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Geometric Transformations

Lecturer: Carol O’Sullivan

Credits: Some slides taken from Robb T. Koether, Hampden-Sydney College

Objectives

- Learn how to carry out transformations
  - Rotation
  - Translation
  - Scaling
  - Combinations!

Computer Graphics Problems

- Much of graphics concerns itself with the problem of displaying 3D objects in 2D screen
- We want to be able to:
  - rotate, translate, scale our objects
  - view them from arbitrary points of view
  - View them in perspective
- Want to display objects in coordinate systems that are convenient for us and to be able to reuse object descriptions
- Road example
  - Cars, tyres
  - View from a helicopter

Matrices

- If you need to rotate a million vertices representing a dinosaur object about some axis, you don’t need to multiply each point by 5 different matrices
  - you simply multiply the 5 matrices together once and multiply each dinosaur point by that one matrix. Huge saving!

Matrices

- Matrix addition

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} a+e & c+g \\ b+f & d+h \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 6 \end{bmatrix}$$

Matrices

- Matrix multiplication

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \text{x} \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} \boxed{ae+cf} & ag+ch \\ be+df & bg+dh \end{bmatrix}$$

Matrices

- Matrix multiplication

$$\begin{bmatrix} \boxed{a \quad c} \\ b \quad d \end{bmatrix} \mathrm{x} \begin{bmatrix} e & \boxed{\begin{matrix} g \\ h \end{matrix}} \\ f & \end{bmatrix} = \begin{bmatrix} ae + cf & \boxed{ag + ch} \\ be + df & bg + dh \end{bmatrix}$$

Matrices

- Matrix multiplication

$$\begin{bmatrix} a & c \\ \boxed{b \quad d} \end{bmatrix} \text{x} \begin{bmatrix} \boxed{e} & g \\ \boxed{f} & h \end{bmatrix} = \begin{bmatrix} ae + cf & ag + ch \\ \boxed{be + df} & bg + dh \end{bmatrix}$$

Matrices

- Matrix multiplication

$$\begin{bmatrix} a & c \\ \boxed{b \quad d} \end{bmatrix} \text{x} \begin{bmatrix} e & \boxed{g} \\ f & \boxed{h} \end{bmatrix} = \begin{bmatrix} ae + cf & ag + ch \\ be + df & \boxed{bg + dh} \end{bmatrix}$$

Matrices

- Matrix multiplication not commutative in most cases
  - **AB** /= **BA**
  - If **AB** = **AC**, it does not necessarily follow that **B** = **C**
- It is associative and distributive
  - (AB)C = A(BC)
  - A(B+C) = AB + AC
  - (A+B)C = AC + BC

$$A = r^2$$

- Transpose A^T of a matrix A is one whose rows are switched with its columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}^T$$

Geometric Transformations

- Many geometric transformations are *linear* and can be represented as a matrix multiplication.
- Function *f* is linear iff:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

- <u>Implications</u>:
  - to transform a line we transform the *end-points*. Points between are *affine combinations* of the transformed endpoints.
  - Given line defined by points *P* and Q, points along transformed line are affine combinations of transformed *P′* and Q′

$$L(t) = P + t(Q - P)$$
$$L'(t) = P' + t(Q' - P')$$

Homogeneous Co-ordinates

- Basis of the homogeneous co-ordinate system is the set of *n* basis vectors and the origin position:

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \text{ and } P_o$$

- All points and vectors are therefore compactly represented using their ordinates:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ a_o \end{bmatrix} \text{or more usually} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Homogeneous Co-ordinates

- Vectors have no positional information and are represented using $a_o = 0$ whereas points are represented with $a_o = 1$:

$$\vec{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0$$

$$P = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + P_o$$

- <u>Examples</u>:

$$\begin{bmatrix} 0.2 \\ 1.3 \\ 2.2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1.0 \\ 1.0 \\ 0.0 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0.2 \\ 1.3 \\ 2.2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1.0 \\ 1.0 \\ 0.0 \\ 0 \end{bmatrix}$$

Points — Associated vectors

4D Vectors

- XYZ and W
- vec4 in GLSL
- For POINTS, set the 4th component to 1.0
- For VECTORS, set the 4th component to 0.0
- Q: Any idea why?

- vec4 (1.0, 5.0, -10.0, 0.0);
- vec4 (1.0, 5.0, -10.0, 1.0);

Homogenous Coordinates

- Using this scheme, every rotation, translation, and scaling operation can be represented by a matrix multiplication, and any combination of the operations corresponds to the products of the corresponding matrices
- Using homogeneous co-ordinates allows us to treat translation in the same way as rotation and scaling

Translation

- Simplest of the operations
  - Add a positive number – moves to the right
  - Add a negative number – moves to the left
- Addition of constant values, causes uniform translations in those directions
- Translations are independent and can be performed in any order (including all at once)
  - Object moved one unit to the right then up
  - Same as if moved one unit up and to the right
  - Net result is motion of sqrt(2) units to the upper-right

Translation

Definition (Translation)

A translation is a displacement in a particular direction

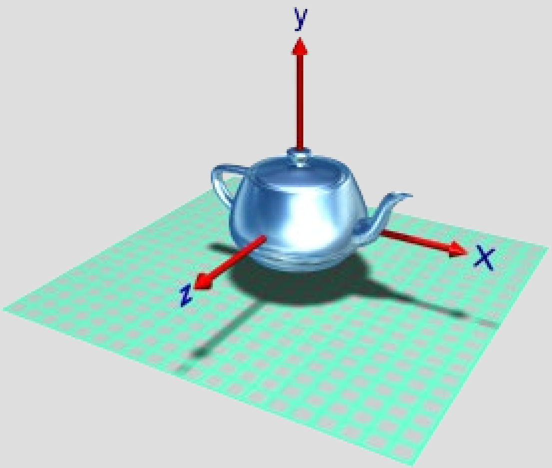
- A translation is defined by specifying the displacements a, b, and c

$$x' = x + a$$
$$y' = y + b$$
$$z' = z + c$$

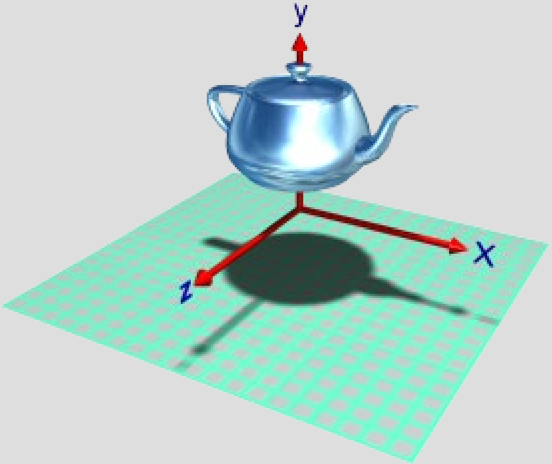
Translation

- Translation *only applies to points*, we never translate vectors.
- <u>Remember</u>: points have homogeneous co-ordinate *w* = *1*

