Thesis

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Contents

Chapter 1

Introduction

In this document I am going to keep notes of everything I read. Hopefully that will help me stay organised.

Chapter 2

Differential Geometry

This chapter will give a rigorous mathematical introduction to differential geometry to lay the groundwork for general relativity. This includes notes on notation.

2.1 Topological Spaces

Topology is the branch of mathematics concerned with extending concepts that are familiar in $\mathbb{R}^n$, such as convergence and continuity, to arbitrary sets. If X is a nonempty set, then a *topology*, $\mathscr{T}$, on X is a collection of subsets of X with the following properties:

1. $X \in \mathscr{T}$ and $\emptyset \in \mathscr{T}$,
2. If $\{U_i\}_{i \in I} \subseteq \mathscr{T}$ for an index set I, then $\bigcup_{i \in I} U_i \in \mathscr{T}$,
3. $T_1 \cap T_2 \in \mathscr{T}$ whenever $T_1, T_2 \in \mathscr{T}$.

That is, a topology $\mathscr{T}$ is a set of subsets of X that is closed under arbitrary unions and finite intersections. The pair $(X, \mathscr{T})$ is called a *topological space*. Often, when the intention is obvious, the topological space is referred to simply as X. The subsets $T \in \mathscr{T}$ are called the *open sets*, and their complements $X \setminus T$, $T \in \mathscr{T}$ are the *closed sets*.

An *open cover* of a subset $S \subseteq X$ is a collection of open sets $\{U_i\}_{i \in I}$ such that $S \subseteq \bigcup_{i \in I} U_i$. The set S is *compact* if every open cover of S has a finite subcover.

An example of a topological space is a *metric space*; a pair (X, d) where X is a set and $d : X \times X \to [0, \infty)$ is a function such that for all $x, y, z \in X$

1. $d(x,y) = 0 \Leftrightarrow x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,y) \leq d(x,z) + d(z,y)$.

In a metric space, an *open ball* centred at a with radius $r \in \mathbb{R}^+$ is the set

$$B(a,r) = \{x \in X | d(a,x) < r\}.$$

A subset of X is called *open* if it is a union of open balls. In a metric space (X,d), the *metric topology*, $\mathscr{T}_m$, is the set of open subsets of X with respect to the metric d.

If X is a topological space, and $x_0 \in X$, then a *neighbourhood* of x_0 is any open subset of X containing x_0. A topological space is called *Hausdorff* if any pair of distinct points have disjoint neighbourhoods. Any metric space (X,d) is a Hausdorff space: for any $x, y \in X$, simply take the neighbourhoods to be the open balls $d(x,r)$ and $d(y,r)$ with $r = \frac{1}{2}d(x,y)$.

A space X is *second countable* if there is a countable set $\mathcal{U}$ of open sets in X such that every open set in X is a union of sets in $\mathcal{U}$.

A mapping f between topological spaces X and Y is *continuous* at $x_0 \in X$ if for all open neighbourhoods U of $f(x_0) \in Y$, the preimage $f^{-1}(U)$ contains an open neighbourhood of x_0. It is said to be *continuous* on X if it is continuous at each point.

A *homeomorphism* between X and Y is a continuous *bijection* f (i.e. f is 1-1 and onto) with a continuous inverse f^{-1}. If such an f exists, X and Y are said to be *homeomorphic*.

2.2 Manifolds

Analysis is a familiar concept in $\mathbb{R}^n$, however on general topological spaces it is not applicable in the same way. A manifold is a structure used to describe and investigate certain topological spaces by identifying them with Euclidean space on "small enough" scales. We shall only define and discuss *smooth* manifolds. To make this idea precise and to use it, we require the following new concepts.

Let M be a topological space. A homeomorphism $x : U \to U'$, for open subsets $U \subseteq M$ and $U' \subseteq \mathbb{R}^n$, is called a *coordinate chart*. We write $x = (x^1(p), x^2(p), ..., x^n(p))$, where the x^i, called *(local) coordinates*, are functions of the point $p \in U$. Strictly speaking, the coordinate chart is a map, but commonly it is used to identify the set U with U' and $p \in U$ with $x(p)$ in $\mathbb{R}^n$.

Usually more than one chart is required to cover the whole space, and these charts must be consistent where they overlap. We say two charts (U, x) and (V, y) are *compatible* if $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ is smooth.

An *atlas* $\mathcal{A}$ on M is a set of coordinate charts on M such that

1. the domains of the charts, U, cover M
2. any two charts are compatible.

If the compatibility criterion above is applied in the reverse order, it is seen that $y \circ x^{-1} = (x \circ y^{-1})^{-1}$ must also be smooth, that is, the maps between the two charts are smooth homeomorphisms with smooth inverses. Such maps are called *diffeomorphisms*. The set of all charts compatible with every chart in an atlas $\mathcal{A}$ is called the *maximal atlas* containing $\mathcal{A}$ and is unique. A maximal atlas may also be called a *smooth structure* on M.

A *(smooth) manifold* M is a second countable Hausdorff space equipped with a maximal atlas $\mathcal{A}$. If the charts in $\mathcal{A}$ map $U \in M \to \mathbb{R}^n$, then n is called the *dimension* of the manifold. Sometimes M is referred to as an *n-manifold*. If N is a closed subset of an n-manifold M, we say that N is a *k-submanifold* of M if N can be covered by charts (x, U) in M such that $x(N \cap U) \subset \mathbb{R}^k \times \mathbf{0}$, where $\mathbf{0}$ is the zero of $\mathbb{R}^{n-k}$.

The *Jacobian* of a map $y : \mathbb{R}^n \to \mathbb{R}^n$, $y(x^1, \ldots, x^n) = (y^1, \ldots, y^n)$ is defined to be the matrix

$$J_{ij} = \frac{\partial y^i}{\partial x^j}.$$

On a manifold M, two charts (U, x) and (V, y) are said to have the same *orientation* if the diffeomorphism

$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$$

has positive Jacobian determinant. An atlas $\mathcal{A}$ is *oriented* if each pair of overlapping charts have the same orientation. A manifold M is *orientable* if it has an oriented atlas.

Let M be a smooth manifold with an atlas $\mathcal{A} = (x_\alpha, U_\alpha)_{\alpha \in I}$ for some index set I. A *partition of unity subordinate to* (U_α) is a family of smooth functions $(\psi_\beta)_{\beta \in J}$, $\psi_\beta : M \to [0, 1]$ with compact support such that:

1. for each $\beta \in J$ there is $\alpha \in I$ such that $\text{supp}\psi_\beta \subset U_\alpha$,
2. for every compact subset $S \in M$, $S \cup \text{supp}\psi_\beta \neq \emptyset$ for only finitely many β,
3. $\sum_{\beta \in J} \psi_\beta(x) = 1$ for all $x \in M$.

Theorem 2.2.1. *Let M be a smooth manifold, and let (x_α, U_α) be an atlas on M. Then there exists a partition of unity subordinate to (U_α).*

For any open cover (U_α) we can find a refinement such that we can use the same index set for the partition of unity as we use for the cover. For a proof of this statement and Theorem 2.2.1 see [**?**].

Examples of manifolds include:

1. $\mathbb{R}^n$ itself, which is covered by a single chart, namely the identity;
2. The n-sphere S^n, which is a submanifold of $\mathbb{R}^{n+1}$.
3. Lie groups (where being a smooth manifold is part of the definition);
4. The product of two manifolds [**?**].

2.3 Calculus on Manifolds

Within the framework of smooth structures it is possible to define concepts from analysis on a manifold M. It is usually convenient to describe these concepts in terms of local coordinates, however it is important to be sure that such definitions are coordinate independent. In this chapter, whenever a concept is defined in terms of a local coordinate chart, the coordinate independence is verified by showing the definition is invariant under a change of coordinates.

2.3.1 The Tangent Space

In $\mathbb{R}^n$ we are used to thinking of a vector as pointing from one point in space to another. In an arbitrary space this idea no longer makes sense. Instead, we must regard a vector as an object associated with a single point on the space. To develop these ideas we think about a vector as acting on functions on M.

A function $f : M \to \mathbb{R}$ is *smooth* at a point $p \in M$ if there is a coordinate chart (U, x) covering p such that the function $\bar{f} = f \circ x^{-1} : x(U) \to \mathbb{R}$ is smooth at $x(p)$. Note that this definition does not depend on the choice of chart. For suppose $f : M \to \mathbb{R}$ is smooth at p for a coordinate chart (U, x) and let (V, y) be another coordinate chart at p. Then, as $x \circ y^{-1}$ is smooth by definition, the function $f \circ y^{-1} = \bar{f} \circ x \circ y^{-1}$ is smooth.

Let $C^\infty(U)$ denote the space of smooth functions $f : U \to \mathbb{R}$. A *vector field* on U is a linear functional $X : C^\infty(U) \to C^\infty(U)$ that satisfies the *Leibniz Rule*,

$$X(fh) = fX(h) + X(f)h,$$

for $f, h \in C^\infty(U)$. Given a coordinate chart $(x^1, \ldots, x^n)$ at p, a vector field X in a neighbourhood U of p can be written

$$X = X^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}, \tag{2.1}$$

where the X^i are functions on U and are referred to as the *components* of X with respect to the coordinate system. The vector field is *smooth* if all of the X^i are smooth. Thus, we are able to extend the definition of a vector field to a global vector field $X : C^\infty(M) \to C^\infty(M)$. The space of all smooth vector fields on M is denoted $\mathcal{T}(M)$. The expression $X^i(p)\frac{\partial}{\partial x^i} = X^i_p\frac{\partial}{\partial x^i}$ is referred to as a *tangent vector* at p.

In equation (2.1) and here on the *Einstein Summation Convention* will apply: when an index appears twice in an expression, summation over that index is implicitly assumed. That is, the expression (2.1), should be interpreted as

$$X = \sum_i X^i \frac{\partial}{\partial x^i}.$$

Let $(x^{1'}, \ldots, x^{n'})$ be another coordinate chart at p. Using the chain rule we obtain the *contravariant transformation law*

$$X = X^i\frac{\partial}{\partial x^i} = X^i\frac{\partial x^{j'}}{\partial x^i}\frac{\partial}{\partial x^{j'}}. \tag{2.2}$$

By defining $X^{j'} = X^i \frac{\partial x^{j'}}{\partial x^i}$ we see that in this alternative coordinate chart the vector field has the same form.

The space of all tangent vectors $X^i_p \frac{\partial}{\partial x^i}$ at a point p is called the *tangent space to* M at p, denoted T_pM. It is clear that T_pM is a vector space and that for a given coordinate chart $(x^1, \ldots, x^n)$ it has basis $\frac{\partial}{\partial x^i}$. For brevity, when the coordinate chart is clear, we sometimes use the notation $\partial_i \equiv \frac{\partial}{\partial x^i}$. The disjoint union of the tangent spaces is denoted TM and is called the *tangent bundle*.

A *smooth curve* on M is a smooth map $\mathcal{C} : I \to M$ for an open interval $I = (t_1, t_2) \subset \mathbb{R}$. The curve passes through a point $p \in M$ if there is $t_p \in I$ so that $\mathcal{C}(t_p) = p$. Under a chart (U, x), the coordinate representation of $\mathcal{C}$ is denoted $\bar{\mathcal{C}} = (x^1(t), \ldots, x^n(t))$.

Let $\mathcal{C} : I \to M$ be a smooth curve with $\mathcal{C}(t_p) = p$, and $f \in C^\infty(M)$, with coordinate representations $\bar{\mathcal{C}}$ and $\bar{f}$ respectively. The composition $f \circ \mathcal{C}$ maps the interval I to $\mathbb{R}$. The *derivative*, df of f along $\mathcal{C}$ at p is

$$\begin{aligned} \frac{d}{dt} f \circ \mathcal{C}\Big|_{t_p} &= \frac{d}{dt} \bar{f}(x^1(t), \ldots, x^n(t)) \\ &= \frac{\partial \bar{f}}{\partial x^i}\Big|_{x_p} \frac{\partial x^i}{\partial t}\Big|_{t_p} \\ &= X_p(f). \end{aligned} \tag{2.3}$$

where $X_p = \frac{\partial x^i}{\partial t}\Big|_{t_p} \frac{\partial}{\partial x^i}$ is the *tangent vector* to $\mathcal{C}$ at p.

2.3.2 The Cotangent Space

Consider a vector space V with basis $\{e_1, \ldots, e_n\}$ so that any vector $v \in V$ can be written $v = v^i e_i$ with respect to that basis. The *dual space* V^* to V is the vector space of linear functionals $\omega : V \to \mathbb{R}$. The basis for V^* is defined to be $\{\varepsilon^1, \ldots, \varepsilon^n\}$ such that

$$\langle \varepsilon^i, e_j \rangle \equiv \varepsilon^i e_j = \delta^i_j,$$

where the symbol $\langle \omega, v \rangle$ denotes the *inner product* of $v \in V$ and $\omega \in V^*$, and δ^i_j is the *Kronecker delta* defined by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The dual space to the tangent space T_pM is called the *cotangent space*, denoted T_p^*M. The elements of T_p^*M are called *covectors* or *1-forms*. For a function $f \in C^\infty(M)$, we define its *exterior derivative*, $\mathrm{d}f$, by the unique 1-form such that

$$\mathrm{d}f(X) = \langle \mathrm{d}f, X \rangle := X(f) \tag{2.4}$$

for all $X \in TM$. Observe that if we apply this definition to each of the simple functions $f = x^1, \ldots x^n$, where the basis for T_pM is $(\partial_1, \ldots, \partial_n)$, we obtain

$$\left\langle \mathrm{d}x^i, \partial_j \right\rangle = \delta^i_j.$$

Thus, the basis for the cotangent space is $\mathrm{d}x^1, \ldots, mboxdx^n$.

The form for a general covector is $\omega = \omega_i \mathrm{d}x^i$. As in the case for vectors, we require that covectors are invariant under coordinate transformations, and so can derive the *covariant transformation property*:

$$\omega = \omega_i \mathrm{d}x^i = \omega_i \frac{\partial x^i}{\partial x^{j'}} \mathrm{d}x^{j'}. \tag{2.5}$$

The components of ω in the new frame are $\omega_{j'} = \omega_i \frac{\partial x^i}{\partial x^{j'}}$.

2.3.3 Tensors

A (r, s)-*tensor* (say "r times contravariant and s times covariant") over a vector space V is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \to \mathbb{R}.$$

The specification that the map is multilinear means that it is linear in each argument. The rank of a tensor is $(r + s)$, that is, the total number of arguments it takes. The space of (r, s)-tensors on a vector space V is denoted by $T^r_s(V)$.

Given tensors $T \in T^r_s$ and $S \in T^p_q$, the *tensor product*, $T \otimes S \in T^{r+p}_{s+q}$ is defined by

$$T \otimes S(\omega^1, \ldots, \omega^{r+p}; v_1, \ldots, v_{s+q}) = T(\omega^1, \ldots, \omega^r; v_1, \ldots, v_s) S(\omega^{r+1}, \ldots, \omega^{r+p}; v_{s+1}, \ldots, v_{s+q})$$

The tensor product is not commutative, but it is associative, that is

$$(T \otimes S) \otimes R = T \otimes (S \otimes R),$$

so a product of three tensors can be written $T \otimes S \otimes R$ without ambiguity.

On the tangent space T_p, a (r, s)-tensor maps an ordered set of r 1-forms and s vectors to $\mathbb{R}$. We can immediately see that vectors are type (1,0)-tensors and 1-forms are type (0,1)-tensors. Scalar functions are tensors of type (0,0). The components of a (r, s)-tensor over T_pM can be determined in coordinates by acting on the basis vectors and 1-forms,

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = T(\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_r}; \partial_{j_1}, \ldots, \partial_{j_s}),$$

where each of the i's and j's run through $1 \ldots n$. Thus, the tensor can be written in expanded form

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}.$$

Under a coordinate change the *tensor transformation law* is

$$T^{i'_1 \ldots i'_r}_{j'_1 \ldots j'_s} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_s}}{\partial x^{j'_s}} T^{i_1 \ldots i_r}_{j_1 \ldots j_s}. \tag{2.6}$$

This is consistent with the transformation laws already seen for vectors (equation (2.2)) and 1-forms (equation (2.5)).

The operation *contraction* is a map performed by summing over one upper and one lower index. Precisely which indices are involved can be conveniently indicated by using repeated indices in local coordinates. For example, on an n-dimensional vector space, contraction over the second upper index and first lower index on a $(3, 2)$-tensor produces:

$$T^{ijk}_{jl} = \sum_{j=1}^{n} T^{ijk}_{jl} = S^{ik}_{l}.$$

Lemma 2.3.1. *Contraction of a tensor is a tensor.*

Proof. In order to prove this we must prove that the resultant tensor obeys the transformation law (2.6). We prove the lemma for the tensor in the above example for simplicity. A proof for a general tensor proceeds in the same way.

For the tensor $T = T^{ijk}_{jl}$, (2.6) gives

$$T^{i'j'k'}_{j'l'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^l}{\partial x^{l'}}T^{ijk}_{jl}.$$

Here we recall that

$$\frac{\partial x^j}{\partial x^n}\frac{\partial x^m}{\partial x^j} = \delta^m_n$$

where δ^m_n is the Kronecker delta. Thus we have

$$T^{i'j'k'}_{j'l'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^l}{\partial x^{l'}}T^{ijk}_{jl}.$$

That is

$$S^{i'k'}_{l'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^l}{\partial x^{l'}}S^{ik}_{l},$$

i.e. S^{ik}_l obeys the tensor transformation law as required. □

Thus, contraction is a mapping from $(r+1, s+1)$-tensors to (r, s)-tensors.

A tensor is *symmetric* in some of its indices if it is unchanged by an interchange of those indices. For example, if

$$S_{ijkl} = S_{jikl},$$

we say S_{ijkl} is symmetric in i and j. We can *symmetrize* a tensor over any number of its indices by summing over the permutations of those indices and dividing by the number of permutations. The resulting tensor will be symmetric. Symmetrization is denoted by round brackets and is given by:

$$T_{(i_1,\ldots,i_n)i_{n+1}\ldots i_k} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{\sigma(i_1,\ldots,i_n)i_{n+1}\ldots i_k}, \tag{2.7}$$

where S_n is the group of permutations on $(1,\ldots,n)$.

A tensor is *skew-symmetric* in some of its indices if it changes sign under a swap of any two of those indices, for example, if

$$S_{ijkl} = -S_{ikjl},$$

we say S_{ijkl} is skew-symmetric in i and j. In a similar fashion to symmetrization, we can *skew* a tensor over a number of its indices by taking the alternating sum over the permutations of those indices and dividing by the number of permutations. This is denoted by square brackets:

$$T_{[i_1,\ldots,i_n]i_{n+1}\ldots i_k} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}\sigma\, T_{\sigma(i_1,\ldots,i_n)i_{n+1}\ldots i_k}, \tag{2.8}$$

where $\mathrm{sgn}\sigma = -1$ if the permutation σ results from an odd number of swaps of indices, and $\mathrm{sgn}\sigma = 1$ if the number of swaps is even. We can also symmetrize or skew over upper indices. Sometimes we may wish to symmetrize or skew over indices which are not adjacent. If so, the indices not included are placed within vertical bars, for example,

$$T_{(i|j|k)} = \frac{1}{2}(T_{ijk} + T_{kji}).$$

2.3.4 Riemannian Manifolds

A *Riemannian metric* on a smooth manifold M is a symmetric, positive definite, $(0,2)$-tensor field g. That is, $g(X,X) > 0$ if $X \in T_pM \neq 0$. A *Riemannian manifold* is a smooth manifold equipped with a Riemannian metric. In coordinates, the metric has the form

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

For the sake of brevity the tensor product symbol is usually dropped and this is written $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

On each tangent space T_pM the Riemannian metric determines an inner product,

$$\langle X_p, Y_p\rangle = g_{ij}X_p^iY_p^j, \tag{2.9}$$

for $X_p, Y_p \in T_pM$. In particular, $\langle \partial_i, \partial_j\rangle = g_{ij}$. For example, in Euclidean space, $\mathbb{R}^n$, the metric is $g_{ij} = \delta_{ij}$. Let $X_0 = (X_0^1, \ldots, X_0^n)$ and $Y_0 = (Y_0^1, \ldots, Y_0^n)$ be two vectors at a point $p_0 \in \mathbb{R}^n$ in Cartesian coordinates. Then from (2.9) we have

$$\langle X_0, Y_0\rangle = \delta_{ij}X_0^iY_0^j = X_0^1Y_0^1 + \cdots + X_0^nY_0^n,$$

which is the familiar dot product.

Theorem 2.3.2. *Every smooth manifold M may be equipped with a Riemannian metric.*

Proof. Let $(x_i, U_i)_{i\in I}$ be an atlas on M, and let $(\psi_i)_{i\in I}$ be a partition of unity subordinate to this atlas. For $p \in U_i$ let $X_p, Y_p \in T_pM$, be given in coordinates by $(X_{pi}^1, \ldots, X_{pi}^n)$ and $(Y_{pi}^1, \ldots, Y_{pi}^n)$ respectively. Then we can define a Riemannian metric on M by

$$g(X_p, Y_p) = \sum_{i\in I} \psi_i X_{pi}^j Y_{pi}^j.$$

□

We define the *inverse metric tensor* g^{ij} by

$$g^{ij}g_{jk} = g_{jk}g^{ij} = \delta^i_k.$$

Clearly this is also symmetric. The metric and inverse metric can be used to perform the important operations of *raising and lowering indices* on a tensor, which allows us to identify the contravariant and covariant components. In particular, we can use the metric and inverse to change vectors to covectors and vice versa:

$$\begin{array}{rcl} X_i & = & g_{ij}X^j \\ \omega^i & = & g^{ij}\omega_j \end{array} .$$

If we take a n-submanifold M of $\mathbb{R}^m$ then our knowledge of the metric in $\mathbb{R}^m$ allows us to construct a metric on M. Let $(u^1, \ldots, u^m)$ be the usual Cartesian coordinates in $\mathbb{R}^m$ and let M have local coordinates $(x^1, \ldots, x^n)$. The *induced metric* is [**?**]

$$g = \sum_{i=1}^{m} (\mathrm{d}u^i)^2 = \sum_{i=1}^{m} \left(\frac{\partial u^i}{\partial x^j} \mathrm{d}x^j \right)^2 . \tag{2.10}$$

2.4 Mappings between Manifolds

Consider two manifolds M and N, and suppose there exists a diffeomorphism $\phi : M \to N$. Let f be a function $f : N \to \mathbb{R}$. We can precompose the diffeomorphism ϕ with f to define a corresponding function on M. This is the *pull back* of f by ϕ:

$$\phi^* f = (f \circ \phi). \tag{2.11}$$

Recalling that a vector field on a manifold is a differential operator on the space of smooth functions on the manifold, we can use ϕ to define vector fields on N in terms of vector fields on M. This is called the *push forward* of a vector field X by ϕ, denoted $\phi_* X$, and is defined by its action on functions $f \in C^\infty(N)$,

$$(\phi_* X) f = X(\phi^* f). \tag{2.12}$$

That is, the action of the vector field $\phi_* X \in \mathcal{T}(N)$ is the action of $X \in \mathcal{T}(M)$ on the pulled back function $\phi^* f$.

The pull back and push forward can be extended to purely contravariant or purely covariant tensors. For a diffeomorphism $\phi : M \to N$, with coordinate representation $\phi(x^1, \ldots x^n) = (\phi^1(x), \ldots, \phi^m(x))$, the general formulae are [**?**]:

$$(\phi^* T)_{i_1,\ldots,i_s} = \frac{\partial \phi^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial \phi^{j_s}}{\partial x^{i_s}} T_{j_1 \ldots j_s} \tag{2.13}$$

$$(\phi_* T)^{j_1,\ldots,j_r} = \frac{\partial \phi^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial \phi^{j_r}}{\partial x^{i_r}} T^{i_1 \ldots i_r} \tag{2.14}$$

If $\phi : M \to N$ is a diffeomorphism and Y is a vector field on N, we put

$$\phi^* Y = (\phi^{-1})_* Y. \tag{2.15}$$

2.5 Isometries

Intuitively, we think of a symmetry on a manifold as a transformation of the manifold under which the geometry "looks the same", that is, the metric remains the same under the transformation. Mathematically, symmetries on a Riemannian manifold M with metric g are given by *isometries*; diffeomorphisms $\phi : M \to M$ for which $\phi^* g = g$. A composition of isometries is clearly also an isometry, as is the inverse of an isometry. Thus, the set of isometries on M forms a group, called the *isometry group*, $\mathcal{I}(M)$, of M. In fact, the isometry group can be shown to be a finite dimensional Lie group acting on M [**?**].

