Lecture notes for Math 104

5. Continuous functions

Instructor: Mengxuan Yang

We will cover the following topics:

- Definition of continuous functions
- Basic properties of continuous functions
- Uniform continuity
- Advanced properties of continuous functions

1 Continuity (Section 17)

Continuous functions are functions that take nearby values at nearby points.

Definition 1.1. *Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in A$. Then f is continuous at c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - f(c)| < \epsilon.$$

A function $f : A \to \mathbb{R}$ is continuous if it is continuous at every point of A, and it is continuous on $B \subset A$ if it is continuous at every point in B.

Note that c must belong to the domain A of f in order to define the continuity of f at c. If c is an isolated point of A, then the continuity condition holds automatically since, for sufficiently small $\delta > 0$, the only point $x \in A$ with $|x - c| < \delta$ is $x = c$, and then $0 = |f(x) - f(c)| < \epsilon$. Thus, a function is continuous at every isolated point of its domain, and isolated points are not of much interest.

If $c \in A$ is an accumulation point of A, then the continuity of f at c is equivalent to the condition that

$$\lim_{x \to c} f(x) = f(c),$$

meaning that the limit of f as $x \to c$ exists and is equal to the value of f at c.

Example 1.1. *If $f : (a, b) \to \mathbb{R}$ is defined on an open interval, then f is continuous on (a, b) if and only if*

$$\lim_{x \to c} f(x) = f(c) \quad \text{for every } a < c < b$$

since every point of (a, b) is an accumulation point.

Example 1.2. *If $f:[a,b]\to\mathbb{R}$ is defined on a closed, bounded interval, then f is continuous on $[a,b]$ if and only if*

$$\begin{array}{ll} \lim_{x\to c} f(x)=f(c) & \text{for every } a<c<b, \\ \lim_{x\to a^+} f(x)=f(a), & \lim_{x\to b^-} f(x)=f(b). \end{array}$$

Example 1.3. *Suppose that*

$$A=\left\{0,1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right\}$$

and $f:A\to\mathbb{R}$ is defined by

$$f(0)=y_0, \quad f\left(\frac{1}{n}\right)=y_n$$

for some values $y_0, y_n\in\mathbb{R}$. Then $1/n$ is an isolated point of A for every $n\in\mathbb{N}$, so f is continuous at $1/n$ for every choice of y_n. The remaining point $0\in A$ is an accumulation point of A, and the condition for f to be continuous at 0 is that

$$\lim_{n\to\infty} y_n=y_0.$$

As for limits, we can give an equivalent sequential definition of continuity.

Theorem 1.1. *If $f:A\to\mathbb{R}$ and $c\in A$ is an accumulation point of A, then f is continuous at c if and only if*

$$\lim_{n\to\infty} f\left(x_n\right)=f(c)$$

for every sequence (x_n) in A such that $x_n\to c$ as $n\to\infty$.

Exercise 1. *Prove Theorem 1.1.*

In particular, f is discontinuous at $c\in A$ if there is sequence (x_n) in the domain A of f such that $x_n\to c$ but $f\left(x_n\right)\nrightarrow f(c)$.

Let's consider some examples of continuous and discontinuous functions to illustrate the definition.

Example 1.4. *The function $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=\sqrt{x}$ is continuous on $[0,\infty)$. To prove that f is continuous at $c>0$, we note that for $0\le x<\infty$,*

$$|f(x)-f(c)|=|\sqrt{x}-\sqrt{c}|=\left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right|\le\frac{1}{\sqrt{c}}|x-c|,$$

so given $\epsilon>0$, we can choose $\delta=\sqrt{c}\epsilon>0$ in the definition of continuity. To prove that f is continuous at 0 , we note that if $0\le x<\delta$ where $\delta=\epsilon^2>0$, then

$$|f(x)-f(0)|=\sqrt{x}<\epsilon.$$

Example 1.5. *The function* $\sin : \mathbb{R} \to \mathbb{R}$ *is continuous on* $\mathbb{R}$*. To prove this, we use the trigonometric identity for the difference of sines and the inequality* $|\sin x| \leq |x|$*:*

$$\begin{aligned} |\sin x - \sin c| &= \left| 2\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| \\ &\leq |x-c|. \end{aligned}$$

It follows that we can take $\delta = \epsilon$ *in the definition of continuity for every* $c \in \mathbb{R}$*.*

Example 1.6. *The sign function* $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$*, defined by*