$$\begin{aligned} x' &= x + a \\ y' &= y + b \\ z' &= z + c \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



translate along ***y***

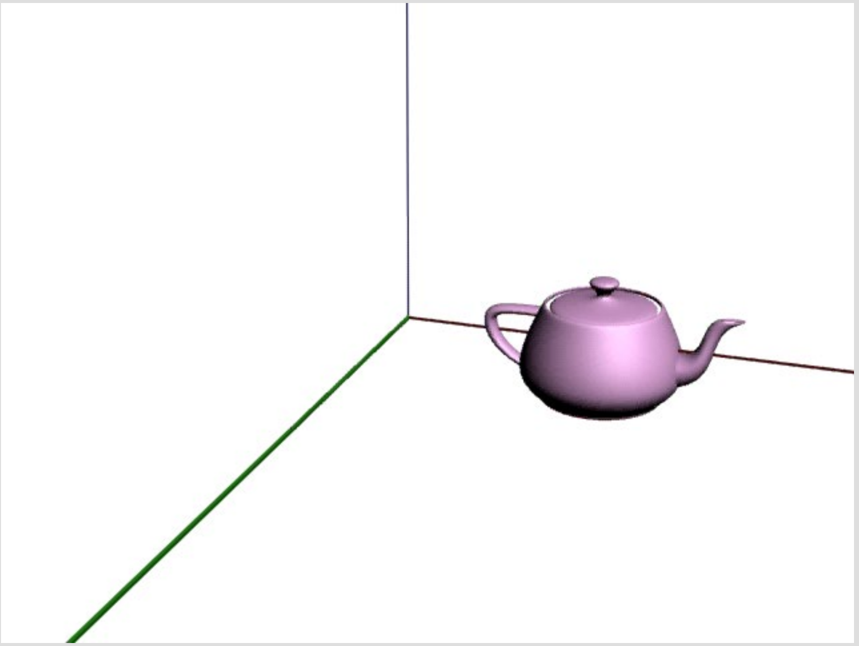


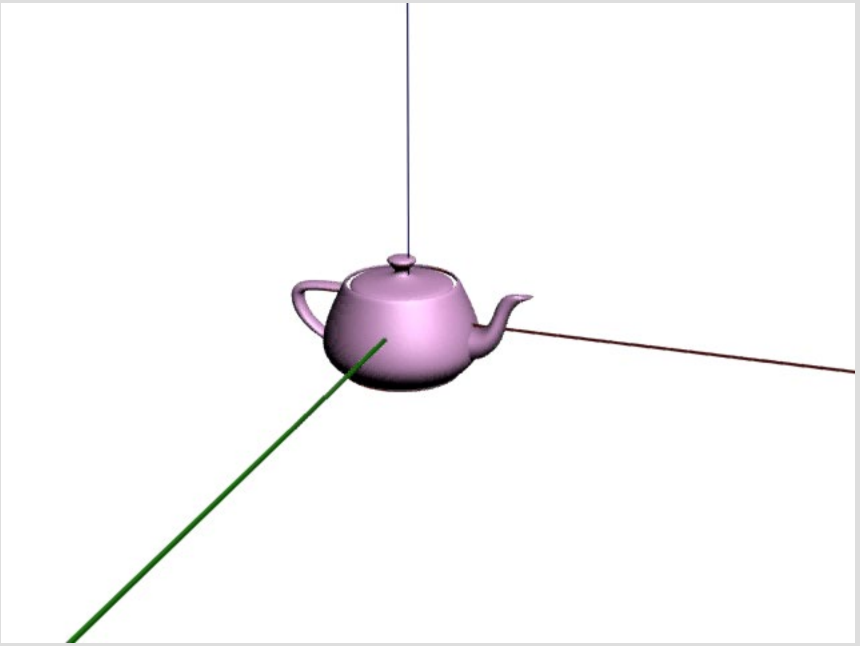
Scaling

- What if we want to make things larger or smaller?
- Have a car model
  - Want one 3 times smaller!

Scaling an object

- 3 times smaller
- Multiply all our coordinates by 1/3
- We get a model that is 1/3 of the size
- However
  - If original coordinates described a car 1 mile from the origin
  - Miniature car would only be 1/3 mile from the origin
- Solution – translation to origin, and then scale, then translate back





Scaling

Definition (Scaling)

A scaling is an expansion or contraction in the x, y, and z directions by scale factors sx, sy, sz and centred at the point (a,b, c)

- Generally we centre the scaling at the origin

$$x' = s_x x$$

$$y' = s_y y$$

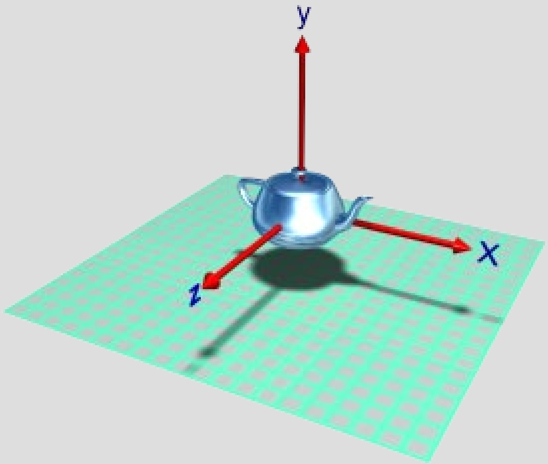
$$z' = s_z z$$

Non-Uniform Scaling

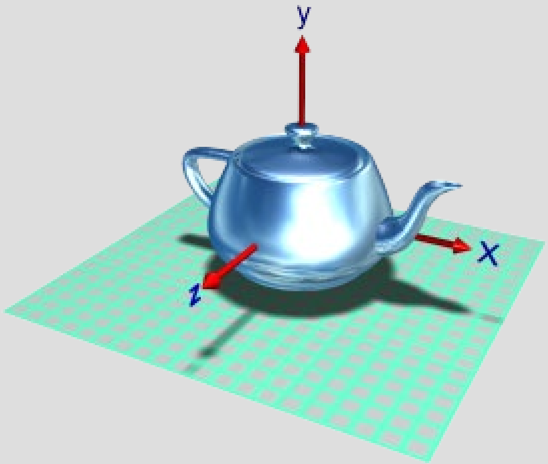
- Make an object twice as big in the x-direction
  - Multiply all x-coordinates by 2, leave y&z unchanged
- 3 times as large in the y-direction
  - Multiply all y-coordinates by 3, leave z&x unchanged