An important example is the Euclidean space $\mathbb{R}^n$ with metric $g_{ij} = \delta_{ij}$. By elementary geometry, the isometry group $\mathcal{I}(\mathbb{R}^n)$ turns out to be the *Euclidean group* $E(n) = \mathcal{O}(n) \times R^n$, where $\mathcal{O}(n)$ is the group of orthogonal matrices (invertible $n \times n$ matrices A such that $A^T A = \mathbb{I}$). The group acts on $x \in \mathbb{R}^n$ by

$$x \mapsto A(x) + b$$

where $b \in \mathbb{R}^n$ is a translation, and $A \in \mathcal{O}(n)$ corresponds to orthogonal transformations, that is, the reflections and rotations. The connected component of $E(n)$ containing the identity forms the continuous subgroup of *orientation preserving* isometries. The structure $SE(n) = \mathcal{SO}(n) \times \mathbb{R}^n$, where $\mathcal{SO}(n) = \{A \in \mathcal{O}(n) | \det A = 1\}$ corresponds to the rotations. The reflections are *orientation reversing*.

A *1-parameter family of diffeomorphisms* is a group of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$ such that $\phi_t \circ \phi_s = \phi_{t+s}$. The identity is ϕ_0 and $\phi_{-t} = \phi_t^{-1}$. On an n-manifold M there is a 1-1 correspondence between 1-parameter families of diffeomorphisms and vector fields [**?**]. Under a

family of diffeomorphisms ϕ_t, each point p traces out a smooth curve $\mathcal{C}_p(t) = (x_p^1(t), \ldots, x_p^n(t))$ with $\mathcal{C}(0) = p$. We can define a vector field $X^i \frac{\partial}{\partial x^i}$ along $\mathcal{C}_p$ (cf. (2.3)) by

$$\frac{dx_p^i}{dt} = X^i. \tag{2.16}$$

This can be done for every point on our manifold and thus we have a vector field on M. The curves $\mathcal{C}$ are called *integral curves* of X. Now, we wish to view this as the diffeomorphisms parameterised by t acting on a point $p \in M$, rather than a curve parameterised by t through the point p, so we write

$$\phi_t(p) = \mathcal{C}_p(t).$$

The vector field X is called the *generator* of the family of diffeomorphisms.

The diffeomorphisms R_θ clearly form a 1-parameter family with parameter θ. The corresponding vector field is

$$X = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}. \tag{2.17}$$

The integral curves of X are concentric circles about the origin.

2.6 The Lie Derivative

On a manifold M we may ask how a tensor T changes as it travels along the integral curves of a vector field. To this end we define the *Lie Derivative*, $\mathcal{L}_X T$, of a tensor field $T \in T_s^r$ along a vector field X by

$$\mathcal{L}_X T := \frac{d}{dt}(\phi_t^* T)|_{t=0} \tag{2.18}$$

where ϕ_t is the 1-parameter family of diffeomorphisms generated by X. The Lie derivative is a well-behaved differential operator, that is, for real numbers a, b and tensors T, S,

1. it is linear:
$$\mathcal{L}_X(aT + bS) = a\mathcal{L}_X T + b\mathcal{L}_X S \tag{2.19}$$

2. it obeys the Leibniz Rule:
$$\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S) \tag{2.20}$$

We use these properties to give explicit formulae for functions, 1-forms and vector fields, then extend them to a general tensor. In the working that follows, M is a Riemannian n-manifold with local coordinates $(x^1, \ldots, x^n)$ in which the vector field is $X = X^i \frac{\partial}{\partial x^i}$.

Theorem 2.6.1. *Let $f : M \to \mathbb{R}$ be a smooth function on M. Then the Lie derivative of f with respect to X is*

$$\mathcal{L}_X(f) = Xf, \tag{2.21}$$

where the right hand side is the action of the vector field on the function as in §2.3.1.

Proof.

$$\begin{aligned} \mathcal{L}_X(f) &= \frac{d}{dt}(\phi_t^* f)|_{t=0} \\ &= \frac{d}{dt}(f \circ \phi_t)|_{t=0} \\ &= X^i \frac{\partial f}{\partial x^i} \qquad \text{(cf. equation (2.3))} \\ &= X(f) \end{aligned}$$

□

Unlike functions, we are unable to write the Lie derivative of a 1-form in a manifestly coordinate free manner. Thus, we must operate in a local chart and check the coordinate independence explicitly.

Theorem 2.6.2. *Let $\omega = \omega_i dx^i$ be a 1-form with respect to the coordinate system $(x^1, \ldots, x^n)$. The Lie derivative of ω with respect to X is*

$$\mathcal{L}_X \omega = \left(\omega_i \frac{\partial X^i}{\partial x^j} + X^i \frac{\partial \omega_j}{\partial x^i} \right) dx^j. \tag{2.22}$$

Proof. From (2.13) we have

$$\phi^* \omega = \omega_i \frac{\partial \phi^i}{\partial x^j} \mathrm{d}x^j.$$

Consider the action of the Lie derivative on the basis 1-forms $\mathrm{d}x^i$:

$$\begin{aligned} \frac{d}{dt}(\phi_t^* \mathrm{d}x^i)|_{t=0} &= \frac{d}{dt}\left(\frac{\partial \phi_t^i}{\partial x^j} \mathrm{d}x^j \right)|_{t=0} \\ &= \frac{d}{dt} \frac{\partial \phi_t^i}{\partial x^i}\bigg|_{t=0} \mathrm{d}x^j \\ &= \frac{\partial X^i}{\partial x^j} \mathrm{d}x^j \qquad \text{(since } \phi_0 = id\text{)}. \end{aligned} \tag{2.23}$$

Now consider the general 1-form $\omega_i \mathrm{d}x^i$. Applying the Leibniz rule,

$$\begin{aligned}\mathcal{L}_X(\omega_i \mathrm{d}x^i) &= \omega_i \mathcal{L}_X(\mathrm{d}x^i) + \mathcal{L}_X(\omega_i)\mathrm{d}x^i \\ &= \omega_i \frac{\partial X^i}{\partial x^j}\mathrm{d}x^j + X^j \frac{\partial \omega_i}{\partial x^j}\mathrm{d}x^i \\ &= \left(\omega_i \frac{\partial X^i}{\partial x^j} + X^i \frac{\partial \omega_j}{\partial x^i}\right)\mathrm{d}x^j,\end{aligned}$$

where in the last term we have swapped dummy indices. □

Now we must confirm the coordinate independence of this result. Let x' be a change of coordinates. Then by equation (2.2) in this coordinate system

$$X = X^{j'}\frac{\partial}{\partial x^{j'}} = X^i \frac{\partial x^{j'}}{\partial x^i}\frac{\partial}{\partial x^{j'}},$$

and by equation (2.5),

$$\omega = \omega_{j'}\mathrm{d}x^{j'} = \omega_i \frac{\partial x^i}{\partial x^{j'}}\mathrm{d}x^{j'}.$$

Substituting this into (2.22) in the x' frame:

$$\begin{aligned}\mathcal{L}_X\omega &= \left(X^{i'}\frac{\partial}{\partial x^{i'}}\omega_{j'} + \omega_{i'}\frac{\partial}{\partial x^{j'}}X^{i'}\right)dx^{j'} \\ &= \left(X^m \frac{\partial x^{i'}}{\partial x^m}\frac{\partial}{\partial x^{i'}}\left(\omega_n \frac{\partial x^n}{\partial x^{j'}}\right) + \omega_n \frac{\partial x^n}{\partial x^{i'}}\frac{\partial}{\partial x^{j'}}\left(X^m \frac{\partial x^{i'}}{\partial x^m}\right)\right)dx^{j'} \\ &= \left(X^m \frac{\partial x^{i'}}{\partial x^m}\left(\frac{\partial \omega_n}{\partial x^{i'}}\frac{\partial x^n}{\partial x^{j'}} + \omega_n \frac{\partial^2 x^n}{\partial x^{i'}\partial x^{j'}}\right) + \omega_n \frac{\partial x^n}{\partial x^{i'}}\left(\frac{\partial X^m}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^m} + X^m \frac{\partial^2 x^{i'}}{\partial x^{i'}\partial x^m}\right)\right)dx^{i'} \\ &= \left(X^m \frac{\partial x^{i'}}{\partial x^m}\frac{\partial \omega_n}{\partial x^{i'}}\frac{\partial x^n}{\partial x^{j'}} + X^m \omega_n \frac{\partial x^{i'}}{\partial x^m}\frac{\partial^2 x^n}{\partial x^{i'}\partial x^{j'}} + \omega_n \frac{\partial x^n}{\partial x^{i'}}\frac{\partial X^m}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^m} + \omega_n X^m \frac{\partial x^n}{\partial x^{i'}}\frac{\partial^2 x^{i'}}{\partial x^{j'}\partial x^m}\right)dx^{j'} \\ &= \left(\left(X^m \frac{\partial \omega_n}{\partial x^{j'}} + \omega_n \frac{\partial X^m}{\partial x^{j'}}\right)\delta^n_m + \omega_n X^m \left(\frac{\partial x^{i'}}{\partial x^m}\frac{\partial^2 x^n}{\partial x^{i'}\partial x^{j'}} + \frac{\partial x^n}{\partial x^{i'}}\frac{\partial^2 x^{i'}}{\partial x^{j'}\partial x^m}\right)\right)dx^{j'}. \qquad (2.24)\end{aligned}$$

Here we notice that

$$\frac{\partial y^i}{\partial x^m}\frac{\partial x^n}{\partial y^i} = \delta^n_m,$$

therefore

$$\frac{\partial}{\partial y^j}\left(\frac{\partial y^i}{\partial x^m}\frac{\partial X^n}{\partial y^i}\right) = \frac{\partial^2 y^i}{\partial y^j \partial x^m}\frac{\partial x^n}{\partial y^i} + \frac{\partial y^i}{\partial x^m}\frac{\partial^2 x^n}{\partial y^j \partial y^i} = 0.$$

Thus the second term in the expression (2.24) vanishes and we have

$$\mathcal{L}_X\omega = \left(X^m\frac{\partial x^{j'}}{\partial x^n}\frac{\partial \omega_m}{\partial x^{j'}} + \omega_m\frac{\partial x^{j'}}{\partial x^n}\frac{\partial X^m}{\partial x^{j'}}\right)\frac{\partial x^n}{\partial x^{j'}}dx^{j'} = \left(X^m\frac{\partial}{\partial x^m}\omega_n + \omega_m\frac{\partial}{\partial x^n}X^m\right)dx^n.$$

Hence proving coordinate independence.

Theorem 2.6.3. *Let Y be a vector field. The Lie derivative of Y with respect to X is*

$$\mathcal{L}_X Y = [X, Y], \tag{2.25}$$

where $[\ ,\ ]$ denotes the Lie bracket:

$$[X, Y] = XY - YX.$$

While this is inherently coordinate independent, the proof requires working in coordinates.

Proof. Consider the left hand side first. Let $Y = \frac{\partial}{\partial x^j}$ and recall $\langle \frac{\partial}{\partial x^j}, \mathrm{d}x^i\rangle = \delta^i_j$. It follows that

$$\mathcal{L}_X\langle\frac{\partial}{\partial x^j}, \mathrm{d}x^i\rangle = \left(\mathcal{L}_X\frac{\partial}{\partial x^j}\right)\mathrm{d}x^i + \left(\mathcal{L}_X\mathrm{d}x^i\right)\frac{\partial}{\partial x^j} = 0,$$

therefore, from (2.23) we have

$$\mathcal{L}_X\frac{\partial}{\partial x^j} = -\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}.$$

Now consider $Y = Y^j\frac{\partial}{\partial x^j}$

$$\begin{aligned}\mathcal{L}_X(Y^j\frac{\partial}{\partial x^j}) &= \mathcal{L}_X(Y^j)\frac{\partial}{\partial x^j} + Y^j\mathcal{L}(\frac{\partial}{\partial x^j})\\ &= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^i\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}\\ &= \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}\end{aligned}$$

(again, swapping dummy indices in the last step).

Now consider the right hand side of (2.25). The action of a product of vector fields on a function $(XY)f$ is $X(Yf)$ so we have

$$\begin{aligned}(XY - YX)f &= X^i\frac{\partial}{\partial x^i}\left(Y^j\frac{\partial f}{\partial x^j}\right) - Y^i\frac{\partial}{\partial x^i}\left(X^j\frac{\partial f}{\partial x^j}\right)\\ &= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^iY^j\frac{\partial^2 f}{\partial x^i\partial x^j} - Y^i\frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^i} - Y^iX^j\frac{\partial^2 f}{\partial x^i\partial x^j}\\ &= \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j},\end{aligned}$$

that is, the same as the left hand side. Thus the result is proved. □

Now we can deduce the Lie derivative of a general tensor. Consider the arbitrary (r, s)-tensor

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}$$

By the linearity (2.19) and the Leibniz rule (2.20), we find Lie derivative of T to be

$$\begin{aligned} \mathcal{L}_X T &= (X T^{i_1 \dots i_r}_{j_1 \dots j_s}) \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \\ &\quad + T^{i_1 \dots i_r}_{j_1 \dots j_s} \left[\sum_{n=1}^{r} \partial_{i_1} \otimes \cdots \otimes (\mathcal{L}_X \partial_{i_n}) \otimes \dots \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \right. \\ &\quad \left. + \sum_{m=1}^{s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes (\mathcal{L}_X \mathrm{d}x^{j_m}) \otimes \cdots \otimes \mathrm{d}x^{j_s} \right]. \end{aligned}$$

This is coordinate independent by virtue of the independence of the tensor product and since we have already shown independence for (1,0)- and (0,1)-tensors.

Of particular interest is the covariant 2-tensor, that is tensors of the same type as the Riemannian metric. The Lie derivative for a covariant 2-tensor $\varphi = \varphi_{ij} dx^i \otimes dx^j$ is:

$$\mathcal{L}_X \varphi = \left(X^i \frac{\partial}{\partial x^i} \varphi_{jk} + \frac{\partial X^i}{\partial x^j} \varphi_{ik} + \frac{\partial X^i}{\partial x^k} \varphi_{ji} \right) dx^j \otimes dx^k \tag{2.26}$$

2.7 The Covariant Derivative

A *linear connection* on M is a map

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

written $\nabla_X Y$, such that

1. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

$$\nabla_X (aY_1 + bY_2) = a \nabla_X Y_1 + b \nabla_X Y_2,$$

2. $\nabla_X Y$ is linear over f in X:

$$\nabla_{(fX_1 + gX_2)} Y = f \nabla_{X_1} Y + g \nabla_{X_2} Y,$$

3. and ∇ obeys the Leibniz rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y.$$

$\nabla_X Y$ is the *covariant derivative* of Y in the direction X.

For a given vector field X, the covariant derivative is a map from the space of vector fields into itself. With respect to a local coordinate system (x, U), we can consider the covariant derivative of the basis vectors along each coordinate direction. This will define a $n \times n$ matrix of components for the resulting vector field:

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k.$$

The Γ^j_{ik} are called the *Christoffel Symbols*. We write $\nabla_{\partial_i} \equiv \nabla_i$. As the following lemma shows, for general vector fields X, Y, the action of a connection is completely determined by its Christoffel symbols [**?**].

Lemma 2.7.1. *Let ∇ be a linear connection on U and let $X, Y \in \mathcal{T}(U)$ be written $X = X^i\partial_i$ and $Y = Y^j\partial_j$ in local coordinates. Then*

$$\nabla_X Y = (XY^k + X^iY^j\Gamma^k_{ij})\partial_k$$

Proof.

$$\begin{aligned}
\nabla_X Y &= \nabla_X(Y^j\partial_j) \\
&= Y^j\nabla_X\partial_j + (XY^j)\partial_j && \text{(Leibniz)} \\
&= Y^j\nabla_{X^i\partial_i}\partial_j + (XY^j)\partial_j \\
&= X^iY^j\nabla_{\partial_i}\partial_j + (XY^j)\partial_j && (C^\infty\text{-linearity in X}) \\
&= X^iY^j\Gamma^k_{ij}\partial_k + (XY^j)\partial_j \\
&= (XY^k + X^iY^j\Gamma^k_{ij})\partial_k
\end{aligned}$$

where in the last line we have simply changed the dummy index. □

In particular, the above lemma tells us

$$\nabla_i Y^j = \partial_i Y^j + \Gamma^j_{ik}Y^k. \tag{2.27}$$

We can use the above expression to derive coordinate transformation properties for the Christoffel symbols. We constructed the Christoffel symbols so that the left hand side of (2.27) transforms as a tensor. Therefore, from equation (2.6), under a coordinate change $x \to x'$ we require

$$\nabla_{i'}Y^{j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\nabla_i Y^j. \tag{2.28}$$

Expanding both sides of (2.28)

$$\begin{aligned}
\frac{\partial}{\partial x^{i'}}Y^{j'} + \Gamma^{j'}_{i'k'}Y^{k'} &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\left(\frac{\partial}{\partial x^i}Y^j + \Gamma^j_{ik}Y^k\right) \\
\frac{\partial x^i}{\partial x^{i'}}\frac{\partial}{\partial x^i}\left(\frac{\partial x^{j'}}{\partial x^j}Y^j\right) + \Gamma^{j'}_{i'k'}\frac{\partial x^{k'}}{\partial x^k}Y^k &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial}{\partial x^i}Y^j + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\Gamma^j_{ik}Y^k \\
\frac{\partial x^i}{\partial x^{i'}}\frac{\partial^2 x^{j'}}{\partial x^i \partial x^j}Y^j + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial}{\partial x^i}Y^j + \Gamma^{j'}_{i'k'}\frac{\partial x^{k'}}{\partial x^k}Y^k &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial}{\partial x^i}Y^j + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\Gamma^j_{ik}Y^k \\
\Gamma^{j'}_{i'k'}\frac{\partial x^{k'}}{\partial x^k}Y^k &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\Gamma^j_{ik}Y^k - \frac{\partial x^i}{\partial x^{i'}}\frac{\partial^2 x^{j'}}{\partial x^i \partial x^j}Y^j \\
\Gamma^{j'}_{i'k'}Y^k &= \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\Gamma^j_{ik}Y^k - \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial^2 x^{j'}}{\partial x^i \partial x^k}Y^k
\end{aligned}$$

Where in the last term the dummy index has been changed $j \to k$. The law must hold for all Y, thus the transformation law for Christoffel symbols is

$$\Gamma^{j'}_{i'k'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\Gamma^j_{ik} - \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial^2 x^{j'}}{\partial x^i \partial x^k}. \tag{2.29}$$

Note that (2.29) is not the tensor transformation law. This is because the Christoffel symbols themselves are not tensors.

The linear connection ∇ can be extended to act over tensor fields so that the resultant connection (also denoted ∇) [**?**]:

1. agrees with the linear connection over the tangent space,
2. gives the ordinary derivative on functions:
$$\nabla_X f = Xf,$$
3. obeys the Leibniz rule when acting on tensor products:
$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S),$$
4. and commutes with contraction
$$\nabla(T^{i_1,\ldots k,\ldots,i_r}_{j_1,\ldots,k,\ldots,j_s}) = (\nabla T)^{i_1,\ldots k,\ldots,i_r}_{j_1,\ldots,k,\ldots,j_s}.$$

Lee [**?**] offers the following Lemma for finding the covariant derivative of a tensor field:

Lemma 2.7.2. *Let ∇ be a linear connection. The components of the covariant derivative of an (r,s)-tensor field T with respect to a coordinate system are given by*

$$\nabla_k T^{i_1\ldots i_r}_{j_1\ldots j_s} = \partial_k T^{i_1\ldots i_r}_{j_1\ldots j_s} + \sum_{l=1}^{r} T^{i_1\ldots p\ldots i_r}_{j_1\ldots j_s}\Gamma^{i_l}_{kp} - \sum_{l=1}^{s} T^{i_1\ldots i_r}_{j_1\ldots p\ldots j_s}\Gamma^{p}_{kj_l} \tag{2.30}$$

A linear connection is *metric compatible* if for all vector fields X, Y and Z the following product rule holds:

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle .$$

Lemma 2.7.3. *A connection ∇ is metric compatible if and only if $\nabla g \equiv 0$*

Proof. Suppose ∇ is metric compatible.

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{2.31}$$

Expanding the left hand side,

$$\begin{aligned} \nabla_X \langle Y, Z\rangle &= \nabla_X(g_{mn}Y^m Z^n) \\ &= (\nabla_X g_{mn})Y^m Z^n + g_{mn}(\nabla_X Y^m)Z^n + g_{mn}Y^m(\nabla_X Z^n) \\ &= (\nabla_X g_{mn})Y^m Z^n + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle . \end{aligned}$$

Therefore,

$$(\nabla_X g_{mn})Y^m Z^n = 0.$$

This is true for all X, Y, Z, so

$$\nabla g = 0$$

□

Therefore, for a metric compatible connection

$$\begin{aligned}\nabla_i(Y_k) &= \nabla_i(g_{jk}Y^j) \\ &= (\nabla_i g_{jk})Y^j + (\nabla_i Y^j)g_{jk} \\ &= (\nabla_i Y^j)g_{jk}.\end{aligned}$$

2.8 Derivation of the Levi-Civita Connection

Consider the equation

$$\mathcal{L}_X g = \frac{\partial}{\partial x^j}(X_k) + \frac{\partial}{\partial x^k}(X_j) - X_l g^{il}(\frac{\partial}{\partial x^j}g_{ik} + \frac{\partial}{\partial x^k}g_{ji} - \frac{\partial}{\partial x^i}g_{jk}) = 0. \tag{2.32}$$

Define as Christoffel symbols

$$\Gamma^l_{jk} = \frac{1}{2}g^{il}(\frac{\partial}{\partial x^j}g_{ik} + \frac{\partial}{\partial x^k}g_{ji} - \frac{\partial}{\partial x^i}g_{jk}) \tag{2.33}$$

It is possible (but tedious) to check that these transform as Christoffel symbols should. By the symmetry of the metric tensor, these Christoffel symbols Γ^l_{jk} are symmetric in j and k. A connection has this property if and only if it is *torsion free*, that is, ∇ satisfies

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{2.34}$$

With these Christoffel symbols, (2.32) becomes:

$$\partial_j(X_k) + \partial_k(X_j) - 2X_i\Gamma^i_{jk} = 0. \tag{2.35}$$

Now define a connection ∇ by:

$$\nabla_j X_k = \partial_j X_k - \Gamma^l_{jk}X_l. \tag{2.36}$$

Which gives

$$\mathcal{L}_X g_{jk} = 2\nabla_{(j}X_{k)} \tag{2.37}$$

This connection is metric compatible: from (2.30),

$$\begin{aligned}\nabla_k g_{ij} &= \partial_k g_{ij} - \Gamma^l_{ki}g_{lj} - \Gamma^l_{kj}g_{il} \\ &= \partial_k g_{ij} - \tfrac{1}{2}g^{ml}g_{lj}(\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ki}) - \tfrac{1}{2}g^{lm}g_{il}(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{kj}) \\ &= \partial_k g_{ij} - \tfrac{1}{2}(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ki}) - \tfrac{1}{2}(\partial_k g_{ji} + \partial_j g_{ki} - \partial_i g_{kj}) \\ &= 0\end{aligned} \tag{2.38}$$

Theorem 2.8.1. ***Fundamental Lemma of Riemannian Geometry*** *Let (M, g) be a Riemannian Manifold. There exists a unique linear connection ∇ on M that is compatible and torsion free.*

Proof. We have already shown existence by construction, so all that is left is to show uniqueness. Following the proof in [**?**], we do this by deriving a formula for ∇. By metric compatibility (2.31) we have the following three equations:

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \end{aligned}$$

Using the torsion free condition (2.34) in each of the above we get

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X,Z]\rangle \\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y,X]\rangle \\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z,Y]\rangle . \end{aligned}$$

Adding the first two equations and subtracting the third

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y,[X,Z]\rangle + \langle Z,[Y,X]\rangle - \langle X,[Z,Y]\rangle .$$

Now we can solve for $\langle \nabla_X Y, Z\rangle$,

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,X]\rangle + \langle X,[Z,Y]\rangle\right).$$

Noticing that the right hand side of the above equation does not depend on the connection, it follows that if ∇^1 and ∇^2 are two connections

$$\left\langle \nabla^1_X Y, Z\right\rangle = \left\langle \nabla^2_X Y, Z\right\rangle$$

for all X, Y and Z. This can only happen if $\nabla^1 = \nabla^2$. □

This connection is called the *metric connection* or the *Levi-Civita connection.*

2.9 Parallel Transport, geodesics and the Riemann curvature tensor

Consider a closed curve $\mathcal{C}(t) = (x^1(t), ...x^n(t))$ which passes through a point $p \in M$ at t_p, i.e. $\mathcal{C}(t_p) = p$. The tangent vector to $\mathcal{C}$ at p is given by $X_p = \frac{dx^i}{dt}\,|_{t_p}\,\partial_i = X^i\partial_i$. A vector V^j

undergoes *parallel transport* along $\mathcal{C}$ if at all points on $\mathcal{C}$

$$X^i \nabla_i V^j = 0. \tag{2.39}$$

This generalises as expected to an arbitrary tensor.

