

# CS7GV2: Mathematics of Light and Sound, M.Sc. in Computer Science.

## Lecture #6: Random Phasor Sums

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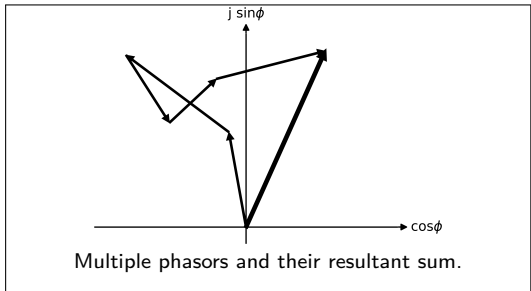
October 21, 2022

## Random walks

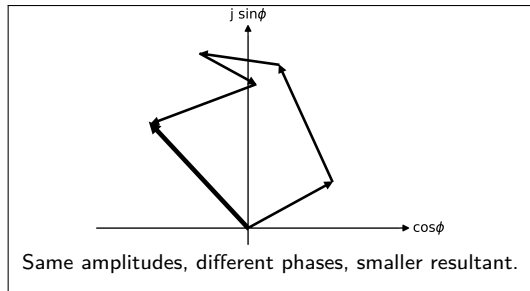
Multiple wave phasors can be summed like vectors to find the *resultant* wave phasor at a point in space and time.

When wave amplitudes are independent and wave phases are independent\* their summation is called a “random walk.”

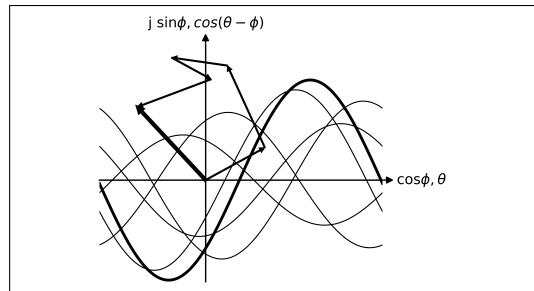
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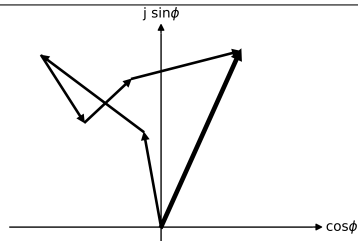
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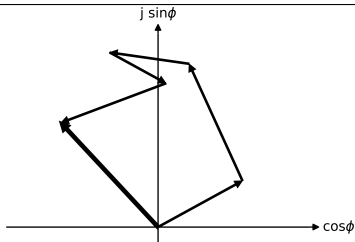
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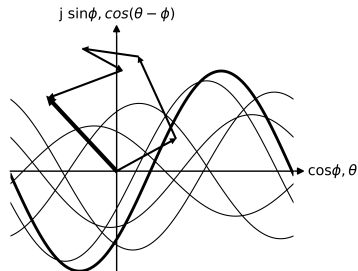
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Multiple phasors and their resultant sum.



Same amplitudes, different phases, smaller resultant.



## Independence and randomness

Independence means that one event or value of a random quantity  $X$  (e.g. a wave's amplitude) has no effect on another,  $Y$  (e.g. its phase,)

$$\mathbf{P}(X|Y) = \mathbf{P}(X \cap Y)/\mathbf{P}(Y)$$

$$\mathbf{P}(X \cap Y) = \mathbf{P}(X|Y)\mathbf{P}(Y)$$

$$\mathbf{P}(X \cap Y) = \mathbf{P}(X)\mathbf{P}(Y) \text{ i.f.f.}$$

$$\mathbf{P}(X) = \mathbf{P}(X|Y) \text{ and } \mathbf{P}(Y) = \mathbf{P}(Y|X).$$

# Independence and randomness

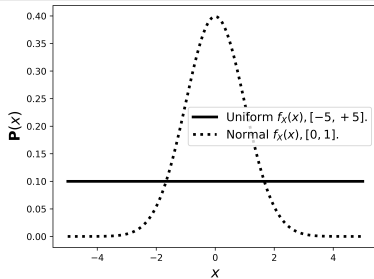
A random quantity is one whose value depends on the outcome of a random phenomenon.

Its occurrence may be known\* to follow a particular probability density function  $f_X$ , or probability mass function  $p_X$ , with descriptive parameters  $\mu, \sigma$ , etc.

