Chapter 1

Functions, Spaces and Norms

A mixed partial derivative of a function can be written in *multi-index notation.* Let $d \in \mathbb{N}$, and $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a d-tuple of non-negative integers. The length of α is given by $|\alpha| \equiv \sum_{i=1}^{d} \alpha_i$. Then for a function $f \in C^{|\alpha|}(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$, the mixed partial deriviative of f is given by

$$D^\alpha f = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_d})^{\alpha_d} f.$$

Here $C^k(\Omega)$, $k \in \mathbb{Z}_+$ is the set of all continuous real valued functions f on Ω such that $D^\alpha f \in C(\Omega)$ $\forall \alpha$ s.t. $|\alpha| \leq k$.

If Ω is a bounded open set, $C^k(\overline{\Omega})$ denotes the set of all $u \in C^k(\Omega)$ such that $D^\alpha u$ can be extended to a continuous function on $\overline{\Omega}$, the closure of Ω, for all α such that $|\alpha| \leq k$.

Definition 1.0.1 *A normed space W is* ***complete*** *if every Cauchy sequence in W converges to an element in W.*

Definition 1.0.2 *A complete normed space is called a* ***Banach space****.*

Definition 1.0.3 *A compelete inner product space is called a* ***Hilbert space****.*

Lemma 1.0.1 ***(Cauchy-Schwarz inequality)*** *Let W be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_W$ and $u, v \in W$. Then*

$$|\langle u, v \rangle_W \leq \|u\|_W \|v\|.$$

Lemma 1.0.2 ***(Triangle Inequality)*** *Let W be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_W$ and $u, v \in W$. Then*

$$\|u + v\|_W \leq \|u\|_W + \|v\|_W$$

1.1 Hölder Continuity

Let $\Omega \subset \mathbb{R}^d$ be open, and $0 < \gamma \leq 1$. A function $u : \Omega \to \mathbb{R}$ is said to be *Hölder continuous with exponent* γ if

$$|u(x) - u(y)| \leq C|x - y|^\gamma \tag{1.1}$$

for some constant C.

Definition 1.1.1 *1. If $u : U \to \mathbb{R}$ is bounded and continuous we write*

$$\|u\|_{C(\overline{\Omega})} \equiv \sup_{x \in U} |U(x)|$$

2.

1.2 Lipschitz Continuity

1.3 Lebesgue Spaces

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and u be a real valued function defined on Ω. For $1 \leq p < \infty$, the **Lebesgue space** $L^p(\Omega)$ is

$$L^p\Omega := \left\{ u : \int_\Omega |u(x)|^p dx < \infty \right\}. \tag{1.2}$$

This space has a natural norm defined by $\|u\|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}$. To define the Lebesgue space for $p = \infty$ we need the following definitions.

Definition 1.3.1 *A subset A of $\mathbb{R}^d$ is said to have* ***measure zero*** *if for every $\epsilon > 0$ there exists a set of open cubes $\{U_k\}_{k=1}^\infty$ such that $A \subset \bigcup_{k=1}^\infty U_k$ and $\sum_{k=1}^\infty vol(U_k) < \epsilon$.*

For example, A could be a set of distinct points. It is always possible to make smaller boxes around this set of points, so A is a set of measure zero.

Definition 1.3.2 *The* ***essential supremum*** *of a measurable function* $u : \Omega \to \mathbb{R}$ *is the smallest* $a \in \mathbb{R}$ *such that the set* $\{\mathbf{x} \in \Omega : u(\mathbf{x}) > a\}$ *has measure zero. If no such* a *exists,* $esssup_{\mathbf{x}\in\Omega} u(\mathbf{x}) = \infty$*.*

$$L^\infty(\Omega) := \left\{ u : \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})| < \infty \right\} \tag{1.3}$$

with norm $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}\{|u(\mathbf{x})|, \mathbf{x} \in \Omega\}$.

Lemma 1.3.1 ***Hö lder inequality*** *Let* $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *with* $1/p + 1/p' = 1$*. Then*

$$\| \int_\Omega u(\mathbf{x})v(\mathbf{x})dx| \leq \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}$$

1.4 Sobolev Spaces

The set of functions $u \in C^\infty(\Omega)$ with compact support is denoted $\mathcal{D}(\Omega)$. The set of **locally integrable functions** is defined as

$$L^1_{loc}(\Omega) := \{u : u \in L^1(K), \text{ for compact } K \subset \Omega\}.$$

Definition 1.4.1 *A function* $f \in L^1_{loc}(\Omega)$ *has a* ***weak derivative*** D^α_w *if there exists a function* $g \in L^1_{loc}(\Omega)$ *such that*

$$\int_\Omega g(\mathbf{x})\phi(\mathbf{x})dx = (-1)^{|\alpha|} \int_\Omega f(\mathbf{x})D^\alpha_q \phi(\mathbf{x})dx, \ \phi \in \mathcal{D}(\Omega).$$

If such a g *exists, we define* $D^\alpha_w f := g$*.*

We can now define the **Sobolev Spaces** $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) := \{u \in L^1_{loc}(\Omega) : \|u\|_{W^{k,p}(\Omega)} < \infty\}$$

where the norm $\|u\|_{W^{k,p}(\Omega)}$ is defined by

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \|D_w^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p} & \text{for } 1 \le p < \infty \\ \max_{|\alpha|\le k} \|D_w^\alpha u\|_{L^p(\Omega)} & \text{for } p = \infty. \end{cases}$$

For the special case $p = 2$ the Sobolev space $W^{k,2}(\Omega)$ is denoted $H^k(\Omega)$, and an inner product is induced on this space by the norm.

Theorem 1.4.1 *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Theorem 1.4.2 *The space $H^k(\Omega)$ is a Hilbert space.*

Definition 1.4.2 *A **functional** is a function from a vector of function space into its underlying scalar field, or a set of functions to the real numbers. A functional on a real vector space V is **linear** if $f(v+w) = f(v) + f(w)$ and $f(cv) = cf(v)$ $\forall$ $v, w \in V$ and $c \in \mathbb{R}$.*

Definition 1.4.3 *Let V be a vector space. The **dual space** of V is the space consisting of all continuous linear functionals on V, and is denoted V'.*

Theorem 1.4.3 ***(Riesz representation theorem)*** *Any continuous functional L on a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ can be represented uniquely as*

$$L(v) = \langle u, v \rangle_H \text{ for some } u \in H$$

Moreover, we have

$$\|L\|_{H'} = \|u\|_H.$$

Lemma 1.4.1 ***(Lax-Milgram)*** *Suppose that V is a real Hilbert space equipped with norm $\|\cdot\|_V$. Let $l(\cdot)$ be a continuous linear functional on V, and $a(\cdot,\cdot)$ a continuous, coercive bilinear functional on $V \times V$. Then there exists a unique $u \in V$ such that*

$$a(u.v) = l(v) \; \forall v \in V.$$

The solution is stable with respect to the right hand side such that

$$\|u\|_V \le C\|l\|_{V'}.$$