Lecture notes for Math 104

9. The Riemann integral

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Let $f : [a, b] \to \mathbb{R}$ be a bounded (not necessarily continuous) function on a compact (closed, bounded) interval. We will define what it means for f to be Riemann integrable on $[a, b]$ and, in that case, define its Riemann integral $\int_a^b f$. The integral of f on $[a, b]$ is a real number whose geometrical interpretation is the signed area under the graph $y = f(x)$ for $a \leq x \leq b$. This number is also called the integral of f.

We will cover the following topics:

- Definition of Riemann integral
- Criterion for integrability
- Examples of Riemann integrable function

1 The supremum and infimum of functions

In this section we collect some results about the supremum and infimum of functions that we use to study Riemann integration.

we define the supremum and infimum of a function, which are the supremum or infimum of its range (Section 4 in the textbook).

Definition 1.1. *If $f : A \to \mathbb{R}$ is a real-valued function, then*

$$\sup_A f = \sup\{f(x) : x \in A\}, \quad \inf_A f = \inf\{f(x) : x \in A\}.$$

A function is bounded if its range is bounded.

Definition 1.2. *If $f : A \to \mathbb{R}$, then f is bounded from above if $\sup_A f$ is finite, bounded from below if $\inf_A f$ is finite, and bounded if both are finite. A function that is not bounded is said to be unbounded.*

Proposition 1.1. *Suppose that $f, g : A \to \mathbb{R}$ and $f \leq g$. Then*

$$\sup_A f \leq \sup_A g, \quad \inf_A f \leq \inf_A g.$$

Proof. If $\sup g = \infty$, then $\sup f \leq \sup g$. Otherwise, if $f \leq g$ and g is bounded from above, then

$$f(x) \leq g(x) \leq \sup_A g \qquad \text{for every } x \in A.$$

Thus, f is bounded from above by $\sup_A g$, so $\sup_A f \leq \sup_A g$. Similarly, $-f \geq -g$ implies that $\sup_A(-f) \geq \sup_A(-g)$, so $\inf_A f \leq \inf_A g$. □

Linear property:

Proposition 1.2. *Suppose that $f : A \to \mathbb{R}$ is a bounded function and $c \in \mathbb{R}$. If $c \geq 0$, then*

$$\sup_A cf = c \sup_A f, \quad \inf_A cf = c \inf_A f.$$

If $c < 0$, then

$$\sup_A cf = c \inf_A f, \quad \inf_A cf = c \sup_A f.$$

Proof. Notice$\{cf(x) : x \in A\} = c\{f(x) : x \in A\}$. □

For sums of functions, we get an inequality.

Proposition 1.3. *If $f, g : A \to \mathbb{R}$ are bounded functions, then*

$$\sup_A(f + g) \leq \sup_A f + \sup_A g, \quad \inf_A(f + g) \geq \inf_A f + \inf_A g.$$

Proof. Since $f(x) \leq \sup_A f$ and $g(x) \leq \sup_A g$ for every $x \in [a, b]$, we have

$$f(x) + g(x) \leq \sup_A f + \sup_A g.$$

Thus, $f + g$ is bounded from above by $\sup_A f + \sup_A g$, so

$$\sup_A(f + g) \leq \sup_A f + \sup_A g.$$

The proof for the infimum is analogous (or apply the result for the supremum to the functions $-f, -g$). □

Finally, we prove some inequalities that involve the absolute value.

Proposition 1.4. *If $f, g : A \to \mathbb{R}$ are bounded functions, then*

$$\left|\sup_A f - \sup_A g\right| \leq \sup_A |f - g|, \qquad \inf_A f - \inf_A g \Bigg| \leq \sup_A \Bigg| f - g \mid .$$

Proof. Since $f = f - g + g$ and $f - g \leq |f - g|$, we get from Proposition 1.1 and Proposition 1.3 that

$$\sup_A f \leq \sup_A (f - g) + \sup_A g \leq \sup_A |f - g| + \sup_A g,$$

so

$$\sup_A f - \sup_A g \leq \sup_A |f - g|.$$

Exchanging f and g in this inequality, we get

$$\sup_A g - \sup_A f \leq \sup_A |f - g|,$$

which implies that

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|.$$

Replacing f by $-f$ and g by $-g$ in this inequality, we get

$$\left| \inf_A f - \inf_A g \right| \leq \sup_A |f - g|,$$

where we use the fact that $\sup(-f) = -\inf f$.

$\square$

Proposition 1.5. *If $f, g : A \to \mathbb{R}$ are bounded functions such that*

$$|f(x) - f(y)| \leq |g(x) - g(y)| \quad \text{for all } x, y \in A,$$

then

$$\sup_A f - \inf_A f \leq \sup_A g - \inf_A g.$$

Proof. The condition implies that for all $x, y \in A$, we have

$$f(x) - f(y) \leq |g(x) - g(y)| = \max[g(x), g(y)] - \min[g(x), g(y)] \leq \sup_A g - \inf_A g,$$

which implies that

$$\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A g - \inf_A g.$$

and

$$\sup\{f(x) - f(y) : x, y \in A\} = \sup_A f - \inf_A f,$$

so the result follows.

$\square$

2 Definition of the integral (Section 32)

The integral is approximated by upper and lower sums based on a partition of an interval.

We say that two intervals are almost disjoint if they are disjoint or intersect only at a common endpoint. For example, the intervals [0,1] and [1,3] are almost disjoint, whereas the intervals $[0, 2]$ and $[1, 3]$ are not.

Definition 2.1. *Let I be a nonempty, compact interval. A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, nonempty, compact subintervals whose union is I.*

A partition of $[a, b]$ with subintervals $I_k = [x_{k-1}, x_k]$ is determined by the set of endpoints of the intervals

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Abusing notation, we will denote a partition P either by its intervals

$$P = \{I_1, I_2, \ldots, I_n\}$$

or by the set of endpoints of the intervals

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}.$$

We'll adopt either notation as convenient; the context should make it clear which one is being used. There is always one more endpoint than interval.