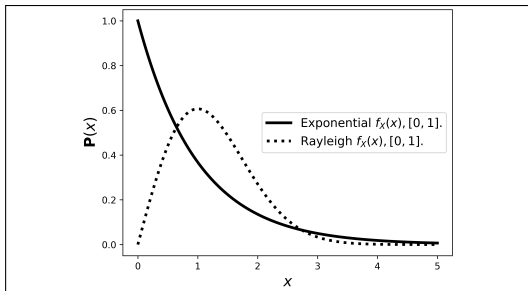
Example PDFs are *Uniform, Normal, Exponential, Poisson, Rayleigh*.



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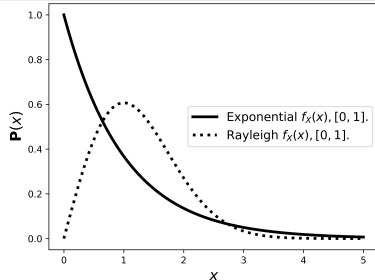
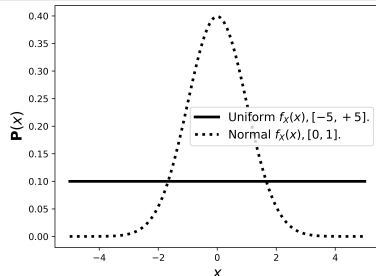
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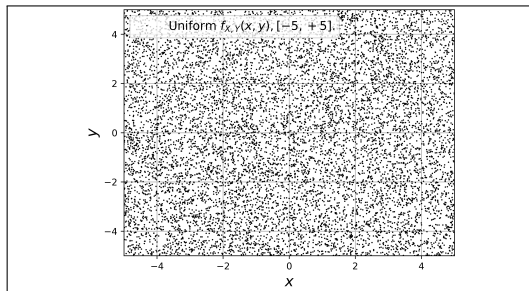
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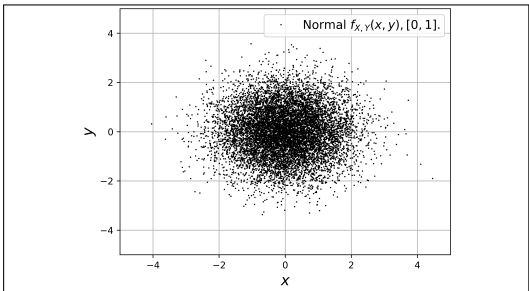
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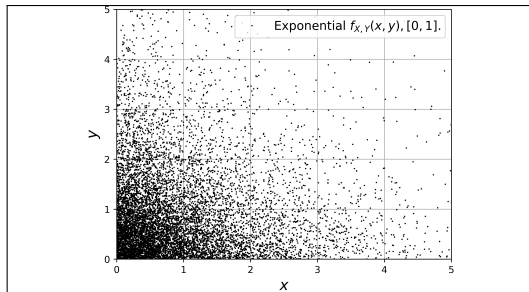
10,000 values  $(x, y)$  chosen randomly



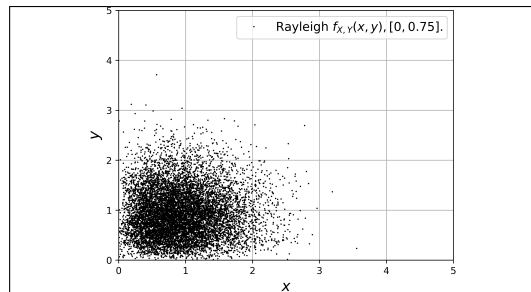
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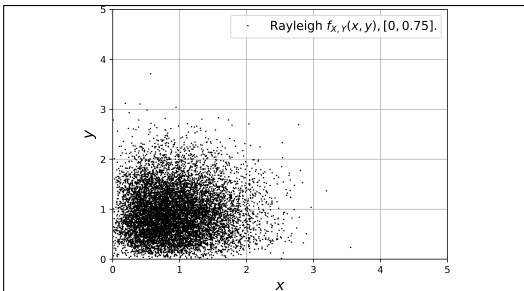
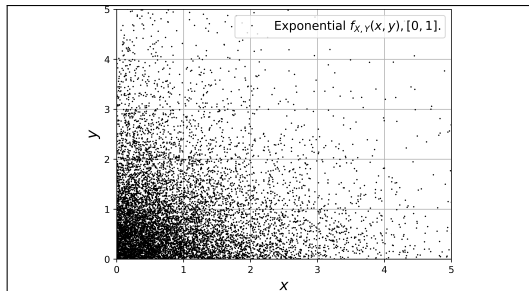
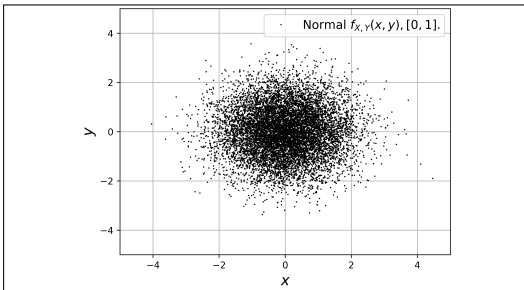
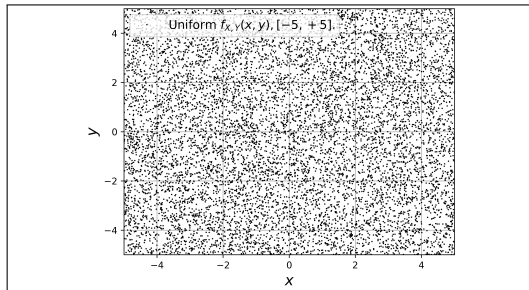
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## Descriptive statistics