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is not continuous at 0 since $\lim_{x\to 0} \operatorname{sgn} x$ *does not exist (see Example 6.8). The left and right limits of* sgn *at 0,*

$$\lim_{x\to 0^-} f(x) = -1, \qquad \lim_{x\to 0^+} f(x) = 1,$$

do exist, but they are unequal. We say that f *has a jump discontinuity at 0.*

Example 1.7. *The function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at 0 since $\lim_{x\to 0} f(x)$ *does not exist (see Example 6.9. . The left and right limits of* f *at 0 do not exist either, and we say that* f *has an essential discontinuity at 0.*

Example 1.8. *The function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $c \neq 0$ *but discontinuous at 0 because* $\lim_{x\to 0} f(x)$ *does not exist.*

Example 1.9. *The function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at every point of $\mathbb{R}$*. (See Figure 1) The continuity at* $c \neq 0$ *is obvious. To prove continuity at 0 , note that for* $x \neq 0$*,*

$$|f(x) - f(0)| = |x\sin(1/x)| \leq |x|,$$

so $f(x) \to f(0)$ as $x \to 0$. If we had defined $f(0)$ to be any value other than 0, then f would not be continuous at 0. In that case, f would have a removable discontinuity at 0 .

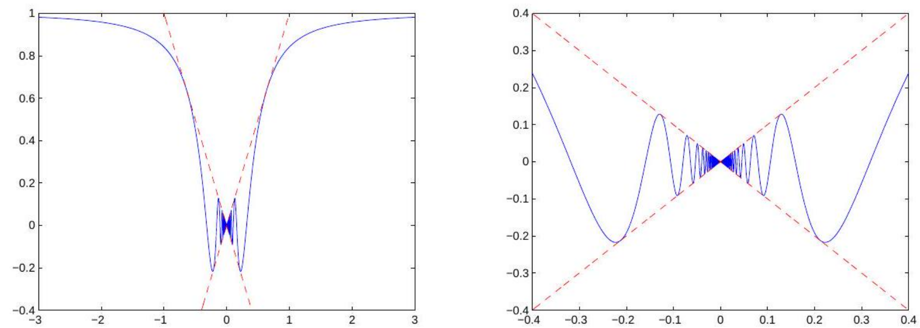

Figure 1. A plot of the function $y = x\sin(1/x)$ and a detail near the origin with the lines $y = \pm x$ shown in red.

Example 1.10. *The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous at every $c \in \mathbb{R}$. If $c \notin \mathbb{Q}$, choose a sequence (x_n) of rational numbers such that $x_n \to c$ (possible since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then $x_n \to c$ and $f(x_n) \to 1$ but $f(c) = 0$. If $c \in \mathbb{Q}$, choose a sequence (x_n) of irrational numbers such that $x_n \to c$; for example if $c = p/q$, we can take

$$x_n = \frac{p}{q} + \frac{\sqrt{2}}{n},$$

since $x_n \in \mathbb{Q}$ would imply that $\sqrt{2} \in \mathbb{Q}$. Then $x_n \to c$ and $f(x_n) \to 0$ but $f(c) = 1$. Alternatively, by taking a rational sequence (x_n) and an irrational sequence $(\tilde{x}_n)$ that converge to c, we can see that $\lim_{x\to c} f(x)$ does not exist for any $c \in \mathbb{R}$.

Example 1.11. *The Thomae function $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \text{ where } p \text{ and } q > 0 \text{ are relatively prime,} \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0. \end{cases}$$

Figure 2 shows the graph of f on $[0, 1]$. The Thomae function is continuous at 0 and every irrational number and discontinuous at every nonzero rational number.

To prove this claim, first suppose that $x = p/q \in \mathbb{Q}\backslash\{0\}$ is rational and nonzero. Then $f(x) = 1/q > 0$, but for every $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains irrational points y such that $f(y) = 0$ and $|f(x) - f(y)| = 1/q$. The definition of continuity therefore fails if $0 < \epsilon \le 1/q$, and f is discontinuous at x.

Second, suppose that $x \notin \mathbb{Q}$ is irrational. Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $1/n < \epsilon$. There are finitely many rational numbers $r = p/q$ in the interval $(x- 1, x+1)$ with p, q relatively prime and $1 \leq q \leq n$; we list them as $\{r_1, r_2, \dots, r_m\}$. Choose