Example 2.1. *The set of intervals*

$$\{[0, 1/5], [1/5, 1/4], [1/4, 1/3], [1/3, 1/2], [1/2, 1]\}$$

is a partition of $[0, 1]$*. The corresponding set of endpoints is*

$$\{0, 1/5, 1/4, 1/3, 1/2, 1\}.$$

We denote the length of an interval $I = [a, b]$ by

$$|I| = b - a.$$

Note that the sum of the lengths $|I_k| = x_k - x_{k-1}$ of the almost disjoint subintervals in a partition $\{I_1, I_2, \ldots, I_n\}$ of an interval I is equal to length of the whole interval. This is obvious geometrically; algebraically, it follows from the telescoping series

$$\begin{aligned}\sum_{k=1}^{n} |I_k| &= \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_2 - x_1 + x_1 - x_0 \\ &= x_n - x_0 \\ &= |I|.\end{aligned}$$

Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with

$$M = \sup_I f, \quad m = \inf_I f.$$

If $P = \{I_1, I_2, \ldots, I_n\}$ is a partition of I, let

$$M_k = \sup_{I_k} f, \quad m_k = \inf_{I_k} f.$$

These suprema and infima are well-defined, finite real numbers since f is bounded. Moreover,

$$m \le m_k \le M_k \le M.$$

Definition 2.2. *Define the upper Riemann sum of f with respect to the partition P by*

$$U(f; P) = \sum_{k=1}^{n} M_k |I_k| = \sum_{k=1}^{n} M_k (x_k - x_{k-1}),$$

and the lower Riemann sum of f with respect to the partition P by

$$L(f; P) = \sum_{k=1}^{n} m_k |I_k| = \sum_{k=1}^{n} m_k (x_k - x_{k-1}).$$

Note that

$$m(b - a) \le L(f; P) \le U(f; P) \le M(b - a).$$

Geometrically, $U(f; P)$ is the sum of the signed areas of rectangles based on the intervals I_k that lie above the graph of f, and $L(f; P)$ is the sum of the signed areas of rectangles that lie below the graph of f.

Let $\Pi(a, b)$, or Π for short, denote the collection of all partitions of $[a, b]$. We define the upper Riemann integral of f on $[a, b]$ by

$$U(f) = \inf_{P \in \Pi} U(f; P).$$

The set $\{U(f; P) : P \in \Pi\}$ of all upper Riemann sums of f is bounded from below by $m(b - a)$, so this infimum is well-defined and finite.

Similarly, the set $\{L(f; P) : P \in \Pi\}$ of all lower Riemann sums is bounded from above by $M(b-a)$, and we define the lower Riemann integral of f on $[a, b]$ by

$$L(f) = \sup_{P \in \Pi} L(f; P).$$

We will prove $L(f) \le U(f)$ later. We define Riemann integrability by their equality.

Definition 2.3. *A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if it is bounded and its upper integral $U(f)$ and lower integral $L(f)$ are equal. In that case, the Riemann integral of f on $[a, b]$, denoted by*

$$\int_a^b f(x)dx, \quad \int_a^b f, \quad \int_{[a,b]} f$$

or similar notations, is the common value of $U(f)$ and $L(f)$.

An unbounded function is not Riemann integrable.

Let us illustrate the definition of Riemann integrability with a number of examples.

Example 2.2. *Define $f : [0, 1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then

$$\int_0^1 \frac{1}{x} dx$$

isn't defined as a Riemann integral becuase f is unbounded. In fact, if

$$0 < x_1 < x_2 < \cdots < x_{n-1} < 1$$

is a partition of $[0, 1]$, then

$$\sup_{[0,x_1]} f = \infty,$$

so the upper Riemann sums of f are not well-defined.

Example 2.3. *The constant function $f(x) = 1$ on $[0, 1]$ is Riemann integrable, and*

$$\int_0^1 1dx = 1.$$

To show this, let $P = \{I_1, I_2, \ldots, I_n\}$ be any partition of $[0, 1]$ with endpoints

$$\{0, x_1, x_2, \ldots, x_{n-1}, 1\}.$$

Since f is constant,

$$M_k = \sup_{I_k} f = 1, \quad m_k = \inf_{I_k} f = 1 \quad \text{for } k = 1, \ldots, n,$$

and therefore

$$U(f; P) = L(f; P) = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 1.$$

Geometrically, this equation is the obvious fact that the sum of the areas of the rectangles over (or, equivalently, under) the graph of a constant function is exactly equal to the area under the graph. Thus, every upper and lower sum of f *on* $[0,1]$ *is equal to 1, which implies that the upper and lower integrals*

$$U(f) = \inf_{P\in\Pi} U(f;P) = \inf\{1\} = 1, \quad L(f) = \sup_{P\in\Pi} L(f;P) = \sup\{1\} = 1$$

are equal, and the integral of f *is 1. More generally, the same argument shows that every constant function* $f(x) = c$ *is integrable and*

$$\int_a^b c\,dx = c(b-a).$$

The following is an example of a discontinuous function that is Riemann integrable.

Example 2.4. *The function*

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

is Riemann integrable, and

$$\int_0^1 f\,dx = 0.$$

To show this, let $P = \{I_1, I_2, \dots, I_n\}$ *be a partition of* $[0,1]$*. Then, since* $f(x) = 0$ *for* $x > 0$*,*

$$M_k = \sup_{I_k} f = 0, \quad m_k = \inf_{I_k} f = 0 \quad \text{for } k = 2, \dots, n.$$

The first interval in the partition is $I_1 = [0, x_1]$*, where* $0 < x_1 \leq 1$*, and*

$$M_1 = 1, \quad m_1 = 0,$$

since $f(0) = 1$ *and* $f(x) = 0$ *for* $0 < x \leq x_1$*. It follows that*

$$U(f;P) = x_1, \quad L(f;P) = 0.$$

Thus, $L(f) = 0$ *and*

$$U(f) = \inf\{x_1 : 0 < x_1 \leq 1\} = 0,$$

so $U(f) = L(f) = 0$ *are equal, and the integral of* f *is 0 . In this example, the infimum of the upper Riemann sums is not attained and* $U(f;P) > U(f)$ *for every partition* P*.*

The next example is a bounded function on a compact interval whose Riemann integral doesn't exist.