Imagine you're standing at the north pole and you walk south until you reach the equator. Then you sidestep 90 degrees around the equator before walking backwards until you return to the north pole. Although on your travels you maintained the direction you were facing, you now appear to be facing at right angles to where you started. If you consider different paths for our example, such as traversing just 45 degrees around the equator instead of 90, it is clear that you would be facing at 45 degrees to your starting position. Thus the difference in angle is path dependent. This is because of the curvature of the earth- if you traversed a closed loop on flat space you would end up facing the same way regardless of path.

By the definition of parallel transport, the covariant derivative of a tensor in a direction along which it is parallelly transported is zero. Therefore, the covariant derivative of a tensor in an arbitrary direction is a measure of the the change in the tensor compared to if it had been parallelly transported and as such is a measure of curvature. The commutator of two covariant derivatives ∇_i and ∇_j is the difference between parallelly transporting the tensor along the x^i then x^j compared to x^j then x^i.

Consider this action on a dual vector field ω_k and a smooth function f.

$$\begin{aligned} \nabla_i\nabla_j(f\omega_k) &= \nabla_i(\omega_k\nabla_j f + f\nabla_j\omega_k) &= \nabla_i\omega_k\nabla_j f + \omega_k\nabla_i\nabla_j f + \nabla_i f\nabla_j\omega_k + f\nabla_i\nabla_j\omega_k \\ \nabla_j\nabla_i(f\omega_k) &= \nabla_j(\omega_k\nabla_i f + f\nabla_i\omega_k) &= \nabla_j\omega_k\nabla_i f + \omega_k\nabla_j\nabla_i f + \nabla_j f\nabla_i\omega_k + f\nabla_i\nabla_j\omega_k \end{aligned} \tag{2.40}$$

Subtract the second equation from the first, and using the torsion free property of the connection find that the commutator gives:

$$\nabla_{[i}\nabla_{j]}(f\omega_k) = f(\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_k$$

Now consider a second 1-form ω'_k such that $\omega'_k = \omega_k$ at a point p. Then we can find a set of 1-forms μ^α_k and smooth functions f_α with $f_\alpha(p) = 0$ such that $\omega'_k - \omega_k = \sum_\alpha f_\alpha \mu^\alpha_k$. Therefore,

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)(\omega'_k - \omega_k) \mid_p = \sum_\alpha f_\alpha(\nabla_i\nabla_j - \nabla_j\nabla_i)\mu^\alpha_k \mid_p = 0$$

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\omega'_k = (\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_k$$

. Therefore, $(\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_k$ can only depend on $\omega_k(p)$ and $\nabla_i\nabla_j - \nabla_j\nabla_i$ is a $(1,3)$ tensor. This defines the *Riemann Curvature Tensor*:

$$R^m_{ijk}\omega_m = (\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_k \tag{2.41}$$

As usual we can lower the m index on the Riemann tensor using the metric. Then it can be seen that the following symmetry properties hold [**?**]:

1. $R_{ijkm} = -R_{jikm}$ (skew in first two indices)
2. $R_{ijkm} = -R_{ijmk}$ (skew in last two indices)
3. $R_{m[kij]} = 0$ (cyclic property)

Some manipulation using properties 1 to 3 will give you another useful symmetry [**?**]:

4. $R_{mkij} = R_{ijmk}$ (symmetric under index pair interchange).

It can be verified (by considering Riemann normal coordinates at an arbitrary point) that the Riemann Tensor has the following property:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \tag{2.42}$$

These relations are known as the *Bianchi identities* and are very important.

The Riemann tensor gives rise to two additional important tensors. The first is the *Ricci curvature tensor* which is defined to be contraction over the first and third components of the Riemann tensor

$$R_{kj} = R^i_{kij}. \tag{2.43}$$

By the symmetry property, of the Riemann curvature tensor, it is clear that the Ricci tensor is symmetric. The second is the *scalar curvature*, which is the trace of the Ricci curvature:

$$R = g^{kj}R_{kj} = R^j_j. \tag{2.44}$$

For an n-manifold, we may at first naively suppose that the Riemann tensor has n^4 independent components. However, consideration of the symmetry properties shows that not all of these are independent. As R_{mjik} is skew in the first two and last two components, this gives us $\binom{n}{2}$ independent components for each pair. The cyclic property imposes $\binom{n}{3}$ constraints. Thus, the number of independent components of the Riemann tensor is

$$\binom{n}{2}^2 - n\binom{n}{3} = \frac{n^2(n^2-1)}{12}. \tag{2.45}$$

For a surface $n = 2$, and there is one independent component. By inspection we see that $(g_{mi}g_{kj} - g_{mj}g_{ki})$ satisfies the symmetry properties, therefore for a surface the Riemann curvature tensor is

$$R_{mkij} = K(g_{mi}g_{kj} - g_{mj}g_{ki}) \tag{2.46}$$

where K is a scalar function. If we calculate the Ricci tensor we get

$$R_{kj} = Kg^{mi}(g_{mi}g_{kj} - g_{mj}g_{ki}) = Kg_{kj}, \tag{2.47}$$

and then the scalar curvature is

$$R = g^{kj}R_{kj} = Kg^{kj}g_{kj} = 2K. \tag{2.48}$$

The function K is called the *Gaussian curvature*. Note that for manifolds of dimension $n > 2$, the Gaussian curvature can still be defined, but is not necessarily scalar. Notice that

$$2\nabla_{[i}F_{j]k} = \nabla_i\nabla_j X_k - \nabla_j\nabla_i X_k, \tag{2.49}$$

therefore, by comparing (2.49) with (**??**) and (2.41), we see that the Riemann tensor can also be written

$$R^m_{kij}X_m = (\nabla_i\nabla_j - \nabla_j\nabla_i)X_k.$$

2.10 Geodesics

A *geodesic* is a path that parallel transports its own tangent vector. Expanding (2.39) gives:

$$\begin{aligned} \tfrac{dx^i}{dt}\nabla_i\tfrac{dx^j}{dt} &= 0 \\ \tfrac{dx^i}{dt}\left(\partial_i\tfrac{dx^j}{dt} + \Gamma^j_{ik}\tfrac{dx^k}{dt}\right) &= 0 \\ \tfrac{d^2x^j}{dt^2} + \Gamma^j_{ik}\tfrac{dx^i}{dt}\tfrac{dx^k}{dt} &= 0 \end{aligned} \tag{2.50}$$

The last equation above is called the *geodesic equation.* In flat space, where we can choose coordinates such that $\Gamma^j_{ik} = 0$, the geodesic equation is simply $\frac{d^2x^j}{dt^2} = 0$, the equation for a straight line. In curved space, geodesics can be thought of as representing the shortest distance between points.

In flat space we know that parallel straight lines never meet, but obviously this is not true for geodisics in curved space. To explore this further consider a 1-parameter family of geodesics $\gamma_s(t)$, $s \in \mathbb{R}$, that do not cross. These define a smooth two dimensional surface on which we can choose the coordinates to be s and t so that the points on the surface are $x^j(s,t)$. The vector fields $T^j = \frac{\partial x^j}{\partial t}$ and $S^j = \frac{\partial x^j}{\partial s}$ represent the the tangent vectors to the geodesics and the deviation vectors respectively. The deviation vectors can be interpreted as pointing from one geodesic to the next. We want to explore how these change, so we define $V^i = (\nabla_T S)^i = T^j\nabla_j S^i$, which we interpret as the relative velocity of the geodesics, and $A^i = (\nabla_T V)^i = T^j\nabla_j V^i$ the relative

acceleraton between geodesics. It can be shown (see, eg. [**?**]) that the relative acceleration between geodesics is proportional to the curvature:

$$A^i = R^i_{jkl} T^j T^k S^l \tag{2.51}$$

This is known as the *geodesic deviation equation.*

2.11 Euclidean and Hyperbolic space

Consider the following family of metrics parameterised by p:

$$g = \frac{1}{(1 + p((x^1)^2 + (x^2)^2))^2}((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2).$$

When $p = 1$, this is the (now familiar) metric for the sphere S^2. When $p = 0$, g is the metric for Euclidean space $\mathbb{R}^2$, and when $p = -1$, and we confine our attention to the unit disc $\|x\| < 1$, we have *hyperbolic space* $\mathbb{H}^2$. These three spaces are are the 2-dimensional versions of the "model spaces" for Riemannian geometry.

The Riemann curvature for each of these spaces satisfies (2.46). As done for the sphere in the previous section, we find the Gaussian curvature by calculating one component of the tensor using the definition (2.41):

$$R^1_{212} = \frac{4p}{(1 + p((x^1)^2 + (x^2)^2))^2}.$$

Using (2.46), we find the Gaussian curvature $K = 2p$. Thus, each of these spaces have constant Gaussian curvature, and therefore are maximally symmetric.

Chapter 3

General Relativity

Einstein's theory of General Relativity, published in two papers in 1915 and 1916, revolutionised physics. It replaced Newton's theory of gravitation with a theory roughly based on Riemannian manifolds, where the classical idea of a gravitational force acting at a distance was replaced by the concept of curvature of space-time. We say roughly because now the metric is no longer positive definite (see §3.1 and §3.4 below). In general relativity, space-time is viewed as a 4-dimensional manifold, three dimensions corresponding to space and one to time. We will adopt the common practice of denoting local coordinates by x^μ, $\mu = 0, 1, 2, 3$, with $\mu = 0$ corresponding to the time dimension, and $\mu = 1, 2, 3$ the spatial dimensions. Sometimes it is desirable to consider the spatial dimensions separately to the temporal dimension. In these situations, the spatial dimensions are given Latin indices, x^i, to indicate running from 1 to 3.

3.1 Pseudo-Riemannian Metrics

A *pseudo-Riemannian metric* on a manifold M is a symmetric (0,2)-tensor field g that is *non-degenerate* at every point $p \in M$. Nondegenerate means that if $X \in T_pM$,

$$g(X, Y) = 0 \; \forall \; Y \in T_pM \;\Leftrightarrow\; X = 0.$$

Compare this with the definition of the Riemannian manifold in §2.3.4. For a pseudo-Riemannian metric, the positive-definite criterion of the Riemannian metric has been relaxed. Indeed, the Riemannian metrics are a subset of the pseudo-Riemannian metrics. The *index*, r, of the metric g is the number of negative eigenvalues of the matrix $g_{\mu\nu}$ representing g. If $r = 1$, the metric is called *Lorentzian*. In general relativity the metrics are Lorentzian. The negative eigenvalue corresponds to the time dimension, and the three positive eigenvalues correspond to the spatial

dimensions.

As an alternative notation, physicists define the *line element*

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

which is used interchangably with the metric. This defines an invariant "distance" between two close events with coordinates x^μ and $x^\mu + dx^\mu$. The fact that this distance is invariant means that it is absolute in a physical sense, all observers will find the same value for this quantity regardless of their respective motions and the coordinates they are using to measure it. This is known as the *principle of equivalence* or *coordinate invariance*. Because of the negative eigenvalue in the metric, the distance ds^2 is not positive definite. From the metric one can distinguish events related to each other in three different ways:

$$\begin{aligned} ds^2 &> 0 \text{ Spacelike separation} \\ ds^2 &< 0 \text{ Timelike separation} \\ ds^2 &= 0 \text{ Null separation} \end{aligned}$$

Many of the properties of Riemannian manifolds carry over to pseudo-Riemannian manifolds. In particular, the fundamental lemma of Riemannian geometry (Theorem 2.8.1) holds for pseudo-Riemannian geometry. Therefore, we still have the Levi-Civita connection and the Riemannian Curvature.

3.2 Einstein's Field Equations

The metric on space time is governed by the *Einstein Field Equations*,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.1}$$

The left hand side of this equation represents the geometry of space-time, and is defined as the *Einstein Tensor* $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor and R the scalar curvature. The right hand side represents the distribution of mass-energy through *stress-energy tensor* $T_{\mu\nu}$, a symmetric 2-tensor which describes the density, momentum and stress at each point in space-time and *Newton's constant of gravitation* ($G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$).

The equation (3.1) is actually a set of 10 coupled, non-linear, partial differential equations. There is no systematic way to solve such a system of equations, and as such very few analytic solutions exist.

3.3 Relativity in 1+1 dimensions

As a simple example we consider relativity in 1 spatial and 1 time dimension, with a spacetime metric g_{ij} with i, j taking the values 0 or 1. Without loss of generality we can assume that the metric tensor is diagonal, and thus the inverse metric components are $g^{00} = 1/g_{00}$ and $g^{11} = 1/g_{11}$. The Christoffel symbols in this case are

$$\begin{array}{ll} \Gamma^0_{00} = \frac{\partial_0 g_{00}}{2g_{00}} & \Gamma^1_{11} = \frac{\partial_1 g_{11}}{2g_{11}} \\ \Gamma^0_{10} = \frac{\partial_1 g_{00}}{2g_{00}} & \Gamma^1_{10} = \frac{\partial_0 g_{11}}{2g_{11}} \\ \Gamma^0_{11} = -\frac{\partial_0 g_{11}}{2g_{00}} & \Gamma^1_{00} = -\frac{\partial_1 g_{00}}{2g_{11}}. \end{array} \tag{3.2}$$

The Riemann tensor has $2^2(2^2-1)/12 = 1$ independent component, which can be written as

$$R^1_{010} = \frac{-2g_{11}(\partial_1^2 g_{00} + \partial_0^2 g_{11}) + \partial_1 g_{00}\partial_1 g_{11} + (\partial_0 g_{11})^2}{4g_{11}^2} + \frac{\partial_0(\partial_0 g_{11}(\partial_0 g_{00} - \partial_1 g_{00})}{4g_{00}g_{11}}. \tag{3.3}$$

By the symmetries of the Riemann Tensor, the other non zero components are

$$R^1_{001} = -R^1_{010},\ R^0_{101} = g^{00}g_{11}R^1_{010},\ R^0_{110} = -g^{00}g_{11}R^1_{010}. \tag{3.4}$$

The Ricci tensor is given by

$$R_{00} = R^1_{010},\ R_{11} = -g^{00}g_{11}R^1_{010},\ R_{10} = 0 = R_{01}. \tag{3.5}$$

We can now calculate the left hand side of 3.1.

$$\begin{array}{lrcl} (i,j) = (0,0), & R^1_{010} - R^1_{010} & = & 0 \\ (i,j) = (1,1), & -g^{00}g_{11}R^1_{010} + g^{00}g_{11}R^1_{010} & = & 0. \end{array} \tag{3.6}$$

This tells us that in 1+1 dimensions, the stress energy tensor T_{ij} must be identically zero, and that therefore there is no gravity in 1+1 dimensions. Note that we could have attained this result directly from the symmetries, however the calculation of the the Christoffel symbols and the Riemann and Ricci tensors may come in useful later.

3.4 Minkowski Space

The *Minkowski metric* is the model metric for Lorentz spaces, in the same way $\mathbb{R}^n$ with metric δ_{ij} is the model Riemannian space. In four dimensions, the Minkowski metric corresponds to flat space-time, is denoted by η, and is given by

$$\eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{3.7}$$

include Minkowski in other coordinates, show that $T_{ij} = 0$

3.5 Black Holes

3.5.1 The Schwarzschild Solution

The Schwarzschild metric corresponds to the space-time in a vacuum around a spherically symmetric massive body, such as the Earth, a star, or a black hole. The metric is given in spherical coordinates $\{t, r, \theta, \phi\}$ by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.8}$$

where m is the mass of the body and $r > 2m$.

3.5.2 The Reissner-Nordstrom Solution

Outside a spherically symmetric charged body the metric is:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.9}$$

This is called the *Reissner-Nordstrom* solution. *Birkhoff's theorem*, 1923, states that these are the only allowable solutions for a spherically symmetric space-time, and in particular there are no time dependent solutions. This is not to say that the source must be static, just that it must be spherically symmetric. Thus, there is no spherically symmetric gravitational radiation.

include Riemann etc tensors for these metrics.

- Kerr metric
- apparent and event horizons (EH= boundary where photons can escape to infinity (nonlocal-whatever that means), AH= smooth closed surface of zero null expansion $\nabla^\mu k_\mu = 0$ (local)) (if an AH exists, it cannot be outside an EH)

3.6 Linearised Gravity

Consider flat Minkowski space plus a small perturbation (corresponding to a weak gravitational field). This is *linearised gravity* and the metric can be written:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1 \tag{3.10}$$

By "small" we mean that in our analysis we only need to consider terms linear in $h_{\mu\nu}$, and we can raise and lower indices using the background flat metric $\eta_{\alpha\beta}$ and its inverse $\eta^{\alpha\beta}$. This leads to the inverse metric:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.11}$$

We can calculate the Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\nu}\left(\partial_\beta h_{\gamma\nu} + \partial_\gamma h_{\beta\nu} - \partial_\nu h_{\gamma\beta}\right), \tag{3.12}$$

and the Riemann Tensor to first order in $h_{\alpha\beta}$

$$\begin{array}{rcl} R^{\alpha}_{\beta\gamma\delta} & = & \frac{1}{2}\eta^{\alpha\nu}(\partial_\gamma\partial_\delta h_{\beta\nu} - \partial_\gamma\partial_\nu h_{\beta\delta} + \partial_\nu\partial_\beta h_{\gamma\delta} - \partial_\delta\partial_\beta h_{\gamma\nu}) \\ & = & \eta^{\alpha\nu}(\partial_\delta\partial_{[\gamma}h_{\beta]\nu} + \partial_\nu\partial_{[\beta}h_{\gamma]\delta}) \end{array} \tag{3.13}$$

Contraction yields the Ricci Tensor,

$$R_{\alpha\beta} = \partial^\gamma\partial_{(\alpha}h_{\beta)\gamma} - \frac{1}{2}\partial_\alpha\partial_\beta h - \frac{1}{2}\Box h_{\alpha\beta} \tag{3.14}$$

where $h = h^{\alpha}_{\alpha}$ and $\Box = \partial^\gamma\partial_\gamma$ is the flat space D'Alembertian. The Ricci scalar is

$$R = \partial^\gamma\partial^\nu h_{\gamma\nu} - \Box h \tag{3.15}$$

Therefore the Einstein tensor becomes

$$\begin{array}{rcl} G_{\alpha\beta} & = & \partial^\gamma\partial_{(\alpha}h_{\beta)\gamma} - \frac{1}{2}\Box h_{\alpha\beta} - \frac{1}{2}\partial_\alpha\partial_\beta h - \frac{1}{2}\eta_{\alpha\beta}\left(\partial^\gamma\partial^\nu h_{\gamma\nu} - \Box h\right) \\ & = & -\frac{1}{2}\Box\bar{h}_{\alpha\beta} + \partial^\gamma\partial_{(\alpha}\bar{h}_{\beta)\gamma} - \frac{1}{2}\eta_{\alpha\beta}\partial^\gamma\partial^\nu\bar{h}_{\gamma\nu} \end{array} \tag{3.16}$$

Where in the final line $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$ is the *trace reversed* metric perturbation, which has trace $\bar{h} = -h$, and we have assumed we are working in $n = 4$ dimensions.

3.6.1 Gauge Transformations

There may be many coordinate systems in which the spacetime can be expressed as a flat background spacetime plus a small perturbation. To see this, let us consider our background spacetime manifold M_b equipped with the Minkowski metric $\eta_{\mu\nu}$ as separate to the the physical spacetime M_p with metric $g_{\mu\nu}$, and consider a diffeomorphism between them $\phi : M_b \to M_p$. Then we can define the perturbation as $h_{\mu\nu} = (\phi * g)_{\mu\nu} - \eta_{\mu\nu}$. If the gravitational fields are weak, then there will exist a ϕ such that $\|h_{\mu\nu}\| << 1$.

Now consider a vector field ξ^μ on M_b. As we saw in §2.5, this generates a one-parameter family of diffeomorphisms $\psi_\epsilon : M_b \to M_b$. If the parameter ϵ is small, and ϕ is such that h is small,

then the resultant perturbation by $(\phi \circ \psi_\epsilon)$ will also be small. This family of diffeomorphisms parameterised by ϵ defines a family of perturbations:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= [(\phi \circ \psi_\epsilon)^* g]_{\mu\nu} - \eta_{\mu\nu} \\ &= [\psi^*_\epsilon(\phi^* g)]_{\mu\nu} - \eta_{\mu\nu} \\ &= \psi^*_\epsilon(h+\eta)_{\mu\nu} - \eta_{\mu\nu} \\ &= \psi^*_\epsilon(h_{\mu\nu}) + \psi^*_\epsilon(\eta_{\mu\nu}) - \eta_{\mu\nu} \\ &= \psi^*_\epsilon(h_{\mu\nu}) + \epsilon \tfrac{\psi^*_\epsilon(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon} \\ &= h_{\mu\nu} + \epsilon \mathcal{L}_\xi \eta_{\mu\nu} \end{aligned} \tag{3.17}$$

We arrive at the final equation above by assuming ϵ is small and taking the first order approximation for ψ^*_ϵ for the first term, and by recalling the definition of the Lie derivative, equation 2.18, for the second term. We then recall equation 2.37 and that the metric connection becomes partial derivatives in our flat space background, to arrive at:

$$h^{(\epsilon)}_{\mu\nu} = h_{\mu\nu} + 2\epsilon \partial_{(\mu}\xi_{\nu)}. \tag{3.18}$$

Substituting this into equation (3.13) leaves the linearised Riemann Tensor, and hence the curvature of the spacetime, unchanged, demonstrating the coordinate freedom in linearised gravity. Such transformations are called *gauge transformations* where the choice of ϵ gives the gauge.

The trace reversed metric perturbation transforms as

$$\bar{h}^{(\epsilon)}_{\alpha\beta} = \bar{h}_{\alpha\beta} + 2\epsilon \partial_{(\alpha}\xi_{\beta)} - \eta_{\alpha\beta}\partial^\mu \xi_\mu$$

Taking the divergence of this gives

$$\begin{aligned} \partial^\beta \bar{h}^{(\epsilon)}_{\alpha\beta} &= \partial^\beta \bar{h}_{\alpha\beta} + \epsilon \partial^\beta \partial_\alpha \xi_\beta + \epsilon \partial^\beta \partial_\beta \xi_\alpha - \epsilon \partial_\alpha \partial^\mu \xi_\mu \\ &= \partial^\beta \bar{h}_{\alpha\beta} + \epsilon \partial^\beta \partial_\beta \xi_\alpha \end{aligned} \tag{3.19}$$

Thus, solving the equation $\partial^\beta \bar{h}_{\alpha\beta} = -\epsilon \partial^\beta \partial_\beta \xi_\alpha$ (which is always possible), gives a gauge in which

$$\partial^\beta \bar{h}^{(\epsilon)}_{\alpha\beta} = 0 \tag{3.20}$$

This gauge is called the *Lorenz gauge* or the *harmonic gauge*, and is clearly not unique.

3.6.2 Gravitational Plane Wave Solutions

Now let us return to the Einstein Equation (3.1).

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

Let us take the vacuum equations $T_{\alpha\beta} = 0$. Substituting (3.20) into the equation for the linearised Einstein tensor (3.16) gives an equation very similar to the electromagnetic wave equations:

$$\Box \bar{h}_{\alpha\beta} = 0 \tag{3.21}$$

This suggests we may attempt a monchromatic plane wave solution of the form $\bar{h}_{\alpha\beta} = A_{\alpha\beta} e^{ik_\sigma x^\sigma}$ where $A_{\alpha\beta}$ is a constant, symmetric $(0,2)$-tensor and k^σ is a 4-vector. To check if this gives a solution we plug in:

$$\begin{aligned} \Box A_{\alpha\beta} \exp^{ik_\sigma x^\sigma} &= \eta^{\mu\nu}\partial_\mu\partial_\nu(A_{\alpha\beta}e^{ik_\sigma x^\sigma}) \\ &= \eta^{\mu\nu}\partial_\mu(ik_\nu A_{\alpha\beta}e^{ik_\sigma x^\sigma}) \\ &= -\eta^{\mu\nu}(k_\mu k_\nu A_{\alpha\beta}e^{ik_\sigma x^\sigma}) \\ &= -(k^\nu k_\nu A_{\alpha\beta}e^{ik_\sigma x^\sigma}) \end{aligned} \tag{3.22}$$

which will be 0 if $k^\nu k_\nu = 0$, that is, the wave vector is null. This implies that these *gravitational waves* propagate at the speed of light. The timelike component of k is the frequency of the wave and usually denoted ω, so that the four vector $k^\nu = (\omega, \mathbf{k})$.

Using the Lorenz gauge condition (3.20) we obtain

$$0 = \partial_\beta(A^{\alpha\beta}e^{ik_\sigma x^\sigma}) = ik_\beta A^{\alpha\beta}e^{ik_\sigma x^\sigma}$$

which is true only if

$$k_\beta A^{\alpha\beta} = 0 \tag{3.23}$$

We say k_β is *orthogonal* or *transverse* to $A^{\alpha\beta}$. We can use our remaining gauge freedom (see [**?**]) to choose a gauge in which the trace

$$A^\alpha_\alpha = 0 \tag{3.24}$$

and for any constant timelike unit vector U^β we have

$$A_{\alpha\beta}U^\beta = 0 \tag{3.25}$$

These conditions together determine the *transverse traceless (TT)* gauge.