$$\delta = \min \{|x - r_k| : k = 1, 2, \dots, n\}$$

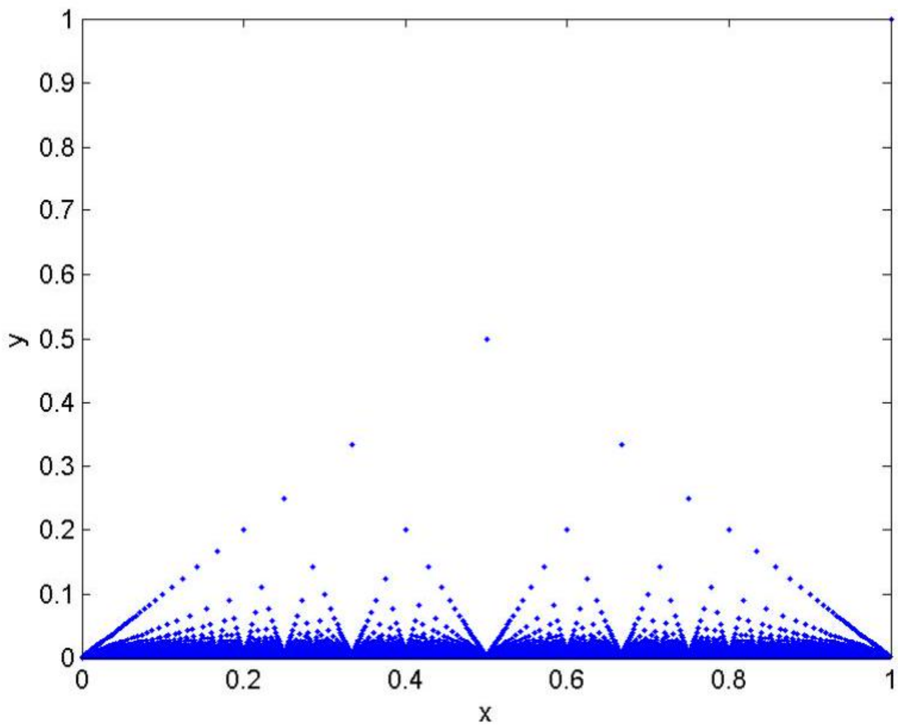

Figure 2. A plot of the Thomae function in Example 1.11 on $[0, 1]$

to be the distance of x to the closest such rational number. Then $\delta > 0$ since $x \notin \mathbb{Q}$. Furthermore, if $|x - y| < \delta$, then either y is irrational and $f(y) = 0$, or $y = p/q$ in lowest terms with $q > n$ and $f(y) = 1/q < 1/n < \epsilon$. In either case, $|f(x) - f(y)| = |f(y)| < \epsilon$, which proves that f is continuous at $x \notin \mathbb{Q}$.

The continuity of f at 0 follows immediately from the inequality $0 \leq f(x) \leq |x|$ for all $x \in \mathbb{R}$.

2 Basic properties of continuous functions (Section 17)

The basic properties of continuous functions follow from those of limits.

Theorem 2.1. *If $f, g : A \to \mathbb{R}$ are continuous at $c \in A$ and $k \in \mathbb{R}$, then kf, $f+g$, and fg are continuous at c. Moreover, if $g(c) \neq 0$ then f/g is continuous at c.*

Example 2.1. *The function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \frac{x + 3x^3 + 5x^5}{1 + x^2 + x^4}$$

is continuous on $\mathbb{R}$ since it is a rational function whose denominator never vanishes.

In addition to forming sums, products and quotients, another way to build up more complicated functions from simpler functions is by composition. We recall that the composition $g \circ f$ of functions f, g is defined by $(g \circ f)(x) = g(f(x))$.

Theorem 2.2. *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at c.*

Proof. Let $\epsilon > 0$ be given. Since g is continuous at $f(c)$, there exists $\eta > 0$ such that

$$|y - f(c)| < \eta \text{ and } y \in B \text{ implies that } |g(y) - g(f(c))| < \epsilon.$$

Next, since f is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - f(c)| < \eta.$$

Combing these inequalities, we get that

$$|x - c| < \delta \text{ and } x \in A \text{ implies that } |g(f(x)) - g(f(c))| < \epsilon,$$

which proves that $g \circ f$ is continuous at c. □

Example 2.2. *The function*

$$f(x) = \begin{cases} 1/\sin x & \text{if } x \neq n\pi \text{ for } n \in \mathbb{Z}, \\ 0 & \text{if } x = n\pi \text{ for } n \in \mathbb{Z} \end{cases}$$

is continuous on $\mathbb{R}\backslash\{n\pi : n \in \mathbb{Z}\}$, since it is the composition of $x \mapsto \sin x$, which is continuous on $\mathbb{R}$, and $y \mapsto 1/y$, which is continuous on $\mathbb{R}\backslash\{0\}$, and $\sin x \neq 0$ when $x \neq n\pi$. It is discontinuous at $x = n\pi$ because it is not locally bounded at those points.

