Lecture notes for Math 104

4. Limits of Functions

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We will cover the following topics:

- Definition of limits of functions
- Left, right, and infinite limits
- Properties of limits

The notes cover Section 20 in the textbook. We first introduce the accumulation point.

Definition 0.1 (accumulation point). *Given a set A. Then c is called an accumulation point of A if for any $\delta > 0$, there exists $a \neq c$ and $a \in A$ such that*

$$|a - c| < \delta\,.$$

1 Limits

We begin with the $\epsilon - \delta$ definition of the limit of a function.

Definition 1.1. *Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then*

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Remark 1.1. *This definition seems different from the definition in the textbook (Section 20 Definition 20.1). But they are actually equivalent (Theorem 20.6 in the textbook or Theorem 1.1 in this note). In my class, I prefer to use Definition 1.1 since it's easy to understand and check.*

Note that it follows directly from the definition that

$$\lim_{x \to c} f(x) = L \quad \text{if and only if} \quad \lim_{x \to c} |f(x) - L| = 0.$$

In defining a limit as $x \to c$, we do not consider what happens when $x = c$, and a function needn't be defined at c for its limit to exist.

Example 1.1. *Let $A = [0, \infty) \setminus \{9\}$ and define $f : A \to \mathbb{R}$ by*

$$f(x) = \frac{x-9}{\sqrt{x}-3}.$$

We claim that

$$\lim_{x \to 9} f(x) = 6.$$

To prove this, let $\epsilon > 0$ be given. If $x \in A$, then $\sqrt{x} - 3 \neq 0$, and dividing this factor into the numerator we get $f(x) = \sqrt{x} + 3$. It follows that

$$|f(x) - 6| = |\sqrt{x} - 3| = \left| \frac{x-9}{\sqrt{x}+3} \right| \leq \frac{1}{3}|x - 9|.$$

Thus, if $\delta = 3\epsilon$, then $x \in A$ and $|x - 9| < \delta$ implies that $|f(x) - 6| < \epsilon$.

Like the limits of sequences, the limits of functions are unique.

Proposition 1.1. *The limit of a function is unique if it exists.*

Proof. Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$. Assume that

$$\lim_{x \to c} f(x) = L_1, \quad \lim_{x \to c} f(x) = L_2$$

where $L_1, L_2 \in \mathbb{R}$. For every $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} &0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L_1| < \epsilon/2, \\ &0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |f(x) - L_2| < \epsilon/2. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, since c is an accumulation point of A, there exists $x \in A$ such that $0 < |x - c| < \delta$. It follows that

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon.$$

Since this holds for arbitrary $\epsilon > 0$, we must have $L_1 = L_2$.

□

The next theorem gives an equivalent sequential characterization of the limit.

Theorem 1.1. *Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then*

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{n \to \infty} f(x_n) = L.$$

for every sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} x_n = c.$$

Proof. First assume that the limit exists and is equal to L. Suppose that (x_n) is any sequence in A with $x_n \neq c$ that converges to c, and let $\epsilon > 0$ be given. From Definition 1.1, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$, and since $x_n \to c$ there exists $N \in \mathbb{N}$ such that $0 < |x_n - c| < \delta$ for all $n > N$. It follows that $|f(x_n) - L| < \epsilon$ whenever $n > N$, so $f(x_n) \to L$ as $n \to \infty$.

To prove the converse, assume that the limit does not exist or is not equal to L. Then there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a point $x \in A$ with $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon_0$. Therefore, for every $n \in \mathbb{N}$ there is an $x_n \in A$ such that

$$0 < |x_n - c| < \frac{1}{n}, \quad |f(x_n) - L| \geq \epsilon_0.$$

It follows that $x_n \neq c$ and $x_n \to c$, but $f(x_n) \nrightarrow L$, so the sequential condition does not hold. This proves the result.

□

A non-existence proof for a limit directly from Definition 1.1 is often awkward. (One has to show that for every $L \in \mathbb{R}$ there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$.) The previous theorem gives a convenient way to show that a limit of a function does not exist.