*Expected value* or *mean* of a continuous random quantity  $X$  with probability density function  $f_X$ ,

$$\mathbf{E}[X] = \int_{-\infty}^{+\infty} x f_X(x) dx.$$

And for a discrete, finite, random quantity  $X$  with probability mass function  $p_X$ ,

$$\mathbf{E}[X] = \sum_{i=1}^N x_i p_X(x_i).$$

## Descriptive statistics

This is the *arithmetic mean* when probability mass function  $p_X$  is uniformly  $1/N$ .

$$\begin{aligned}\mathbf{E}[X] &= \sum_{i=1}^N x_i \frac{1}{N} \\ &= \frac{1}{N} \sum_{i=1}^N x_i \\ &= (x_1 + x_2 + \dots + x_N)/N\end{aligned}$$

# Descriptive statistics

Linearity of expectation,

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y].$$

If  $Y = aX + b$  for  $a, b \in \mathbb{R}$ ,

$$\mathbf{E}[Y] = a\mathbf{E}[X] + b.$$

If  $X, Y$  independent,

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y].$$

# Descriptive statistics

*Variance* is mean distance squared to the mean  
(when uniform,)

$$\begin{aligned}\sigma_X^2 &= \mathbf{E}[(X - \mathbf{E}[X])^2] \\ &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\ &= \mathbf{E}[X^2] \text{ when } \mathbf{E}[X] = 0.\end{aligned}$$

*Standard deviation*  $\sigma_X = \sqrt{\sigma_X^2}$ .

[cf. SCFT STD vs MAD.]

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*Standard deviation*  $\sigma_X = \sqrt{\sigma_X^2}$ .

[cf. SCFT STD vs MAD.]

## Random phasor sum

Defined as a weighted sum of random phasors:

$$\begin{aligned}\frac{1}{\sqrt{N}} \sum_{n=1}^N a_n e^{j\phi_n} &= \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathbf{a}_n = A e^{j\theta} \\ &= \mathbf{A} \quad (\text{the "resultant."})\end{aligned}$$

## Random phasor sum

$$\begin{aligned}\mathbf{E}[\operatorname{Re}\{\mathbf{A}\}] &= \mathbf{E}\left[\frac{1}{\sqrt{N}} \sum_{n=1}^N a_n \cos \phi_n\right] \\ &= \frac{1}{\sqrt{N}} \sum \mathbf{E}[a_n \cos \phi_n] \\ &= \frac{1}{\sqrt{N}} \sum \mathbf{E}[a_n] \mathbf{E}[\cos \phi_n] \\ &= 0.\end{aligned}$$

Similarly,  $\mathbf{E}[\operatorname{Im}\{\mathbf{A}\}] = 0$ .

