Lecture notes for Math 104

7. power series

Instructor: Mengxuan Yang

Power series are one of the most useful type of series in analysis. For example, Taylor expansion. We will cover the following topics:

- Definition of power series
- Convergence of power series (radius of convergence)
- Calculation of power series
- Differentiability of power series

We omit the integrability of power series in the textbook (Section 26) and leave it to the future.

1 Definition

(Section 23) A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$$

where the constants a_n are some coefficients.

If all but finitely many of the a_n are zero, then the power series is a polynomial function, but if infinitely many of the a_n are nonzero, then we need to consider the convergence of the power series.

The basic facts are these: Every power series has a radius of convergence $0 \leq R \leq \infty$, which depends on the coefficients a_n:

- The power series converges absolutely in $|x| < R$ and diverges in $|x| > R$.
- The convergence is uniform on every interval $|x| < \rho$ where $0 \leq \rho < R$.
- If $R > 0$, then the sum of the power series is infinitely differentiable in $|x| < R$, and its derivatives are given by differentiating the original power series term-by-term.

Definition 1.1. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers and $c \in \mathbb{R}$. The power series centered at c with coefficients a_n is the series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Example 1.1. *The following are power series centered at 0:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots \\
&\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots \\
&\sum_{n=0}^{\infty} (n!) x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \dots \\
&\sum_{n=0}^{\infty} (-1)^n x^{2^n} = x - x^2 + x^4 - x^8 + \dots
\end{aligned}$$

An example of a power series centered at 1 is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

2 Radius of convergence

(Section 23) First, we prove that every power series has a radius of convergence.

Theorem 2.1 (Radius of convergence)**.** *Let*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

be a power series.

There is a non-negative, extended real number $0 \le R \le \infty$ such that the series converges absolutely for $0 \le |x-c| < R$ and diverges for $|x-c| > R$.

Furthermore, if $0 \le \rho < R$, then the power series converges uniformly on the interval $|x-c| \le \rho$, and the sum of the series is continuous in $|x-c| < R$.

Proof. We assume without loss of generality that $c = 0$. Suppose the power series

$$\sum_{n=0}^{\infty} a_n x_0^n$$

converges for some $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Then its terms converge to zero, so they are bounded and there exists $M \ge 0$ such that

$$|a_n x_0^n| \le M \quad \text{for } n = 0, 1, 2, \dots.$$

If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M r^n, \quad r = \left|\frac{x}{x_0}\right| < 1.$$

Comparing the power series with the convergent geometric series $\sum M r^n$, we see that $\sum a_n x^n$ is absolutely convergent. Thus, if the power series converges for some $x_0 \in \mathbb{R}$, then it converges absolutely for every $x \in \mathbb{R}$ with $|x| < |x_0|$.

Let

$$R = \sup \left\{ |x| \geq 0 : \sum a_n x^n \text{ converges } \right\}.$$

If $R = 0$, then the series converges only for $x = 0$. If $R > 0$, then the series converges absolutely for every $x \in \mathbb{R}$ with $|x| < R$, since it converges for some $x_0 \in \mathbb{R}$ with $|x| < |x_0| < R$. Moreover, the definition of R implies that the series diverges for every $x \in \mathbb{R}$ with $|x| > R$. If $R = \infty$, then the series converges for all $x \in \mathbb{R}$.

Finally, let $0 \leq \rho < R$ and suppose $|x| \leq \rho$. Choose $\sigma > 0$ such that $\rho < \sigma < R$. Then $\sum |a_n \sigma^n|$ converges, so $|a_n \sigma^n| \leq M$, and therefore

$$|a_n x^n| = |a_n \sigma^n| \left|\frac{x}{\sigma}\right|^n \leq |a_n \sigma^n| \left|\frac{\rho}{\sigma}\right|^n \leq M r^n,$$

where $r = \rho/\sigma < 1$. Since $\sum M r^n < \infty$, the M-test (Theorem 5.2) implies that the series converges uniformly on $|x| \leq \rho$, and then it follows from Theorem 9.16 that the sum is continuous on $|x| \leq \rho$. Since this holds for every $0 \leq \rho < R$, the sum is continuous in $|x| < R$.

□

The following definition, therefore, makes sense for every power series.

Definition 2.1. *If the power series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges for $|x - c| < R$ *and diverges for* $|x - c| > R$*, then* $0 \leq R \leq \infty$ *is called the radius of convergence of the power series.*

Theorem 2.1 does not say what happens at the endpoints $x = c \pm R$, and in general the power series may converge or diverge there.

We refer to the set of all points where the power series converges as its interval of convergence, which is one of

$$(c - R, c + R), \quad (c - R, c + R], \quad [c - R, c + R), \quad [c - R, c + R].$$

We won't discuss here any general theorems about the convergence of power series at the endpoints (e.g., the Abel theorem). Also note that a power series need not converge uniformly on $|x - c| < R$.