Example 2.5. *The Dirichlet function* $f : [0,1] \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \backslash \mathbb{Q}. \end{cases}$$

That is, f *is one at every rational number and zero at every irrational number.*

This function is not Riemann integrable. If $P = \{I_1, I_2, \dots, I_n\}$ *is a partition of* $[0,1]$*, then*

$$M_k = \sup_{I_k} f = 1, \quad m_k = \inf_{I_k} = 0,$$

since every interval of non-zero length contains both rational and irrational numbers. It follows that

$$U(f; P) = 1, \quad L(f; P) = 0$$

for every partition P *of* $[0,1]$*, so* $U(f) = 1$ *and* $L(f) = 0$ *are not equal. The Dirichlet function is discontinuous at every point of* $[0,1]$*, and the moral of the last example is that the Riemann integral of a highly discontinuous function need not exist. Nevertheless, some fairly discontinuous functions are still Riemann integrable.*

2.1 Refinements of partitions.

As the previous examples illustrate, a direct verification of integrability from Definition 11.11 is unwieldy even for the simplest functions because we have to consider all possible partitions of the interval of integration. To give an effective analysis of Riemann integrability, we need to study how upper and lower sums behave under the refinement of partitions.

Definition 2.4. *A partition* $Q = \{J_1, J_2, \dots, J_m\}$ *is a refinement of a partition* $P = \{I_1, I_2, \dots, I_n\}$ *if every interval* I_k *in* P *is an almost disjoint union of one or more intervals* J_ℓ *in* Q*.*

Equivalently, if we represent partitions by their endpoints, then Q is a refinement of P if $Q \supset P$, meaning that every endpoint of P is an endpoint of Q.

Example 2.6. *Consider the partitions of* $[0,1]$ *with endpoints*

$$P = \{0, 1/2, 1\}, \quad Q = \{0, 1/3, 2/3, 1\}, \quad R = \{0, 1/4, 1/2, 3/4, 1\}.$$

Thus, P, Q*, and* R *partition* $[0,1]$ *into intervals of equal length* $1/2, 1/3$*, and* $1/4$*, respectively. Then* Q *is not a refinement of* P *but* R *is a refinement of* P*.*

Example 2.7. *Given two partitions, neither one need be a refinement of the other. However, two partitions* P, Q *always have a common refinement; the smallest one is* $R = P \cup Q$*, meaning that the endpoints of* R *are exactly the endpoints of* P *or* Q *(or both).*

As we show next, refining partitions decreases upper sums and increases lower sums. (The proof is easier to understand than it is to write out - draw a picture!)

Theorem 2.1. *Suppose that $f : [a, b] \to \mathbb{R}$ is bounded, P is a partitions of $[a, b]$, and Q is refinement of P. Then*

$$U(f; Q) \leq U(f; P), \quad L(f; P) \leq L(f; Q).$$

Proof. Let

$$P = \{I_1, I_2, \dots, I_n\}, \quad Q = \{J_1, J_2, \dots, J_m\}$$

be partitions of $[a, b]$, where Q is a refinement of P, so $m \geq n$. We list the intervals in increasing order of their endpoints. Define

$$M_k = \sup_{I_k} f, \quad m_k = \inf_{I_k} f, \quad M'_\ell = \sup_{J_\ell} f, \quad m'_\ell = \inf_{J_\ell} f.$$

Since Q is a refinement of P, each interval I_k in P is an almost disjoint union of intervals in Q, which we can write as

$$I_k = \bigcup_{\ell=p_k}^{q_k} J_\ell$$

for some indices $p_k \leq q_k$. If $p_k < q_k$, then I_k is split into two or more smaller intervals in Q, and if $p_k = q_k$, then I_k belongs to both P and Q. Since the intervals are listed in order, we have

$$p_1 = 1, \quad p_{k+1} = q_k + 1, \quad q_n = m.$$

If $p_k \leq \ell \leq q_k$, then $J_\ell \subset I_k$, so

$$M'_\ell \leq M_k, \quad m_k \geq m'_\ell \quad \text{for } p_k \leq \ell \leq q_k.$$

Using the fact that the sum of the lengths of the J-intervals is the length of the corresponding I-interval, we get that

$$\sum_{\ell=p_k}^{q_k} M'_\ell |J_\ell| \leq \sum_{\ell=p_k}^{q_k} M_k |J_\ell| = M_k \sum_{\ell=p_k}^{q_k} |J_\ell| = M_k |I_k|.$$

It follows that

$$U(f; Q) = \sum_{\ell=1}^{m} M'_\ell |J_\ell| = \sum_{k=1}^{n} \sum_{\ell=p_k}^{q_k} M'_\ell |J_\ell| \leq \sum_{k=1}^{n} M_k |I_k| = U(f; P).$$

Similarly,

$$\sum_{\ell=p_k}^{q_k} m'_\ell |J_\ell| \geq \sum_{\ell=p_k}^{q_k} m_k |J_\ell| = m_k |I_k|,$$

and

$$L(f;Q) = \sum_{k=1}^{n} \sum_{\ell=p_k}^{q_k} m'_\ell \, |J_\ell| \geq \sum_{k=1}^{n} m_k \, |I_k| = L(f;P),$$

which proves the result.

□

It follows from this theorem that all lower sums are less than or equal to all upper sums, not just the lower and upper sums associated with the same partition.

Proposition 2.1. *If $f : [a,b] \to \mathbb{R}$ is bounded and P, Q are partitions of $[a,b]$, then*

$$L(f;P) \leq U(f;Q).$$

Proof. Let R be a common refinement of P and Q. Then, by Theorem 2.1,

$$L(f;P) \leq L(f;R), \quad U(f;R) \leq U(f;Q).$$

It follows that

$$L(f;P) \leq L(f;R) \leq U(f;R) \leq U(f;Q).$$

□

An immediate consequence of this result is that the lower integral is always less than or equal to the upper integral.

Proposition 2.2. *If $f : [a,b] \to \mathbb{R}$ is bounded, then*

$$L(f) \leq U(f).$$

Proof. Let

$$A = \{L(f;P) : P \in \Pi\}, \quad B = \{U(f;P) : P \in \Pi\}.$$

From Proposition 2.1, $L \leq U$ for every $L \in A$ and $U \in B$, so $\sup A \leq \inf B$, or $L(f) \leq U(f)$.

□

3 The Cauchy criterion for integrability (Section 32)

The following theorem gives a criterion for integrability that is analogous to the Cauchy condition for the convergence of a sequence.