Now let us set the vector $U^\beta = \delta^\beta_0$ as the time basis vector, which is always possible under a Lorentz transformation, and orient the coordinate axes so that the wave propagates along the x^3 direction $k^\beta \to (\omega, 0, 0, \omega)$. Then the TT conditions require that the tensor $A_{\alpha\beta}$ is of the following form

$$A^{TT}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.26}$$

Therefore the wave is characterised by two components A_{xx} and A_{xy} which we call h_+ and $h_\times$ respectively.

To understand the action of these waves, consider a collection of slowly moving test particles. Let these particles have four velocities described by a vector field $U^\mu(x)$ and separation vectors $S^\mu(x)$. We invoke the geodesic deviation equation (2.51):

$$U^\alpha \nabla_\alpha U^\beta \nabla_\beta S^\mu = A^\mu = R^\mu_{\alpha\beta\nu} U^\alpha U^\beta S^\nu$$

For our slowly moving test particles we have $\tau = x^0 = t$, so the left hand side of this equation becomes $\frac{\partial^2}{\partial t^2} S^\mu$. As we are only interested in results up to first order, we may write $U^\mu = (1, 0, 0, 0)$ which means for the right hand side we only need to calculate $R^\mu_{00\nu}$. From the linearised Riemann tensor (3.13) and the transverse traceless conditions above we see that $R^\mu_{00\nu} = \frac{1}{2}\frac{\partial^2}{\partial t^2} h^{TT\mu}_\nu$. The geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\nu \frac{\partial^2}{\partial t^2} h^{TT\mu}_\nu \tag{3.27}$$

so from the structure of h^{TT} it is clear that only S^1 and S^2 are affected- that is that the test particles are only disturbed in directions perpendicular to the wave that is travelling in the x^3 direction.

Solving (3.27) for the case $h_\times = 0$ yields

$$\begin{array}{rcl} S^1 & = & (1 + \frac{1}{2} h_+ \exp(i k_\sigma x^\sigma)) S^1_{init} \\ S^2 & = & (1 - \frac{1}{2} h_+ \exp(i k_\sigma x^\sigma)) S^2_{init} \end{array} \tag{3.28}$$

That is, particles separated in the x^1 direction will oscillate in the x^1 direction, and particles x^2 direction will oscillate in the x^2 direction. If we had a ring of particles, the ring would oscillate in the shape of an + and so this is called the + polarisation mode. Similarly, if we consider $h_+ = 0$ we find

$$\begin{array}{rcl} S^1 & = & S^1_{init} + \frac{1}{2} h_\times \exp(i k_\sigma x^\sigma) S^2_{init} \\ S^2 & = & S^2_{init} + \frac{1}{2} h_\times \exp(i k_\sigma x^\sigma) S^1_{init} \end{array} \tag{3.29}$$

A ring of particles woulds oscillate in a $\times$. This is called the $\times$-polarisation mode. We can define polarisation tensors $e^+_{\alpha\beta}$ and $e^\times_{\alpha\beta}$, so that a general gravitational wave in the TT gauge can be written

$$h^{TT}_{\alpha\beta} = h_+ e^+_{\alpha\beta} + h_\times e^+_{\alpha\beta}. \tag{3.30}$$

Generation of Gravitational Waves

To discuss the generation of gravitational waves we require the full Einstein Field Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.31}$$

As we are no longer looking at vacuum solutions, $T_{\mu\nu}$ does not vanish and we cannot assume the transverse traceless gauge. However, we can still use the trace reversed metric perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \tag{3.32}$$

and the Lorenz gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$ so we still obtain the Einstein tensor in the form

$$G_{\mu\nu} = \frac{1}{2}\Box\bar{h}_{\mu\nu}, \tag{3.33}$$

and the Einstein equations become a wave equation for each component

$$\Box\bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu}. \tag{3.34}$$

This can be solved by means of a Green's function satisfying

$$\Box G(x^\alpha) = \delta^{(4)}(x^\alpha), \tag{3.35}$$

so that

$$\bar{h}_{\mu\nu}(x^\alpha) = 16\pi \int G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha) d^4\mathbf{y}. \tag{3.36}$$

As can be found in any text on partial differential equations, the Green's function required is of the form

$$G(x^\sigma) = \frac{-1}{4\pi r}\delta(t-r)\Theta(t) \tag{3.37}$$

where Θ is the Heaviside function

$$\Theta(t) = \begin{cases} 1 & \text{if} t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.38}$$

The argument of the delta function is sometimes denoted $t_r = t - r$ for "retarded time" to indicate that the gravitational wave at a source is the sum of contributions from sources on its past light cone.

Suppose the source is slow moving and isolated, and at a distance r such that the approximation $|\mathbf{x} - \mathbf{y}| \approx r$ is valid.

- derivation of the Green's function
- quadrupole moment
- binary system

- gravitational wave frequency = twice orbital frequency
- https://www.icts.res.in/media/uploads/Talk/Document/boyle_icts_lec2.pdf
- http://www.tat.physik.uni-tuebingen.de/~kokkotas/Teaching/NS.BH.GW_files/GW_Physics.pdf
- http://webs.um.es/bussons/GW_lecture_KG.pdf
- http://www.ego-gw.it/public/events/vesf/presentations2007/hendry.pdf
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- http://eagle.phys.utk.edu/guidry/astro616/lectures/lecture_ch21.pdf
- http://arxiv.org/pdf/1209.0667.pdf
- http://mathpages.com/rr/s6-08/6-08.htm
- http://arxiv.org/pdf/0903.0338.pdf
- https://www.astro.umd.edu/~miller/teaching/mexico/lecture1.pdf
- Post Newtonian expansion- chirp mass and innermost stable circular orbit

Chapter 4

Formalism of Numerical Relativity

TO ADD:
Form of equations: BSSN, Generalised Harmonic
Singularity Treatment: Moving punctures, Excision
Numerical Methods: Finite differencing, spectral

4.1 Introduction

The Einstein field equations are written in such a way that space and time are treated with equal footing. This is very elegant, but in order to study the evolution of a system we must re-write them as a Cauchy problem. The natural way to do this is to view spacetime as a family of 3D spatial hypersurfaces evolving in time. A spacetime in which this is possible is called "globally hyperbolic", and contain no closed timelike curves. Although it is a loss of mathematical generality, we assume that all physically realistic spacetimes have this property.

4.2 Splitting Space and Time

Consider a time-orientable spacetime (M, g), and define on it a global time function $t : M \to \mathbb{R}$ satisfying

$$\|\nabla_\alpha t\| = g^{\alpha\beta}\nabla_\alpha t \nabla_\beta t \equiv -\frac{1}{\alpha^2} < 0. \tag{4.1}$$

Our spacelike hypersurfaces Σ_t are the level sets of this function:

$$\Sigma_t = \{p \in M | t(p) = t\}\,. \tag{4.2}$$

We define the unit normal to the slices in terms of the 1-form $\nabla_\alpha t$

$$n^\alpha \equiv -\alpha g^{\alpha\beta}\nabla_\beta t \tag{4.3}$$

where the negative sign is chosen so that n^α is in the direction of increasing t. By construction, n^α is normalised and timelike,

$$n^\alpha n_\alpha = \alpha^2 g^{\alpha\beta}\nabla_\alpha t \nabla_\beta t = -1, \tag{4.4}$$

so may be thought of as the 4-velocities of observers whose worldlines are always normal to the spacelike hypersurfaces. These are called "Eulerian observers" and have acceleration given by

$$a_\alpha = n^\beta \nabla_\beta n_\alpha. \tag{4.5}$$

Contracting all free indices of an arbitrary tensor with the expression $-n^\alpha n_\beta$ will return the timelike components of the tensor, and so we call this the *normal projection operator*. If the timelike components of a tensor vanish, the tensor is described as *spatial*.

We can now use the normal to construct the spatial metric on the slices $\gamma_{\alpha\beta}$ by calculating the spacetime distance with $g_{\alpha\beta}$ and killing off the timelike components:

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta. \tag{4.6}$$

This metric is purely spatial ($\gamma_{\alpha\beta}n^\alpha = 0$), positive definite, and has inverse

$$\gamma^{\alpha\beta} = g^{\alpha\beta} + n^\alpha n^\beta. \tag{4.7}$$

We construct the *spatial projection operator* to project 4-dimensional tensors into the spatial hypersurfaces by raising one index of the spatial metric

$$\gamma^\alpha_\beta = g^\alpha_\beta + n^\alpha n_\beta = \delta^\alpha_\beta + n^\alpha n_\beta. \tag{4.8}$$

To project a tensor into the spatial surface, each free index must be contracted with the projection operator. The indices on the resultant tensor can be raised or lowered with either $g_{\alpha\beta}$ or $\gamma_{\alpha\beta}$. Further, the projected tensor is purely spatial, which can be seen for an arbitrary vector v^α by contracting the projected vector with the normal n_α

$$\gamma^\alpha_\beta v^\beta = n_\alpha \delta^\alpha_\beta v^\beta + n_\alpha n^\alpha n_\beta v^\beta = n_\beta v^\beta - n_\beta v^\beta = 0 \tag{4.9}$$

Using this and normal projection operator we can now decompose any tensor into its spatial and timelike parts. For example, our arbitrary vector v^α can be decomposed as

$$v^\alpha = \delta^\alpha_\beta v^\beta = (\gamma^\alpha_\beta - n^\alpha n_\beta)v^\beta. \tag{4.10}$$

The covariant derivative on the slice can be induced from the four dimensional covariant derivative by projecting it into Σ. For a general tensor

$$D_\alpha T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = \gamma^\beta_\alpha \gamma^{\mu_1}_{\rho_1} \ldots \gamma^{\mu_n}_{\rho_n} \ldots \gamma^{\sigma_1}_{\nu_1} \ldots \gamma^{\sigma_m}_{\nu_m} \nabla_\beta T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}. \tag{4.11}$$

It is straightforward to show that this is compatible with the 3-metric, that is $D_\alpha \gamma_{\mu\nu} = 0$.

The 3-dimensional Riemann tensor is defined in terms of the 3-dimensional covariant derivative in the usual way,

$$R^\delta_{\gamma\beta\alpha} v_\delta = 2 D_{[\alpha} D_{\beta]} v_\gamma \tag{4.12}$$

for any spatial vector v_γ, with the additional requirement that

$$R^\delta_{\gamma\beta\alpha} n_\delta = 0. \tag{4.13}$$

Thus, it, along with the Ricci tensor and Ricci scalar, are all purely spatial. However, these only tell us about the curvature intrinsic to the 3-dimensional slice Σ_t. We have lost information about curvature in the fourth dimension. This information is contained in the extrinsic curvature.

The *extrinsic curvature* represents how the hypersurfaces are are curved with respect to the 4-dimensional spacetime by measuring how normal vectors differ from point to point. It is defined by projecting gradients of the normal vector into the slice Σ:

$$K_{\alpha\beta} \equiv -\gamma^\mu_\alpha \gamma^\nu_\beta \nabla_\mu n_\nu. \tag{4.14}$$

Clearly this is purely spatial, and it can be shown [**?**] that it is symmetric and can be written as

$$K_{\alpha\beta} = -\frac{1}{2} \mathcal{L}_{\mathbf{n}} \gamma_{\alpha\beta}, \tag{4.15}$$

which makes clear another interpretation of the extrinsic curvature as the rate at which the hypersurface deforms as measured by Eulerian observers.

The relationship between the four dimensional Riemann tensor ${}^{(4)}R^\delta_{\gamma\beta\alpha}$, its 3-dimensional counterpart and the extrinsic curvature is found by taking spatial and mixed projections of ${}^{(4)}R^\delta_{\gamma\beta\alpha}$. The normal projection of all four indices vanishes identically due to symmetries of the Riemann tensor, and these also cause the mixed projection of one index into the spatial direction and three in the normal direction to cancel. Therefore we need to consider the three remaining projections. The completely spatial projection yields *Gauss' equation*,

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\rho_\gamma \gamma^\sigma_\delta {}^{(4)}R_{\mu\nu\rho\sigma} = R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma} K_{\beta\delta} - K_{\alpha\delta} K_{\beta\gamma}. \tag{4.16}$$

Projecting one index in the normal direction and the others spatially results in *Codazzi's equation*,

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\rho_\gamma n^\sigma {}^{(4)}R_{\mu\nu\rho\sigma} = D_\beta K_{\alpha\gamma} - D_\alpha K_{\beta\gamma}. \tag{4.17}$$

Lastly, two spatial and two normal projections gives *Ricci's equation*

$$\mathcal{L}_{\mathbf{n}} K_{\alpha\beta} = n^{\delta} n^{\gamma} \gamma_{\alpha}^{\nu} \gamma_{\beta}^{\rho} {}^{(4)}R_{\delta\rho\gamma\nu} - \frac{1}{\alpha} D_{\alpha} D_{\beta} \alpha - K_{\beta}^{\gamma} K_{\alpha\gamma}. \tag{4.18}$$

For a derivation of these equations see [**?**].

4.3 Projections of the Einstein Field Equations

We take contractions of the Gauss, Codazzi and Ricci equations to express the 4-dimensional Ricci tensor and scalar in terms of their 3-dimensional counterparts and use these to rewrite the Einstein Field Equations in $3+1$ form.

First we contract Gauss' equation 4.16 twice:

$$\begin{aligned} g^{\beta\delta} g^{\alpha\gamma} \gamma_{\alpha}^{\mu} \gamma_{\beta}^{\nu} \gamma_{\gamma}^{\rho} \gamma_{\delta}^{\sigma} {}^{(4)}R_{\mu\nu\rho\sigma} &= g^{\beta\delta} g^{\alpha\gamma} (R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma} K_{\beta\delta} - K_{\alpha\delta} K_{\beta\gamma}) \\ \gamma^{\mu\rho} \gamma^{\nu\sigma} {}^{(4)}R_{\mu\nu\rho\sigma} &= R + K^2 - K_{\alpha\beta} K^{\alpha\beta}, \end{aligned} \tag{4.19}$$

where $K = K_{\alpha}^{\alpha}$ is the trace of the extrinsic curvature. Expanding the left hand side as $(g^{\mu\rho} + n^{\mu} n^{\rho})(g^{\nu\sigma} + n^{\nu} n^{\sigma}) {}^{(4)}R_{\mu\nu\rho\sigma}$ yields

$${}^{(4)}R + 2 n^{\mu} n^{\rho} {}^{(4)}R_{\mu\rho} = R + K^2 - K_{\alpha\beta} K^{\alpha\beta}, \tag{4.20}$$

because the expression $n^{\mu} n^{\rho} n^{\nu} n^{\sigma} {}^{(4)}R_{\mu\nu\rho\sigma}$ vanishes due to the symmetries of the Riemann tensor. We now project the Einstein Equations in the normal direction:

$$n^{\mu} n^{\rho} {}^{(4)}R_{\mu\rho} - \frac{1}{2} n^{\mu} n^{\rho} g_{\mu\rho} {}^{(4)}R = n^{\mu} n^{\rho} {}^{(4)}R_{\mu\rho} + \frac{1}{2} {}^{(4)}R = 8\pi n^{\mu} n^{\rho} T_{\mu\rho}. \tag{4.21}$$

We define the energy density ρ to be the energy density as measured by Eulerian observers:

$$\rho = n_{\mu} n_{\rho} T^{\mu\rho}. \tag{4.22}$$

Then combining 4.20 and 4.21 gives us the *Hamiltonian constraint*

$$R + K^2 - K_{\alpha\beta} K^{\alpha\beta} = 16\pi\rho. \tag{4.23}$$

This expression contains only information about the spatial metric and the extrinsic curvature, and their spatial derivatives. It contains no time derivatives, but imposes conditions on $\gamma_{\alpha\beta}$ and $K_{\alpha\beta}$ that must be satisfied, hence why it is called a constraint. The Codazzi equation 4.17 leads to another constraint.

We proceed by contracting the Codazzi equation once:

$$\begin{aligned} g^{\alpha\gamma}\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\rho}_{\gamma}n^{\sigma\,(4)}R_{\mu\nu\rho\sigma} &= g^{\alpha\gamma}(D_\beta K_{\alpha\gamma} - D_\alpha K_{\beta\gamma}) \\ \gamma^{\mu\rho}\gamma^{\nu}_{\beta}n^{\sigma\,(4)}R_{\mu\nu\rho\sigma} &= D_\beta K - D_\alpha K^{\alpha}_{\beta}. \end{aligned} \tag{4.24}$$

By again substituting $\gamma^{\mu\rho} = g^{\mu\rho} - n^\mu n^\rho$, and eliminating $n^\mu n^\rho n^{\sigma\,(4)}R_{\mu\nu\rho\sigma}$ using symmetries, we obtain

$$\gamma^{\nu}_{\beta}n^{\sigma\,(4)}R_{\nu\sigma} = D_\beta K - D_\alpha K^{\alpha}_{\beta}. \tag{4.25}$$

Now we perform a mixed projection on the Einstein equation,

$$\gamma^{\nu}_{\beta}n^{\sigma\,(4)}R_{\nu\sigma} - \frac{1}{2}\gamma^{\nu}_{\beta}n^{\sigma}g_{\nu\sigma}{}^{(4)}R = 8\pi\gamma^{\nu}_{\beta}n^{\sigma}T_{\nu\sigma}. \tag{4.26}$$

The second term on the left hand side vanishes because $\gamma^{\nu}_{\beta}n_\nu = 0$. Combining equations 4.25 and 4.26 results in the *momentum constraint*,

$$D_\alpha K^{\alpha}_{\beta} - D_\beta K = 8\pi S_\beta, \tag{4.27}$$

where we define the momentum density measured by a normal observer to be

$$S_\beta = -\gamma^{\nu}_{\beta}n^{\sigma}T_{\nu\sigma}. \tag{4.28}$$

Note that the momentum constraint 4.27 again has no time derivatives.

Next we turn our attention to the Ricci equation 4.18. We start by making the following expansion:

$$n^\delta n^\gamma \gamma^{\nu}_{\alpha}\gamma^{\rho\,(4)}_{\beta}R_{\delta\rho\gamma\nu} = (\gamma^{\delta\gamma} - g^{\delta\gamma})\gamma^{\nu}_{\alpha}\gamma^{\rho\,(4)}_{\beta}R_{\delta\rho\gamma\nu} = g^{\mu\sigma}\gamma^{\delta}_{\mu}\gamma^{\rho}_{\beta}\gamma^{\gamma}_{\sigma}\gamma^{\nu\,(4)}_{\alpha}R_{\delta\rho\gamma\nu} - \gamma^{\nu}_{\alpha}\gamma^{\rho\,(4)}_{\beta}R_{\rho\nu}. \tag{4.29}$$

We can make a substitution in the first term on the right hand side of 4.29 using Gauss' equation 4.16

$$\begin{aligned} g^{\mu\sigma}\gamma^{\delta}_{\mu}\gamma^{\rho}_{\beta}\gamma^{\gamma}_{\sigma}\gamma^{\nu\,(4)}_{\alpha}R_{\delta\rho\gamma\nu} &= g^{\mu\sigma}(R_{\mu\beta\sigma\alpha} + K_{\mu\sigma}K_{\beta\alpha} - K_{\mu\alpha}K_{\beta\sigma}) \\ &= R_{\beta\alpha} + KK_{\beta\alpha} - K^{\sigma}_{\alpha}K_{\beta\sigma}. \end{aligned} \tag{4.30}$$

We use a spatial projection of the Einstein equations to make a substition for the second term on the right hand side of 4.29, recalling **??** (*put the trace of the EFE in the GR chapter*)

$$\begin{aligned} \gamma^{\nu}_{\alpha}\gamma^{\rho\,(4)}_{\beta}R_{\rho\nu} &= \gamma^{\nu}_{\alpha}\gamma^{\rho}_{\beta}(8\pi T_{\rho\nu} + \tfrac{1}{2}g_{\rho\nu}{}^{(4)}R \\ &= 8\pi\gamma^{\nu}_{\alpha}\gamma^{\rho}_{\beta}(T_{\rho\nu} - \tfrac{1}{2}g_{\rho\nu}T). \end{aligned} \tag{4.31}$$

We define the spatial stress to be

$$S_{\alpha\beta} = \gamma^{\nu}_{\alpha}\gamma^{\rho}_{\beta}T_{\rho\nu}, \tag{4.32}$$

with trace $S = S^\alpha_\alpha$. Now consider the term

$$\begin{aligned} \gamma^\nu_\alpha \gamma^\rho_\beta g_{\rho\nu} T &= \gamma_{\alpha\beta} g^{\gamma\delta} T_{\gamma\delta} \\ &= \gamma_{\alpha\beta}(\gamma^{\gamma\delta} - n^\gamma n^\delta) T_{\gamma\delta} \\ &= \gamma_{\alpha\beta}(S - \rho). \end{aligned} \tag{4.33}$$

Substituting this into 4.31 and combining with 4.30, the Ricci Equation 4.18 becomes

$$\mathcal{L}_{\mathbf{n}} K_{\alpha\beta} = R_{\beta\alpha} + K K_{\beta\alpha} - 2K^\sigma_\alpha K_{\beta\sigma} - 8\pi(S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(S - \rho)) - \frac{1}{\alpha} D_\alpha D_\beta \alpha. \tag{4.34}$$

However, the derivative $\mathcal{L}_{\mathbf{n}}$ is not a natural derivative to use as it is not dual to the one-form $\nabla_\alpha t$ which was the foundation of this 3+1 framework. Instead, consider a timelike vector field t^α that is dual to $\nabla_\alpha t$, i.e.,

$$t^\alpha \nabla_\alpha t = 1. \tag{4.35}$$

We can decompose t^α into a spatial component β^α, and a component parallel to the normal,

$$t^\alpha = c n^\alpha + \beta^\alpha. \tag{4.36}$$

Then, by 4.35,

$$-\alpha c g^{\alpha\beta} \nabla_\beta t \nabla_\alpha t + \beta^\alpha \nabla_\alpha t = 1. \tag{4.37}$$

As β^α is purely spatial, $\beta^\alpha \nabla_\alpha t = 0$, thus from 4.1 we see $c = \alpha$. We can use this to replace $\mathcal{L}_{\mathbf{n}}$:

$$\mathcal{L}_{\mathbf{t}} K_{\alpha\beta} = \mathcal{L}_{\alpha\mathbf{n}+\beta} K_{\alpha\beta} = \alpha \mathcal{L}_{\mathbf{n}} K_{\alpha\beta} + \mathcal{L}_\beta K_{\alpha\beta}. \tag{4.38}$$

With this, 4.34 becomes

$$\mathcal{L}_{\mathbf{t}} K_{\alpha\beta} = -D_\alpha D_\beta \alpha + \alpha(R_{\beta\alpha} + K K_{\beta\alpha} - 2K^\sigma_\alpha K_{\beta\sigma}) - 8\pi\alpha(S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(S - \rho)) - \mathcal{L}_\beta K_{\alpha\beta}. \tag{4.39}$$

This is the *evolution equation for the extrinsic curvature.*

Finally, we use 4.38 in 4.15 to find the *evolution equation for the spatial metric*:

$$\mathcal{L}_{\mathbf{t}} \gamma_{\alpha\beta} = -2\alpha K_{\alpha\gamma} + \mathcal{L}_\beta \gamma_{\alpha\beta}. \tag{4.40}$$

4.4 Choosing Coordinates

We choose global coordinates x^α such that $x^0 \equiv t$, the global time function introduced at the beginning of this chapter, and x^i where $i = 1, 2, 3$ denote the spatial coordinates on a particular slice. Here, and in the remainder of this thesis, Greek letters $\alpha, \beta, \ldots$ will be used for indices

that take the values $0, \ldots, 3$, while denoted with Latin letters $i, j, \ldots$ will take values $1, \ldots 3$. The motivation for this will become clear shortly.

With these choices,

$$\nabla_\alpha t = (\partial_0 t, \partial_1 t, \partial_2 t, \partial_3 t) = (1, 0, 0, 0), \tag{4.41}$$

and therefore

$$n_\alpha = (-\alpha, 0, 0, 0) \tag{4.42}$$

Once coordinates have been chosen on a slice Σ_t, we need to know how they propagate to the subsequent slice Σ_{t+dt}.

We choose that points on the same integral curves $\mathcal{C}(t)$ of t^α, the timelike vector field introduced above, have the same spatial coordinates, that is, $\mathcal{C}(t) = (\mathcal{C}^0(t), \mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3)$ where the $\mathcal{C}^i$, $i = 1, 2, 3$ are constant. As such, the vector β^α gives us the relative velocity between the Eulerian observers and the lines that correspond to constant spatial coordinates, and hence is called the *shift vector*.