Scale

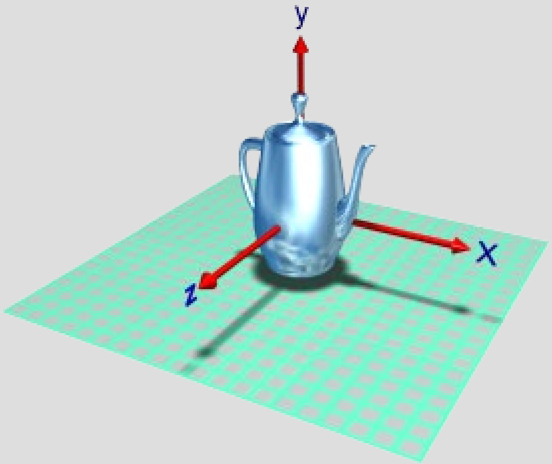
- all vectors are scaled from the origin:



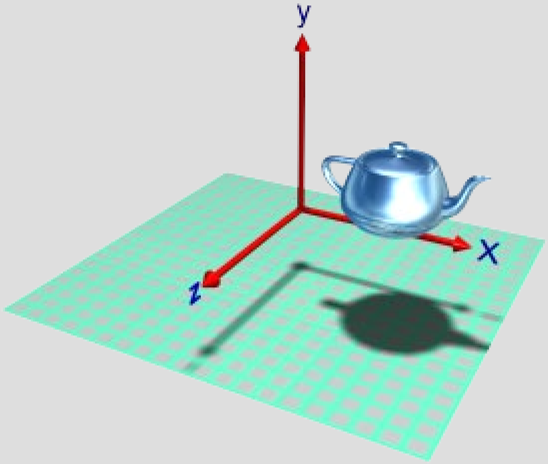
Original



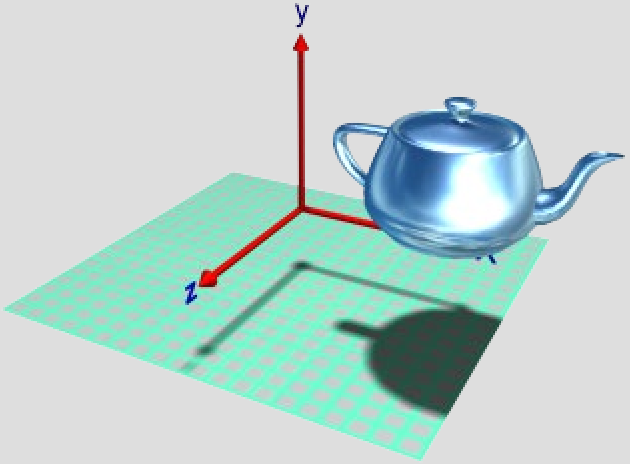
scale all axes



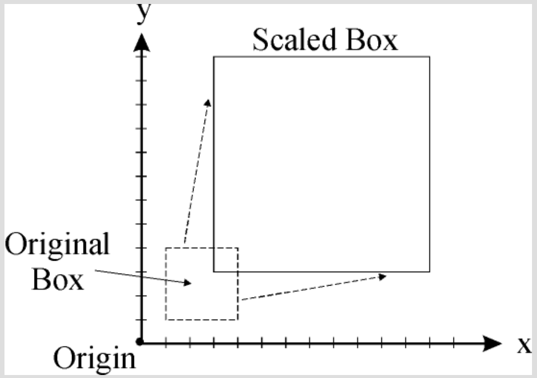
scale Y axis



offset from origin



distance from origin also scales



Scale

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}' = \mathbf{S}\mathbf{v}$$

We would also like to scale points thus we need a *homogeneous transformation* for consistency:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} 1/s_x & 0 & 0 & 0 \\ 0 & 1/s_y & 0 & 0 \\ 0 & 0 & 1/s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

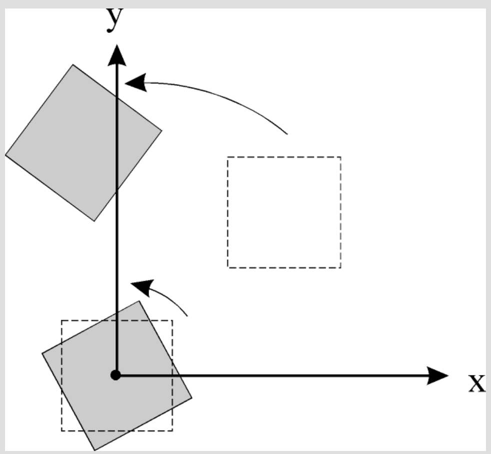
Rotation

Definition (Rotation)