Theorem 3.1 (MOST USEFUL!)**.** *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition P of $[a,b]$, which may depend on ϵ, such that*

$$U(f;P) - L(f;P) < \epsilon.$$

Proof. First, suppose that the condition holds. Let $\epsilon > 0$ and choose a partition P that satisfies the condition. Then, since $U(f) \leq U(f;P)$ and $L(f;P) \leq L(f)$, we have

$$0 \leq U(f) - L(f) \leq U(f;P) - L(f;P) < \epsilon.$$

Since this inequality holds for every $\epsilon > 0$, we must have $U(f) - L(f) = 0$, and f is integrable.

Conversely, suppose that f is integrable. Given any $\epsilon > 0$, there are partitions Q, R such that

$$U(f;Q) < U(f) + \frac{\epsilon}{2}, \quad L(f;R) > L(f) - \frac{\epsilon}{2}.$$

Let P be a common refinement of Q and R. Then, by Theorem 2.1,

$$U(f;P) - L(f;P) \leq U(f;Q) - L(f;R) < U(f) - L(f) + \epsilon.$$

Since $U(f) = L(f)$, the condition follows.

□

If $U(f;P) - L(f;P) < \epsilon$, then $U(f;Q) - L(f;Q) < \epsilon$ for every refinement Q of P, so the Cauchy condition means that a function is integrable if and only if its upper and lower sums get arbitrarily close together for all sufficiently refined partitions. It is worth considering in more detail what the Cauchy condition in Theorem 3.1 implies about the behavior of a Riemann integrable function.

Definition 3.1. *The oscillation of a bounded function f on a set A is*

$$\operatorname*{osc}_A f = \sup_A f - \inf_A f.$$

If $f : [a,b] \to \mathbb{R}$ is bounded and $P = \{I_1, I_2, \dots, I_n\}$ is a partition of $[a,b]$, then

$$U(f;P) - L(f;P) = \sum_{k=1}^{n} \sup_{I_k} f \cdot |I_k| - \sum_{k=1}^{n} \inf_{I_k} f \cdot |I_k| = \sum_{k=1}^{n} \operatorname*{osc}_{I_k} f \cdot |I_k|.$$

A function f is Riemann integrable if we can make $U(f;P) - L(f;P)$ as small as we wish. This is the case if we can find a sufficiently refined partition P such that the oscillation of f on most intervals is arbitrarily small, and the sum of the lengths of the remaining intervals (where the oscillation of f is large) is arbitrarily small.

Thus, roughly speaking, a function is Riemann integrable if it oscillates by an arbitrarily small amount except on a finite collection of intervals whose total length is arbitrarily small.

One direct consequence of the Cauchy criterion is that a function is integrable if we can estimate its oscillation by the oscillation of an integrable function.

Proposition 3.1. *Suppose that $f, g : [a, b] \to \mathbb{R}$ are bounded functions and g is integrable on $[a, b]$. If there exists a constant $C \geq 0$ such that*

$$\operatorname*{asc}_I f \leq C \operatorname*{asc}_I g$$

on every interval $I \subset [a, b]$, then f is integrable.

Proof. If $P = \{I_1, I_2, \ldots, I_n\}$ is a partition of $[a, b]$, then

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^{n} \operatorname*{osc}_{I_k} f \cdot |I_k| \\ &\leq C \sum_{k=1}^{n} \operatorname*{osc}_{I_k} g \cdot |I_k| \\ &\leq C[U(g; P) - L(g; P)]. \end{aligned}$$

Thus, f satisfies the Cauchy criterion in Theorem 3.1 if g does, which proves that f is integrable if g is integrable.

□

4 Continuous and monotonic functions (Section 33)

The Cauchy criterion leads to the following fundamental result that every continuous function is Riemann integrable.

Theorem 4.1. *A continuous function $f : [a, b] \to \mathbb{R}$ on a compact interval is Riemann integrable.*

Proof. A continuous function on a compact set is bounded, so we just need to verify the Cauchy condition in Theorem 11.23

Let $\epsilon > 0$. A continuous function on a compact set is uniformly continuous, so there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b - a} \quad \text{for all } x, y \in [a, b] \text{ such that } |x - y| < \delta.$$

Choose a partition $P = \{I_1, I_2, \ldots, I_n\}$ of $[a, b]$ such that $|I_k| < \delta$ for every k; for example, we can take n intervals of equal length $(b-a)/n$ with $n > (b-a)/\delta$.

Since f is continuous, it attains its maximum and minimum values M_k and m_k on the compact interval I_k at points x_k and y_k in I_k. These points satisfy $|x_k - y_k| < \delta$, so

$$M_k - m_k = f(x_k) - f(y_k) < \frac{\epsilon}{b - a}.$$

The upper and lower sums of f therefore satisfy

$$\begin{aligned}
U(f;P) - L(f;P) &= \sum_{k=1}^{n} M_k \, |I_k| - \sum_{k=1}^{n} m_k \, |I_k| \\
&= \sum_{k=1}^{n} (M_k - m_k) \, |I_k| \\
&< \frac{\epsilon}{b-a} \sum_{k=1}^{n} |I_k| \\
&< \epsilon
\end{aligned}$$

and Theorem 3.1 implies that f is integrable.

□

Another class of integrable functions consists of monotonic (increasing or decreasing) functions.

Theorem 4.2. *A monotonic function $f : [a,b] \to \mathbb{R}$ on a compact interval is Riemann integrable.*

Proof. Suppose that f is monotonic increasing, meaning that $f(x) \le f(y)$ for $x \le y$. Let $P_n = \{I_1, I_2, \dots, I_n\}$ be a partition of $[a,b]$ into n intervals $I_k = [x_{k-1}, x_k]$, of equal length $(b-a)/n$, with endpoints

$$x_k = a + (b-a)\frac{k}{n}, \quad k = 0, 1, \dots, n.$$

Since f is increasing,

$$M_k = \sup_{I_k} f = f\left(x_k\right), \quad m_k = \inf_{I_k} f = f\left(x_{k-1}\right).$$

Hence, summing up a telescoping series, we get

$$\begin{aligned}
U\left(f;P_n\right) - L\left(U;P_n\right) &= \sum_{k=1}^{n} \left(M_k - m_k\right)\left(x_k - x_{k-1}\right) \\
&= \frac{b-a}{n} \sum_{k=1}^{n} \left[f\left(x_k\right) - f\left(x_{k-1}\right)\right] \\
&= \frac{b-a}{n} [f(b) - f(a)].
\end{aligned}$$

It follows that $U\left(f;P_n\right) - L\left(U;P_n\right) \to 0$ as $n \to \infty$, and Theorem 3.1 implies that f is integrable.