A way to calculate the radius of convergence of a power series:

Theorem 2.2. *Suppose that $a_n \neq 0$ for all sufficiently large n and the limit*

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists or diverges to infinity. Then the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of convergence R.

Proof. Let

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n} \right| = |x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

By the ratio test, the power series converges if $0 \leq r < 1$, or $|x-c| < R$, and diverges if $1 < r \leq \infty$, or $|x-c| > R$, which proves the result.

□

The root test gives an expression for the radius of convergence of a general power series.

Theorem 2.3 (Hadamard)**.** *The radius of convergence R of the power series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

where $R = 0$ if the limsup diverges to ∞, and $R = \infty$ if the limsup is 0.

Proof. Let

$$r = \limsup_{n\to\infty} |a_n (x-c)^n|^{1/n} = |x-c| \limsup_{n\to\infty} |a_n|^{1/n}.$$

By the root test, the series converges if $0 \leq r < 1$, or $|x-c| < R$, and diverges if $1 < r \leq \infty$, or $|x-c| > R$, which proves the result.

□

This theorem provides an alternate proof of Theorem 10.3 from the root test; in fact, our proof of Theorem 10.3 is more-or-less a proof of the root test.

We consider an example of power series and their radius of convergence.

Example 2.1. *The geometric series*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

has a radius of convergence:

$$R = \lim_{n\to\infty} \frac{1}{1} = 1.$$

In fact,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1,$$

and diverges for $|x| > 1$.

At $x = 1$*, the series becomes*

$$1 + 1 + 1 + 1 + \dots$$

and at $x = -1$ *it becomes*

$$1 - 1 + 1 - 1 + 1 - \dots,$$

so the series diverges at both endpoints $x = \pm 1$*. Thus, the interval of convergence of the power series is* $(-1, 1)$.

3 Algebraic operations on power series

We can add, multiply, and divide power series in a standard way. For simplicity, we consider the power series centered at 0.

Proposition 3.1. *If* $R, S > 0$ *and the functions*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{in } |x| < R, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{in } |x| < S$$

are sums of convergent power series, then

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n \quad \text{in } |x| < T,$$

$$(fg)(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{in } |x| < T,$$

where $T = \min(R, S)$ *and*

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k.$$

It may happen that the radius of convergence of the power series for $f + g$ or fg is larger than the radius of convergence of the power series for f, g. For example, if $g = -f$, then the radius of convergence of the power series for $f + g = 0$ is ∞ whatever the radius of convergence of the power series for f.

The reciprocal of a convergent power series that is nonzero at its center also has a power series expansion.

4 Differentiation of power series

(Section 26) We can, differentiate power series, and they behaves as nicely as one can imagine in this respect. The sum of a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

is infinitely differentiable inside its interval of convergence, and its derivative

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

is given by term-by-term differentiation.

Theorem 4.1. *Suppose that the power series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of convergence R. Then the power series

$$\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

also has a radius of convergence R.

Proof. Assume without loss of generality that $c = 0$, and suppose $|x| < R$. Choose ρ such that $|x| < \rho < R$, and let

$$r = \frac{|x|}{\rho}, \quad 0 < r < 1.$$

To estimate the terms in the differentiated power series by the terms in the original series, we rewrite their absolute values as follows:

$$\left| n a_n x^{n-1} \right| = \frac{n}{\rho} \left(\frac{|x|}{\rho} \right)^{n-1} |a_n \rho^n| = \frac{n r^{n-1}}{\rho} |a_n \rho^n| .$$

The ratio test shows that the series $\sum n r^{n-1}$ converges, since

$$\lim_{n\to\infty} \left[\frac{(n+1)r^n}{n r^{n-1}} \right] = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n} \right) r \right] = r < 1,$$

so the sequence $\left(nr^{n-1}\right)$ is bounded, by M say. It follows that

$$\left|na_n x^{n-1}\right| \leq \frac{M}{\rho} \left|a_n \rho^n\right| \qquad \text{for all } n \in \mathbb{N}.$$

The series $\sum |a_n \rho^n|$ converges, since $\rho < R$, so the comparison test implies that $\sum na_n x^{n-1}$ converges absolutely.