A rotation turns about a point (a,b) through an angle θ

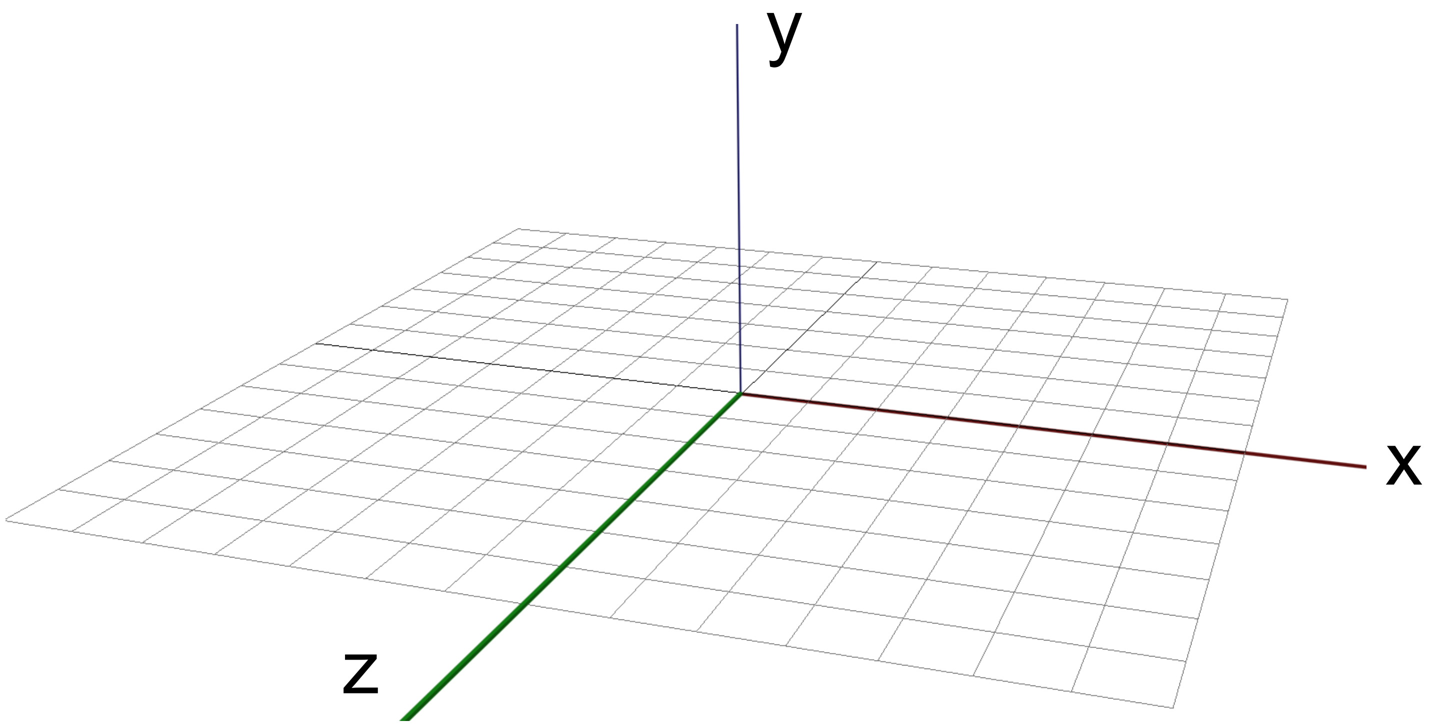
- Generally, we rotate about the origin
- Using the z-axis as the axis of rotation, the equations are:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$



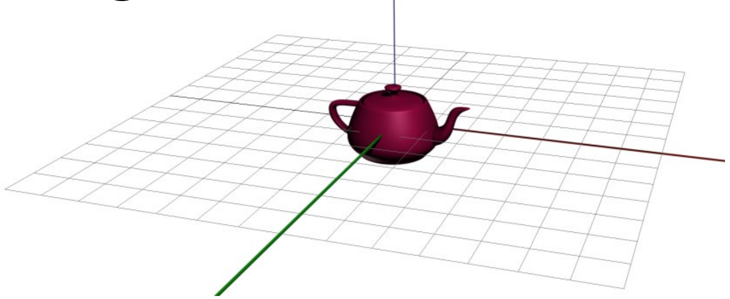
Rotation - idea

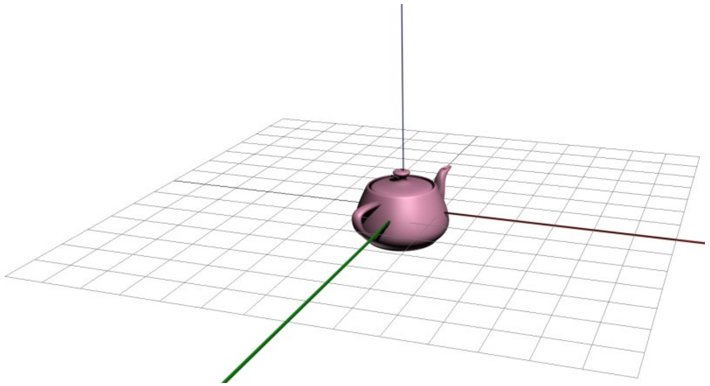
- Visualise rotation about an axis:
  - Put your eye on that axis in the positive direction and look towards the origin
  - Then, a positive rotation corresponds to a counter-clockwise rotation

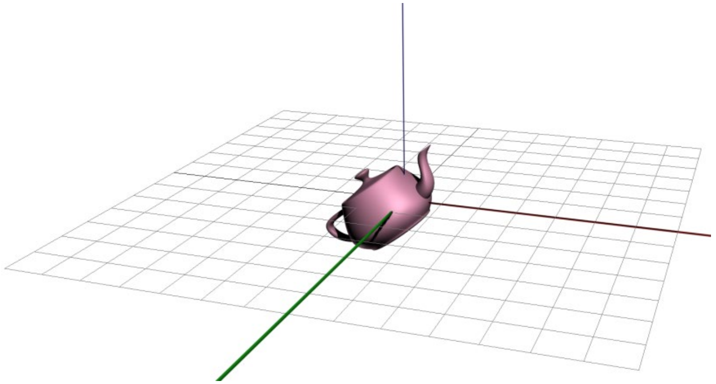


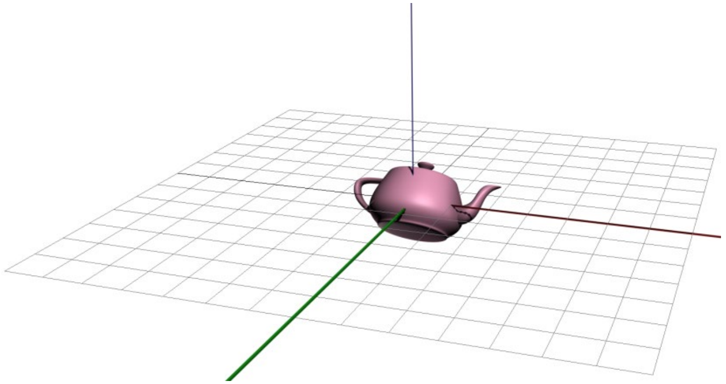
Which Axis?