The proof for a monotonic decreasing function f is similar, with

$$\sup_{I_k} f = f\left(x_{k-1}\right), \quad \inf_{I_k} f = f\left(x_k\right).$$

□

5 Linearity, monotonicity, and additivity (Section 33)

The integral has the following three basic properties.
(1) Linearity:

$$\int_a^b cf = c\int_a^b f, \quad \int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(2) Monotonicity: if $f \le g$, then

$$\int_a^b f \le \int_a^b g.$$

(3) Additivity: if $a < c < b$, then

$$\int_a^c f + \int_c^b f = \int_a^b f$$

In this section, we prove these properties and derive a few of their consequences.

5.1 Linearity

We begin by proving linearity.

Theorem 5.1. *If $f : [a,b] \to \mathbb{R}$ is integrable and $c \in \mathbb{R}$, then cf is integrable and*

$$\int_a^b cf = c\int_a^b f.$$

Proof. Suppose that $c \ge 0$. Then for any set $A \subset [a,b]$, we have

$$\sup_A cf = c\sup_A f, \quad \inf_A cf = c\inf_A f,$$

so $U(cf;P) = cU(f;P)$ for every partition P. Taking the infimum over the set Π of all partitions of $[a,b]$, we get

$$U(cf) = \inf_{P\in\Pi} U(cf;P) = \inf_{P\in\Pi} cU(f;P) = c\inf_{P\in\Pi} U(f;P) = cU(f).$$

Similarly, $L(cf;P) = cL(f;P)$ and $L(cf) = cL(f)$. If f is integrable, then

$$U(cf) = cU(f) = cL(f) = L(cf),$$

which shows that cf is integrable and

$$\int_a^b cf = c\int_a^b f.$$

Now consider $-f$. Since

$$\sup_A(-f) = -\inf_A f, \quad \inf_A(-f) = -\sup_A f,$$

we have

$$U(-f; P) = -L(f; P), \quad L(-f; P) = -U(f; P).$$

Therefore

$$\begin{aligned}
U(-f) &= \inf_{P\in\Pi} U(-f; P) = \inf_{P\in\Pi}[-L(f; P)] = -\sup_{P\in\Pi} L(f; P) = -L(f), \\
L(-f) &= \sup_{P\in\Pi} L(-f; P) = \sup_{P\in\Pi}[-U(f; P)] = -\inf_{P\in\Pi} U(f; P) = -U(f).
\end{aligned}$$

Hence, $-f$ is integrable if f is integrable and

$$\int_a^b (-f) = -\int_a^b f.$$

Finally, if $c < 0$, then $c = -|c|$, and a successive application of the previous results shows that cf is integrable with $\int_a^b cf = c\int_a^b f$.

□

Next, we prove the linearity of the integral with respect to sums.

Theorem 5.2. *If $f, g : [a, b] \to \mathbb{R}$ are integrable functions, then $f + g$ is integrable, and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof. We first prove that if $f, g : [a, b] \to \mathbb{R}$ are bounded, but not necessarily integrable, then

$$U(f + g) \leq U(f) + U(g), \quad L(f + g) \geq L(f) + L(g).$$

Suppose that $P = \{I_1, I_2, \ldots, I_n\}$ is a partition of $[a, b]$. Then

$$\begin{aligned}
U(f + g; P) &= \sum_{k=1}^{n} \sup_{I_k}(f + g) \cdot |I_k| \\
&\leq \sum_{k=1}^{n} \sup_{I_k} f \cdot |I_k| + \sum_{k=1}^{n} \sup_{I_k} g \cdot |I_k| \\
&\leq U(f; P) + U(g; P).
\end{aligned}$$

Let $\epsilon > 0$. Since the upper integral is the infimum of the upper sums, there are partitions Q, R such that

$$U(f; Q) < U(f) + \frac{\epsilon}{2}, \quad U(g; R) < U(g) + \frac{\epsilon}{2},$$

and if P is a common refinement of Q and R, then

$$U(f;P) < U(f) + \frac{\epsilon}{2}, \quad U(g;P) < U(g) + \frac{\epsilon}{2}.$$

It follows that

$$U(f+g) \leq U(f+g;P) \leq U(f;P) + U(g;P) < U(f) + U(g) + \epsilon.$$

Since this inequality holds for arbitrary $\epsilon > 0$, we must have $U(f+g) \leq U(f) + U(g)$.

Similarly, we have $L(f+g;P) \geq L(f;P) + L(g;P)$ for all partitions P, and for every $\epsilon > 0$, we get $L(f+g) > L(f) + L(g) - \epsilon$, so $L(f+g) \geq L(f) + L(g)$.

For integrable functions f and g, it follows that

$$U(f+g) \leq U(f) + U(g) = L(f) + L(g) \leq L(f+g).$$

Since $U(f+g) \geq L(f+g)$, we have $U(f+g) = L(f+g)$ and $f+g$ is integrable. Moreover, there is equality throughout the previous inequality, which proves the result.

□

The product of integrable functions is also integrable, as is the quotient provided it remains bounded. Unlike the integral of the sum, however, there is no way to express the integral of the product $\int fg$ in terms of $\int f$ and $\int g$.

Theorem 5.3. *If $f, g : [a,b] \to \mathbb{R}$ are integrable, then $fg : [a,b] \to \mathbb{R}$ is integrable. If, in addition, $g \neq 0$ and $1/g$ is bounded, then $f/g : [a,b] \to \mathbb{R}$ is integrable.*

Proof. First, we show that the square of an integrable function is integrable. If f is integrable, then f is bounded, with $|f| \leq M$ for some $M \geq 0$. For all $x, y \in [a,b]$, we have

$$\left|f^2(x) - f^2(y)\right| = |f(x) + f(y)| \cdot |f(x) - f(y)| \leq 2M|f(x) - f(y)|.$$

Taking the supremum of this inequality over $x, y \in I \subset [a,b]$, we get that

$$\sup_I \left(f^2\right) - \inf_I \left(f^2\right) \leq 2M \left[\sup_I f - \inf_I f\right],$$

meaning that

$$\operatorname*{OSc}_I \left(f^2\right) \leq 2M \operatorname*{osc}_I f.$$

Thus, f^2 is integrable if f is integrable.

