *imaginary* part,

$$a + b\mathrm{j} \in \mathbb{C} \quad \text{for } a, b \in \mathbb{R}.$$

Imaginary unit j is defined as,

$$\mathrm{j}^2 = -1 \quad \text{so } \mathrm{j} = \pm\sqrt{-1}.$$

(In engineering, $\mathrm{j} = -\mathrm{i}$ is used to denote imaginary to avoid confusion with electrical current I.)

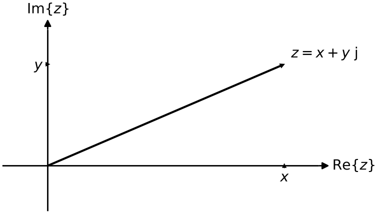
Complex Numbers

They allow expressions that wouldn't be possible otherwise, e.g. roots of $(x+1)^2 = -9$ are at $x = -1 \pm 3\mathrm{j}$.

$$(-1 \pm 3\mathrm{j} + 1)^2 =$$
$$(\pm 3\mathrm{j})^2 = (\pm 3)^2 \mathrm{j}^2 =$$
$$(+3)^2 \mathrm{j}^2 \text{ and } (-3)^2 \mathrm{j}^2 =$$
$$(9)(-1) = -9.$$

Complex Numbers

They can be used to associate numbers that go together, such as point vector coordinates (x, y).



Complex Numbers

But some consideration required, e.g compensation for $\mathrm{j}^2 = -1$ to express vector magnitude,

$$\begin{aligned} \text{for}\quad z &= x + y\,\mathrm{j}\,, \\ \text{complex conjugate}\quad \bar{z} &= x - y\,\mathrm{j}\,, \\ \text{magnitude squared}\quad |z|^2 &= z\bar{z} = x^2 + y^2, \\ \text{magnitude}\quad |z| &= \sqrt{z\bar{z}}. \end{aligned}$$

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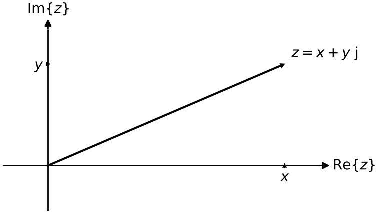

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Euler's Formula

Euler's constant $\mathrm{e} \approx 2.71828$. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$.

e^x is called the natural exponential function and can be written $\exp x$ or $\exp(x)$.

Euler's formula expresses sinusoidal functions through the natural exponential function,

$$\mathrm{e}^{\mathrm{j}\,\theta} = \cos\theta + \mathrm{j}\,\sin\theta.$$

Euler's Formula

The value of any smooth function f at point b in the neighbourhood of point a can be expressed through the Taylor Series,

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a)^1 + \frac{f''(a)}{2!}(b-a)^2 + \dots$$

At $x{=}a{=}0$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}\mathrm{e}^x{=}\mathrm{e}^0{=}1$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^x{=}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x{=}$ $\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^x{=}\mathrm{e}^0 = 1$, etc.,

$$\text{so,}\;\; \exp(b) = \exp(0) + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$$

Euler's Formula

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$\begin{aligned} \sin\theta &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1} \\ &= \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \end{aligned}$$

Euler's Formula

$$
\begin{aligned}
e^{j\,\theta} = \exp(j\,\theta) &= 1 + \frac{(j\,\theta)^1}{1!} + \frac{(j\,\theta)^2}{2!} + \frac{(j\,\theta)^3}{3!} \dots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \\
&\qquad + j\,(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots) \\
&= \cos\theta + j\,\sin\theta.
\end{aligned}
$$

(Hence the expression $e^{i\,\pi} - 1 = 0$.)

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$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1}$$

$$= \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$\begin{aligned}
\mathrm{e}^{\mathrm{j}\,\theta} = \exp(\mathrm{j}\,\theta) &= 1 + \frac{(\mathrm{j}\,\theta)^1}{1!} + \frac{(\mathrm{j}\,\theta)^2}{2!} + \frac{(\mathrm{j}\,\theta)^3}{3!} \ldots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \\
&\quad + \mathrm{j}\,\Big(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\Big) \\
&= \cos\theta + \mathrm{j}\,\sin\theta.
\end{aligned}$$

(Hence the expression $\mathrm{e}^{\mathrm{i}\,\pi} - 1 = 0$.)