original



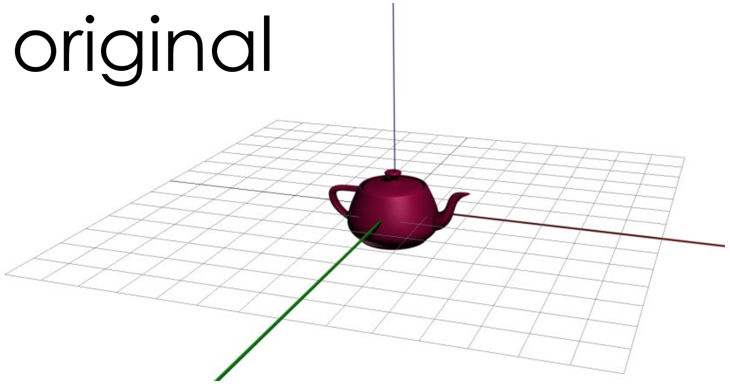




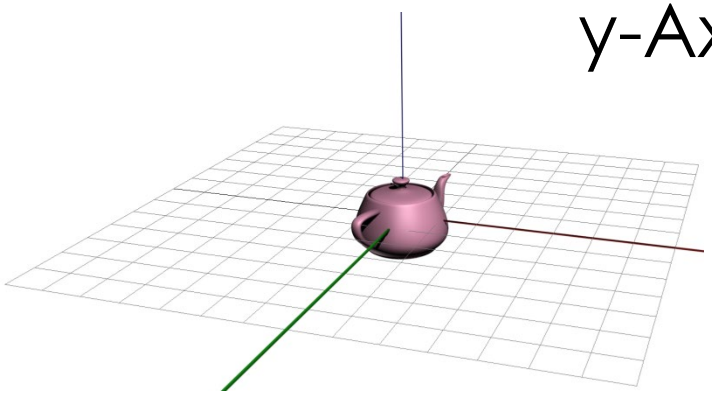


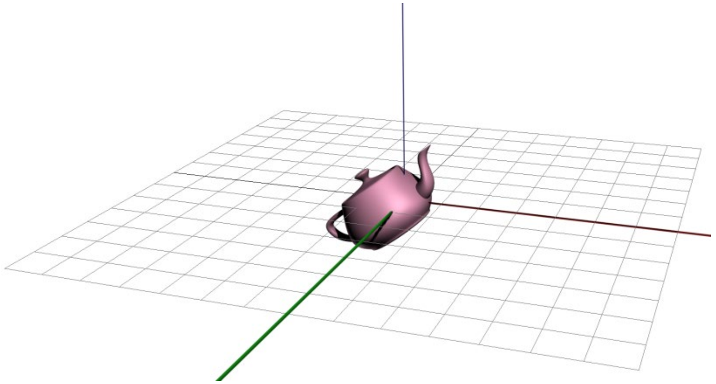
Which Axis?