Since the integral is linear, we then see from the identity

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

that fg is integrable if f, g are integrable. We remark that the trick of representing a product as a difference of squares isn't a new one: the ancient Babylonian apparently used this identity, together with a table of squares, to compute products.

In a similar way, if $g \neq 0$ and $|1/g| \leq M$, then

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| = \frac{|g(x) - g(y)|}{|g(x)g(y)|} \leq M^2|g(x) - g(y)|.$$

Taking the supremum of this equation over $x, y \in I \subset [a, b]$, we get

$$\sup_I \left(\frac{1}{g}\right) - \inf_I \left(\frac{1}{g}\right) \leq M^2 \left[\sup_I g - \inf_I g\right],$$

meaning that $\operatorname{osc}_I(1/g) \leq M^2 \operatorname{osc}_I g$, and Proposition 11.25 implies that $1/g$ is integrable if g is integrable. Therefore $f/g = f \cdot (1/g)$ is integrable.

□

5.2 Monotonicity

Next, we prove the monotonicity of the integral.

Theorem 5.4. *Suppose that $f, g : [a, b] \to \mathbb{R}$ are integrable and $f \leq g$. Then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. First suppose that $f \geq 0$ is integrable. Let P be the partition consisting of the single interval $[a, b]$. Then

$$L(f; P) = \inf_{[a,b]} f \cdot (b - a) \geq 0,$$

so

$$\int_a^b f \geq L(f; P) \geq 0.$$

If $f \geq g$, then $h = f - g \geq 0$, and the linearity of the integral implies that

$$\int_a^b f - \int_a^b g = \int_a^b h \geq 0,$$

which proves the theorem. One immediate consequence of this theorem is the following simple, but useful, estimate for integrals.

□

Theorem 5.5. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and*

$$M = \sup_{[a,b]} f, \quad m = \inf_{[a,b]} f.$$

Then

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Proof. Since $m \leq f \leq M$ on $[a, b]$, Theorem 11.36 implies that

$$\int_a^b m \leq \int_a^b f \leq \int_a^b M,$$

which gives the result.

□

A further consequence is the intermediate value theorem for integrals, which states that a continuous function on a compact interval is equal to its average value at some point in the interval.

Theorem 5.6. *If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $x \in [a, b]$ such that*

$$f(x) = \frac{1}{b-a} \int_a^b f.$$

Proof. Since f is a continuous function on a compact interval, the extreme value theorem implies it attains its maximum value M and its minimum value m. From Theorem 5.5

$$m \leq \frac{1}{b-a} \int_a^b f \leq M$$

By the intermediate value theorem, f takes on every value between m and M, and the result follows.

□

We can estimate the absolute value of an integral by taking the absolute value under the integral sign. This is analogous to the corresponding property of sums:

$$\left| \sum_{k=1}^n a_n \right| \leq \sum_{k=1}^n |a_k| \,.$$

Theorem 5.7. *If f is integrable, then $|f|$ is integrable and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. First, suppose that $|f|$ is integrable. Since

$$-|f| \leq f \leq |f|,$$

we get from Theorem 11.36 that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|, \qquad \text{or} \qquad \left| \int_a^b f \right| \leq \int_a^b |f|.$$

To complete the proof, we need to show that $|f|$ is integrable if f is integrable. For $x, y \in [a, b]$, the reverse triangle inequality gives

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|.$$

This implies

$$\sup_I |f| - \inf_I |f| \le \sup_I f - \inf_I f,$$

meaning that $\operatorname{osc}_I |f| \le \operatorname{osc}_I f$. Proposition 3.1 then implies that $|f|$ is integrable if f is integrable.

$\square$

Finally, we prove a useful positive result for the integral of continuous functions.

Proposition 5.1. *If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f \ge 0$ and $\int_a^b f = 0$, then $f = 0$.*

Proof. Suppose for contradiction that $f(c) > 0$ for some $a \le c \le b$. For definiteness, assume that $a < c < b$. (The proof is similar if c is an endpoint.) Then, since f is continuous, there exists $\delta > 0$ such that

$$|f(x) - f(c)| \le \frac{f(c)}{2} \quad \text{for } c - \delta \le x \le c + \delta,$$

where we choose δ small enough that $c - \delta > a$ and $c + \delta < b$. It follows that

$$f(x) = f(c) + f(x) - f(c) \ge f(c) - |f(x) - f(c)| \ge \frac{f(c)}{2}$$

for $c - \delta \le x \le c + \delta$. Using this inequality and the assumption that $f \ge 0$, we get

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f \ge 0 + \frac{f(c)}{2} \cdot 2\delta + 0 > 0.$$

This contradiction proves the result.

$\square$

5.3 Additivity.

Finally, we prove additivity. This property refers to additivity with respect to the interval of integration, rather than linearity with respect to the function being integrated.

Theorem 5.8. *Suppose that $f : [a, b] \to \mathbb{R}$ and $a < c < b$. Then f is Riemann integrable on $[a, b]$ if and only if it is Riemann integrable on $[a, c]$ and $[c, b]$. Moreover, in that case,*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. Suppose that f is integrable on $[a, b]$. Then, given $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f; P) - L(f; P) < \epsilon$. Let $P' = P \cup \{c\}$ be the refinement of P obtained by adding c to the endpoints of P. (If $c \in P$, then

$P' = P$.) Then $P' = Q \cup R$ where $Q = P' \cap [a, c]$ and $R = P' \cap [c, b]$ are partitions of $[a, c]$ and $[c, b]$ respectively. Moreover,

$$U(f; P') = U(f; Q) + U(f; R), \quad L(f; P') = L(f; Q) + L(f; R).$$

It follows that

$$\begin{aligned} U(f; Q) - L(f; Q) &= U(f; P') - L(f; P') - [U(f; R) - L(f; R)] \\ &\le U(f; P) - L(f; P) < \epsilon, \end{aligned}$$

which proves that f is integrable on $[a, c]$. Exchanging Q and R, we get the proof for $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then there are partitions Q of $[a, c]$ and R of $[c, b]$ such that

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2}, \quad U(f; R) - L(f; R) < \frac{\epsilon}{2}.$$

Let $P = Q \cup R$. Then

$$U(f; P) - L(f; P) = U(f; Q) - L(f; Q) + U(f; R) - L(f; R) < \epsilon,$$

which proves that f is integrable on $[a, b]$.