Corollary 1.1. *Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x\to c} f(x)$ does not exist if either of the following conditions holds:*

(1) There are sequences $(x_n), (y_n)$ in A with $x_n, y_n \neq c$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c, \text{ but } \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n).$$

(2) There is a sequence (x_n) in A with $x_n \neq c$ such that $\lim_{n\to\infty} x_n = c$ but the sequence $(f(x_n))$ diverges.

Example 1.2. *Define the sign function* $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ *by*

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Then the limit

$$\lim_{x\to 0} \operatorname{sgn} x$$

doesn't exist. To prove this, note that $(1/n)$ is a non-zero sequence such that $1/n \to 0$ and $\operatorname{sgn}(1/n) \to 1$ as $n \to \infty$, while $(-1/n)$ is a non-zero sequence such that $-1/n \to 0$ and $\operatorname{sgn}(-1/n) \to -1$ as $n \to \infty$. Since the sequences of sgn-values have different limits, Corollary 1.1 implies that the limit does not exist.

Example 1.3. *The limit*

$$\lim_{x \to 0} \frac{1}{x},$$

corresponding to the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *given by* $f(x) = 1/x$, *doesn't exist. For example, if* (x_n) *is the non-zero sequence given by* $x_n = 1/n$, *then* $1/n \to 0$ *but the sequence of values* (n) *diverges to* ∞.

Example 1.4. *The limit*

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right),$$

corresponding to the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *given by* $f(x) = \sin(1/x)$, *doesn't exist. (See Figure 1) For example, the non-zero sequences* $(x_n), (y_n)$ *defined by*

$$x_n = \frac{1}{2\pi n}, \quad y_n = \frac{1}{2\pi n + \pi/2}$$

both converge to zero as $n \to \infty$, *but the limits*

$$\lim_{n \to \infty} f(x_n) = 0, \quad \lim_{n \to \infty} f(y_n) = 1$$

are different.

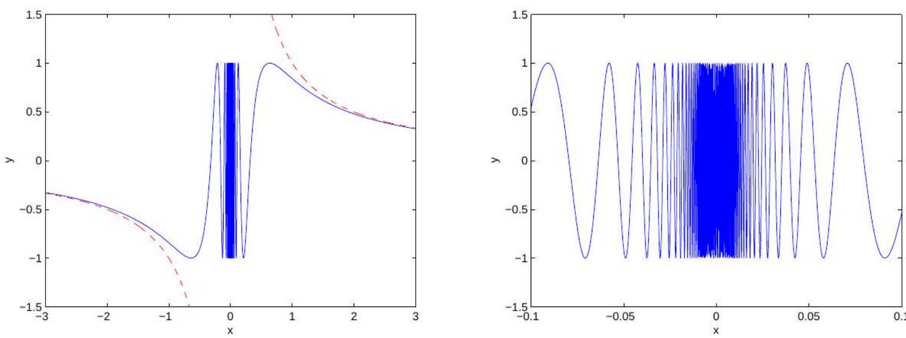

Figure 1. A plot of the function $y = \sin(1/x)$, with the hyperbola $y = 1/x$ shown in red, and detail near the origin.

2 Left, right, and infinite limits

We can define other kinds of limits in an obvious way. We list some of them here and give examples, whose proofs are left as an exercise. All these definitions can be combined in various ways and have obvious equivalent sequential characterizations.

Definition 2.1. *(Right and left limits). Let* $f : A \to \mathbb{R}$, *where* $A \subset \mathbb{R}$. *If* $c \in \mathbb{R}$ *is an accumulation point of* $\{x \in A : x > c\}$, *then* f *has the right limit*

$$\lim_{x \to c^+} f(x) = L,$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$c < x < c + \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

If $c \in \mathbb{R}$ *is an accumulation point of* $\{x \in A : x < c\}$, *then* f *has the left limit*

$$\lim_{x \to c^-} f(x) = L,$$

if for every $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$c - \delta < x < c \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

The existence and equality of the left and right limits imply the existence of the limit.