Invoking the duality condition 4.35 once more, we find

$$t^\alpha = (\frac{\mathrm{d}\mathcal{C}^0}{\mathrm{d}t}, 0, 0, 0) = (1, 0, 0, 0), \tag{4.43}$$

and thus the Lie derivative along t^α reduces to the partial

$$\mathcal{L}_t = \partial_t = \partial_0. \tag{4.44}$$

The result 4.42 tells us that the timelike contravariant components of any purely spatial tensor must vanish. For example, consider a purely spatial $(1, 1)$ tensor S^α_β:

$$n_\alpha S^\alpha_\beta = -\alpha S^0_\beta = 0, \tag{4.45}$$

thus $S^0_\beta = 0$ for $\beta = 0, ...3$. In particular, the timelike components of the inverse spatial metric vanish, $\gamma^{0\alpha} = 0$, and we can write the spatial component of the vector field t^α as $\beta^\alpha = (0, \beta^i)$, $i = 1, 2, 3$. This allows us to write the spacetime metric and its inverse in terms of α, β and γ. Recalling 4.36 and 4.43, we find

$$n^0 = \alpha^{-1} \text{ and } n^i = -\alpha^{-1}\beta^i, \ i = 1, 2, 3, \tag{4.46}$$

and inserting this into 4.7 gives us the inverse spacetime metric

$$g^{\alpha\beta} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}, \tag{4.47}$$

which can be inverted to find

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}. \tag{4.48}$$

In line element form, this is

$$ds^2 = -\alpha^2 dt^s + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{4.49}$$

This expression makes clear the interpretation of the function α. For Eulerian observers, $dx^i = -\beta^i dt$. Substituting this into 4.49 gives us

$$\mathrm{d}\tau^2 = -ds^2 = \alpha^2 dt^2, \tag{4.50}$$

and we can now see that α is the rate of change of proper time τ with respect to coordinate time t for Eulerian observers. For this reason, the function α is called the *lapse*.

We have seen that the timelike contravariant components of purely spatial tensors vanish. This is not true for the timelike covariant components, however these can be determined from the spatial components. To see why, consider a purely spatial tensor $S_{\gamma\delta}$:

$$\begin{aligned} S_{\gamma\delta} &= \gamma^\alpha_\gamma \gamma^\beta_\delta S_{\alpha\beta} \\ &= g_{\gamma\mu} g_{\delta\nu} \gamma^{\alpha\mu} \gamma^{\beta_\nu} S_{\alpha\beta} \\ &= g_{\gamma\mu} g_{\delta\nu} (\gamma^{0\mu}\gamma^{0\nu} S_{00} + \gamma^{i\mu}\gamma^{0\nu} S_{i0} + \gamma^{0\mu}\gamma^{j\nu} S_{0j} + \gamma^{i\mu}\gamma^{j\nu} S_{ij}) \\ &= g_{\gamma\mu} g_{\delta\nu} \gamma^{i\mu}\gamma^{j\nu} S_{ij}. \end{aligned} \tag{4.51}$$

We can explicitly write the timelike covariant components in terms of the spacelike components:

$$S_{00} = \beta^i\beta^j S_{ij}, \quad S_{i0} = \beta^k S_{ik}, \quad S_{0j} = \beta^k S_{kj}. \tag{4.52}$$

Since in these coordinates spatial tensors are determined entirely from their spacelike components, the constraint and evolution equations can be written in terms of these components alone. The Hamiltonian constraint 4.23 becomes

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{4.53}$$

and the momentum constraint 4.27,

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i. \tag{4.54}$$

The evolution equation for the extrinsic curvature 4.39 becomes

$$\begin{aligned} \partial_t K_{ij} = \; & \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - D_iD_j\alpha - 8\pi\alpha(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S - \rho)) \\ & + \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k, \end{aligned} \tag{4.55}$$

where the final three terms come from the Lie derivative, and similarly the evolution equation for the spatial metric becomes

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i. \tag{4.56}$$

The equations 4.53, 4.54, 4.55 and 4.56 are called the *ADM Equations* after Arnowitt, Deser and Misner who derived them in 1962 (although they weren't the first to do so).

4.5 Gauge Conditions

- must avoid singularities
- must counteract grid stretching

Let us now consider our options for the gauge conditions, which can be chosen freely. Recall the lapse function gives the proper time as measured by Eulerian observers (those who are moving normal to the hypersurface). The acceleration of these observers is given by

$$a^\mu = n^\nu \nabla_\nu n^\mu \tag{4.57}$$

which implies the proper acceleration is given in terms of the lapse function as

$$a_i = \partial_i \ln \alpha. \tag{4.58}$$

The evolution of the volume elements associated with the Eulerian observers can be found using the definition of the extrinsic curvature:

$$\nabla_\mu n^\mu = -K. \tag{4.59}$$

4.5.1 Geodesic slicing

Take the coordinate time t to coincide with the time of Eulerian observers, that is, choose $\alpha = 1$. From 4.58, this implies $a_i = 0$ and that the Eulerian observers are in freefall (therefore follow geodesics). In a non-uniform gravitational field, the "observers" will collide and the coordinate system will become singular.

4.5.2 Maximal slicing

Require the volume elements remain constant- from 4.59:

$$K = \partial_t K = 0 \tag{4.60}$$

- BSSN
- Initial Data
- Moving Punctures
- Weyl Tensor- GW extraction

4.5.3 Bona-Masso

4.6 Initial Data

To find the initial data for our system we must solve the constraint equations **??** and **??**. These form a system of four coupled partial differential equations of elliptic type.

4.6.1 York-Lichnerowicz conformal decomposition

Suppose a 3-metric $\bar{\gamma}_{ij}$ is known, and consider a transformation of the form

$$\gamma_{ij} = \phi^4 \bar{\gamma}_{ij}$$

where ϕ is some positive scaling factor. Such a transformation is called a "conformal transformation" as it can be shown to preserve angles. $\bar{\gamma}_{ij}$ is called the "conformal metric". The extrinsic curvature is separated into it's trace

$$K = \gamma^{ij} K_{ij}$$

and trace free parts

$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij} K$$

A conformal transformation is performed on A_{ij}

$$A_{ij} = \phi^{-10} \bar{A}_{ij}$$

where the tenth power is chosen for later convenience.

4.7 Aside? Post Newtonian methods

Post-Newtonian methods are used in low curvature and low velocity regimes, such as the inspiral phase of a binary coalescence. It is less computationally expensive than a full numerical calculation. In post-Newtonian methods, solutions are constructed iteratively starting with the Newtonian solution. At each step a correction term of order v^2 is added to to the previous step. Essentially this is a Taylor expansion in the parameter v^2 around the the Newtonian solution.

4.8 perturbative methods

4.9 Dealing with singluarities

Area inside the apparent horizon is causally disconnected from the outside, therefore boundary conditions are not important. However in practice the excision occurs well inside the horizon where the Einstein equations are highly non-linear. Further, the excised region moves on the grid which can be complicated to perform. Also, the apparent horizon must be found, this is expensive. (Rezzolla ppt).

Chapter 5

State of the Art (this chapter is a total mess)

- During inspiral and after the merger during the ringdown, can use perturbative methods (McWilliams- that's all he said about it) Due to the high computational cost of NR BBH simulations, the early inspiral can instead be modelled using approximate analytic techniques based on the post- Newtonian method, and the late inspiral and merger can be simulated with NR. PN results are accurate in the regime where the black holes are far apart and moving slowly. (Hinder?)
- choices of α and β lead to different numerical properties- for 4 decades most research was dedicated to finding stable choices (McWilliams).

5.1 Extreme Mass Ratios

Usually in astrophysics, extreme mass ratios are on the order of 10^6. These kinds of systems can be explored using perturbative methods, so in numerical relativity the focus is more around the 10^2 level. The time scale of the orbit is $M_1 + M_2$, while the size of the time step is M_small, this makes it difficult [**?**].

The smallest mass ratio evolved to date is $q = 10^{-2}$ by Lousto and Zlochower in 2011 [**?**]. They used Cactus and the Einstein toolkit, with the Carpet mesh refinment driver to evolve the BBH system through the last two orbits, the merger and through to the Kerr black hole remnant. Their success was due to improvements in the moving puncture numerical technique.

5.2 Longest Waveforms

The longest NR BBH waveform so far produced lasts for 15 orbits and includes the merger and ringdown phases, and is described in Boyle et al. [39] and Scheel et al. [40]. This waveform, from an equal mass binary of non-spinning black holes, was generated using the SpEC code and, due to its length and quoted accuracy, has been used in a number of studies comparing NR and PN results [41âĂŞ46].As a result of this work, simulations of BBH mergers with mass ratios q = m1/m2 âĽě 1 (where m1 âĽě m2 are the masses of the individual holes) of q = 2 and dimensionless spins up to 0.4 are now possible with the SpEC code, and these are presented in Ref. [47]Additionally, simulations with dimensionless spins of 0.44 anti-aligned with the orbital angular momentum are presented in Chu et al. [48]. In a talk by H. Pfeiffer [49], a series of long unequal mass simulations performed with the SpEC code was presented, with 15 orbits up to q = 4 and 8 orbits up to q = 6. (Hinder?)

5.3 Highest Spin

Extreme spin $a = S/M^2 \leq 0.95$ by Lovelace, Scheel and Szilagyi 2011. Equal mass black holes with spin anti-aligned with orbital angular momentum (this reduces total angular momentum of the system). Something happens with Bowen York initial data that limits the spin to $\approx$ 0.93 which is only 60% of possible rotational energy (look into this). The authors fixed this by combining the conformal thin sandwich with a conformally curved metric. (McWilliams) Mathematically, Kerr black holes have a maximum dimensionless spin of 1, and there is a good probability that highly spinning black holes exist in nature [57âĂŞ59]. Accretion models suggest .95, some observations suggest $>$ .98. Typically, black holes with dimensionless spins as high as 0.6âĂŞ0.8 can be evolved with only a moderate increase in computational cost over the non-spinning case. The theoretical maximum has almost been reached in Dain et al. [60], where âĹij 7.5 orbits of black holes with dimensionless spins 0.92 are evolved.

5.4 Christodoulou Memory

Non-linear propagation of GW results in the emission of GWs generated by GWs, known as Christodoulou memory. The effect is tiny, but potentially detectable by LISA. It's difficult to simulate because the errors in simulation need to be smaller than the already tiny effect. Pollney and Reisswig 2010 accurately calculated this effect using multipatch techniques and Cauchy characteristic extraction. (McWilliams)

5.5 General Relativity Magneto Hydro Dynamics

This work helps with finding EM counterparts to GW radiation. Studying Gamma Ray Bursts [?]. Four codes capable of GRMHD: WhiskyMHD, SACRA, LSU-LIU-BYU-PI collaboration code, Illinois group code.

Samurai project: compare the waveforms from fice different codes for the last 4 orbits and merger of a binary, equal mass, non-spinning black holes in circular orbit. These codes were BAM, CCATIE, Hahndol, MayaKranc and SpEC. The first fours use the BSSN fomrulation of the Einstein equations while SpEC uses the Generalised Harmonic formulation.SpEC uses a pseudo-spectral evolution scheme, whereas the others use finite differencing method.

The XiRel project [63, 64] was started in or- der to improve the performance of the publicly available Carpet [36, 37] adaptive mesh refinement infrastructure. As a result of recent work, Carpet now scales efficiently up to 2048 processing cores, and as it is used for BBH simulations at AEI, GaTech, LSU, RIT and UIUC,

Hyperbolic BBH encounters can be thought of as orbits of eccentricity e $>$ 1. In a Newtonian system, such a configuration would result in scattering of one black hole off the potential of the other, but in full GR, for a sufficiently small impact parameter, the black holes be- come gravitationally bound due to the emission of energy through gravitational waves and merge quickly [109]

Current evolutions using the SpEC code in the Generalised Harmonic formulation such as those in Refs. [39, 40] use an outer boundary condition designed to satisfy the Einstein constraint equations as well as to min- imise the incoming gravitational radiation [31, 149, 150], though these are not designed to be mathematically well- posed (a necessary condition for formal stability of the problem under small perturbations).

Mathematically, gravitational radiation from a BBH inspiral and merger is defined at future null infinity; i.e. the part of the spacetime towardswhich all null rays, such as light or GWs, propagate.GWs must be read off at a finite spatial radius, and this introduces an error in the waveform. An obvious solution is to compute the radiation at several radii and extrapolate the waveform to infinite radius as a function of retarded time fromthe source. It was shown in Hannam et al. Ref. [155] that this is not a trivial pro- cedure, and a detailed analysis and successful extrapo- lation of the inspiral portion was presented in Boyle et al. Ref. [39]. A failure to extrapolate waveforms from the SpEC code can lead to a phase error of 0.5 radians [39], or a mismatch of âĹij 10âĹŠ2 [156], if using waveforms extracted at r = 50M only.

Instead of performing a traditional Cauchy evolution, where the solution is evolved along timelike directions, it is possible to evolve along null directions. This is called characteristic evolution. This cannot be done straightfor- wardly in the neighbourhood of the black holes as it leads to caustics in the solution, but it can be done in regions sufficiently far from them. The Cauchy-Characteristic Extraction (CCE) method combines the two approaches.

For numerical relativity evolution purposes, the four-dimensional BBH spacetime is usually foliated with three-dimensional spacelike slices. Each of these slices can be split into two regions with very different compu- tational requirements. The region around the black holes has a complicated geometry reflecting the shapes of the horizons, but the region far from the black holes con- sists only of gravitational radiation. Since the radiation propagates essentially radially, it requires a constant an- gular resolution to resolve it. The majority of BBH NR codes today use Cartesian-type coordinates everywhere in the grid. These have the advantage of simplicity, as only a single coordinate patch is required to cover the en- tire simulation domain. However, they are not efficient, as they lead to an increasing angular resolution with radius.

While most NR codes use finite differencing methods with global Cartesian coordinates, the SpEC code uses multiple coordinate patches as well as spectral methods. For simple equations, these methods can be shown to be significantly more accurate (exponential rather than polynomial convergence) for a given computational cost, and indeed, the quoted accuracy and efficiency on the waveforms from the SpEC code is impressive.

5.6 Cactus

Flesh provides APIs for thorns to communicate with each other, performs administrative tasks and build and run time. Each thorn profides three configuration files (plus two optional files)

Chapter 6

Data analysis

Our data model is of a continuous time stochastic process $X(t)$ over an interval $[0, T]$, sampled at n points $t_1, \ldots, t_n \subset T$. We assume an additive noise model, that is

$$X_t = h_t + N_t, \quad t \in t_1, \ldots, t_n, \tag{6.1}$$

where $X_t = X(t)$, h_t is a sample from a deterministic signal waveform at time t, and $\{N_t : t \in [0, T]\}$ is the detector noise, a zero mean Gaussian random process with covariance matrix Σ_N.

6.1 Hypothesis Testing

Gravitational wave detection can be cast as a hypothesis testing problem. The *null hypothesis*, H_0, is that the signal is absent and the data consists entirely of noise with probability distribution P_0. The *alternative hypothesis*, H_1, is that a signal is present and the data has probability distribution P_1. This can be written

$$\begin{aligned} H_0 &: X_t = N_t \sim P_0, & t \in t_1, \ldots, t_n \\ H_1 &: X_t = h_t + N_t \sim P_1, & t \in t_1, \ldots, t_n. \end{aligned} \tag{6.2}$$

A *hypothesis test* or *decision rule* δ is a partition of the observation set $\mathcal{X}$ into two subsets $\mathcal{X}_0$ and $\mathcal{X}_1 = \mathcal{X}_0^c$. If the data $x \in \mathcal{X}_0$, we accept the null hypothesis H_0. If $x \in \mathcal{X}_1$, we claim a detection!

We now need a way to choose a decision rule δ that is optimal in some sense. There are several several ways to do this; here the Neyman-Pearson approach is described. The Bayesian and Minimax approaches are also common, and can be found in [poor].

6.1.1 Neyman-Pearson Approach

There are two types of errors that can occur in testing H_0 against H_1. A *type I error* or *false alarm* occurs if we choose H_1 when H_0 is true, and the probability of this occuring for a decision rule δ is called the *false alarm probability* denoted $P_F(\delta)$. The second error occurs if we choose H_0 when H_1 is true and is called a *type II error* or *miss*. We can define a *miss probability*, $P_M(\delta)$, however it is more common to talk in terms of the *detection probability* or *power* of the test $P_D(\delta) = 1 - P_M(\delta)$. The Neyman-Pearson optimum test maximises the power of the test, subject to a chosen constraint on the false alarm probability:

$$\delta_{NP} = \underset{\delta}{\operatorname{argmax}} P_D(\delta) \text{ subject to } P_F(\delta) \leq \alpha. \tag{6.3}$$

The bound α is called the *significance level* of the test. We seek to devise a function $\phi : \mathcal{X} \to [0, 1]$ representing the probability of choosing H_1 given some data $x \in \mathcal{X}$. The detection and false alarm probabilities for a decision rule associated with such a function are as follows:

$$P_D(\delta) = E_1[\phi(\mathcal{X})] = \int_{\mathcal{X}} \phi(x) p_1(x) d\mathbf{x} \tag{6.4}$$

$$P_F(\delta) = E_0[\phi(\mathcal{X})] = \int_{\mathcal{X}} \phi(x) p_0(x) d\mathbf{x} \tag{6.5}$$

The existence and uniqueness of such a test function is given by the Neyman-Pearson Lemma.

Lemma 6.1.1. *Consider the hypothesis pair 6.2. Suppose P_0 and P_1 have densities p_0 and p_1 respectively and $\alpha > 0$. Then*

1. *Existence: For every $\alpha \in (0, 1)$, there is a function $\phi(x)$ associated to a test δ_{NP} which is of the form*

$$\phi(x) = \begin{cases} 1 & \text{if } p_1(x) > \lambda p_0(x) \\ \gamma & \text{if } p_1(x) = \lambda p_0(x) \\ 0 & \text{if } p_1(x) < \lambda p_0(x) \end{cases} \tag{6.6}$$

   where $\eta \geq 0$, $0 \leq \gamma \leq 1$ and such that $P_F(\delta_{NP}) = \alpha$.

2. *Optimality: Let δ' be any decision rule satisfying $P_F(\delta') \leq \alpha$, and that it is associated with some function ϕ'. Then $P_D(\delta_{NP}) \geq P_D(\delta')$, where δ_{NP} is as defined by the function 6.6.*

3. *Uniqueness: Suppose that δ''_{NP} is any α-level Neyman-Pearson decision rule for H_0 versus H_1. Then the function $\phi''(x)$ associated with δ''_{NP} must be of the form 6.6, except for possibly on a subset of $\mathcal{X}$ having zero probability under H_0 and H_1.*

Proof. 1. Let η be the smallest number such that

$$P_0(p_1(X) > \eta p_0(X)) \leq \alpha. \tag{6.7}$$

If $P_0(p_1(X) > \eta p_0(X)) < \alpha$, choose

$$\gamma = \frac{\alpha - P_0(p_1(X) > \eta p_0(X))}{P_0(p_1(X) = \eta p_0(X))}, \tag{6.8}$$

otherwise γ can be chosen arbitrarily. Note that for a test of the form 6.6, the false alarm probability is the expectation of the function ϕ under H_0:

$$P_F(\delta_{NP}) = E_0[\phi] = 1 \times P_0(p_1(X) > \eta p_0(X)) + \gamma P_0(p_1(X) = \eta p_0(X)) = \alpha. \tag{6.9}$$

2. Note that $\forall x \in \mathcal{X}$

$$[\phi(x) - \phi'(x)][p_1(x) - \lambda p_0(x)] \geq 0,$$

and therefore,

$$\int_{\mathcal{X}} [\phi(x) - \phi'(x)][p_1(x) - \lambda p_0(x)] d\mathbf{x} \geq 0. \tag{6.10}$$

Expanding this and rearranging terms gives

$$\int_{\mathcal{X}} \phi(x) p_1(x) d\mathbf{x} - \int_{\mathcal{X}} \phi'(x) p_1(x) d\mathbf{x} \geq \lambda \left[\int_{\mathcal{X}} \phi(x) p_0(x) d\mathbf{x} - \int_{\mathcal{X}} \phi'(x) p_0(x) d\mathbf{x} \right], \tag{6.11}$$

which, using 6.4 and 6.5, becomes

$$P_D(\delta_{NP}) - P_D(\delta') \geq \lambda \left[\underbrace{P_F(\delta_{NP})}_{=\alpha} - \underbrace{P_F(\delta')}_{\leq \alpha} \right] \geq 0. \tag{6.12}$$

Thus, $P_D(\delta_{NP}) \geq P_D(\delta')$.

3. By definition, we must have $P_D(\delta''_{NP}) = P_D(\delta_{NP})$. Therefore, by 6.12 we have

$$0 \geq \lambda \left[\alpha - P_F(\delta''_{NP}) \right] \geq 0 \tag{6.13}$$

or, $P_F(\delta''_{NP}) = \alpha$. Reversing the steps in part 2 above, we find

$$\int_{\mathcal{X}} [\phi(x) - \phi''(x)][p_1(x) - \lambda p_0(x)] d\mathbf{x} = 0. \tag{6.14}$$

Since we know the integrand to be non-negative, this implies that it is zero except possibly on a set of zero probability under H_0 and H_1. Therefore, $\phi(x)$ and $\phi''(x)$ differ only on the set $\{x \in \mathcal{X} | p_1(x) - \lambda p_0(x) = 0\}$, which implies ϕ'' is also of the form 6.6, differing only in γ.

□

We define the quantity

$$L(x) = \frac{p_1(x)}{p_0(x)}, \quad x \in \mathcal{X}, \tag{6.15}$$

to be the *likelihood ratio*. Then, by rearranging the conditions in 6.6, we see that a Neyman-Pearson test involves comparing the likelihood ratio for an observed value of X to a threshold λ.

6.2 Introduction

For a known waveform the most efficient data analysis tool is matched filtering. But often the waveforms of GW are poorly known, limiting the matched filter's applicability.

A linear time frequency transform correlates the signal with a family of waveforms that are well concentrated in time and space. These waveforms are called *time frequency atoms*.

6.3 Fourier Transforms

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e{-}i\omega t dt$$

6.4 Windowed Fourier Transforms

Gabor atoms

$$g_{u,\xi}(t) = g(t-u)e^{i\xi t}$$

When g is Gaussian, best localisation and the atoms are called *Gabor functions*. Fourier transform of Gabor atom

$$\hat{g}_{u,\xi}(\omega) = \hat{g}(\omega - \xi)e^{-iu(\omega-\xi)}$$

Windowed Fourier transform:

$$Sf(u,\xi) = \int_{-\infty}^{+\infty} f(t)g *_{u,\xi} (t)dt = \int_{-\infty}^{+\infty} f(t)g(t-u)e^{-i\xi t}dt$$

Also can be written as frequency integral

$$Sf(u,\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\hat{g}^*_{u,\xi}(\omega)d\omega$$

6.5 wavelet transforms

Wavelet transforms are described in terms of families of basis functions $\psi_{a,b}(x)$ which are dilations and translations of a real function $\psi(x)$ of zero average:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)$$

A wavelet transform of a function f has a different time frequency resolution to a windowed Fourier transfrom. It is computed by correlating f with a wavelet atom

$$Wf(a,b) = \int_{-\infty}^{+\infty} f(t)\psi^*_{a,b}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\hat{\psi}^*_{a,b}(\omega)d\omega$$

6.6 Notes from "method for detection and reconstruction of gravitational wave transients with networks of advanced detectors

6.6.1 Low latency search

Designed to identify potentially significant candidates in real time (within minutes). Streaming? Promptly share sky location with partner telescopes to search for EM counterparts eg Gamma ray burst and afterglow or kilonova which will fade within seconds to days.

6.6.2 Archive analysis

Establish significance of event and identify progenitors. Slower and more detailed.

6.6.3 Inverse problem for data from a network of detectors

Network of K detectors. Time series data from the detectors is transformed with the Wilson-Debauchies-Meyer (WDM) transform so that each detector k presents data $x_k[i]$ in a general time-frequency (TF) domain, where i is the data sampling (TF pixel) index.

Detector noise assumed to be Gaussian, described by WDM power spectral density $S_k[i]$ estimated for every data sample. The noise-scaled (whitened) data is defined as $\omega_k[i] = x_k[i]/\sqrt{S_k[i]}$

TF series combined to obtain energy TF maps $E[i] = \sum_k \omega_k^2[i]$ where the $E[i]$ are maximised over all possible time-of-flight delays in the network (i think this means it's lining up the signals based on the travel time of GW from one detector to another. Obviously this depends on where

in the sky the GW is coming from). The energy maps are are used to identify TF areas with excess energy above baseline detector noise using an appropriate clustering algorithm. The inverse problem is to extract the signal waveform, polarisation and sky location. (Note that this slightly differs from my definition of the inverse problem which is to determine the parameters from the waveform. Many steps to the inverse problem).