## Random phasor sum

$$\sigma_{\text{Re}\{\mathbf{A}\}}^2 = \mathbf{E}[\text{Re}\{\mathbf{A}\}^2].$$

$$\begin{aligned}\text{Re}\{\mathbf{A}\}^2 &= 1/\sqrt{N}(a_1 \cos \phi_1 + a_2 \cos \phi_2 + \dots) \times \\ &\quad 1/\sqrt{N}(a_1 \cos \phi_1 + a_2 \cos \phi_2 + \dots) \\ &= 1/N \sum_n \sum_m a_n a_m \cos \phi_n \cos \phi_m.\end{aligned}$$

$$\begin{aligned}\mathbf{E}[\text{Re}\{\mathbf{A}\}^2] &= 1/N \sum_n \sum_m \mathbf{E}[a_n a_m] \times \\ &\quad \mathbf{E}[\cos \phi_n \cos \phi_m] \\ &= 1/N \sum_n \mathbf{E}[a_n^2] \mathbf{E}[\cos^2 \phi_n]\end{aligned}$$



## Random phasor sum

$$\begin{aligned} & \text{(since for } n \neq m, \mathbf{E}[\cos \phi_n \cos \phi_m] \\ & \quad = \mathbf{E}[\cos \phi_n] \mathbf{E}[\cos \phi_m] = 0) \\ & = 1/N \sum_n \mathbf{E}[a_n^2] \times \mathbf{E}[1/2 + 1/2 \cos 2\phi_n] \\ & \quad \text{(since } \cos^2 \phi = (1 + \cos 2\phi)/2) \\ & = 1/N \sum_n \mathbf{E}[a_n^2]/2. \end{aligned}$$

$$\text{Similarly, } \sigma_{\text{Im}\{\mathbf{A}\}}^2 = 1/N \sum_n \mathbf{E}[a_n^2]/2.$$

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Similarly,  $\mathbf{E}[\operatorname{Im}\{\mathbf{A}\}] = 0$ .

$$\begin{aligned}\sigma_{\operatorname{Re}\{\mathbf{A}\}}^2 &= \mathbf{E}[\operatorname{Re}\{\mathbf{A}\}^2]. \\ \operatorname{Re}\{\mathbf{A}\}^2 &= \frac{1}{\sqrt{N}}(a_1 \cos \phi_1 + a_2 \cos \phi_2 + \dots) \times \\ &\quad \frac{1}{\sqrt{N}}(a_1 \cos \phi_1 + a_2 \cos \phi_2 + \dots) \\ &= \frac{1}{N} \sum_n \sum_m a_n a_m \cos \phi_n \cos \phi_m. \\ \mathbf{E}[\operatorname{Re}\{\mathbf{A}\}^2] &= \frac{1}{N} \sum_n \sum_m \mathbf{E}[a_n a_m] \times \\ &\quad \mathbf{E}[\cos \phi_n \cos \phi_m] \\ &= \frac{1}{N} \sum_n \mathbf{E}[a_n^2] \mathbf{E}[\cos^2 \phi_n]\end{aligned}$$

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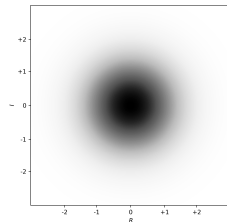
# Large numbers

Central Limit Theorem says that the probability density of the *sum* of  $N$  independent, identically-distributed, random quantities approaches Normal as  $N \rightarrow \infty$ , ,

$$f_{R,I}(r, i) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{r^2 + i^2}{2\sigma^2}\right\}$$

Where  $R = \text{Re}\{\mathbf{A}\}$  and  $I = \text{Im}\{\mathbf{A}\}$  and  $\sigma^2 = \sigma_R^2 = \sigma_I^2$ .  
[cf. SCFT p. 125.]

# Large numbers



Normal  $f_{R,i}(r, i), [0, 1]$ .

# Large numbers

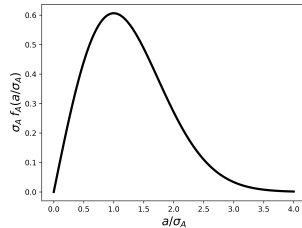
Through transformation of variables,<sup>\*</sup> marginal statistics for  $A$  and  $\theta$  are found as Rayleigh and Uniform respectively,

$$f_A(a) = a/\sigma^2 \exp\left\{\frac{a^2}{2\sigma^2}\right\}$$

$$f_\theta(\phi) = 1/2\pi$$

$$\mathbf{E}[A] = \sqrt{\pi/2} \sigma, \quad \sigma_A = (2 - \pi/2)\sigma^2$$

# Large numbers



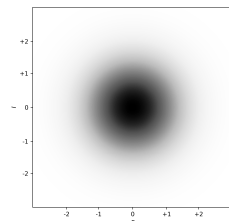
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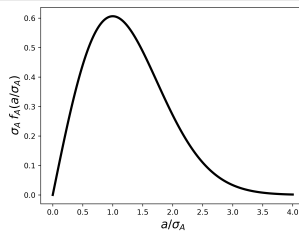
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