Example 2.3. *The function*

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $\mathbb{R}\backslash\{0\}$, since it is the composition of $x \mapsto 1/x$, which is continuous on $\mathbb{R}\backslash\{0\}$, and $y \mapsto \sin y$, which is continuous on $\mathbb{R}$.

Example 2.4. *The function*

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is continuous on $\mathbb{R}\backslash\{0\}$ *since it is a product of functions that are continuous on* $\mathbb{R}\backslash\{0\}$*. As shown in Example 1.9,* f *is also continuous at 0, so* f *is continuous on* $\mathbb{R}$*.*

3 Uniform continuity (Section 19)

Uniform continuity is a subtle but powerful strengthening of continuity.

Definition 3.1. *Let* $f : A \to \mathbb{R}$*, where* $A \subset \mathbb{R}$*. Then* f *is uniformly continuous on* A *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|x - y| < \delta \text{ and } x, y \in A \text{ implies that } |f(x) - f(y)| < \epsilon.$$

The key point of this definition is that δ depends only on ϵ, not on x, y. A uniformly continuous function on A is continuous at every point of A, but the converse is not true.

To explain this point in more detail, note that if a function f is continuous on A, then given $\epsilon > 0$ and $c \in A$, there exists $\delta(\epsilon, c) > 0$ such that

$$|x - c| < \delta(\epsilon, c) \text{ and } x \in A \text{ implies that } |f(x) - f(c)| < \epsilon.$$

If for some $\epsilon_0 > 0$ we have

$$\inf_{c \in A} \delta(\epsilon_0, c) = 0$$

however we choose $\delta(\epsilon_0, c) > 0$ in the definition of continuity, then no $\delta_0(\epsilon_0) > 0$ depending only on ϵ_0 works simultaneously for every $c \in A$. In that case, the function is continuous on A but not uniformly continuous.

Before giving some examples, we state a sequential condition for uniform continuity to fail.

Proposition 3.1. *A function* $f : A \to \mathbb{R}$ *is not uniformly continuous on* A *if and only if there exists* $\epsilon_0 > 0$ *and sequences* $(x_n), (y_n)$ *in* A *such that*

$$\lim_{n\to\infty} |x_n - y_n| = 0 \text{ and } |f(x_n) - f(y_n)| \geq \epsilon_0 \text{ for all } n \in \mathbb{N}.$$

Proof. If f is not uniformly continuous, then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there are points $x, y \in A$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon_0$. Choosing $x_n, y_n \in A$ to be any such points for $\delta = 1/n$, we get the required sequences.

Conversely, if the sequential condition holds, then for every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$. It follows that the uniform continuity condition in Definition 3.1 cannot hold for any $\delta > 0$ if $\epsilon = \epsilon_0$, so f is not uniformly continuous. □

Example 3.1. *Define $f : [0,1] \to \mathbb{R}$ by $f(x) = x^2$. Then f is uniformly continuous on $[0,1]$. To prove this, note that for all $x, y \in [0,1]$ we have*

$$\left|x^2 - y^2\right| = |x+y||x-y| \leq 2|x-y|,$$

so we can take $\delta = \epsilon/2$ in the definition of uniform continuity. Similarly, $f(x) = x^2$ is uniformly continuous on any bounded set.

Example 3.2. *The function $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbb{R}$. We have already proved that f is continuous on $\mathbb{R}$ (it's a polynomial). To prove that f is not uniformly continuous, let*

$$x_n = n, \quad y_n = n + \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

but

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \qquad \text{for every } n \in \mathbb{N}.$$

It follows from Proposition 3.1 that f is not uniformly continuous on $\mathbb{R}$. The problem here is that in order to prove the continuity of f at c, given $\epsilon > 0$ we need to make $\delta(\epsilon, c)$ smaller as c gets larger, and $\delta(\epsilon, c) \to 0$ as $c \to \infty$.

Example 3.3. *The function $f : (0,1] \to \mathbb{R}$ defined by*

$$f(x) = \frac{1}{x}$$

is continuous but not uniformly continuous on $(0,1]$. It is continuous on $(0,1]$ since it's a rational function whose denominator x is nonzero in $(0,1]$. To prove that f is not uniformly continuous, we define $x_n, y_n \in (0,1]$ for $n \in \mathbb{N}$ by

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n+1}.$$

Then $|x_n - y_n| \to 0$ as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = (n+1) - n = 1 \qquad \text{for every } n \in \mathbb{N}.$$

It follows from Proposition 3.1 that f is not uniformly continuous on $(0,1]$. The problem here is that given $\epsilon > 0$, we need to make $\delta(\epsilon, c)$ smaller as c gets closer to 0 , and $\delta(\epsilon, c) \to 0$ as $c \to 0^+$.