Data vector $\mathbf{x}[i] = x_1[i], \ldots, x_K[i]$ recorded by detector at time of a GW signal $\mathbf{h}[i] = [h_+[i], h_\times[i]]$ coming from sky location θ and ϕ is a superposition of network response $\mathcal{F}\mathbf{h}[i]$ and noise $\mathbf{n}[\mathbf{h}]$:

$$\mathbf{x}[i] = \mathcal{F}\mathbf{h}[i] + \mathbf{n}[\mathbf{h}]$$

where $\mathcal{F}$ is the network antenna pattern matrix:

$$\mathcal{F} = \begin{bmatrix} F_{1+}(\theta,\phi) & F_{1\times}(\theta,\phi) \\ \cdots & \cdots \\ F_{K+}(\theta,\phi) & F_{K\times}(\theta,\phi) \end{bmatrix}$$

(Antenna patterns often also include a transformation by polarisation angle Ψ, but this is equivalent to a rotation of the wave frame and can be included in the definition of $\mathbf{h}$).

BX-LBTN-KYW2-9V7B-9K2D

6.6.4 Matched Filtering

from [?] Searches for GWs from CBC events employ matched filtering as the first step.

"whitening": Optimal strategy, weight detector output and template wavefor by inverse of amplitude spectral density of detector noise

discretely sampled time series whitened data: $\overrightarrow{s} = s_i$ whitened template waveform: $\overrightarrow{h}_\alpha = h_{\alpha i}$ noise: $\overrightarrow{n} = n_i$ zero-mean $< n_i >= 0$, unit-variance Gaussian random process $< n_i, n_j >= \delta_{ij}$

output of matched filter is given by the vector inner product

$$\rho_\alpha = \overrightarrow{h}^*_\alpha \cdot \overrightarrow{s} \tag{6.16}$$

search over phase by using complex-valued templates where $\Re\overrightarrow{h}_\alpha$ contains the cosine like phase and $\Im\overrightarrow{h}_\alpha$ contains the sine-like phase.

6.7 Jaranowski et.al

Random Variables

A *probability space* is a triple $(\mathcal{X}, \mathcal{O}, P)$ where

- $\mathcal{X}$ is a sample space, usually an n-dimensional Euclidean space $\mathbb{R}^n$ or a discrete set $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$, or for continuous process, a function space.
- $\mathcal{O}$ is a σ-algebra of subsets of $\mathcal{X}$ called observation events, these are the set of values that data can take.
  - if $\mathcal{X} = \mathbb{R}^n$, consider open intervals

    $$\{\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n : a_1 < y_1 < b_1, \ldots a_n < y_n < b_n\} \tag{6.17}$$

    where a_i and b_i are real numbers. Then usually $\mathcal{O}$ is the smalles σ-algebra containing all such intervals. This is known as the *Borel* σ-algebra and is denoted by B^n.
  - if $\mathcal{X}$ is a discrete set Γ, the σ-algebra $\mathcal{O}$ is the set of all subsets of Γ. This is called the *power set* of Γ and is denoted by 2^Γ.
- P is a probability distribution (measure) on $\mathcal{O}$. This is a function assigning a number $P(A) \geq 0$ to any set $A \in \mathcal{O}$ such that $P(\emptyset) = 0$ and $P(\mathcal{X}) = 1$. For probability spaces $(\mathbb{R}^n, B^n, P)$ usually there exists a *probability density function* $p : \mathbb{R}^n \to [0, \infty)$ such that

  $$P(A) = \int_A p(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{6.18}$$

  where $\mathbf{x}$ are some coordinates on $\mathbb{R}^n$.

σ-algebra:

1. $\mathcal{X} \in \mathcal{O}$
2. if $A \in \mathcal{O}$, then $A^c := \mathcal{X} - A \in \mathcal{O}$
3. for any countable family A_k of members of $Ocal$, their union is in $\mathcal{O}$

Together these imply that the intersection of a finite or countable family of members of $\mathcal{O}$ belong to $\mathcal{O}$.

A *random variable* X is a real-valued function $X : \mathcal{X} \to \mathbb{R}$. The sets $\{\xi \in \mathcal{X} : X(\xi) \leq x\} \in \mathcal{O} \forall x \in \mathbb{R}$.

The *cumulative distribution function* of X is

$$P_X(x) := P(X \leq x). \tag{6.19}$$

It is non-decreasing and right-continuous.

A *continuous* random function is one with a continuous cumulative distribution function. If it is differentiable, then the probability density function is

$$p_X(x) = \frac{\mathrm{d}P_X(x)}{\mathrm{d}x}. \tag{6.20}$$

This is always non-negative as the cdf is non-decreasing. It is also normalised

$$\int_{-\infty}^{\infty} p_X(x)\mathrm{d}x = 1 \tag{6.21}$$

The *expectation value* or *mean* of a continuous random variable is

$$E\{X\} := \int -\infty^{\infty} x p_X(x)\mathrm{d}x \tag{6.22}$$

The *variance* of the random variable X is

$$\mathrm{Var}\{X\} \equiv \sigma_X^2 := E\{(X - E\{X\})^2\}. \tag{6.23}$$

The *covariance* between two random variables X and Y defined on the same probability space is

$$\mathrm{Cov}\{XY\} := E\{(X - E\{X\})(Y - E\{Y\})\} = E\{XY\} - E\{X\}E\{Y\}. \tag{6.24}$$

Random variables for which $E\{XY\} = E\{X\}E\{Y\}$ are called *uncorrelated*. $E\{XY\} = 0$ they are called *orthogonal*.

X, Y continuous. $p_{X,Y}$ the *joint* pdf. *independent* if $p_{X,Y} = p_X(x)p_Y(y)$. The *conditional* pdf of X given $Y = y$ is

$$p_{X|Y}(x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}. \tag{6.25}$$

A *complex* random variable $Z = X + iY$ has pdf p_Z defined as the joint pdf of the two real random variables X and Y

$$p_Z(z) := p_{X,Y}(x, y). \tag{6.26}$$

Mean

$$E\{Z\} := E\{X\} + iE\{Y\} \tag{6.27}$$

Variance

$$\mathrm{Var}\{Z\} := E\{|Z - E\{Z\}|^2\} = E\{|Z|^2\} - |E\{Z\}|^2. \tag{6.28}$$

Stochastic Processes

Let $T \subset \mathbb{R}$. A *stochastic* process $X(t)$ is a family of random variables labelled by the numbers $t \in T$, all defined on the same probability space $(\mathcal{X}, \mathcal{O}, P)$. For each finite subset $t_1, \ldots, t_n \subset T$ the random variables $X(t_1), \ldots, X(t_n)$ have a joint n-dimensional cumulative distribution function F defined by

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = P(x(t_1) \leq x_1, \ldots x(t_n) \leq t_n). \tag{6.29}$$

When T is a set of discrete points the stochastic process is called a *random sequence*, and if the variable t is time it's called a *time series*.

The stochastic process is *Gaussian* if the cdf $F_{t_1,\ldots,t_n}$ is Gaussian for any n and any $t_1, \ldots, t_n \in T$. *Stationary* if for all finite dimensional cdfs and all τ

$$F_{t_1+\tau,\ldots,t_n+\tau}(x_1, \ldots, x_n) = F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) \tag{6.30}$$

The *moments* of the probability distribution are

$$\mu_{m_1 \ldots m_n} := E\{X(t_1)^{m_1} \cdots X(t_n)^{m_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{m_1} \cdots x_n^{m_n} \mathrm{d}F_{t_1,\ldots,t_n}(x_1, \ldots, x_n). \tag{6.31}$$

This is an n-fold *Stieltjes integral*. When F is differentiable, it can be written in terms of the joint pdf

$$f_{t_1 \ldots t_n}(x_1 \ldots x_n) = \frac{\partial^n \mathrm{d}F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)}{\partial x_1 \ldots \partial x_n} \tag{6.32}$$

and the above integral can be written

$$\mu_{m_1 \ldots m_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{m_1} \cdots x_n^{m_n} f_{t_1 \ldots t_n}(x_1 \ldots x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{6.33}$$

The first order moment is called the *mean* value of the stochastic process and is denoted by $m(t) := E\{X(t)\} = \mu_1$, if the process is stationary this is constant. The second order moment is called the *autocorrelation function* and is denoted by $K(t, s) := E\{X(t)X(s)\}$, if the process is stationary it depends only on the time difference $K(t, s) = R(t - s)$ for some function R s.t. $R(0) > 0$, $R(-\tau) = R(\tau)$ and $|R(\tau)| < R(0)$.

For a complex stationary process, $R(t) := E\{X(t+\tau)X(t)^*\}$, and $R(-\tau) = R(\tau)^*$. Consider a complex stationary stochastic process

$$X(t) = X_0 U(t), \tag{6.34}$$

where X_0 is a random variable with $E\{X_0\} = 0$, and $U : T \to \mathbb{C}$ with $\int_{-\infty}^{\infty} U(t)dt = 0$. Stationarity gives $R(\tau) = E\{X(t+\tau)X(t)^*\} = E\{X_0 U(t+\tau) X_0^* U(t)^*\} = E\{|X_0|^2\} U(t+\tau)U(t)^*$ is

independent of t. If we take $\tau = 0$, this means $|U(t)|^2 = r^2$, a constant, thus $U(t) = r\exp(i\phi(t))$. For non-zero τ, $r^2\exp(i\phi(t+\tau))\exp(-i\phi(t)) = r^2\exp(i(\phi(t+\tau) - \phi(t))$ is independent of t, implying $\phi'(t) = \omega$ for some constant ω, and therefore $\phi = \omega t + \theta$. Therefore, $X(t) = X_0\exp(i\omega t)$, where the r and $\exp(\theta)$ have been "absorbed" into the X_0. The autocorrelation function $R(\tau) = E\{|X_0|^2\}\exp(i\omega\tau) = b_0\exp(i\omega\tau)$, where $b_0 = E\{|X_0|^2\} = E\{|X_0 - E\{X_0\}|^2\}$ is the variance of X_0.

A more general complex stochastic process can be written as a superposition

$$X(t) = \sum_{k=1}^{\infty} X_k \exp(i\omega_k t), \tag{6.35}$$

where for each complex r.v. X_k, $E\{X_k\} = 0$ and $E\{X_k X_l^*\} = 0$ for $k \neq l$. The autocorrelation function is then

$$R(\tau) = \sum_{k=1}^{\infty} b_k \exp(i\omega_k \tau), \tag{6.36}$$

with each variance $b_k < \infty$. Such a stationary process is called a *process with discrete spectrum*, where the set $\{\omega_k\}$ is the *spectrum*.

Any complex stationary stochastic process can be obtained as the limit of a sequence of processes with discrete spectra.

Cramèr Representation Theorem Let $X(t)$, $-\infty < t < \infty$, be a zero-mean stochastically continuous complex stationary process. Then there exists an orthogonal process $Z(\omega)$ such that for all t, $X(t)$ may be written in the form,

$$X(t) = \int_{-\infty}^{\infty} \exp(i\omega t)\mathrm{d}Z(\omega), \tag{6.37}$$

the integral being defined in the mean-square sense. The process $Z(\omega)$ has the following properties:

1. $E\{\mathrm{d}Z(\omega)\} = 0 \forall \omega$
2. $E\{|\mathrm{d}Z(\omega)|^2\} = \mathrm{d}F(\omega)\forall\omega$, where $F(\omega)$ is the (non-normalised) integrated spectrum of $X(t)$
3. for any two distinct frequencies ω, ω'

$$\mathrm{Cov}\{\mathrm{d}Z(\omega), \mathrm{d}Z(\omega')\} = E\{\mathrm{d}Z(\omega)\mathrm{d}Z(\omega')^*\} = 0 \tag{6.38}$$

Wiener-Khinchin Theorem A necessary and sufficient condition for $\rho(\tau)$ to be the autocorrelation function of a stochastically continuous stationary process $X(t)$ is that there exists a function $F(\omega)$ having the properties of a cdf on $(-\infty, \infty)$ such that for all τ, $\rho(\tau)$ may be expressed in the form

$$\rho(\tau) = \int_{-\infty}^{\infty} \exp(i\omega\tau) \mathrm{d}F(\omega) \tag{6.39}$$

If F differentiable everywhere,

$$\mathrm{d}F(\omega) = S(\omega)\mathrm{d}\omega \tag{6.40}$$

S is called the *two-sided spectral density* of the stationary stochastic process. In this case, the correlation between the Fourier components are

$$E\{\tilde{X}(\omega)\tilde{X}(\omega')^*\} = S(\omega)\delta(\omega' - \omega) \tag{6.41}$$

In the discrete-time case, a time series $X_k = X_0 U(k)$, $k \in \mathbb{Z}$, is stationary iff it is of the form

$$X_k = X_0 \exp(i\omega_0 k). \tag{6.42}$$

The autocorrelation function is $R(l) = E\{X_k X_{k+l}\}$. The Cramèr representation and Wiener-Khinchen theorems become

$$\begin{array}{rcl} x_k & = & \int_{-\pi}^{\pi} \exp(i\omega k)\mathrm{d}Z(\omega) \\ R(l) & = & \int_{-\pi}^{\pi} \exp(i\omega l)\mathrm{d}F(\omega) \end{array} \tag{6.43}$$

If F is differentiable with derivative S, we can write

$$R(l) = \int_{-\pi}^{\pi} \exp(i\omega l) S(\omega)\mathrm{d}\omega, \tag{6.44}$$

which can be solved to find

$$S(\omega) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \exp(-i\omega l) R(l). \tag{6.45}$$

Hypothesis Testing

Bayesian Approach Assign *costs* C_{ij}, $i, j = 0, 1$ incurred by choosing hypothesis H_i when H_j is true. We say these costs are *uniform* if $C_{ij} = 0$ when $i = j$ and $C_{ij} = 1$ when $i \neq j$.

Conditional Risk R_j of a decision rule δ is

$$R_j(\delta) = C_{0j} P_j(\mathcal{R}) + C_{1j} P_1(\mathcal{R}') \tag{6.46}$$

where $P_j(\mathcal{R}) = P(X = x \in \mathcal{R} | H_j) = \int_{\mathcal{R}} p_j(x) dx$ is the probability distribution of the data wehen hypothesis H_j is true.

A priori probabilities/ priors π_0, $\pi_1 = 1 - \pi_0$ of hypotheses H_0 and H_1.
Bayes' Risk average cost of rule δ:

$$\begin{aligned} r(\delta) &= \pi_0 R_0(\delta) + \pi_1 R_1(\delta) \\ &= \pi_0 C_{00} + \pi_1 \end{aligned} \tag{6.47}$$

Matched filter

$$x(t) = s(t) + n(t) \tag{6.48}$$

- x =data
- s =signal, deterministic
- n =noise, additive, continuous Gaussian stochastic process, zero mean.

Consider it to be continuous for theory, obtain results by suitable sampling of continuous expressions.

Observed on interval $[0, T_0]$. Can be shown that log-likelihood is (the Cameron-Martin formula)

$$\ln \Lambda[x] = \int_0^{T_0} q(t)x(t)\mathrm{d}t - \frac{1}{2}\int_0^{T_0} q(t)s(t)\mathrm{d}t \tag{6.49}$$

where q is the soln to

$$s(t) = \int_0^{T_0} K(t, t')q(t')\mathrm{d}t'. \tag{6.50}$$

and K is the autocorrelation function of the noise. This means the likelihood ratio test is equivalent to comparing

$$G := \int_0^{T_0} q(t)x(t)\mathrm{d}t \tag{6.51}$$

to a threshold G_0. Hypothesis test expectations:

$$E_0\{G\} = 0, E_1\{G\} = \int_0^{T_0} q(t)s(t)\mathrm{d}t. \tag{6.52}$$

$$\begin{aligned} Var\{G\} &= \int_0^{T_0}\int_0^{T_0} q(t_1)q(t_2)E\{n(t_1)n(t_2)\}\mathrm{d}t_1\mathrm{d}t_2 \\ &= \int_0^{T_0} q(t_1)q(t_2)K(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2 \\ &= int_0^{T_0} q(t)s(t)\mathrm{d}t = \rho^2 \end{aligned} \tag{6.53}$$

ρ is the *signal to noise ratio.*

Since data x is Gaussian, and G is linear in x, pdf's when signal is absent or present are

$$p_0(G) = \frac{1}{\sqrt{2\pi\rho^2}} \exp(-\frac{G^2}{2\rho^2} p_1(G) = \frac{1}{\sqrt{2\pi\rho^2}} \exp(-\frac{(G-\rho^2)^2}{2\rho^2} \tag{6.54}$$

Chapter 7

Model reduction for Gravitational Wave data analysis

Chapter 8

Reduced Basis Methods for Parameterised Partial Differential Equations

Numerical methods are used to solve PDEs, however for complicated problems the number of parameters required may be quite large, resulting in prohibitive computational cost. The reduced basis method is a means of reducing the number of degrees of freedom in order to reduce costs and make solving the problem feasible. Sometimes we are interested in the field that is the solution to the PDE, perhaps a termperature function. Other times we are intersted in a function of this field, perhaps the maximum temperature. This method evaluates the relationship between the input parameters and the output function, hopefully bypassing many of the expensive computations. This is useful in many real-time and multiple query contexts.

8.1 Problem Formulation

Consider a PDE with $P \geq 1$ parameters and let $\mathcal{D} \subset \mathbb{R}^P$ denote the parameter domain such that $\overrightarrow{\mu} \in \mathcal{D}$ is a particular parameter P-tuple. These parameters describe properties of the system. In the case of binary coalescence systems there are eight parameters: the mass of each of the compact objects, the spin magnitude, and the spin orientation (2 parameters for each object).

The solutions to the PDE system reside on a manifold defined as

$$\mathcal{M} = \{u(\mu) : \mu \in \mathcal{D}\} \tag{8.1}$$

The PDE can be written

$$F_\mu(u(\mu)) = 0 \tag{8.2}$$

where F_μ is a parameter dependent differential operator and $u(\mu)$ is the solution to the system for parameter μ.

Recall (from DG chapter) the dual space V^* of a vector space V is the space of (bounded?) linear functionals (aka linear forms) $g : V \to \mathbb{R}$ over V. The norm over V^* is

$$||g||_{V*} \equiv \sup_{v \in V} \frac{g(v)}{||v||_V}$$

The Reisz Representation Theorem tells us that for $g \in V^*$ $\exists!$ w_g s.t. $(w_g, v)_V = g(v)$ $\forall v \in V$. We can then invoke the Cauchy Schwarz inequality to show that $||g||_{V^*} = ||w_g||_V$.

For a bilinear form $b : V \times V \to \mathbb{R}$ we define the symmetric part $b_s(w, v) = \frac{1}{2}(b(w, v)+b(v, w))$, and the skew symmetric part $b_{ss} = \frac{1}{2}(b(w, v) - b(v, w))$. It is *coercive* if

$$\alpha = \inf_{w \in V} \frac{b(w, w)}{||w||_V^2} > 0 \tag{8.3}$$

Where α is the *coercivity constant*. It is continuous if

$$\gamma = \sup_{w \in V} \sup_{v \in V} \frac{b(w, v)}{||w||_V ||v||_V} < \infty \tag{8.4}$$

where γ is the *continuity constant*.

$g : V \times \mathcal{D} \to \mathbb{R}$ is a *parametric linear form* if $\forall \mu \in \mathcal{D}$, $g(\cdot; \mu) : V \to \mathbb{R}$ is linear and continuous if $\forall \mu \in \mathcal{D}$, $g(\cdot; \mu) \in V^*$. A parametric bilinear form is defined similarly, and is coercive if

$$\alpha(\mu) = \inf_{w \in V} \frac{b(w, w; \mu)}{||w||_V^2} > 0 \quad \forall \mu \in \mathcal{D} \tag{8.5}$$

Then we define $\alpha_0 \equiv \min_{\mu \in \mathcal{D}} \alpha(\mu)$. A parametric bilinear form is continuous if

$$\gamma(\mu) = \sup_{w \in V} \sup_{v \in V} \frac{b(w, v; \mu)}{||w||_V ||v||_V} < \infty \quad \forall \mu \in \mathcal{D} \tag{8.6}$$

and we define $\gamma_0 \equiv \max_{\mu \in \mathcal{D}} \gamma(\mu)$.

A parametric bilinear form has *affine parameter dependence* if it can be written in the form

$$b(w, v; \mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu) b^q(w, v) \tag{8.7}$$

for some finite Q_b, and where $\Theta_b^q : \mathcal{D} \to \mathbb{R}$ are smooth and $b^q : V_1 \times V_2 \to \mathbb{R}$ are parameter independent continuous bilinear forms. A coercive affine parametric bilinear form is *parametrically coercive* if b_s can be written

$$b_s(w, v; \mu) = \sum_{q=1}^{Q_{b_s}} \Theta_{b_s}^q(\mu) b_s^q(w, v)$$

with $\Theta^q_{b_s}(\mu) > 0 \quad \forall \mu \in \mathcal{D}$ and $b^q_s(,v) \geq 0 \quad \forall v \in V$.

The *inf-sup stability "constant"* of a parametric bilinear form is defined as

$$\beta(\mu) = \inf_{w \in V_1} \sup_{v \in V_2} \frac{(w, v; \mu)}{||w||_{V_1} ||v||_{V_2}} \geq 0. \tag{8.8}$$

If $\exists \beta_0 > 0$ s.t. $\beta(\mu) \geq \beta_0 \; \forall \mu \in \mathcal{D}$, then we say b is *inf-sup stable* over $V_1 \times V_2$.

The *supremising operator* $T_\mu : V_1 \to V_2$ associated with b is defined as

$$T_\mu w = \arg \sup_{v \in V_2} \frac{b(w, v; \mu)}{||v||_{V_2}} \tag{8.9}$$

Note that $(T_\mu w, v)_{V_2} = b(w, v; \mu) \quad \forall v \in V_2$. If b admits an affine representation, then

$$T_\mu w = \sum_{q=1}^{Q_b} \Theta^q_b(\mu) \mathbb{T}^q w \tag{8.10}$$

where $\mathbb{T}^q : V_1 \to V_2$ is given by $(\mathbb{T}^q w, v)_{V_2} = b^q(v, w) \quad \forall v \in V_2$.

8.2 Parameter Domains and Grids

Consider our parameter domain $\mathcal{D} \subset \mathbb{R}^P$, and a parameter value $\mu \in \mathcal{D}$. We define $\mathcal{D}_{box} \subset \mathbb{R}^P$ to be the smallest P dimensional parallelipiped such that $\mathcal{D} \subset \mathcal{D}_{box}$, $\mathcal{D} \equiv [\mu_1^{min}, \mu_1^{max}] \times \ldots [\mu_P^{min}, \mu_P^{max}]$. If $\mu^{min} = \min_{p \in 1 \ldots P} \mu_p^{min} > 0$ we can perform a logarithmic transformation $\hat{\mu_p} = \ln \mu_p$, and $\hat{\mathcal{D}}, \hat{\mathcal{D}}_{box}$ are the images of $\mathcal{D}, \mathcal{D}_{box}$ under this transformation.

Three ways to draw points over $\mathcal{D}$ are as follows:

Monte Carlo sampling to create a uniform distribution of m points over $\mathcal{D}$. To create a linear grid $G^{lin}_{[MC;m]}$ parameter values are generated by setting

$$\mu_p = \mu_p^{min} + \text{rand}(\mu_p^{max} - \mu_p^{min}) \tag{8.11}$$

where rand is a random variable uniformly distributed over $[0, 1]$. Any $\mu = [\mu_1 \ldots \mu_P] \notin \mathcal{D}$ is rejected. To create a uniform distribution over the logarithmic space $\hat{\mathcal{D}}$, we sample the points as

$$\mu_p = \mu_p^{min} \exp \left\{ \text{rand} \times \ln \left(\frac{\mu_p^{max}}{\mu_p^{min}} \right) \right\}. \tag{8.12}$$

1D deterministic grids: $G^{lin}_{[z_1,z_2;m]}, G^{ln}_{[z_1,z_2;m]} : z_1 \in \mathbb{R} > z_2 \in \mathbb{R}, m \in \mathbb{N}$:

$$\begin{array}{rcl} G^{lin}_{[z_1,z_2;m]} & = & \{z_1 + \frac{i-1}{m-1}(z_2 - z_1), 1 \leq i \leq m\} \\ G^{ln}_{[z_1,z_2;m]} & = & \{z_1 \exp\left(\frac{i-1}{m-1} \ln(z_2/z_1)\right), 1 \leq i \leq m, z_1, z_2 > 0\} \end{array} \tag{8.13}$$

Chebyshev Gauss-Lobatto points

$$\begin{array}{rcl} G^{lin,Cheb}_{[z_1,z_2;m]} & = & \{z_1 + \frac{1}{2}\left(1 - \cos\pi\left(\frac{i-1}{m-1}\right)\right)(z_2 - z_1), 1 \leq i \leq m\} \\ G^{ln,Cheb}_{[z_1,z_2;m]} & = & \{z_1 exp\left(\frac{1}{2}\left(1 - \cos\pi\left(\frac{i-1}{m-1}\right)\ln(z_2/z_1)\right)\right), 1 \leq i \leq m, z_1, z_2 > 0\} \end{array} \tag{8.14}$$

1D grids can be combined to create multi-dimensional grids.