Finally, if f is integrable, then with the partitions P, Q, R as above, we have

$$\begin{aligned} \int_a^b f &\le U(f; P) = U(f; Q) + U(f; R) \\ &< L(f; Q) + L(f; R) + \epsilon \\ &< \int_a^c f + \int_c^b f + \epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_a^b f &\ge L(f; P) = L(f; Q) + L(f; R) \\ &> U(f; Q) + U(f; R) - \epsilon \\ &> \int_a^c f + \int_c^b f - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\int_a^b f = \int_a^c f + \int_c^b f$.

□

We can extend the additivity property of the integral by defining an oriented Riemann integral.

Definition 5.1. *If $f : [a, b] \to \mathbb{R}$ is integrable, where $a < b$, and $a \le c \le b$, then*

$$\int_b^a f = -\int_a^b f, \quad \int_c^c f = 0.$$

With this definition, the additivity property in Theorem 11.44 holds for all $a, b, c \in \mathbb{R}$ for which the oriented integrals exist.

6 Further existence results (Section 33)

In this section, we prove several further useful conditions for the existence of the Riemann integral.

First, we show that changing the values of a function at finitely many points doesn't change its integrability of the value of its integral.

Proposition 6.1. *Suppose that $f, g : [a, b] \to \mathbb{R}$ and $f(x) = g(x)$ except at finitely many points $x \in [a, b]$. Then f is integrable if and only if g is integrable, and in that case*

$$\int_a^b f = \int_a^b g.$$

Proof. It is sufficient to prove the result for functions whose values differ at a single point, say $c \in [a, b]$. The general result then follows by repeated application of this result.

Since f, g differ at a single point, f is bounded if and only if g is bounded. If f, g are unbounded, then neither one is integrable. If f, g are bounded, we will show that f, g have the same upper and lower integrals. The reason is that their upper and lower sums differ by an arbitrarily small amount with respect to a partition that is sufficiently refined near the point where the functions differ.

Suppose that f, g are bounded with $|f|, |g| \le M$ on $[a, b]$ for some $M > 0$. Let $\epsilon > 0$. Choose a partition P of $[a, b]$ such that

$$U(f; P) < U(f) + \frac{\epsilon}{2}.$$

Let $Q = \{I_1, \dots, I_n\}$ be a refinement of P such that $|I_k| < \delta$ for $k = 1, \dots, n$, where

$$\delta = \frac{\epsilon}{8M}.$$

Then g differs from f on at most two intervals in Q. (This could happen on two intervals if c is an endpoint of the partition.) On such an interval I_k we have

$$\left| \sup_{I_k} g - \sup_{I_k} f \right| \le \sup_{I_k} |g| + \sup_{I_k} |f| \le 2M,$$

and on the remaining intervals, $\sup_{I_k} g - \sup_{I_k} f = 0$. It follows that

$$|U(g; Q) - U(f; Q)| < 2M \cdot 2\delta < \frac{\epsilon}{2}.$$

Using the properties of upper integrals and refinements, we obtain that

$$U(g) \le U(g; Q) < U(f; Q) + \frac{\epsilon}{2} \le U(f; P) + \frac{\epsilon}{2} < U(f) + \epsilon$$

Since this inequality holds for arbitrary $\epsilon > 0$, we get that $U(g) \le U(f)$. Exchanging f and g, we see similarly that $U(f) \le U(g)$, so $U(f) = U(g)$.

An analogous argument for lower sums (or an application of the result for upper sums to $-f, -g$) shows that $L(f) = L(g)$. Thus $U(f) = L(f)$ if and only if $U(g) = L(g)$, in which case $\int_a^b f = \int_a^b g$.

∎

The next proposition allows us to deduce the integrability of a bounded function on an interval from its integrability on slightly smaller intervals.

Proposition 6.2. *Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and integrable on $[a, r]$ for every $a < r < b$. Then f is integrable on $[a, b]$ and*

$$\int_a^b f = \lim_{r \to b^-} \int_a^r f.$$

Proof. Since f is bounded, $|f| \leq M$ on $[a, b]$ for some $M > 0$. Given $\epsilon > 0$, let

$$r = b - \frac{\epsilon}{4M}$$

(where we assume ϵ is sufficiently small that $r > a$). Since f is integrable on $[a, r]$, there is a partition Q of $[a, r]$ such that

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2}.$$

Then $P = Q \cup \{b\}$ is a partition of $[a, b]$ whose last interval is $[r, b]$. The boundedness of f implies that

$$\sup_{[r,b]} f - \inf_{[r,b]} f \leq 2M.$$

Therefore

$$\begin{aligned} U(f; P) - L(f; P) &= U(f; Q) - L(f; Q) + \left(\sup_{[r,b]} f - \inf_{[r,b]} f \right) \cdot (b - r) \\ &< \frac{\epsilon}{2} + 2M \cdot (b - r) = \epsilon, \end{aligned}$$

so f is integrable on $[a, b]$ by Theorem 11.23 Moreover, using the additivity of the integral, we get

$$\left| \int_a^b f - \int_a^r f \right| = \left| \int_r^b f \right| \leq M \cdot (b - r) \to 0 \quad \text{as } r \to b^-.$$

□

An obvious analogous result holds for the left endpoint.

As a corollary of this result and the additivity of the integral, we prove a generalization of the integrability of continuous functions to piecewise continuous functions.

Theorem 6.1. *If $f : [a, b] \to \mathbb{R}$ is a bounded function with finitely many discontinuities, then f is Riemann integrable.*

Proof. By splitting the interval into subintervals with the discontinuities of f at an endpoint and using Theorem 5.8. we see that it is sufficient to prove the result if f is discontinuous only at one endpoint of $[a, b]$, say at b. In that case, f is continuous and therefore integrable on any smaller interval $[a, r]$ with $a < r < b$, and Proposition 6.1 implies that f is integrable on $[a, b]$.