Conversely, suppose $|x| > R$. Then $\sum |a_n x^n|$ diverges (since $\sum a_n x^n$ diverges)

$$\left|na_n x^{n-1}\right| \geq \frac{1}{|x|} \left|a_n x^n\right|$$

for $n \geq 1$, so the comparison test implies that $\sum na_n x^{n-1}$ diverges. Thus the series have the same radius of convergence. □

Theorem 4.2. *Suppose that the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \qquad \text{for } |x-c| < R$$

has radius of convergence $R > 0$. Then f is differentiable in $|x-c| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} na_n (x-c)^{n-1} \qquad \text{for } |x-c| < R.$$

Proof. The term-by-term differentiated power series converges in $|x-c| < R$ by Theorem 4.1. We denote its sum by

$$g(x) = \sum_{n=1}^{\infty} na_n (x-c)^{n-1}.$$

Let $0 < \rho < R$. Then, the power series for f and g both converge uniformly in $|x-c| < \rho$. Thus, we conclude that f is differentiable in $|x-c| < \rho$ and $f' = g$. Since this holds for every $0 \leq \rho < R$, it follows that f is differentiable in $|x-c| < R$ and $f' = g$, which proves the result. □

Repeated application of Theorem 4.2 implies that the sum of a power series is infinitely differentiable inside its interval of convergence and its derivatives are given by term-by-term differentiation of the power series.

Furthermore, we can get an expression for the coefficients a_n in terms of the function f; they are simply the Taylor coefficients of f at c.

Theorem 4.3. *If the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R > 0$, then f is infinitely differentiable in $|x-c| < R$ and

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Proof. We assume $c = 0$ without loss of generality. Applying Theorem 10.22 to the power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \ldots$$

k times, we find that f has derivatives of every order in $|x| < R$, and

$$\begin{aligned}
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \ldots, \\
f''(x) &= 2a_2 + (3 \cdot 2) a_3 x + \cdots + n(n-1) a_n x^{n-2} + \ldots, \\
f'''(x) &= (3 \cdot 2 \cdot 1) a_3 + \cdots + n(n-1)(n-2) a_n x^{n-3} + \ldots, \\
&\vdots \\
f^{(k)}(x) &= (k!) a_k + \cdots + \frac{n!}{(n-k)!} x^{n-k} + \ldots,
\end{aligned}$$

where all of these power series have radius of convergence R. Setting $x = 0$ in these series, we get

$$a_0 = f(0), \quad a_1 = f'(0), \quad \ldots \quad a_k = \frac{f^{(k)}(0)}{k!}, \quad \ldots$$

which proves the result (after replacing 0 by c). $\square$

One consequence of this result is that power series with different coefficients cannot converge to the same sum.

Corollary 4.1. *If two power series*

$$\sum_{n=0}^{\infty} a_n (x - c)^n, \quad \sum_{n=0}^{\infty} b_n (x - c)^n$$

have nonzero-radius of convergence and are equal in some neighborhood of 0 , then $a_n = b_n$ for every $n = 0, 1, 2, \ldots$.

Proof. If the common sum in $|x - c| < \delta$ is $f(x)$, we have

$$a_n = \frac{f^{(n)}(c)}{n!}, \quad b_n = \frac{f^{(n)}(c)}{n!},$$

since the derivatives of f at c are determined by the values of f in an arbitrarily small open interval about c, so the coefficients are equal.

$\square$

5 The exponential function

The power series

$$E(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots + \frac{1}{n!} x^n + \ldots$$

has radius of convergence ∞. It therefore defines an infinitely differentiable function $E : \mathbb{R} \to \mathbb{R}$.

Term-by-term differentiation of the power series implies that

$$E'(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{(n-1)!}x^{(n-1)} + \ldots,$$

so $E' = E$. Moreover $E(0) = 1$.

Proposition 5.1. *For every $x, y \in \mathbb{R}$,*

$$E(x)E(y) = E(x+y).$$

Proof. We have

$$E(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \quad E(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

Multiplying these series term-by-term, we get

$$\begin{aligned} E(x)E(y) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j y^k}{j!k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{n-k} y^k}{(n-k)!k!}. \end{aligned}$$

From the binomial theorem,

$$\sum_{k=0}^{n} \frac{x^{n-k} y^k}{(n-k)!k!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k = \frac{1}{n!}(x+y)^n.$$

Hence,

$$E(x)E(y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y),$$

which proves the result. □

Next, we prove that the exponential is characterized by the properties $E' = E$ and $E(0) = 1$. This is a simple uniqueness result for an initial value problem for a linear ordinary differential equation.

Proposition 5.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that*

$$f' = f, \quad f(0) = 1.$$

Then $f = E$.

In view of this result, we now write $E(x) = e^x$. The following proposition shows that e^x grows faster than any power of x as $x \to \infty$.

Proposition 5.3. *Suppose that n is a non-negative integer. Then*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Proof. The terms in the power series of e^x are positive for $x > 0$, so for every $k \in \mathbb{N}$

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} > \frac{x^k}{k!} \quad \text{for all } x > 0.$$

Taking $k = n + 1$, we get for $x > 0$ that

$$0 < \frac{x^n}{e^x} < \frac{x^n}{x^{(n+1)}/(n+1)!} = \frac{(n+1)!}{x}.$$

Since $1/x \to 0$ as $x \to \infty$, the result follows. $\square$