This is the lecture note for MATH 104, which should be used together with Ch8.pdf

[!PDF|yellow] Ch8, p.1 > The convergence is uniform on every interval |x| < where 0 < R.

you see the difference? you need to specify a ρ instead of directly using an R

[!PDF|yellow] Ch8, p.2 > very power series has a radius of convergence.

The key is that everyone has it!

Check the key results! 1. the series converges absolutely for $|x - c| < R$ 2. diverges for $|x - c| > R$ 3. for the given $\rho \in [0, R)$ 1. the series converges uniformly on $|x - c| < \rho$ 2. the sum of series is continuous on $|x - c| < \rho$ > [!PDF|red] Ch8, p.3 > > lso note that a power series need not converge uniformly on |x − c| < R. > > Keep in mind that 3.1 is different from 1!

[!PDF|yellow] Ch8, p.3 > hus, if the power series converges for some x0 R, then it converges absolutely for every x R with |x| < |x0|.

The key in proving the convergence is to prove that given x_0, all $x \in [0, x_0)$ converges absolutely

[!PDF|red] Ch8, p.3 > then it follows from Theorem 9.16 that the sum is continuous on |x|

If a sequence of continuous functions (f_n) converges uniformly to a function f on an interval, then the limit function f is also continuous on that interval.

[!PDF|yellow] Ch8, p.4 > Theorem 2.2. Suppose that an = 0 for all sufficiently large n and the limit R = lim n→∞ an an+1 exists or diverges to infinity. Then the power series ∞X n=0 an(x − c)n has a radius of convergence R.

The key is to implement the ratio test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Then $\sum_{i=1}^{\infty} a_n$ converges absolutely

[!PDF|yellow] Ch8, p.4 > Theorem 2.3 (Hadamard). The radius of convergence R of the power series ∞X n=0 an(x − c)n is given by R = 1 lim supn→∞ |an|1/n where R = 0 if the limsup diverges to ∞, and R = ∞ if the limsup is 0.

This one is crucial!

[!PDF|yellow] Ch8, p.5 > Proposition 3.1.

These properties hold within the $T = \min\{R, S\}$ but the radius of convergence for the new series may be larger.

[!PDF|yellow] Ch8, p.6 > The reciprocal of a convergent power series that is nonzero at its center also has a power series expansion.

We now try to solve for the new coefficients

we know that

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

and $f(0) \neq 0$, so we know there is a

$$f(x) \cdot \frac{1}{f(x)} = 1$$

then lets think of 1 as a series, then we know $c_0 = 1, c_{i \geq 0} = 0$. We set

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n x^n$$

then we know for the product,

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

for $n = 0$, we have

$$a_0 b_0 = 1, b_0 = \frac{1}{a_0}$$

then for all $n > 0$, we know

$$0 = \sum_{k=0}^{n} a_{n-k} b_k$$

assume we have known $\{b_k\}_{k=0}^{n-1}$, then the new b_n can be obtained simply from

$$b_n a_0 + \sum_{k=0}^{n-1} a_{n-k} b_k = 0 \implies b_n = -\frac{1}{a_0} \sum_{k=0}^{n-1} a_{n-k} b_k$$

> But still no information about the $\frac{1}{f}$'s radius of convergence!

[!PDF|yellow] Ch8, p.6 > Theorem 4.1. Suppose that the power series ∞X n=0 an(x − c)n has a radius of convergence R. Then the power series ∞X n=1 nan(x − c)n−1 also has a radius of convergence R.

i.e. the derivative of a convergent series has the same radius of convergence R

For the proof of it,

> [!PDF|yellow] Ch8, p.6 > The ratio test show
>
> Notice that we're talking about $\sum_{n=0}^{\infty} nr^{n-1}$ instead of $\sum_{n=1}^{\infty} na_n x^{n-1}$

Then the convergence gives boundedness: $\{nr^{n-1}\}_{n=0}^{\infty}$ is bounded by M - given $\sum_{n=0}^{\infty}|a_n\rho^n|$ converges (by definition) - then we know

$$|na_n x^{n-1}| \leq \frac{M}{\rho}|a_n\rho^n| \implies \sum_{n=0}^{\infty} na_n x^{n-1} \text{ converges abs.}$$

> [!PDF|yellow] Ch8, p.7 > > Theorem 4.2. Suppose that the power series > > It exists, we have one , it should be equal: $f' = g$ > But notice the steps: > 1. this holds for $0 < \rho < R$ > 2. this holds for $\forall \rho < R \implies [0, R)$

> [!PDF|yellow] Ch8, p.7 > nfinitely differentiable
>
> with the same R

> [!PDF|yellow] Ch8, p.7 > Theorem 4.3. If the power series f (x) = ∞X n=0 an(x − c)n has radius of convergence R > 0, then f is infinitely differentiable in |x − c| < R and an = f (n)(c) n!
>
> This one is interesting!

The Taylor expansion results come directly from setting $x = 0(c)$

> [!PDF|yellow] Ch8, p.8 > Corollary 4.1. If two power series ∞X n=0 an(x − c)n, ∞X n=0 bn(x − c)n have nonzero-radius of convergence and are equal in some neighborhood of 0 , then an = bn for every n = 0, 1, 2, ...
>
> The corollary shows that if $f = g$ then we must have this one-to-one
>
> $$a_n = b_n = \frac{f^{(n)}(c)}{n!}$$
>
> [!PDF|yellow] Ch8, p.9 > Proposition 5.1. For every x, y R, E(x)E(y) = E(x + y).
>
> Very quick thought: use $c_n = \sum_{k=0}^{n} a_{n-k}b_k$
>
> [!PDF|yellow] Ch8, p.10 > Proposition 5.3. Suppose that n is a non-negative integer. Then lim x→∞ xn ex = 0.
>
> This is the fact that $\exp(x)$ has the higher order than any polynomial