$\square$

Example 6.1. *Define* $f : [0, 2\pi] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \sin(1/\sin x) & \text{if } x \neq 0, \pi, 2\pi, \\ 0 & \text{if } x = 0, \pi, 2\pi \end{cases}$$

Then f *is bounded and continuous except at* $x = 0, \pi, 2\pi$*, so it is integrable on* $[0, 2\pi]$ *(see Figure 1). This function doesn't have jump discontinuities, but Theorem* 11.53 *still applies.*

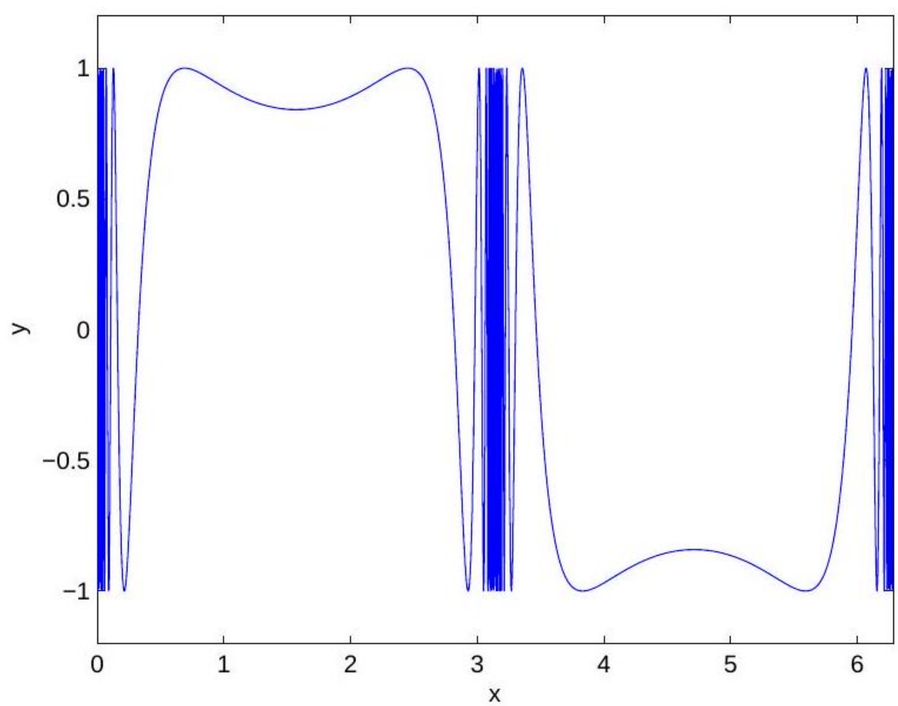

Figure 1. Graph of the Riemann integrable function $y = \sin(1/\sin x)$ *in Example* 6.1

Example 6.2. *Define* $f : [0, 1/\pi] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \operatorname{sgn}[\sin(1/x)] & \text{if } x \neq 1/n\pi \text{ for } n \in \mathbb{N}, \\ 0 & \text{if } x = 0 \text{ or } x \neq 1/n\pi \text{ for } n \in \mathbb{N}, \end{cases}$$

where sgn is the sign function,

$$\operatorname{sgn} x = \begin{cases} 1 & \textit{if } x > 0 \\ 0 & \textit{if } x = 0 \\ -1 & \textit{if } x < 0 \end{cases}$$

Then f oscillates between 1 and -1 a countably infinite number of times as $x \to 0^+$(see Figure 2). It has jump discontinuities at $x = 1/(n\pi)$ and an essential discontinuity at $x = 0$. Nevertheless, it is Riemann integrable. To see this, note that f is bounded on $[0, 1]$ and piecewise continuous with finitely many discontinuities on $[r, 1]$ for every $0 < r < 1$. Theorem 6.1 implies that f is Riemann integrable on $[r, 1]$, and then Proposition 6.2 implies that f is integrable on $[0, 1]$.

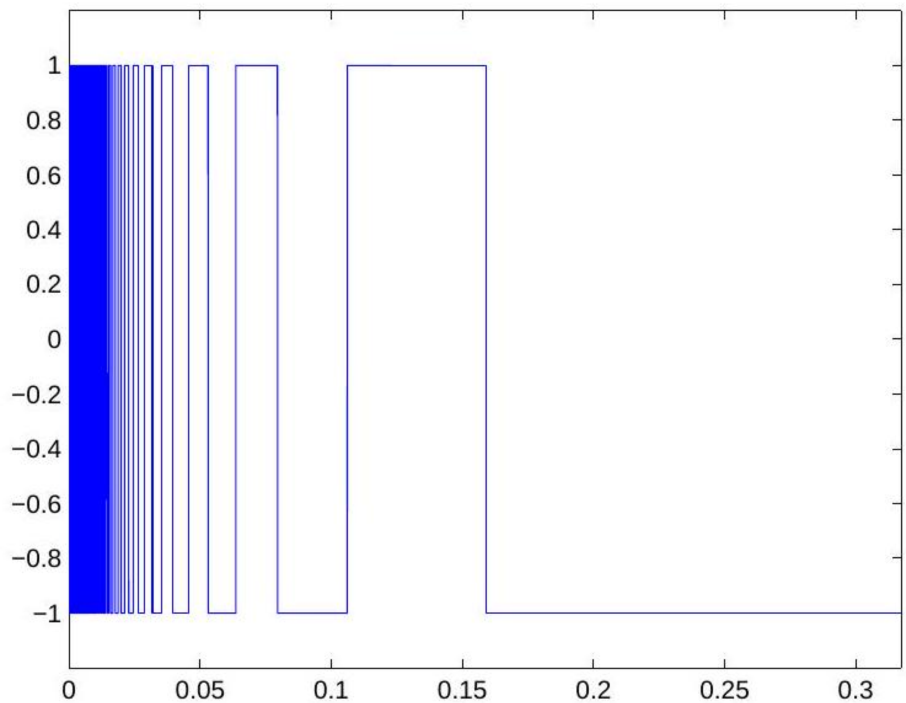

Figure 2. Graph of the Riemann integrable function $y = \operatorname{sgn}(\sin(1/x))$ in Example 6.2