Lecture 8

> *This is the lecture note for MATH 104, which should be used together with [Ch8.pdf]*

Ch8, p.1

> *The convergence is uniform on every interval $|x| < \rho$ where $0 \le \rho < R$.*

you see the difference? you need to specify a ρ instead of directly using an R

Ch8, p.2

> *very power series has a radius of convergence.*

The key is that everyone has it!

Check the key results!

1. the series converges absolutely for $|x-c| < R$
2. diverges for $|x-c| > R$
3. for the given $\rho \in [0, R)$
   1. the series converges uniformly on $|x-c| < \rho$
   2. the sum of series is continuous on $|x-c| < \rho$

   Ch8, p.3

   > *lso note that a power series need not converge uniformly on $|x - c| < R$.*

   Keep in mind that 3.1 is different from 1!

Ch8, p.3

> *hus, if the power series converges for some $x0 \in R$, then it converges absolutely for every $x \in R$ with $|x| < |x0|$.*

The key in proving the convergence is to prove that given x_0, all $x \in [0, x_0)$ converges absolutely

Ch8, p.3

> *then it follows from Theorem 9.16 that the sum is continuous on $|x| \le \rho$*

If a sequence of continuous functions (f_n) converges uniformly to a function f on an interval, then the limit function f is also continuous on that interval.

Ch8, p.4

> *Theorem 2.2. Suppose that an̸ = 0 for all sufficiently large n and the limit R = lim n→∞ an an+1 exists or diverges to infinity. Then the power series ∞X n=0 an(x − c)n has a radius of convergence R.*

The key is to implement the ratio test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

Then $\sum_{i=1}^{\infty} a_n$ converges absolutely

Ch8, p.4

> *Theorem 2.3 (Hadamard). The radius of convergence R of the power series ∞X n=0 an(x − c)n is given by R = 1 lim supn→∞ | an | 1/n where R = 0 if the limsup diverges to ∞, and R = ∞ if the limsup is 0.*

This one is crucial!

Ch8, p.5

> *Proposition 3.1.*

These properties hold within the $T = \min\{R, S\}$ but the radius of convergence for the new series may be larger.

Ch8, p.6

> *The reciprocal of a convergent power series that is nonzero at its center also has a power series expansion.*

We now try to solve for the new coefficients

we know that $$
f(x) = \sum*{i=0}^{\infty} a*{i} x^{i}

$$and\$f(0) \neq 0\$, soweknowthereisa\$\$f(x)\cdot\frac{1}{f(x)} = 1$$

then lets think of 1 as a series, then we know $c_0 = 1, c_{i\geq 0} = 0$. We set $$
\frac{1}{f(x)} = \sum*{n=0}^{\infty}b*{n}x^{n}

$$thenweknowfortheproduct,$$

c*{n} = \sum*{k=0}^{n} a*{n-k}b*{k}

$$for\$n = 0\$, wehave\$\$a_0 b_0 = 1, b_0 = \frac{1}{a_0}$$

then for all $n > 0$, we know $$
0 = \sum*{k=0}^{n} a*{n-k}b_{k}

$$assumewehaveknown\$\{b_k\}_{k=0}^{n-1}\$, thenthenew\$b_n\$canbeobtainedsimplyfrom\$\$b_n a_0 + \sum_{k=0}^{n-1} a_{n-k}b_k = 0 \; =$$

> *But still no information about the $\frac{1}{f}$'s radius of convergence!*

> [Ch8, p.6]
>
> > *Theorem 4.1. Suppose that the power series ∞X n=0 an(x − c)n has a radius of convergence R. Then the power series ∞X n=1 nan(x − c)n−1 also has a radius of convergence R.*
>
> i.e. the derivative of a convergent series has the same radius of convergence R

For the proof of it,

> [Ch8, p.6]
>
> > *The ratio test show*
>
> Notice that we're talking about $\sum_{n=0}^{\infty} nr^{n-1}$ instead of $\sum_{n=1}^{\infty} na_n x^{n-1}$

Then the convergence gives boundedness: $\{nr^{n-1}\}_{n=0}^{\infty}$ is bounded by M

- given $\sum_{n=0}^{\infty}|a_n \rho^n|$ converges (by definition)
- then we know $$
  \lvert na*{n}x^{n-1} \rvert \leq \frac{M}{\rho} \lvert a*{n}\rho^{n} \rvert \implies
  \sum*{n=0}^{\infty} n a*{n} x^{n-1} \text{ converges abs.}

  > [!PDF|yellow] [[Ch8.pdf#page=7&selection=144,0,146,29&color=yellow|Ch8, p.7]] > > Theorem 4.2. Su

> [Ch8, p.7]
>
> > *nfinitely differentiable*
>
> with the same R

Ch8, p.7

> *Theorem 4.3. If the power series f (x) = ∞X n=0 an(x − c)n has radius of convergence R > 0, then f is infinitely differentiable in | x − c | < R and an = f (n)(c) n!*

This one is interesting!

The Taylor expansion results come directly from setting $x = 0(c)$

Ch8, p.8

> *Corollary 4.1. If two power series ∞X n=0 an(x − c)n, ∞X n=0 bn(x − c)n have nonzero-radius of convergence and are equal in some neighborhood of 0 , then an = bn for every n = 0, 1, 2, ...*

The corollary shows that if $f = g$ then we must have this one-to-one $$
a*{n} = b*{n} = \frac{f^{(n)}(c)}{n!}

> [!PDF|yellow] [[Ch8.pdf#page=9&selection=73,0,101,1&color=yellow|Ch8, p.9]] > > Proposition 5.1. For ev