Chapter 1

Reduced basis methods for parameterised PDEs

1.1 Forward model

Let $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^N$ be a parameter vector, and $\Omega \in \mathbb{R}^d$ be our function domain. The forward model $\mathcal{M} : \mathcal{P} \times \Omega \rightarrow$

1.2 Statistical Inverse Problems

In a deterministic setting, regularisation and optimisation techniques are used to find a single point estimate of the parameter $\mathbf{p}$. A statistical formaulation returns a probability density function over parameter space describing the relative likelihood of observation-consistent parameters, called the 'posterior distribution', $\pi(\mathbf{p}|\mathbf{y}_d)$.

Here we infer properties of the probability distribution of model parameters $\mathbf{p} \in \mathbb{R}^N$ from observations $\mathbf{y}_d \in \mathbb{R}^m$. We adopt a Bayesian approach where a prior probability density $\pi(\mathbf{p})$ of $\mathbf{p}$ is given. We also assume that our model of the observations provides us with a likelihood function $\pi(\mathbf{y}_d \mid \mathbf{p})$ of the data $\mathbf{y}_d$. The data $\mathbf{y}_d$ is treated as a random vector $\mathbf{y}$ with probability distribution $\pi(\mathbf{y} \mid \mathbf{p})$ and we denote the expectation as

$$\mathbf{y}(\mathbf{p}) := E(\mathbf{y} \mid \mathbf{p}).$$

Bayes' law leads to a formula for the posterior probability density of $\mathbf{p}$ as

$$\pi(\mathbf{p} \mid \mathbf{y}_d) \propto \pi(\mathbf{p})\, \pi(\mathbf{y}_d \mid \mathbf{p}). \tag{1.1}$$

This probability density could then be further explored for example, in order to find marginals or moments of the posterior or even the MAP (maximum a posteriori) estimate which is of the form

$$\mathbf{p}_{\text{MAP}} = \arg\max_{\mathbf{p}\in\mathbb{R}^N} \pi(\mathbf{p} \mid \mathbf{y}_d). \tag{1.2}$$

In any case, the exploration of the posterior $\pi(\mathbf{p} \mid \mathbf{y}_d)$ requires its numerical evaluation typically many times. In our case each evaluation of the likelihood (and thus the posterior) requires an expensive numerical simulation.

1.2.1 Reduced basis method

In order to decrease the computational load required we reduce the parameter space from $\mathbb{R}^N$ to $\mathbb{R}^{N_r}$ as any exploration or computation in a lower dimensional space is cheaper. For this we generate a sequence of parameter vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots$ in $\mathbb{R}^N$ which are defined using a greedy algorithm

$$\mathbf{p}_{k+1} = \arg\max_{\mathbf{p}\in\mathbb{R}^N} \left(\frac{1}{2} \|\mathbf{y}(\mathbf{p}) - \mathbf{y}(Q_k\mathbf{p})\|^2 + \beta \log \pi(\mathbf{p}) \right), \tag{1.3}$$

where the $Q_k : \mathbb{R}^N \to \mathbb{R}^N$ are orthogonal projections with $\text{range}(Q_k) = \text{span}\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ for $k \geq 1$ and $Q_0 := 0$. This choice introduces the new $\mathbf{p}_{k+1}$ to make sure that a large number of observations $\mathbf{y}_d$ can be approximated in the range of Q_{k+1} while exploring parameter vectors which are sufficiently likely according to the prior.