The non-uniformly continuous functions in the last two examples were unbounded. However, even bounded continuous functions can fail to be uniformly continuous if they oscillate arbitrarily quickly.

Example 3.4. *Define* $f:(0,1]\to\mathbb{R}$ *by*

$$f(x)=\sin\left(\frac{1}{x}\right)$$

Then f *is continuous on* $(0,1]$ *but it isn't uniformly continuous on* $(0,1]$*. To prove this, define* $x_n, y_n \in (0,1]$ *for* $n\in\mathbb{N}$ *by*

$$x_n=\frac{1}{2n\pi},\quad y_n=\frac{1}{2n\pi+\pi/2}.$$

Then $|x_n-y_n|\to 0$ *as* $n\to\infty$*, but*

$$|f(x_n)-f(y_n)|=\sin\left(2n\pi+\frac{\pi}{2}\right)-\sin 2n\pi=1 \qquad \text{for all } n\in\mathbb{N}.$$

It isn't a coincidence that these examples of non-uniformly continuous functions have domains that are either unbounded or not closed. We will prove later that a continuous function on a compact (closed and bounded) set is uniformly continuous.

4 Continuous functions on compact sets (Section 18, 19)

Definition 4.1. *(compact on* $\mathbb{R}$*) A set* $A\subset\mathbb{R}$ *is compact if it is closed and bounded.*

Theorem 4.1. *A set* A *is compact (bounded and closed) if and only if* A *is sequentially compact, meaning that for any sequences* (x_n) *of* A*, there exists a subsequence* (x_{n_k}) *of* (x_n) *such that* x_{n_k} *converges to some point* $a\in A$*.*

Proof. Exercise.

Hint: $\Rightarrow$: using The Bolzano-Weierstrass theorem (Lecture note 2) to find the convergent subsequence.

$\Leftarrow$: using proof by contradiction. □

Continuous functions on compact sets have especially nice properties. For example, they are bounded and attain their maximum and minimum values, and they are uniformly continuous.

Since a closed, bounded interval is compact, these results apply, in particular, to continuous functions $f:[a,b]\to\mathbb{R}$.

First, we prove that the continuous image of a compact set in $\mathbb{R}$ is compact.

Theorem 4.2. *If* $K\subset\mathbb{R}$ *is compact and* $f:K\to\mathbb{R}$ *is continuous, then* $f(K)$ *is compact.*

Proof. We will give the proof using sequences .

We show that $f(K)$ is sequentially compact. Let (y_n) be a sequence in $f(K)$. Then $y_n = f(x_n)$ for some $x_n \in K$. Since K is compact, the sequence (x_n) has a convergent subsequence (x_{n_i}) such that

$$\lim_{i\to\infty} x_{n_i} = x$$

where $x \in K$. Since f is continuous on K,

$$\lim_{i\to\infty} f(x_{n_i}) = f(x).$$

Writing $y = f(x)$, we have $y \in f(K)$ and

$$\lim_{i\to\infty} y_{n_i} = y.$$

Therefore every sequence (y_n) in $f(K)$ has a convergent subsequence whose limit belongs to $f(K)$, so $f(K)$ is compact.

$\square$

Example 4.1. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = \frac{1}{1+x^2}.$$

Then $[0,\infty)$ *is closed but* $f([0,\infty)) = (0,1]$ *is not.*

The following result is one of the most important property of continuous functions on compact sets.

Theorem 4.3 (Weierstrass extreme value)**.** *If* $f : K \to \mathbb{R}$ *is continuous and* $K \subset \mathbb{R}$ *is compact, then* f *is bounded on* K *and* f *attains its maximum and minimum values on* K*.*

Proof. The image $f(K)$ is compact from Theorem 4.1 implies that $f(K)$ is bounded and the maximum M and minimum m belong to $f(K)$. Therefore there are points $x, y \in K$ such that $f(x) = M, f(y) = m$, and f attains its maximum and minimum on K. $\square$

Example 4.2. *Define* $f : [0,1] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f *is unbounded on* $[0,1]$ *and has no maximum value* (f *does, however, have a minimum value of 0 attained at* $x = 0$)*. In this example,* $[0,1]$ *is compact but* f *is discontinuous at 0, which shows that a discontinuous function on a compact set needn't be bounded.*

Example 4.3. *Define $f : (0, 1] \to \mathbb{R}$ by $f(x) = 1/x$. Then f is unbounded on $(0, 1]$ with no maximum value (f does, however, have a minimum value of 1 attained at $x = 1$). In this example, f is continuous but the half-open interval $(0, 1]$ isn't compact, which shows that a continuous function on a non-compact set needn't be bounded.*

Example 4.4. *Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = x$. Then*

$$\inf_{x\in(0,1)} f(x) = 0, \qquad \sup_{x\in(0,1)} f(x) = 1$$

but $f(x) \neq 0, f(x) \neq 1$ for any $0 < x < 1$. Thus, even if a continuous function on a non-compact set is bounded, it needn't attain its supremum or infimum.