original

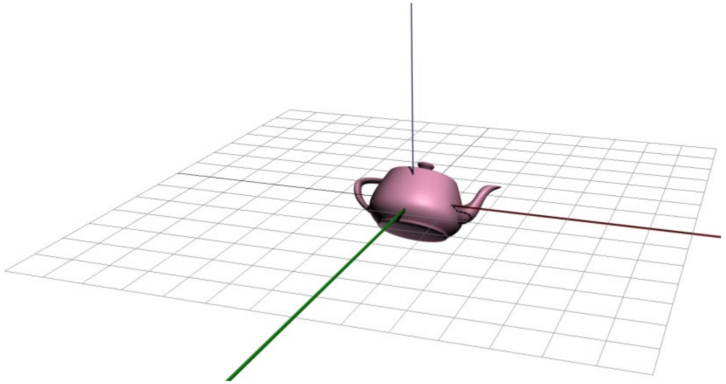


y-Axis





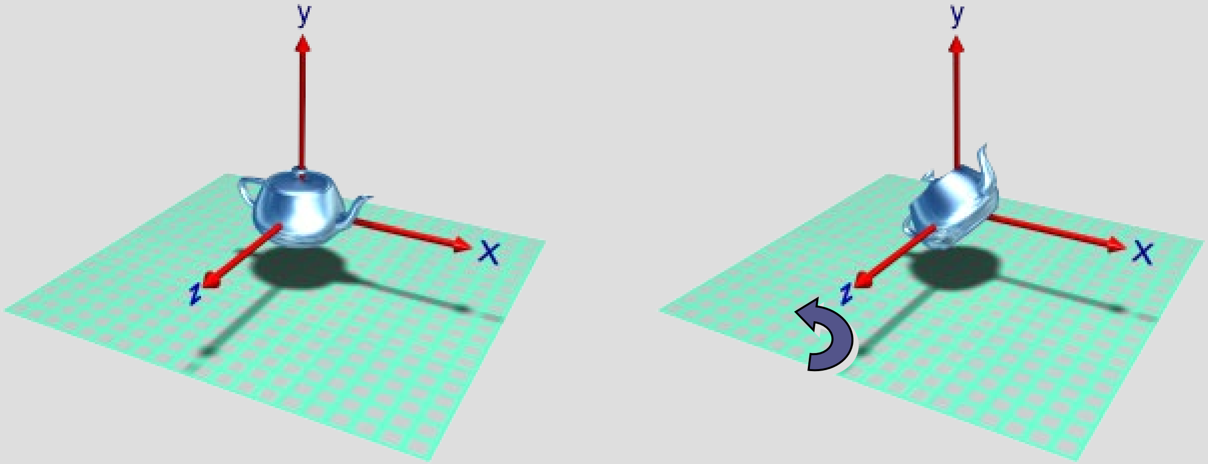
z-axis



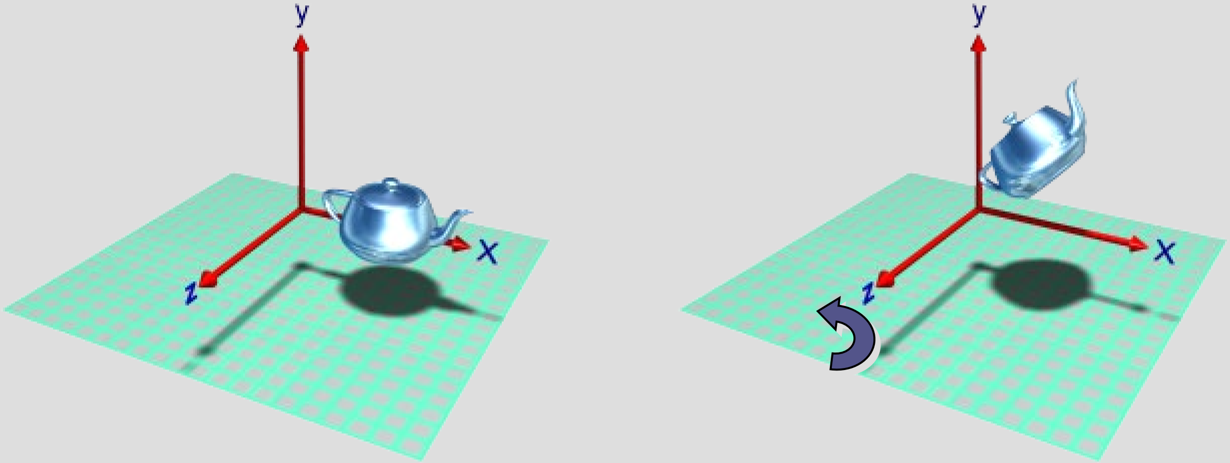
(-) x-axis

Rotation

- Rotations are *anti-clockwise* about the *origin:*



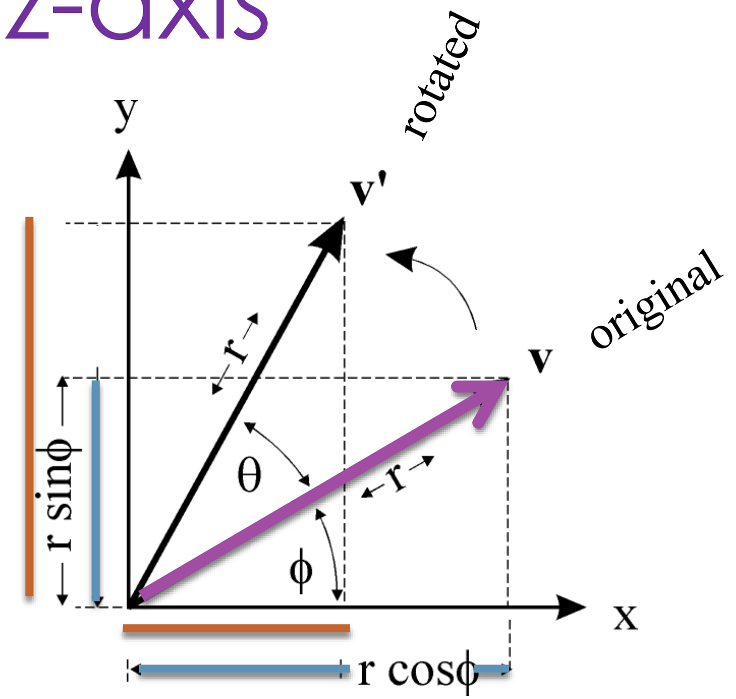
rotation of 45^o about the Z axis



offset from origin rotation

Rotation about the z-axis

$$\mathbf{v} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} \qquad \mathbf{v}' = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$$



$$\text{expand } (\phi+\theta) \Rightarrow \begin{cases} x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\text{but} \quad \begin{array}{l} x = r\cos\phi \\ y = r\sin\phi \end{array} \Rightarrow \begin{array}{l} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{array}$$

Rotation

- 2D rotation of θ about origin: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- 3D homogeneous rotations:

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- <u>Note</u>: $\begin{matrix} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{matrix} \Rightarrow \mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta) = \mathbf{R}^{T}(\theta)$
- If $\mathbf{M}^{-1} = \mathbf{M}^{T}$ then **M** is *orthonormal*. All orthonormal matrices are rotations about the origin.

Which axis is this a rotation around?

Model (World) Matrix

1.00	0.00	0.00	0.00
0.00	-0.83	0.56	0.00
0.00	-0.56	-0.83	0.00
0.00	0.00	0.00	1.00

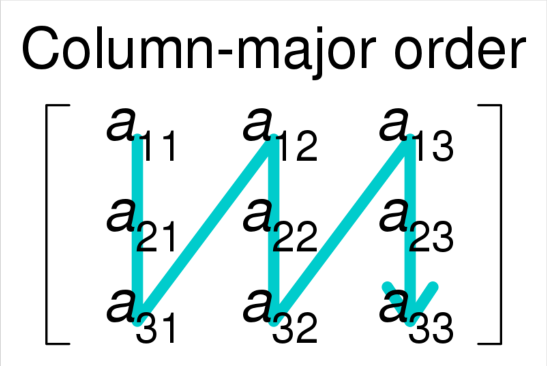
$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer

So it’s a rotation around the X axis, as the first row multiplied by the vertex keeps the x value the same

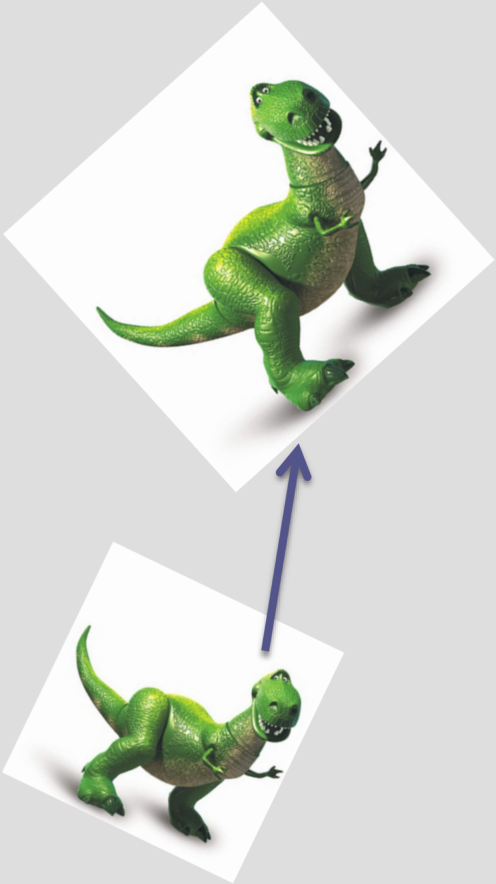
Vertex Shader for Rotation

```
// Remember: these matrices are column-major*
(unlike typical c-programming array filling)

    mat4 rx = mat4( 1.0,  0.0,  0.0, 0.0,
                    0.0,  c.x,  s.x, 0.0,
                    0.0, -s.x,  c.x, 0.0,
                    0.0,  0.0,  0.0, 1.0 );

     mat4 ry = mat4( c.y, 0.0, -s.y, 0.0,
                     0.0, 1.0,  0.0, 0.0,
                     s.y, 0.0,  c.y, 0.0,
                     0.0, 0.0,  0.0, 1.0 );

//note – theta will be in radians in C
//Right-hand rule for rotation directions
//glUniformMatrix4v – set flag to “false”
```



Combining Rotation, Translation, & Scaling

- Often advantageous to combine various transformations to form a more complex transformation
- If we do the algebra – things get complicated quickly
- Easier method – matrices