8.3 Parametric Scalar and vector fields

Consider a parametric scalar or vector field $w : \Omega \times \mathcal{D} \to \mathbb{R}^d$ where $d = 1$ for a scalar field, or the dimension of Ω for a vector field. The *parametric derivative*, also known as the *sensitivity derivative*, measures changes in w as the parameter changes.

$$(D_\sigma w)(\mathbf{x};\mu) = \frac{\partial^\sigma w}{\partial\mu_1^{\sigma_1}\cdots\partial\mu_P^{\sigma_P}} \tag{8.15}$$

where $\sigma = (\sigma_1, \cdots, \sigma_P)$ is an index vector of non negative integers and we denote by $|\sigma| = \sum_{i=1}^P \sigma_i$ the order of the derivative. $I^{P,n} = \{\sigma \in \mathbb{N}_0^P || \sigma| \leq n\}$.

A parametric scalar field, or one component of a parametric vector field is *separable* over Ω if for some finite M

$$w(\mathbf{x};\mu) = \sum_{j=1}^{M} h^j(\mathbf{x})g^j(\mu), \qquad \forall \mathbf{x} \in \Omega, \mu \in \mathcal{D} \tag{8.16}$$

for $h^j : \Omega \to \mathbb{R}$, $g^j : \mathcal{D} \to \mathbb{R}$, $1 \leq j \leq M$. w is $\mathbf{x}$-affine separable over Ω if each $h^j(\mathbf{x})$ is affine in $\mathbf{x}$.

8.4 Problem statement

8.4.1 Exact

We consider a parametric PDE over a physical domain $\Omega \in \mathbb{R}^d$ with boundary $\partial\Omega$, where the field variables $w \in X^e$, where X^e is our exact solution space, may be scalar $w : \Omega \to \mathbb{R}$ or vector-valued $w : \Omega \to \mathbb{R}^d$. We denote by Γ_i^D boundary measurable segments of $\partial\Omega$ on which we impose Dirichlet boundary conditions.

We construct the exact solution space X^e as the Cartesian product of scalar spaces $Y_i^e \equiv v \in H^1(\Omega)|v|_{\Gamma_i^D} = 0$ where $1 \leq i \leq d$ if the field variable is vector-valued, or simply $i = 1$ if the field variable is scalar, in which case the subscript can be omitted. An element $w \in X^e$ is

denoted $w = (w_1, ..., w_d)$. We have a number of options for the inner product and norm with which to equip X^e.

Energy norm (for coercive and continuous a):

$$(((w, v))) = a(w, v), \qquad \forall w, v \in X^e \tag{8.17}$$

$$|||w||| = \sqrt{a(w, w)}, hspace1cm \forall w \in X^e \tag{8.18}$$

We can then define the parametric field variable as $u : \Omega \times \mathcal{D} \to \mathbb{R}$ where for a given $\mu \in \mathcal{D}$, $u(\cdot; \mu) \in X^e$.

Parametric weak form

Suppose we are given parametric linear forms f and l that are bounded over X^e, and an affine parametric bilinear form a that is inf-sum stable and continuous over X^e, and which we assume is parametrically coercive. Then the problem is given $\mu \in \mathcal{D}$ we find $u^e(\cdot; \mu) \in X^e$ such that

$$a(u^e(\mu), v; \mu) = f(v; \mu), \qquad \forall v \in X^e \tag{8.19}$$

and evaluate our output of interest

$$s^e(\mu) = l(u^e(\mu); \mu). \tag{8.20}$$

A problem is *compliant* if

1. $l(\cdot; \mu) = f(\cdot; \mu), \forall \mu \in \mathcal{D}$ that is the output functional and "load/source are identical, and
2. a is symmetric.

8.4.2 "Truth" Approximation

We introduce a family of *conforming* approximation spaces $X^{\mathcal{N}} \subset X^e$ of dimension $\mathcal{N}$. As in the exact case we form $X^{\mathcal{N}}$ as a Cartesian product of scalar approximations $Y_i^{\mathcal{N}_i} \subset Y_i^e$ of dimension $\mathcal{N}_i$ such that $X^{\mathcal{N}} = Y_i^{\mathcal{N}_1} \times \cdots \times Y_i^{\mathcal{N}_d}$, $\mathcal{N} = \sum_i \mathcal{N}_i$. We choose a set of basis functions $\psi_k^{\mathcal{N}} \in X^{\mathcal{N}}$, $1 \leq k \leq \mathcal{N}$. The inner products and norms are inherited from the exact case. We require that the family of truth subspaces $X^{\mathcal{N}}$ satisfies the approximation condition that for any $\epsilon > 0$, $\exists \mathcal{N}$ such that the error in the best fit to $u(\mu)$ in $X^{\mathcal{N}}$ is less than or equal to ϵ for all μ in $\mathcal{D}$:

$$\max_{\mu \in \mathcal{D}} \inf_{w \in X^{\mathcal{N}}} ||u(\mu) - w||_{X^e} \to 0 \text{ as } \mathcal{N} \to \infty \tag{8.21}$$

The family of approximations to the exact problems is given by finding $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ such that

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v; \mu) \qquad \forall v \in X^{\mathcal{N}} \tag{8.22}$$

and evaluating

$$s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu); \mu) \tag{8.23}$$

for a given $\mu \in \mathcal{D}$. This is a standard Galerkin projection. We implicitly assume that we can represent the exact geometry of the domain, and that quadratures are exact. From 8.21 we have that as $\mathcal{N} \to \infty$, $u^{\mathcal{N}}(\cdot; \mu) \to u^e(\cdot; \mu)$. We define the error

$$\epsilon^{\mathcal{N}} = \max_{\mu \in \mathcal{D}} ||u(\cdot; \mu) - u^{\mathcal{N}}(\cdot; \mu)||_{X^e} \tag{8.24}$$

then $\epsilon^{\mathcal{N}} \to 0$ as $\mathcal{N} \to \infty$.

If we expand the solution $u^{\mathcal{N}}(\mu)$ in our basis $\psi_k^{\mathcal{N}}$ as

$$u^{\mathcal{N}}(\mathbf{x}; \mu) = \sum_{j=1}^{\mathcal{N}} u_j^{\mathcal{N}}(\mu) \psi_j^{\mathcal{N}}(\mathbf{x}), \tag{8.25}$$

then

$$\underline{u}^{\mathcal{N}}(\mu) \equiv [u_1^{\mathcal{N}} \cdots u_{\mathcal{N}}^{\mathcal{N}}]^T \in \mathbb{R}^{\mathcal{N}} \tag{8.26}$$

and the problem can be written

$$\underline{A}^{\mathcal{N}}(\mu) \underline{u}^{\mathcal{N}}(\mu) = \underline{F}^{\mathcal{N}}(\mu) \tag{8.27}$$

where the stiffness matrix $\underline{A}^{\mathcal{N}}(\mu) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ has elements given by

$$A_{ij}^{\mathcal{N}}(\mu) = a(\psi_j^{\mathcal{N}}, \psi_i^{\mathcal{N}}; \mu) \tag{8.28}$$

and the components of the load/source vector (and in the compliant case, output vector), are given by

$$F_i^{\mathcal{N}}(\mu) = f(\psi_i^{\mathcal{N}}; \mu). \tag{8.29}$$

Invoking the affine properties of f and a, we can write these as follows:

$$\begin{gathered} \underline{A}^{\mathcal{N}}(\mu) = \textstyle\sum_{q=1}^{Q_a} \Theta_a^q(\mu) \mathbb{A}^{\mathcal{N}_q}, \\ \mathbb{A}_{ij}^{\mathcal{N}_q} = a^q(\psi_j^{\mathcal{N}}, \psi_i^{\mathcal{N}}) \\ 1 \le i, j \le \mathcal{N},\ 1 \le q \le Q_a \end{gathered} \tag{8.30}$$

and

$$\begin{gathered} \underline{F}^{\mathcal{N}}(\mu) = \textstyle\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \mathbb{F}^{\mathcal{N}_q}, \\ \mathbb{F}_i^{\mathcal{N}_q} = f^q(\psi_i^{\mathcal{N}}), \\ 1 \le i \le \mathcal{N},\ 1 \le q \le Q_f \end{gathered} \tag{8.31}$$

Note that $\mathbb{F}^{\mathcal{N}_q}$ and $\mathbb{A}^{\mathcal{N}_q}$ are parameter independent. We also introduce another parameter independent matrix $\mathbb{X}^{\mathcal{N}}$:

$$\mathbb{X}^{\mathcal{N}}_{ij} = (\psi^{\mathcal{N}}_j, \psi^{\mathcal{N}}_i)_{X^{\mathcal{N}}},\ 1 \le i, j \le \mathcal{N} \tag{8.32}$$

so that for any two members $w, v \in X^{\mathcal{N}}$ written in terms of the $\psi^{\mathcal{N}}$ basis, the $X^{\mathcal{N}}$- inner product can be written

$$(w, v)_{X^{\mathcal{N}}} = \underline{w}^T \mathbb{X}^{\mathcal{N}} \underline{v} \tag{8.33}$$

For a family of approximation spaces, we define the "truth approximation space" $X^{\mathcal{N}}$ to correspond to $\mathcal{N} = \mathcal{N}_t$, which is typically quite large, such that $\epsilon^{\mathcal{N}_t}$ is smaller (in the worst case scenario) than some error threshold. We can then consider the parametrically induced manifold $\mathcal{M} \equiv \{u^{\mathcal{N}_t}(\mu) | \mu \in \mathcal{D}\}$, which is typically of comparably low dimension. We seek to represent the solution $u^{\mathcal{N}}(\cdot; \mu)$ in terms of elements of the space span$\{\mathcal{M}^{\mathcal{N}_t}\}$.

8.5 Reduced Basis Approximation

8.5.1 Lagrangian approach

For $\mu \in \mathcal{D}$, approximate $u^{\mathcal{N}_t} = u^{\mathcal{N}_t}(\mathbf{x}; \mu)$ by a linear combination of N precomputed "snapshots" on $\mathcal{M}^{\mathcal{N}_t}$: $u^1, \cdots, u^N$. We specify the maximum dimension of the reduced basis space $N_{\max}$. We introduce a set of parameter points $\mu^n \in \mathcal{D}$ and define our RB samples sets

$$S_N = \{\mu_1, \cdots \mu_N\} \tag{8.34}$$

Note that these sets are nested $S_1 \subset S_2 \subset \cdots \subset S_{N_{max}} \subset \mathcal{D}$. The "snapshots" are then

$$u^n \equiv u(\cdot; \mu^n) \tag{8.35}$$

in terms of which we define our Lagrange RB spaces:

$$W_N = \text{span}\{u^n, 1 \le n \le N\}\ 1 \le N \le N_{\max} \tag{8.36}$$

These spaces are hierarchical. It is also possible to define the RB spaces in other ways, for example the Taylor RB spaces are generated by both the field u^n and the field sensitivity derivatives- the derivative of $u(\cdot; \mu^n)$ with respect to μ- at a particular point in $\mathcal{D}$, while the Hermite RB spaces combine both of these and consider the field and sensitivity derivatives at several points in $\mathcal{D}$.

The snapshots can be written in terms of the finite element basis function $\psi^{\mathcal{N}_t}_i$ as per equation 8.25. We can then construct an orthogonal basis for the Lagrange RB space using Gram Schmidt orthogonalisation with $\mathbb{X}$ the inner product matrix defined in 8.32.:

We can now define "basis" matrices $\mathbb{Z}_N^{\mathcal{N}_t} \in \mathbb{R}^{\mathcal{N}_t \times N}$ as

$$\mathbb{Z}_{Njn}^{\mathcal{N}_t} = \hat{u}_j^n \tag{8.37}$$

That is the nth column of $\mathbb{Z}_{N_{\max}}^{\mathcal{N}_t}$ contains the vector of truth approximation finite element basis coefficients associated with the nth RB basis function. Note that $\mathbb{Z}_n^{\mathcal{N}_t}$ is a principal submatrix of $\mathbb{Z}_{n+1}$. We can express the orthogonality condition as

$$\mathbb{Z}_{N_{\max}}^T \mathbb{X} \mathbb{Z}_{N_{\max}} = \mathbb{I}_{N_{\max}} \tag{8.38}$$

We can now express our problem in the RB approximation space. We look for $u_{X_N} \in X_N$ such that

$$a(u_{X_N}, v; \mu) = f(v; \mu), \ \forall v \in X_N \tag{8.39}$$

and evaluate

$$s_{X_N}(\mu) = f(u_{X_N}(\mu); \mu) \tag{8.40}$$

8.6 Reduced basis for binary coalescences

Binary coalescence systems are described by eight parameters: the mass of each of the compact objects, the spin magnitude, and the spin orientation (2 parameters for each object).

Search via matched filtering- comparing possible waveform templates to actual data from detector. For low mass systems the inspiral part of the coalescence is at detectable frequencies. At higher masses, the merger part is (this requires NR techniques). These are expensive, therefore we search for the most relevant points in the parameter space. Searching is also expensive AND time critical so that events can be matched with EM astronomy.

GW is dependent on time (or frequency) and P parameters $\overrightarrow{\mu} = \mu_1, ..., \mu_P$. Denote a GW by $h_{\overrightarrow{\mu}}$. $\mathcal{H}$ is the space of all normalised GWs for a source. This space is non-linear, but can be represented by a linear space to arbitrarily high accuracy.

Approximate $\mathcal{H}$ with best linear combinations of members $\Psi_i \equiv h_{\overrightarrow{\mu_i}}$ of a catalogue $C_N = \{\Psi_i\}_{i=1}^N$. The reduced basis space is $W_N = span(C_N)$.

Optimal method: Choose waveforms to make this catalog so that the error in representing $\mathcal{H}$ with W_N is minimised over the choice of N catalogue members. Optimal error given by Kolmogorov N width:

$$d_N(\mathcal{H}) = \min_{C_N} \max_{\overrightarrow{\mu}} \min_{u \in W_N} \|u - h_{\overrightarrow{\mu}}\|$$

Where the norm is calculated from the complex inner product such that for two waveforms F and G in Fourier space

$$\langle F, G \rangle \equiv \int_{f_L}^{f_U} \frac{F^*(f)G(f)}{S_n(f)} df$$

where $S_n(f)$ is the power spectral density of the detector.

In reality, this approach is too computationally expensive. Use greedy method.

8.7 Greedy Algorithm

A greedy algorithm solves a problem by making the locally optimal choice at each stage in the hope that this will lead to a global optimum.

Choose a waveform with arbitrary parameter value. Basis vector $e_1 = h_{\overrightarrow{\mu_1}}$ is identified with this waveform, and the first catalogue $C_1 = \Psi_1 = h_{\overrightarrow{\mu_1}}$. Subsequent catalogues are built by adding another waveform by finding

$$\overrightarrow{\mu_j} = \underset{\mu}{\operatorname{argmin}} \|h_\mu - P_{j-1}(h_\mu)\| \tag{8.41}$$

where $P_N(h_\mu) = \sum_{i=1}^{N} e_i \langle e_i, h_\mu \rangle$ is the representation of h_μ using the catalogue C_N obtained by finding orthogonal projection of h_μ onto W_N. Note that this creates a hierarchical sequence of catalogues $C_1 \subset C_2$....

The *greedy error* ϵ_N is the maximum error for a catalog C_N, given by

$$\epsilon_N \equiv \max_{\mu} \|h_{\overrightarrow{\mu}} - P_N(h_\mu\| \tag{8.42}$$

It can be shown that if the decay of the N-width can be bounded by an exponential, $d_N(\mathcal{H}) \leq A \exp^{-cN^\alpha}$ for some real c and α, then $\epsilon_N \leq \tilde{A} \exp^{-dN^\beta}$ where d and β depend on c and α.

Chapter 9

Model Reduction for Compact Binary Coalescence

The highly anticipated detection of gravitational waves in 2015 from the coalescence of two stellar mass black holes heralded the beginning of a new era of astronomy. For the foreseeable future, compact binary coalescenses (CBCs) will remain the most detectable sources of gravitational waves. The the detection of these events requires high fidelity waveforms for use in the matched filtering pipeline. However, calculating these waveforms can take weeks to months of supercomputer resources, so adequately surveying the high dimensional parameter space is a computationally infeasible undertaking.

A gravitational waveform is represented in terms of its two polarisations $h_+(t;\lambda)$ and $h_\times(t;\lambda)$ by $h(t;\lambda) = h_+(t;\lambda) + ih_\times(t;\lambda)$, where $\lambda \in \mathcal{P} \subset \mathbb{R}^8$ is a vector describing the binary source parameters (masses, spin magnitudes, spin directions). We denote the space of gravitational waveforms by X. An inner product on this space is given by the complex scalar product

$$< h(\cdot;\lambda_1), h(\cdot;\lambda_2) >= \int_{t_{min}}^{t_{max}} h^*(t;\lambda_1)h(t;\lambda_2)dt \tag{9.1}$$

with an inherited norm given by $\|h(\cdot;\lambda)\|^2 =< h(\cdot;\lambda), h(\cdot;\lambda) >$. We may write $h(\lambda)$ to denote the full waveform, i.e. $h : \mathcal{P} \to X$, or alternatively $h(t;\lambda)$, in which case $h(t;\lambda)$ denotes the value at time t, i.e. $h : \mathbb{R} \times \mathcal{P} \to \mathbb{C}$.

Question What is the space X? What space do h_+,$h_\times$ reside in?

The space X may have very high dimension, however the space of gravitational waveforms resides on a parametrically induced manifold $\mathcal{M} = \{h(\lambda) : \lambda \in \mathcal{P}\}$, which may be of comparitively low dimension. Thus, for arbitrary λ we seek to represent $h(\lambda)$ as a linear combination of

some basis for span$\{\mathcal{M}\} \subset X$, that is

$$h(t;\lambda) \approx \sum_{i=1}^{N} A_i(\lambda) e_i(t) \tag{9.2}$$

for some basis e_i, and as small an N as possible. This raises some immediate questions.

1. How do we optimally choose the the best N-dimensional subspace
2. How do we choose a basis for this subspace
3. How do we best combine this basis to approximate $h(\lambda)$?

To answer the first question, we set the benchmark by defining the Kolmogorov spaces, X_N^{Kol} as

$$X_N^{Kol} = \arg \inf_{X_N \subset X} \left(\sup_{\lambda \in \mathcal{P}} \inf_{h_N \in X_N} \|h(\lambda) - h_N\|_X \right) \tag{9.3}$$

where X_N denotes any N-dimensional subspace of N. The Kolmogorov N-width is then

$$\epsilon_N^{Kol} = \sup_{\lambda \in \mathcal{P}} \inf_{h_N \in X_N^{Kol}} \|h(\lambda) - h_N\|_X. \tag{9.4}$$

Conceptually, the Kolmogorov spaces are those that minimise the error between $h(\lambda)$ for any $\lambda \in \mathcal{P}$ and the "closest" element of that space, and the Kolmogorov N-width is the supremum of that error. This is a very strong definition, however actually finding such a space is impractical.

9.1 Empirical Interpolation Method

We seek to find approximations to elements of $\mathcal{M}$ by developing an operator $\mathcal{I}$ to interpolate the waveforms $h(\cdot;\lambda)$. The method takes as input a tolerance ϵ, and optionally a maximal number of iterations N_{max}. After N iterations, the output is a set of N basis functions $\{e_i\}_{i=1}^N$ and interpolation points $\{T_i\}_{i=1}^N$. Given these, the interpolant $\mathcal{I}_N h(\cdot;\lambda)$ of $h(\lambda)$ can be expressed as

$$\mathcal{I}_N h(t;\lambda) = \sum_{i=1}^{N} A_i(\lambda) e_i(t). \tag{9.5}$$

The interpolant satisfies the N interpolation constraints

$$\mathcal{I}_N h(T_i;\lambda) = h(T_i;\lambda), \tag{9.6}$$

Algorithm 9.1 The Empirical Interpolation Method (Continuous)

Input: N_{max}, ϵ

1: $N = 0$, $\epsilon' = \epsilon + 1$, $\mathcal{I}_0 h(t;\lambda) = 0$
2: $\lambda_1 = \underset{\lambda\in\mathcal{P}}{\operatorname{argmax}} \|h(\cdot;\lambda)\|_{L^\infty(\mathbb{R})}$
3: **while** $N < N_{max}$ and $\epsilon' > \epsilon$ **do**
4: $\quad N \leftarrow N + 1$
5: $\quad r(t) = h(t;\lambda_N) - \mathcal{I}_{N-1} h(t;\lambda_N)$
6: $\quad T_N = \underset{t\in\mathbb{R}}{\operatorname{argmax}} \|r(t)\|$
7: $\quad e_N(t) = r(t)/r(T_N)$
8: $\quad [\epsilon', \lambda_{N+1}] = \underset{\lambda\in\mathcal{P}}{\operatorname{argmax}} \|h(\cdot;\lambda) - \mathcal{I}_N h(t;\lambda)\|_{L^\infty(\mathbb{R})}$
9: **end while**

Output: $\{e_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N$

leading to the following matrix equation

$$\mathbb{V}\mathbf{A}(\lambda) = \mathbf{h}(\lambda), \tag{9.7}$$

where $(\mathbb{V})_{ij} = e_j(T_i)$, $(\mathbf{A}(\lambda))_i = A_i(\lambda)$, and $(\mathbf{h})_i = h(T_i;\lambda)$. The algorithm proceeds as follows.

This produces a sequence of hierarchical spaces $X_1 \subset X_2 \subset \dots X_N$ where $X_n = \operatorname{span} e_{i=1}^n$.

We start with a training set of parameter values $\Xi_{train} = \{\lambda_i\}_{i=1}^M \subset \mathcal{P}$ and associated waveforms $\{h(\lambda_i)\}$. The EIM method proceeds by identifying a set of parameter values $\{\Lambda_1, ...\Lambda_N\}$ and associated waveforms $\{h_1(t)...h_N(t)\}$ where $h_i(t) = h(t;\Lambda_i)$ to form a basis for the greedy subspace X_N.

The spaces constructed are hierarchical, that is, for $n < n'$, $X_n \subset X_{n'}$.

An ideal greedy algorithm would

Explore elementary effects.

Chapter 10

Reduced basis methods for parameterised PDEs

10.1 Forward model

Let $\lambda \in \mathcal{P} \subset \mathbb{R}^N$ be a parameter vector, and $\Omega \in \mathbb{R}^d$ be our function domain. The forward model $\mathcal{M} : \mathcal{P} \times \Omega \rightarrow$

10.2 Statistical Inverse Problems

In a deterministic setting, regularisation and optimisation techniques are used to find a single point estimate of the parameter $\mathbf{p}$. A statistical formaulation returns a probability density function over parameter space describing the relative likelihood of observation-consistent parameters, called the 'posterior distribution', $\pi(\mathbf{p}|\mathbf{y}_d)$.

Here we infer properties of the probability distribution of model parameters $\mathbf{p} \in \mathbb{R}^N$ from observations $\mathbf{y}_d \in \mathbb{R}^m$. We adopt a Bayesian approach where a prior probability density $\pi(\mathbf{p})$ of $\mathbf{p}$ is given. We also assume that our model of the observations provides us with a likelihood function $\pi(\mathbf{y}_d \mid \mathbf{p})$ of the data $\mathbf{y}_d$. The data $\mathbf{y}_d$ is treated as a random vector $\mathbf{y}$ with probability distribution $\pi(\mathbf{y} \mid \mathbf{p})$ and we denote the expectation as

$$\mathbf{y}(\mathbf{p}) := E(\mathbf{y} \mid \mathbf{p}).$$

Bayes' law leads to a formula for the posterior probability density of $\mathbf{p}$ as

$$\pi(\mathbf{p} \mid \mathbf{y}_d) \propto \pi(\mathbf{p})\, \pi(\mathbf{y}_d \mid \mathbf{p}). \tag{10.1}$$

This probability density could then be further explored for example, in order to find marginals or moments of the posterior or even the MAP (maximum a posteriori) estimate which is of the form

$$\mathbf{p}_{\text{MAP}} = \arg\max_{\mathbf{p}\in\mathbb{R}^N} \pi(\mathbf{p} \mid \mathbf{y}_d). \tag{10.2}$$

In any case, the exploration of the posterior $\pi(\mathbf{p} \mid \mathbf{y}_d)$ requires its numerical evaluation typically many times. In our case each evaluation of the likelihood (and thus the posterior) requires an expensive numerical simulation.

10.2.1 Reduced basis method

In order to decrease the computational load required we reduce the parameter space from $\mathbb{R}^N$ to $\mathbb{R}^{N_r}$ as any exploration or computation in a lower dimensional space is cheaper. For this we generate a sequence of parameter vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots$ in $\mathbb{R}^N$ which are defined using a greedy algorithm

$$\mathbf{p}_{k+1} = \arg\max_{\mathbf{p}\in\mathbb{R}^N} \left(\frac{1}{2}\|\mathbf{y}(\mathbf{p}) - \mathbf{y}(Q_k\mathbf{p})\|^2 + \beta \log \pi(\mathbf{p}) \right), \tag{10.3}$$

where the $Q_k : \mathbb{R}^N \to \mathbb{R}^N$ are orthogonal projections with $\text{range}(Q_k) = \text{span}\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ for $k \geq 1$ and $Q_0 := 0$. This choice introduces the new $\mathbf{p}_{k+1}$ to make sure that a large number of observations $\mathbf{y}_d$ can be approximated in the range of Q_{k+1} while exploring parameter vectors which are sufficiently likely according to the prior.