Phasors

A sinusoid can be described as $A \cos(\omega t + \phi)$ in terms that are *variable*:

- time t, $-\infty < t < +\infty$
- angular frequency $\omega = 2\pi\,\nu$
- temporal frequency ν

and *constant*:

- amplitude A
- phase ϕ

Phasors

A phasor is used to encode the constants,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi} = A\,(\cos\phi + \mathrm{j}\ \sin\phi).$$

Multiply by exponential function encoding the variables $\mathrm{e}^{\mathrm{j}\,\omega t}$ to get a sinusoid,

$$A\,\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t} = A\,\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}.$$

$$A\ \mathrm{Re}\left\{\mathrm{e}^{\mathrm{j}\,(\phi+\omega t)}\right\} = A\ \cos(\omega t + \phi).$$

Phasors

Sum of two sinusoids with the same ang. freq. ω,

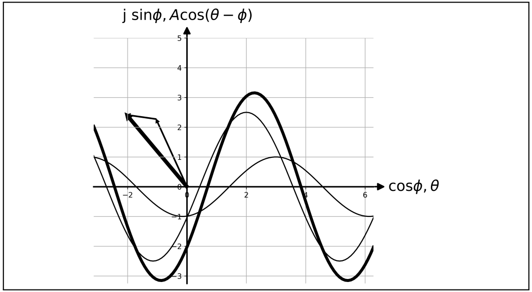
$$A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) =$$

$$\mathrm{Re}\left\{A_1 \, \mathrm{e}^{\mathrm{j}\,(\omega t+\phi_1)} + A_2 \, \mathrm{e}^{\mathrm{j}\,(\omega t+\phi_2)}\right\} =$$

$$\mathrm{Re}\left\{(A_1 \, \mathrm{e}^{\mathrm{j}\,\phi_1} + A_2 \, \mathrm{e}^{\mathrm{j}\,\phi_2}) \, \mathrm{e}^{\mathrm{j}\,\omega t}\right\}.$$

Which is the sum of phasors encoding the constants multiplied by an exponential function encoding the variables.

Phasors



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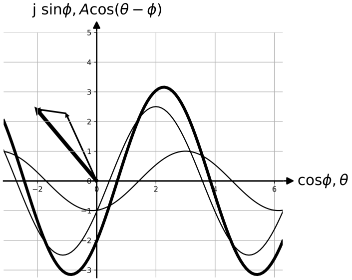

Phasor calculus

The derivative with respect to time of the exponential function of the variables can be expressed as a phasor,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t} = \mathrm{e}^{t} \text{ so } \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\omega t} = \omega \mathrm{e}^{\omega t} \text{ and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{j}\,\omega t} = \mathrm{j}\,\omega\,\mathrm{e}^{\mathrm{j}\,\omega t} = \omega \mathrm{e}^{\mathrm{j}\,\pi/2}\,\mathrm{e}^{\mathrm{j}\,\omega t}$$

since $\mathrm{j} = 0 + \mathrm{j}\,1 = \cos\frac{\pi}{2} + \mathrm{j}\,\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{j}\,\pi/2}$.

Phasor calculus

To differentiate: multiply by $\mathrm{j}\,\omega = \omega e^{\mathrm{j}\,\pi/2}$.

To integrate: multiply by $\frac{1}{\mathrm{j}\,\omega} = \frac{1}{\omega} e^{-\mathrm{j}\,\pi/2}$.

Phasor calculus

For example, find the derivative with respect to time of a sinusoid expressed using a phasor,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\omega t}\right) &= A\mathrm{e}^{\mathrm{j}\,\phi}(\mathrm{j}\,\omega)\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= A\mathrm{e}^{\mathrm{j}\,\phi}\mathrm{e}^{\mathrm{j}\,\pi/2}\omega\mathrm{e}^{\mathrm{j}\,\omega t}\\ &= \omega A\mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)}\mathrm{e}^{\mathrm{j}\,\omega t}.\end{aligned}$$

Phasor calculus

$$\begin{aligned} \mathrm{Re}\left\{\omega A \mathrm{e}^{\mathrm{j}\,(\phi+\pi/2)} \mathrm{e}^{\mathrm{j}\,\omega t}\right\} &= \omega A \cos(\omega t + \phi + \pi/2) \\ &= \omega A \sin(\omega t + \phi). \end{aligned}$$

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