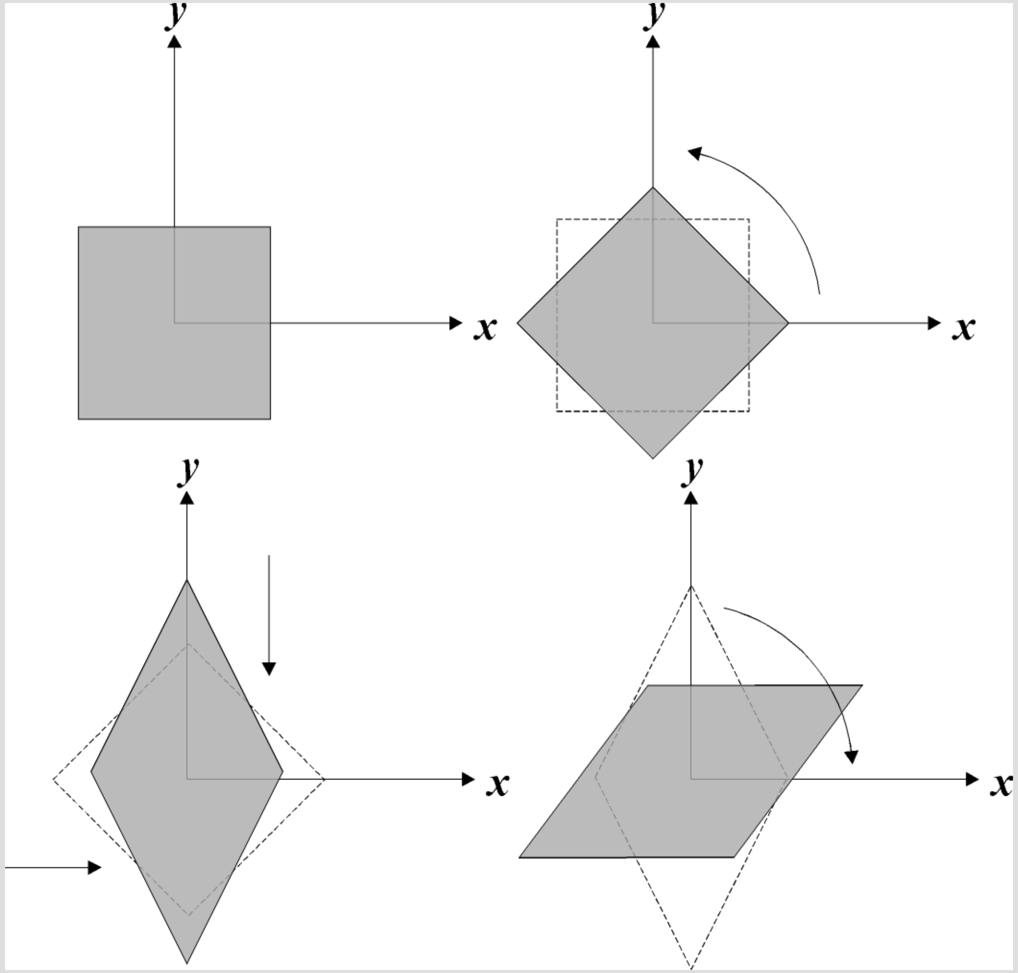


Homogenous Coordinates

- Using this scheme, every rotation, translation, and scaling operation can be represented by a matrix multiplication, and
- any combination of the operations corresponds to the products of the corresponding matrices

Affine Transformations

- All *affine transformations* are combinations of rotations, scaling and translations.



Transformation Composition

- It is common for graphics programs to apply more than one transformation to an object
  - Take vector v_1, Scale it (**S**), then rotate it (**R**)
  - First, $v_2 = \mathbf{S}v_1$, then, $v_3 = \mathbf{R}v_2$
  - $v_3 = \mathbf{R}(\mathbf{S}v_1)$
  - Since matrix multiplication is associative: $v_3 = (\mathbf{RS})v_1$
- In other words, we can represent the effects of transforms by two matrices in a single matrix of the same size by multiplying the two matrices: $\mathbf{M} = \mathbf{RS}$

Transformation Composition

- More complex transformations can be created by *concatenating* or *composing* individual transformations together.

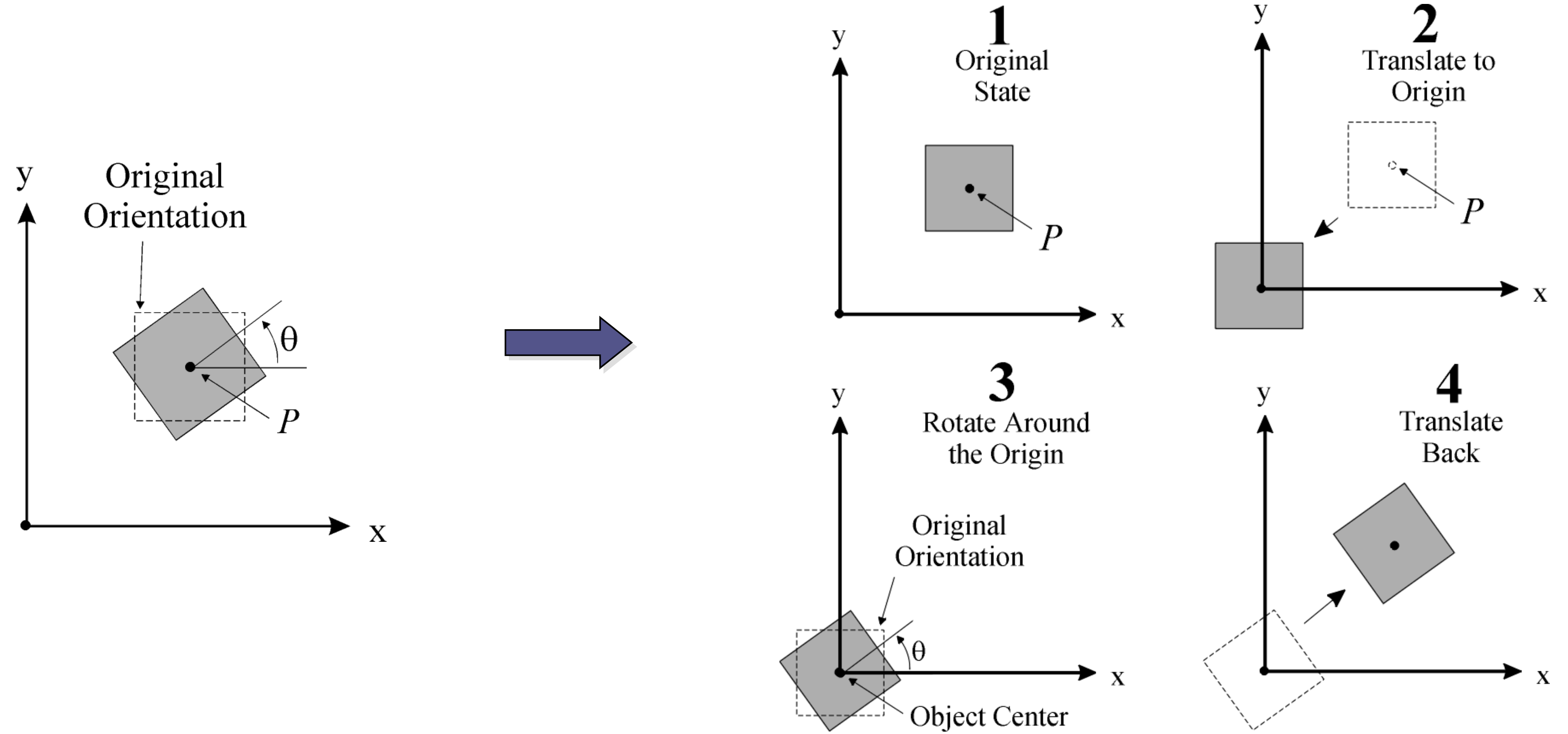
$$\mathbf{M} = \mathbf{T} \circ \mathbf{R} \circ \mathbf{S} \circ \mathbf{T} = \mathbf{TRST} \qquad \mathbf{v}' = \mathbf{T}[\mathbf{R}[\mathbf{S}[\mathbf{Tv}]]] = \mathbf{Mv}$$