Proposition 2.1. *Suppose that* $f : A \to \mathbb{R}$, *where* $A \subset \mathbb{R}$, *and* $c \in \mathbb{R}$ *is an accumulation point of both* $\{x \in A : x > c\}$ *and* $\{x \in A : x < c\}$. *Then*

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L.$$

Proof. It follows immediately from the definitions that the existence of the limit implies the existence of the left and right limits with the same value. Conversely, if both left and right limits exists and are equal to L, then given $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$c - \delta_1 < x < c \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon,$$
$$c < x < c + \delta_2 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Choosing $\delta = \min(\delta_1, \delta_2) > 0$, we get that

$$|x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon,$$

which show that the limit exists.

□

Next, we introduce some convenient definitions for various kinds of limits involving infinity. We emphasize that ∞ and $-\infty$ are not real numbers (what is $\sin \infty$, for example?) and all these definitions have precise translations into statements that involve only real numbers.

Definition 2.2 (Limits as $x \to \pm\infty$)**.** *Let* $f : A \to \mathbb{R}$, *where* $A \subset \mathbb{R}$. *If* A *is not bounded from above, then*

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$ *there exists an* $M \in \mathbb{R}$ *such that*

$$x > M \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

If A is not bounded from below, then

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists an $m \in \mathbb{R}$ such that

$$x < m \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Exercise 1. *Given $f : \mathbb{R} \to \mathbb{R}$, prove*

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f\left(\frac{1}{t}\right), \quad \lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f\left(\frac{1}{t}\right).$$

(assume all the limits exist)

Example 2.1. *We have*

$$\lim_{x \to \infty} \frac{x}{\sqrt{1 + x^2}} = 1, \quad \lim_{x \to -\infty} \frac{x}{\sqrt{1 + x^2}} = -1.$$

Definition 2.3. *(Divergence to $\pm\infty$). Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then*

$$\lim_{x \to c} f(x) = \infty$$

if for every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } f(x) > M,$$

and

$$\lim_{x \to c} f(x) = -\infty$$

if for every $m \in \mathbb{R}$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } f(x) < m.$$

The notation $\lim_{x \to c} f(x) = \pm\infty$ is simply shorthand for the property stated in this definition; it does not mean that the limit exists, and we say that f diverges to $\pm\infty$.

Example 2.2. *We have*

$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \quad \lim_{x \to \infty} \frac{1}{x^2} = 0.$$

Example 2.3. *We have*

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

How would you define these statements precisely? Note that

$$\lim_{x \to 0} \frac{1}{x} \neq \pm\infty,$$

since $1/x$ takes arbitrarily large positive (if $x > 0$) and negative (if $x < 0$) values in every two-sided neighborhood of 0 .

Example 2.4. *None of the limits*

$$\lim_{x \to 0^+} \frac{1}{x} \sin\left(\frac{1}{x}\right), \quad \lim_{x \to 0^-} \frac{1}{x} \sin\left(\frac{1}{x}\right), \quad \lim_{x \to 0} \frac{1}{x} \sin\left(\frac{1}{x}\right)$$

is ∞ or $-\infty$, since $(1/x)\sin(1/x)$ oscillates between arbitrarily large positive and negative values in every one-sided or two-sided neighborhood of 0.

Example 2.5. *We have*

$$\lim_{x \to \infty} \left(\frac{1}{x} - x^3\right) = -\infty, \quad \lim_{x \to -\infty} \left(\frac{1}{x} - x^3\right) = \infty.$$

How would you define these statements precisely and prove them?

3 Properties of limits

3.1 Order properties

As for limits of sequences, limits of functions preserve (non-strict) inequalities.