Example 4.5. *Define $f : [0, 2/\pi] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} x + x\sin(1/x) & \text{if } 0 < x \leq 2/\pi, \\ 0 & \text{if } x = 0. \end{cases}$$

(See Figure 3.)

Then f is continuous on the compact interval $[0, 2/\pi]$, so by Theorem 4.2 it attains its maximum and minimum. For $0 \leq x \leq 2/\pi$, we have $0 \leq f(x) \leq 1/\pi$ since $|\sin 1/x| \leq 1$. Thus, the minimum value of f is 0, attained at $x = 0$. It is also attained at infinitely many other interior points in the interval,

$$x_n = \frac{1}{2n\pi + 3\pi/2}, \qquad n = 0, 1, 2, 3, \ldots,$$

where $\sin(1/x_n) = -1$. The maximum value of f is $1/\pi$, attained at $x = 2/\pi$.

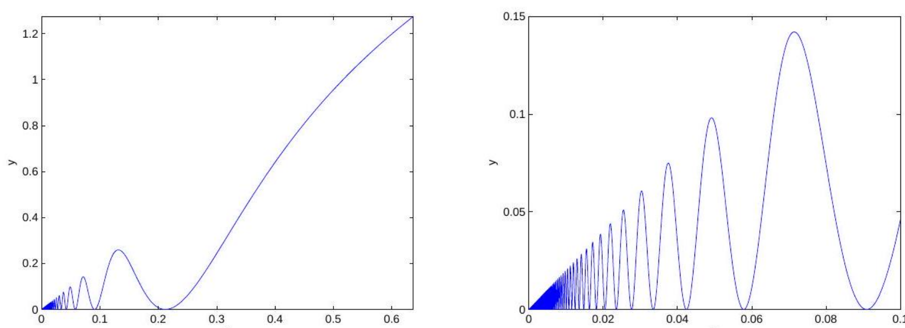

Figure 3. A plot of the function $y = x + x\sin(1/x)$ on $[0, 2/\pi]$ and detail near the origin.

Finally, we prove that continuous functions on compact sets are uniformly continuous.

Theorem 4.4. *If $f : K \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, then f is uniformly continuous on K.*

Proof. Suppose for contradiction that f is not uniformly continuous on K. Then from Proposition 3.1 there exists $\epsilon_0 > 0$ and sequences $(x_n), (y_n)$ in K such that

$$\lim_{n\to\infty} |x_n - y_n| = 0 \text{ and } |f(x_n) - f(y_n)| \geq \epsilon_0 \text{ for every } n \in \mathbb{N}.$$

Since K is compact, there is a convergent subsequence (x_{n_i}) of (x_n) such that

$$\lim_{i\to\infty} x_{n_i} = x \in K.$$

Moreover, since $(x_n - y_n) \to 0$ as $n \to \infty$, it follows that

$$\lim_{i\to\infty} y_{n_i} = \lim_{i\to\infty} [x_{n_i} - (x_{n_i} - y_{n_i})] = \lim_{i\to\infty} x_{n_i} - \lim_{i\to\infty} (x_{n_i} - y_{n_i}) = x,$$

so (y_{n_i}) also converges to x. Then, since f is continuous on K,

$$\lim_{i\to\infty} |f(x_{n_i}) - f(y_{n_i})| = \left|\lim_{i\to\infty} f(x_{n_i}) - \lim_{i\to\infty} f(y_{n_i})\right| = |f(x) - f(x)| = 0,$$

but this contradicts the non-uniform continuity condition

$$|f(x_{n_i}) - f(y_{n_i})| \geq \epsilon_0.$$

Therefore f is uniformly continuous. $\square$

5 The intermediate value theorem (Section 18)

The intermediate value theorem states that a continuous function on an interval takes on all values between any two of its values.

We first prove a special case:

Theorem 5.1 (Intermediate value)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. If $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$, then there is a point $a < c < b$ such that $f(c) = 0$.*

Proof. Assume for definiteness that $f(a) < 0$ and $f(b) > 0$. (If $f(a) > 0$ and $f(b) < 0$, consider $-f$ instead of f.) The set

$$E = \{x \in [a, b] : f(x) < 0\}$$

is nonempty, since $a \in E$, and E is bounded from above by b. Let

$$c = \sup E \in [a, b],$$

which exists by the completeness of $\mathbb{R}$. We claim that $f(c) = 0$.