Numerical methods are used to solve PDEs, however for complicated problems the number of parameters required may be quite large, resulting in prohibitive computational cost. The reduced basis method is a means of reducing the number of degrees of freedom in order to reduce costs and make solving the problem feasible. Sometimes we are interested in the field that is the solution to the PDE, perhaps a termperature function. Other times we are intersted in a function of this field, perhaps the maximum temperature. This method evaluates the relationship between the input parameters and the output function, hopefully bypassing many of the expensive computations. This is useful in many real-time and multiple query contexts.

Consider a parameterised model $\mathcal{M} : \mathcal{P} \to \mathcal{U} \to \mathcal{Y}$ of some phenomenon, where given a parameter value $\mathbf{p}$ in the parameter space $\mathcal{P} \subset \mathbb{R}^N$, the model predicts a state $\mathbf{u}(\mathbf{p})$ in the state space $\mathcal{U} \subset \mathbb{R}^K$ from which we can extract output $\mathbf{y}(\mathbf{p}) \in \mathcal{Y} \subset \mathbb{R}^M$. Suppose in an experimental setting we obtain noisy observations $\mathbf{y}_d \in \mathcal{Y}$. The inverse problem is to estimate $\mathbf{p}$ given $\mathbf{y}_d$, a problem that is often ill-posed.

In order to decrease the computational load required we reduce the parameter space from $\mathbb{R}^N$ to $\mathbb{R}^{N_r}$ as any exploration or computation in a lower dimensional space is cheaper. For this we generate a sequence of parameter vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots$ in $\mathbb{R}^N$ which are defined using a greedy

algorithm

$$\mathbf{p}_{k+1} = \arg\max_{\mathbf{p}\in\mathbb{R}^N} \left(\frac{1}{2}\|\mathbf{y}(\mathbf{p}) - \mathbf{y}(Q_k\mathbf{p})\|^2 + \beta \log \pi(\mathbf{p}) \right), \tag{10.4}$$

where the $Q_k : \mathbb{R}^N \to \mathbb{R}^N$ are orthogonal projections with $\text{range}(Q_k) = \text{span}\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ for $k \geq 1$ and $Q_0 := 0$. This choice introduces the new $\mathbf{p}_{k+1}$ to make sure that a large number of observations $\mathbf{y}_d$ can be approximated in the range of Q_{k+1} while exploring parameter vectors which are sufficiently likely according to the prior. This method is discussed in detail in [**?**]parameter.

10.3 One dimensional wire

Consider a simple one dimensional case. We have a wire fixed at ends $x = 0$ and $x = 1$, under constant tension $T = 1$, and subject to a transverse load $f(x; p)$, where $p \in \mathcal{P}$ is our parameter. Let $u(x, t; p)$ denote the transverse displacement of the wire at point x and time t, with boundary condition $u(0, t; p) = u(1, t; p) = 0$. We find the initial condition $u_0(x; p) = u(x, 0; p) \in H_0[0, 1]$ by solving the one dimensional Poisson equation

$$-\frac{\partial^2 u_0}{\partial x^2}(x) = f(x; p). \tag{10.5}$$

The system time evolution is governed by the one dimensional wave equation, where the wire is assumed to be released from rest

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{10.6}$$

The observations $\mathbf{y}_i \in \mathbb{R}^j$ are a set of time series vectors of the displacement amplitudes at some points $x_i \in [0, 1]$ at j sample times.

10.3.1 Exact Solution

In this example we can explicitly compute the operator $\mathcal{M}$. The solution to (10.5) is

$$u_0(x; p) = \int_0^1 G(x, \xi) f(\xi; p) d\xi, \tag{10.7}$$

Where $G(x, \xi)$ is Green's function given by

$$G(x, \xi) = \begin{cases} \xi(1 - x) \text{ if } 0 \leq \xi \leq x \\ x(1 - \xi) \text{ if } x \leq \xi \leq 1. \end{cases} \tag{10.8}$$

The general solution to (10.6) is

$$u(x,t;p) = \sum_{n=1}^{\infty} \beta_n(p) \sin(n\pi x) \cos(cn\pi t). \tag{10.9}$$

We find by substituting $t = 0$ that the β_n are the Fourier sine series coefficients of u_0:

$$\beta_n(p) = 2 \int_0^1 u_0(x;p) \sin(n\pi x) dx. \tag{10.10}$$

The output $\mathbf{y}$ is found by substituting the observation points $\mathbf{x}$ into (10.7):

$$\mathbf{y}(t) = \sum_{n=1}^{\infty} \beta_n(p) \cos(cn\pi t) \sin(\mathbf{x}n\pi). \tag{10.11}$$

10.3.2 Numerical Solution

We now develop the operator $\mathcal{M}$ for the numerical solution to the system.

Initial condition

Consider a finite differencing scheme. Our domain $[0,1]$ is divided into n subintervals of length $h = \Delta x = 1/n$ so that the endpoints of the intervals are given by $x_j = j\Delta x = jh$ for $j = 0, 1, \cdots n$. Using a Taylor expansion of u around x we find

$$\begin{array}{lcl} -u_{xx}(x) = \frac{-u(x+h)+2u(x)-u(x-h)}{h^2} & = & f(x) \text{ by eq 10.5} \\ \rightarrow \frac{-u_{j+1}+2u_j-u_{j-1}}{h^2} = f(x_j) & & \end{array} \tag{10.12}$$

where $u_i = u(x_i)$. By setting the endpoints according to our boundary conditions $u_0 = u_n = 0$ and defining

$$\hat{u} = [u_1, ..., u_{n-1}]^T \ , \ \hat{f} = [f(x_1), ..., f(x_{n-1})], A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{10.13}$$

the initial value problem then becomes finding the solution to the matrix eq

$$A\hat{u} = h^2 \hat{f} \tag{10.14}$$

the solution of which is

$$\hat{u}_0(p) = A^{-1} h^2 \hat{f}(p) \tag{10.15}$$

Evolution

Defining $\hat{u}(x,t)$ similarly to above, 10.6 becomes

$$\frac{d^2\hat{u}}{dt^2} + c^2 A\hat{u} = 0 \tag{10.16}$$

with initial conditions $\hat{u}(0,p) = \hat{u}_0(p)$ and $\hat{u}'(0,p) = \hat{u}_1(p)$. This admits the solution

$$\hat{u}(t;p) = \cos(cA^{\frac{1}{2}}t)A^{-1}\hat{f}(p) \tag{10.17}$$

10.3.3 Reduced basis construction

To start, let us consider just the evolution equation (10.6). We construct the initial condition as follows

$$u_0 = \sum_{n=1}^{N} p_n \sin(2\pi n x)/n^2, \tag{10.18}$$

where the parameter $\mathbf{p} = [p_n]$ and (10.17) becomes

$$\hat{u}(t;p) = \cos(cA^{\frac{1}{2}}t) \begin{bmatrix} \sin(2\pi\mathbf{x}) & \cdots & \sin(2N\pi\mathbf{x})/N^2 \end{bmatrix} \mathbf{p}. \tag{10.19}$$

The number of gridpoints is fixed at $n = 100$. Let us take the output to be the displacement over times $t_1, \cdots, t_m$ just at one gridpoint x_s.

$$\mathbf{y} = \begin{bmatrix} \mathbf{e}_s^T \cos(cA^{\frac{1}{2}}t_1)\sin(2\pi\mathbf{x}) & \cdots & \mathbf{e}_s^T \cos(cA^{\frac{1}{2}}t_1)\sin(2N\pi\mathbf{x})/N^2 \\ \vdots & \ddots & \vdots \\ \mathbf{e}_s^T \cos(cA^{\frac{1}{2}}t_m)\sin(2\pi\mathbf{x}) & \cdots & \mathbf{e}_s^T \cos(cA^{\frac{1}{2}}t_m)\sin(2N\pi\mathbf{x})/N^2 \end{bmatrix} \mathbf{p} \equiv M\mathbf{p} \tag{10.20}$$

where $\mathbf{e}_s = [0, \ldots, 1 \ldots, 0]$ with the 1 in the s^{th} position.

We investigate how well we are able to approximate the output $\mathbf{y}(\mathbf{p})$ by reduced models $\mathbf{y}(Q_k\mathbf{p})$. We define approximations to the parameter vectors $\mathbf{p}_k$ defined in equation (10.4) by

$$\mathbf{p}_{k+1} = \arg\max_{\mathbf{p}\in\mathcal{M}_k} \frac{1}{2}\|\mathbf{y}(\mathbf{p}) - \mathbf{y}(Q_k\mathbf{p})\|^2. \tag{10.21}$$

where each $\mathcal{M}_k$ contains m independent samples of the uniform distribution over $\mathcal{P}$ and the the elements of $\mathcal{M}_k$ and $\mathcal{M}_j$ are independent for $j \neq k$. Note that this allows us to drop the $\beta \log \pi(\mathbf{p})$ term that appears in equation (10.4). We denote the approximations of $\mathbf{p}_k$ by the same symbol $\mathbf{p}_k$ for simplicity. Here $|\mathcal{M}_k| = 100$, and $Q_k = P_k(P_k^T P_k)^{-1}P_k^T$ where

$$P_k = [\mathbf{p}_1 \cdots \mathbf{p}_k].$$

To quantify how well the reduced model approximation $\mathbf{y}(Q_k\mathbf{p})$ can recover the full model $\mathbf{y}(\mathbf{p})$ for arbitrary $\mathbf{p}$, we use the relative L_2 norm

$$\frac{\sqrt{\int_{\mathcal{P}} ||\mathbf{y}(\mathbf{p}) - \mathbf{y}(Q_k\mathbf{p})||^2 \, d\omega}}{\sqrt{\int_{\mathcal{P}} ||\mathbf{y}(\mathbf{p})||^2 \, d\omega}},$$

where ω is a probability measure over $\mathcal{P}$. We approximate this relative root mean squared error by Monte Carlo quadrature as

$$\rho_k = \frac{\sqrt{\frac{1}{m}\sum_{\mathbf{p}\in\mathcal{M}_{k+1}} ||\mathbf{y}(\mathbf{p}) - \mathbf{y}(Q_k\mathbf{p})||^2}}{\sqrt{\frac{1}{m}\sum_{\mathbf{p}\in\mathcal{M}_{k+1}} ||\mathbf{y}(\mathbf{p})||^2}}. \tag{10.22}$$

Reduced bases were constructed for full basis dimensions $N = 5, 10, 15, 20, 25, 30, 40, 50, 60, 70, 80, 90, 100$. Figure **??** shows ρ_k vs k for each N, calculated using (10.22), and figure **??** shows the number of reduced basis vectors required to achieve $\rho_k < 0.05$. This shows that for more complex problems (a higher N), more reduced basis functions are required to achieve the error threshold, but the relative number of basis functions required decreases. In the case of $N = 100$, a reduced model error of $\rho_k < 0.05$ is achieved with 48 basis vectors, a reduction in the parameter space dimension of more than half.

Inverse Experiment

Next we investigate how well the reduced basis model we have constructed is able to reproduce the initial conditions on the wire. Simulated data $\mathbf{y}_d = \mathbf{y}(\mathbf{p}_0)$ is produced using a sample parameter value $\mathbf{p}_0$ from the uniform distribution over the parameter space $\mathcal{P}$. The problem in the full model case is to find the parameter $\mathbf{p}$ that satisfies

$$\mathbf{p} = \underset{\mathbf{p}'\in\mathcal{P}}{\arg\min} ||\mathbf{y}_d - \mathbf{y}(\mathbf{p}')||^2. \tag{10.23}$$

In the reduced model this becomes finding the reduced parameter $\mathbf{p}_r$ that satisfies

$$\mathbf{p}_r = \underset{\mathbf{p}'_r\in\mathcal{P}_r}{\arg\min} ||\mathbf{y}_d - \mathbf{y}(\mathbf{p}'_r)||^2. \tag{10.24}$$

For each of the full basis dimensions $N = 5, 10, 15, 20, 25, 30, 40, 50, 60, 70, 80, 90, 100$, ten simulated data sets were constructed. Equations (10.23) and (10.24) were solved using Nelder-Mead minimisation with an initial guess of $\mathbf{p}_0 + \Delta\mathbf{p}_0$ for the full case and $\mathbf{p}_0 + Q_k\Delta\mathbf{p}_0$ in the reduced

case, where the error $\Delta\mathbf{p}_0$ is randomly sampled from the uniform distribution over $\mathcal{P}$ and in each case normalised such that $\|\Delta\mathbf{p}_0\| = 0.05$ and $\|Q_k\Delta\mathbf{p}_0\| = 0.05$ respectively.

The number of iterations and function calls required to solve (10.23) and (10.24), averaged over the ten simulations, are shown in Figures **??** and **??** respectively. These plots show an exponential increase in iterations and function calls required to solve the optimisation problem in the full basis, and a sub-exponential relationship between problem dimension and the number of iterations and function calls in the reduced basis space. Further work is required to determine the nature of this relationship.

10.4 Conclusions and further work

These results show the potential for improving the computational efficiency in solving inverse problems using reduced basis methods. Further work is required to investigate the sensitivity of this approach to errors in the initial guess in the optimisation algorithm and noise in the data. The greedy method of constructing the reduced basis should be compared to alternative methods such as proper orthogonal decomposition.

Appendix A

Lie Derivative of a covariant 2-tensor

Appendix B

Functions Spaces Norms

Appendix C

Functions, Spaces and Norms

A mixed partial derivative of a function can be written in *multi-index notation*. Let $d \in \mathbb{N}$, and $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a d-tuple of non-negative integers. The length of α is given by $|\alpha| \equiv \sum_{i=1}^{d} \alpha_i$. Then for a function $f \in C^{|\alpha|}(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$, the mixed partial deriviative of f is given by

$$D^\alpha f = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_d})^{\alpha_d} f.$$

Here $C^k(\Omega)$, $k \in \mathbb{Z}_+$ is the set of all continuous real valued functions f on Ω such that $D^\alpha f \in C(\Omega)$ $\forall \alpha$ s.t. $|\alpha| \leq k$.

If Ω is a bounded open set, $C^k(\overline{\Omega})$ denotes the set of all $u \in C^k(\Omega)$ such that $D^\alpha u$ can be extended to a continuous function on $\overline{\Omega}$, the closure of Ω, for all α such that $|\alpha| \leq k$.

Definition C.0.1. *A normed space W is **complete** if every Cauchy sequence in W converges to an element in W.*

Definition C.0.2. *A complete normed space is called a **Banach space**.*

Definition C.0.3. *A compelete inner product space is called a **Hilbert space**.*

Lemma C.0.1. ***(Cauchy-Schwarz inequality)** Let W be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_W$ and $u, v \in W$. Then*

$$|\langle u, v \rangle_W \leq \|u\|_W \|v\|.$$

Lemma C.0.2. ***(Triangle Inequality)** Let W be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_W$ and $u, v \in W$. Then*

$$\|u + v\|_W \leq \|u\|_W + \|v\|_W$$

C.1 Hölder Continuity

Let $\Omega \subset \mathbb{R}^d$ be open, and $0 < \gamma \leq 1$. A function $u : \Omega \to \mathbb{R}$ is said to be *Hölder continuous with exponent* γ if

$$|u(x) - u(y)| \leq C|x - y|^\gamma \tag{C.1}$$

for some constant C.

Definition C.1.1. *1. If $u : U \to \mathbb{R}$ is bounded and continuous we write*

$$\|u\|_{C(\overline{\Omega})} \equiv \sup_{x \in U} |U(x)|$$

2.

C.2 Lipschitz Continuity

C.3 Lebesgue Spaces

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and u be a real valued function defined on Ω. For $1 \leq p < \infty$, the **Lebesgue space** $L^p(\Omega)$ is

$$L^p\Omega := \left\{ u : \int_\Omega |u(x)|^p dx < \infty \right\}. \tag{C.2}$$

This space has a natural norm defined by $\|u\|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}$. To define the Lebesgue space for $p = \infty$ we need the following definitions.

Definition C.3.1. *A subset A of $\mathbb{R}^d$ is said to have **measure zero** if for every $\epsilon > 0$ there exists a set of open cubes $\{U_k\}_{k=1}^\infty$ such that $A \subset \bigcup_{k=1}^\infty U_k$ and $\sum_{k=1}^\infty vol(U_k) < \epsilon$.*

For example, A could be a set of distinct points. It is always possible to make smaller boxes around this set of points, so A is a set of measure zero.

Definition C.3.2. *The **essential supremum** of a measurable function $u : \Omega \to \mathbb{R}$ is the smallest $a \in \mathbb{R}$ such that the set $\{\mathbf{x} \in \Omega : u(\mathbf{x}) > a\}$ has measure zero. If no such a exists, $esssup_{\mathbf{x}\in\Omega} u(\mathbf{x}) = \infty$.*

$$L^\infty(\Omega) := \left\{ u : \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})| < \infty \right\} \tag{C.3}$$

with norm $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}\{|u(\mathbf{x})|, \mathbf{x} \in \Omega\}$.

Lemma C.3.1. ***Hö lder inequality*** *Let $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $1/p + 1/p' = 1$. Then*

$$\| \int_\Omega u(\mathbf{x})v(\mathbf{x})dx| \leq \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}$$

C.4 Sobolev Spaces

The set of functions $u \in C^\infty(\Omega)$ with compact support is denoted $\mathcal{D}(\Omega)$. The set of **locally integrable functions** is defined as

$$L^1_{loc}(\Omega) := \{u : u \in L^1(K), \text{ for compact } K \subset \Omega\}.$$

Definition C.4.1. *A function $f \in L^1_{loc}(\Omega)$ has a* ***weak derivative*** *D^α_w if there exists a function $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega g(\mathbf{x})\phi(\mathbf{x})dx = (-1)^{|\alpha|}\int_\Omega f(\mathbf{x})D^\alpha_q\phi(\mathbf{x})dx, \ \phi \in \mathcal{D}(\Omega).$$

If such a g exists, we define $D^\alpha_w f := g$.

We can now define the **Sobolev Spaces** $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) := \{u \in L^1_{loc}(\Omega) : \|u\|_{W^{k,p}(\Omega)} < \infty\}$$

where the norm $\|u\|_{W^{k,p}(\Omega)}$ is defined by

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha_w u\|^p_{L^p(\Omega)}\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \max_{|\alpha|\leq k} \|D^\alpha_w u\|_{L^p(\Omega)} & \text{for } p = \infty. \end{cases}$$

For the special case $p = 2$ the Sobolev space $W^{k,2}(\Omega)$ is denoted $H^k(\Omega)$, and an inner product is induced on this space by the norm.

Theorem C.4.1. *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Theorem C.4.2. *The space $H^k(\Omega)$ is a Hilbert space.*

Definition C.4.2. *A* ***functional*** *is a function from a vector of function space into its underlying scalar field, or a set of functions to the real numbers. A functional on a real vector space V is* ***linear*** *if $f(v + w) = f(v) + f(w)$ and $f(cv) = cf(v)$ $\forall$ $v, w \in V$ and $c \in \mathbb{R}$.*

Definition C.4.3. *Let V be a vector space. The* ***dual space*** *of V is the space consisting of all continuous linear functionals on V, and is denoted V'.*

Theorem C.4.3. ***(Riesz representation theorem)*** *Any continuous functional L on a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ can be represented uniquely as*

$$L(v) = \langle u, v \rangle_H \text{ for some } u \in H$$

Moreover, we have

$$\|L\|_{H'} = \|u\|_H.$$

Lemma C.4.4. ***(Lax-Milgram)*** *Suppose that V is a real Hilbert space equipped with norm $\|\cdot\|_V$. Let $l(\cdot)$ be a continuous linear functional on V, and $a(\cdot, \cdot)$ a continuous, coercive bilinear functional on $V \times V$. Then there exists a unique $u \in V$ such that*

$$a(u.v) = l(v) \ \forall v \in V.$$

The solution is stable with respect to the right hand side such that

$$\|u\|_V \leq C\|l\|_{V'}.$$

Appendix D

PDEs

Fix an integer $k \geq 1$, and let Ω denote a subset of $\mathbb{R}^d$.

Definition D.0.1. *A **k-th order partial differential equation** is an expression of the form*

$$F(D^k u(x), D^{k-1}u(x), \cdots, Du(x), u(x), x) = 0 \tag{D.1}$$

where

$$F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \cdots \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is the unknown.

We *solve* D.1 if we find all u satisfying D.1. We may also require u to satisfy boundary conditions on some part $\Gamma \subset \partial\Omega$

Definition D.0.2. *1. D.1 is **linear** if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

*for given functions $a_\alpha(x)$, f. This linear PDE is **homogeneous** if $f \equiv 0$.*

*2. D.1 is **semilinear** if it has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0$$

3. D.1 is ***quasilinear*** *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\dots,Du,u,x)D^\alpha u + a_0(D^{k-1}u,\dots,Du,u,x) = 0$$

4. D.1 is ***fully nonlinear*** *if it depends nonlinearly upon the highest order derivatives.*

Definition D.0.3. *A* ***k-th order system of PDEs*** *is an expression of the form*

$$\mathbf{F}(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \cdots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0} \tag{D.2}$$

where

$$F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$$

is given and

$$\mathbf{u} : U \to \mathbb{R}^m, \mathbf{u} = (u^1, \dots, u^m)$$

is the unknown.

We now turn our attention to linear second order PDEs. As per the definitions above, this can be written in the form

$$\sum_{i,j=1}^{d} a_{ij}u_{x_ix_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu = f \tag{D.3}$$

where a_{ij}, b_i, c_i and f are functions of $\mathbf{x} = (x_1, \dots, x_d \in \Omega \subset \mathbb{R}^d$. We can assume without loss of generality that $a_{ij} = a_{ji}$.

Definition D.0.4. *Let A be the symmetric matrix with entries a_{ij}. Then the equation D.3 is called*

1. ***elliptic*** *if all eigenvalues of A have the same sign,*
2. ***hyperbolic*** *if $(d-1)$ eigenvalues of A have the same sign and one is of opposite sign,*
3. ***parabolic*** *if one eigenvalue is 0 and the others are of the same sign.*

Many practical problems have discontinuous right hand side function f. In these cases, the problem does not have a classical solution u.

Definition D.0.5. *Let a_ij,b,$c \in L^\infty(\Omega)$, $i,j = 1\dots,d$ and $f \in L^2(\Omega)$. A function $u \in H_0^1(\Omega)$ satisfying*

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}(\mathbf{x})\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j}dx + \sum_{i=1}^{d}\int_\Omega b_i(\mathbf{x})\frac{\partial u}{\partial x^i}vdx + \int_\Omega c(\mathbf{x})uvdx = \int_\Omega f(\mathbf{x})v(\mathbf{x})dx,\ \forall v \in H_0^1(\Omega)$$

is called a ***weak solution*** *of D.3. All partial derivatives are understood in the weak sense.*

A classical solution is also a weak solution. however a weak solution may not be smooth enough to be a classical solution. We define the folowing bilinear form

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}(\mathbf{x}) \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} dx + \sum_{i=1}^{d} \int_\Omega b_i(\mathbf{x}) \frac{\partial u}{\partial x^i} v dx + \int_\Omega c(\mathbf{x}) uv dx, \tag{D.4}$$

and linear form

$$l(v) = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) dx, \ \forall v \in H_0^1(\Omega). \tag{D.5}$$

With this notation the problem becomes to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = l(v), \ \forall v \in H_0^1(\Omega). \tag{D.6}$$

The existence of a unique solution is guaranteed by the Lax-Milgram lemma. Consider the following quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$:

$$J(v) = \frac{1}{2} a(u,v) - l(v), \ \forall v \in H_0^1(\Omega). \tag{D.7}$$

Then if u is the weak solution of D.6, then it is the unique minimiser of $J(\cdot)$ over $H_0^1(\Omega)$. Conversely, if u minimises $J(\cdot)$ over $H_0^1(\Omega)$, then u is the unique solution to D.6.

D.1 Cauchy Problems

We now turn our attention to *initial value* or *Cauchy* problems, in particular those of the form

$$\begin{array}{rcl} \frac{\partial}{\partial t} u(t,x) & = & F(D^k u(t,x), D^{k-1} u(t,x), \cdots, Du(t,x), u(t,x), x) \\ u(0,x) & = & f(x), \end{array} \tag{D.8}$$

where F is a linear, constant coefficient differential operator of order $k = 1, 2$.

The Einstein Equations are hyperbolic equations. The protypical hyperbolic equation is the wave equation

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta u = 0. \tag{D.9}$$