Theorem 3.1. *Suppose that $f, g : A \to \mathbb{R}$ and c is an accumulation point of A. If*

$$f(x) \le g(x) \quad \textit{for all } x \in A,$$

and $\lim_{x \to c} f(x), \lim_{x \to c} g(x)$ exist, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Proof. Let

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M.$$

Suppose for contradiction that $L > M$, and let

$$\epsilon = \frac{1}{2}(L - M) > 0.$$

From the definition of the limit, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |f(x) - L| < \epsilon \quad & \text{if } x \in A \text{ and } 0 < |x - c| < \delta_1, \\ |g(x) - M| < \epsilon \quad & \text{if } x \in A \text{ and } 0 < |x - c| < \delta_2. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$. Since c is an accumulation point of A, there exists $x \in A$ such that $0 < |x - a| < \delta$, and it follows that

$$\begin{aligned} f(x) - g(x) &= [f(x) - L] + L - M + [M - g(x)] \\ &> L - M - 2\epsilon \\ &> 0, \end{aligned}$$

which contradicts the assumption that $f(x) \le g(x)$.

□

Next, we state a useful "sandwich" or "squeeze" criterion for the existence of a limit.

Theorem 3.2. *Suppose that $f, g, h : A \to \mathbb{R}$ and c is an accumulation point of A. If*

$$f(x) \le g(x) \le h(x) \quad \text{for all } x \in A$$

and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then the limit of $g(x)$ as $x \to c$ exists and

$$\lim_{x\to c} g(x) = L.$$

Exercise 2. *Prove Theroem 3.2.*

Example 3.1. *We have*

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \quad \text{for all } x \ne 0$$

and

$$\lim_{x\to 0}(-1) = -1, \quad \lim_{x\to 0} 1 = 1,$$

but

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist.}$$

(where is the problem?)

3.2 Algebraic properties.

Limits of functions respect algebraic operations.

Theorem 3.3. *Suppose that $f, g : A \to \mathbb{R}$, c is an accumulation point of A, and the limits*

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M$$

exist. Then

$$\begin{array}{ll} \lim_{x\to c} kf(x) = kL & \text{for every } k \in \mathbb{R}, \\ \lim_{x\to c}[f(x) + g(x)] = L + M, & \\ \lim_{x\to c}[f(x)g(x)] = LM, & \\ \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M} & \text{if } M \ne 0. \end{array}$$

Proof. We prove the results for sums and products from the definition of the limit, and leave the remaining proofs as an exercise. All of the results also follow from the corresponding results for sequences.

First, we consider the limit of $f + g$. Given $\epsilon > 0$, choose δ_1, δ_2 such that

$$
\begin{aligned}
&0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon/2, \\
&0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \epsilon/2,
\end{aligned}
$$

and let $\delta = \min(\delta_1, \delta_2) > 0$. Then $0 < |x - c| < \delta$ implies that

$$
|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M| < \epsilon,
$$

which proves that $\lim(f + g) = \lim f + \lim g$.

To prove the result for the limit of the product, first note that from the local boundedness of functions with a limit (Proposition 6.18 there exists $\delta_0 > 0$ and $K > 0$ such that $|g(x)| \le K$ for all $x \in A$ with $0 < |x - c| < \delta_0$. Choose $\delta_1, \delta_2 > 0$ such that

$$
\begin{aligned}
&0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon/(2K), \\
&0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \epsilon/(2|L| + 1).
\end{aligned}
$$

Let $\delta = \min(\delta_0, \delta_1, \delta_2) > 0$. Then for $0 < |x - c| < \delta$ and $x \in A$,

$$
\begin{aligned}
|f(x)g(x) - LM| &= |(f(x) - L)g(x) + L(g(x) - M)| \\
&\le |f(x) - L||g(x)| + |L||g(x) - M| \\
&< \frac{\epsilon}{2K} \cdot K + |L| \cdot \frac{\epsilon}{2|L| + 1} \\
&< \epsilon,
\end{aligned}
$$

which proves that $\lim(fg) = \lim f \lim g$. $\square$