- Matrix multiplication is *non-commutative* $\Rightarrow$ **order is vital**
- We can create an affine transformation representing rotation about a point P_R:

= *translate to origin, rotate about origin, translate back to original location*

$$\mathbf{M} = \mathbf{T}(P_R)\mathbf{R}(\theta)\mathbf{T}(-P_R)$$

Rotation about a point



Transformation Composition

Rotation in **XY** plane by *q* degrees anti-clockwise about point *P*

$$\mathbf{M} = \mathbf{T}(P)\mathbf{R}(\theta)\mathbf{T}(-P)$$

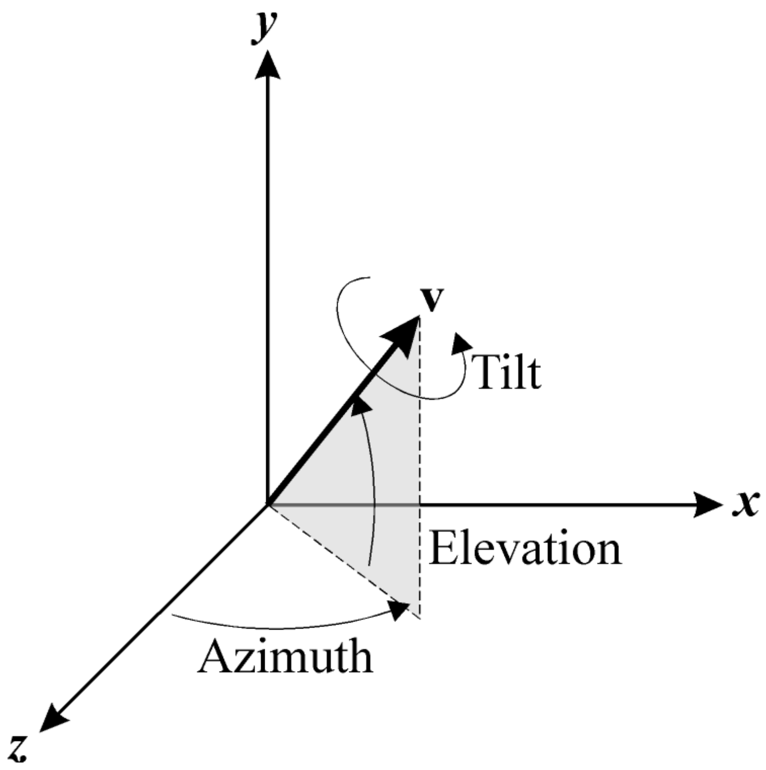
$$= \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

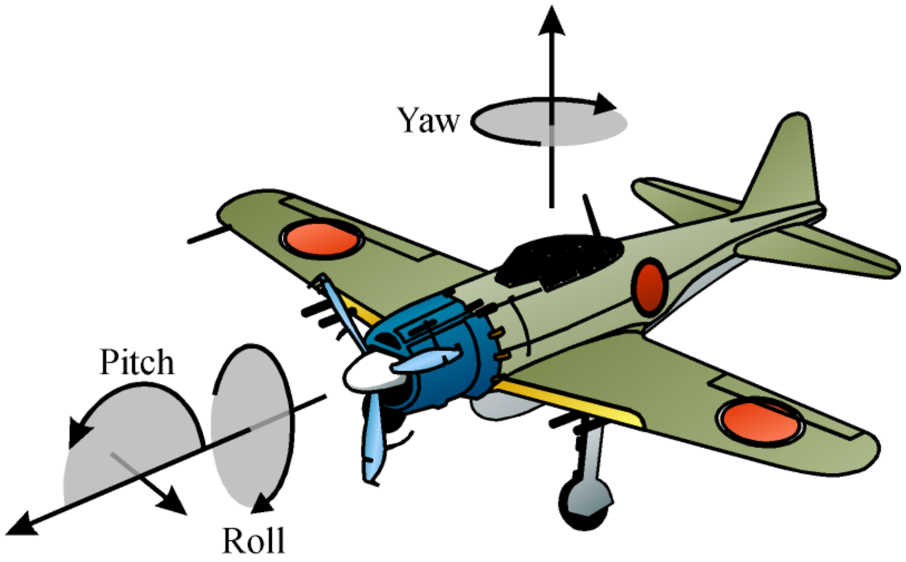
$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & P_x - P_x\cos\theta + P_y\sin\theta \\ \sin\theta & \cos\theta & 0 & P_y - P_x\sin\theta - P_y\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

- *Euler angles* represent the angles of rotation about the co-ordinate axes required to achieve a given orientation (θ_x, θ_y, θ_z)
- The resulting matrix is: $\mathbf{M} = \mathbf{R}(\theta_x)\mathbf{R}(\theta_y)\mathbf{R}(\theta_z)$
- Any required rotation may be described (*though not uniquely*) as a composition of 3 rotations about the coordinate axes.
- Remember rotation *does not commute* $\Rightarrow$ order is important

Rotational DOF





Sometimes known as *roll*, *pitch* and *yaw*

Extra Reading

- **Chapter 3: Geometric Objects and Transformations**
- Interactive Computer Graphics: A Top Down Approach with OpenGL, 6th Edition (or other) Angel
- **Chapter 4: Math for 3D Graphics**
- OpenGL Superbible, 6th Edition
- **Elementary Linear Algebra**, Anton
- "Homogeneous Coordinates and Computer Graphics" by Tom Davis
- http://www.geometer.org/mathcircles/cghomogen.pdf