Suppose for contradiction that $f(c) \neq 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \text{ and } x \in [a, b] \text{ implies that } |f(x) - f(c)| < \frac{1}{2}|f(c)|.$$

If $f(c) < 0$, then $c \neq b$ and

$$f(x) = f(c) + f(x) - f(c) < f(c) - \frac{1}{2}f(c)$$

for all $x \in [a, b]$ such that $|x - c| < \delta$, so $f(x) < \frac{1}{2}f(c) < 0$. It follows that there are points $x \in E$ with $x > c$, which contradicts the fact that c is an upper bound of E.

If $f(c) > 0$, then $c \neq a$ and

$$f(x) = f(c) + f(x) - f(c) > f(c) - \frac{1}{2}f(c)$$

for all $x \in [a, b]$ such that $|x - c| < \delta$, so $f(x) > \frac{1}{2}f(c) > 0$. It follows that there exists $\eta > 0$ such that $c - \eta \geq a$ and

$$f(x) > 0 \text{ for } c - \eta \leq x \leq c.$$

In that case, $c - \eta < c$ is an upper bound for E, since c is an upper bound and $f(x) > 0$ for $c - \eta \leq x \leq c$, which contradicts the fact that c is the least upper bound. This proves that $f(c) = 0$. Finally, $c \neq a, b$ since f is nonzero at the endpoints, so $a < c < b$.

□

We give some examples to show that all of the hypotheses in this theorem are necessary.

Example 5.1. *Let $K = [-2, -1] \cup [1, 2]$ and define $f : K \to \mathbb{R}$ by*

$$f(x) = \begin{cases} -1 & \text{if } -2 \leq x \leq -1 \\ 1 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Then $f(-2) < 0$ and $f(2) > 0$, but f doesn't vanish at any point in its domain. Thus, in general, Theorem 5.1 *fails if the domain of f is not a connected interval $[a, b]$.*

Example 5.2. *Define $f : [-1, 1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \end{cases}$$

Then $f(-1) < 0$ and $f(1) > 0$, but f doesn't vanish at any point in its domain. Here, f is defined on an interval but it is discontinuous at 0. Thus, in general, Theorem 5.1 fails for discontinuous functions. As one immediate consequence of the intermediate value theorem, we show that the real numbers contain the square root of 2.

Example 5.3. *Define the continuous function* $f : [1, 2] \to \mathbb{R}$ *by*

$$f(x) = x^2 - 2.$$

Then $f(1) < 0$ *and* $f(2) > 0$*, so Theorem 5.1 implies that there exists* $1 < c < 2$ *such that* $c^2 = 2$*. Moreover, since* $x^2 - 2$ *is strictly increasing on* $[0, \infty)$*, there is a unique such positive number, and we have proved the existence of* $\sqrt{2}$*.*

The general statement of the Intermediate Value Theorem follows immediately from this special case.

Theorem 5.2 (Intermediate value theorem VERY IMPORTANT!)**.** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is a continuous function on a closed, bounded interval. Then for every* d *strictly between* $f(a)$ *and* $f(b)$ *there is a point* $a < c < b$ *such that* $f(c) = d$*.*

Proof. Suppose, for definiteness, that $f(a) < f(b)$ and $f(a) < d < f(b)$. (If $f(a) > f(b)$ and $f(b) < d < f(a)$, apply the same proof to $-f$, and if $f(a) = f(b)$ there is nothing to prove.) Let $g(x) = f(x) - d$. Then $g(a) < 0$ and $g(b) > 0$, so Theorem 5.1 implies that $g(c) = 0$ for some $a < c < b$, meaning that $f(c) = d$. □

As one consequence of our previous results, we prove that a continuous function maps compact intervals to compact intervals.

Theorem 5.3. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is a continuous function on a closed, bounded interval. Then* $f([a, b]) = [m, M]$ *is a closed, bounded interval.*

Proof. Theorem 4.2 implies that $m \le f(x) \le M$ for all $x \in [a, b]$, where m and M are the maximum and minimum values of f on $[a, b]$, so $f([a, b]) \subset [m, M]$. Moreover, there are points $c, d \in [a, b]$ such that $f(c) = m$, $f(d) = M$.

Let $J = [c, d]$ if $c \le d$ or $J = [d, c]$ if $d < c$. Then $J \subset [a, b]$, and Theorem 5.2 implies that f takes on all values in $[m, M]$ on J. It follows that $f([a, b]) \supset [m, M]$, so $f([a, b]) = [m, M]$. □

First we give an example to illustrate the theorem.

Example 5.4. *Define* $f : [-1, 1] \to \mathbb{R}$ *by*

$$f(x) = x - x^3.$$

Then, using calculus to compute the maximum and minimum of f*, we find that*

$$f([-1, 1]) = [-M, M], \quad M = \frac{2}{3\sqrt{3}}.$$

This example illustrates that $f([a, b]) \neq [f(a), f(b)]$ *unless* f *is increasing.*

Next we give some examples to show that the continuity of f and the interval $[a, b]$ are essential for Theorem 5.3 to hold.

Example 5.5. *Let* $\operatorname{sgn} : [-1, 1] \to \mathbb{R}$ *be the sign function defined in Example* 6.8 *Then f is a discontinuous function on a compact interval $[-1, 1]$, but the range $f([-1, 1]) = \{-1, 0, 1\}$ consists of three isolated points and is not an interval.*

Example 5.6. *In Example 5.5, the function $f : K \to \mathbb{R}$ is continuous on a compact set K but $f(K) = \{-1, 1\}$ consists of two isolated points and is not an interval.*

Example 5.7. *The continuous function $f = \frac{1}{1+x^2}$ maps the unbounded, closed interval $[0, \infty)$ to the half-open interval $(0, 1]$.*

The last example shows that a continuous function may map a closed but unbounded interval to an interval that isn't closed (or open).

6 Monotonic functions (**)

Monotonic functions have continuity properties that are not shared by general functions.

Definition 6.1. *Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is increasing if*

$$f(x_1) \le f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2,$$

strictly increasing if

$$f(x_1) < f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2,$$

decreasing if

$$f(x_1) \ge f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2,$$

and strictly decreasing if

$$f(x_1) > f(x_2) \quad \text{if } x_1, x_2 \in I \text{ and } x_1 < x_2.$$

Theorem 6.1. *If $f : I \to \mathbb{R}$ is monotonic on an interval I, then the left and right limits of f*

$$\lim_{x \to c^-} f(x), \quad \lim_{x \to c^+} f(x),$$

exist at every interior point c of I.

Proof. Assume for definiteness that f is increasing. (If f is decreasing, we can apply the same argument to $-f$ which is increasing). We will prove that

$$\lim_{x \to c^-} f(x) = \sup E, \quad E = \{f(x) \in \mathbb{R} : x \in I \text{ and } x < c\}.$$

The set E is nonempty since c in an interior point of I, so there exists $x \in I$ with $x < c$, and E bounded from above by $f(c)$ since f is increasing. It follows that $L = \sup E \in \mathbb{R}$ exists. (Note that L may be strictly less than $f(c)$!)

Suppose that $\epsilon > 0$ is given. Since L is a least upper bound of E, there exists $y_0 \in E$ such that $L - \epsilon < y_0 \leq L$, and therefore $x_0 \in I$ with $x_0 < c$ such that $f(x_0) = y_0$. Let $\delta = c - x_0 > 0$. If $c - \delta < x < c$, then $x_0 < x < c$ and therefore $f(x_0) \leq f(x) \leq L$ since f is increasing and L is an upper bound of E. It follows that

$$L - \epsilon < f(x) \leq L \quad \text{if } c - \delta < x < c,$$

which proves that $\lim_{x\to c^-} f(x) = L$.

A similar argument, or the same argument applied to $g(x) = -f(-x)$, shows that

$$\lim_{x\to c^+} f(x) = \inf\{f(x) \in \mathbb{R} : x \in I \text{ and } x > c\}.$$

We leave the details as an exercise. □

Corollary 6.1. *Every discontinuity of a monotonic function $f : I \to \mathbb{R}$ at an interior point of the interval I is a jump discontinuity, meaning:*

$\lim_{x\to c} f(x)$ *doesn't exist, but both the left and right limits*

$$\lim_{x\to c^-} f(x), \quad \lim_{x\to c^+} f(x)$$

exist and are different.

Proof. If c is an interior point of I, then the left and right limits of f at c exist by the previous theorem. Moreover, assuming for definiteness that f is increasing, we have

$$f(x) \leq f(c) \leq f(y) \quad \text{for all } x, y \in I \text{ with } x < c < y,$$

and since limits preserve inequalities

$$\lim_{x\to c^-} f(x) \leq f(c) \leq \lim_{x\to c^+} f(x).$$

□

If the left and right limits are equal, then the limit exists and is equal to the left and right limits, so

$$\lim_{x\to c} f(x) = f(c),$$

meaning that f is continuous at c.

One can show that a monotonic function has, at most, a countable number of discontinuities, and it may have a countably infinite number, but we omit the proof.