Lecture notes for Math 104

6. sequences and series of functions

Instructor: Mengxuan Yang

In this chapter, we define and study the convergence of sequences and series of functions. We will cover the following topics:

- Pointwise convergence
- Uniform convergence
- Properties of uniform convergence

We omit the integrability of convergence sequences in the textbook (Section 25) and leave it to the future.

1 Pointwise convergence (Section 24)

Pointwise convergence defines the convergence of functions in terms of the convergence of their values at each point of their domain.

Definition 1.1. *Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ pointwise on A if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in A$.*

We say that the sequence (f_n) converges pointwise if it converges pointwise to some function f, in which case

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Pointwise convergence is, perhaps, the most obvious way to define the convergence of functions, and it is one of the most important. Nevertheless, as the following examples illustrate, it is not as well-behaved as one might initially expect.

Example 1.1. *Suppose that $f_n : (0, 1) \to \mathbb{R}$ is defined by*

$$f_n(x) = \frac{n}{nx + 1}.$$

Then, since $x \neq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x + 1/n} = \frac{1}{x},$$

so $f_n \to f$ *pointwise where* $f : (0,1) \to \mathbb{R}$ *is given by*

$$f(x) = \frac{1}{x}.$$

We have $|f_n(x)| < n$ *for all* $x \in (0,1)$*, so each* f_n *is bounded on* $(0,1)$*, but the pointwise limit* f *is not. Thus, pointwise convergence does not, in general, preserve boundedness.*

Example 1.2. *Suppose that* $f_n : [0,1] \to \mathbb{R}$ *is defined by* $f_n(x) = x^n$*. If* $0 \le x < 1$*, then* $x^n \to 0$ *as* $n \to \infty$*, while if* $x = 1$*, then* $x^n \to 1$ *as* $n \to \infty$*. So* $f_n \to f$ *pointwise where*

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Although each f_n *is continuous on* $[0,1]$*, the pointwise limit* f *is not (it is discontinuous at 1). Thus, pointwise convergence does not, in general, preserve continuity.*

Example 1.3. *Define* $f_n : \mathbb{R} \to \mathbb{R}$ *by*

$$f_n(x) = \frac{\sin nx}{n}.$$

Then $f_n \to 0$ *pointwise on* $\mathbb{R}$*. The sequence* (f_n') *of derivatives* $f_n'(x) = \cos nx$ *does not converge pointwise on* $\mathbb{R}$*; for example,*

$$f_n'(\pi) = (-1)^n$$

does not converge as $n \to \infty$*. Thus, in general, one cannot differentiate a pointwise convergent sequence. This behavior isn't limited to pointwise convergent sequences and happens because the derivative of a small, rapidly oscillating function can be large.*

2 Uniform convergence (Section 24)

In this section, we introduce a stronger notion of convergence of functions than pointwise convergence, called uniform convergence.

Definition 2.1. *Suppose that* (f_n) *is a sequence of functions* $f_n : A \to \mathbb{R}$ *and* $f : A \to \mathbb{R}$*. Then* $f_n \to f$ *uniformly on* A *if, for every* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that*

$$n > N \text{ implies that } |f_n(x) - f(x)| < \epsilon \text{ for all } x \in A.$$

Feel the difference between the definitions of pointwise convergence and uniform convergence.

When the domain A of the functions is understood, we will often say $f_n \to f$ uniformly instead of uniformly on A.

The crucial point in this definition is that N depends only on ϵ and not on $x \in A$, whereas for a pointwise convergent sequence N may depend on both ϵ and x.

A uniformly convergent sequence is always pointwise convergent (to the same limit), but the converse is not true. If a sequence converges pointwise, it may happen that for some $\epsilon > 0$ one needs to choose arbitrarily large N 's for different points $x \in A$, meaning that the sequences of values converge arbitrarily slowly on A. In that case a pointwise convergent sequence of functions is not uniformly convergent.

Example 2.1. *The sequence $f_n(x) = x^n$ in Example 1.2 converges pointwise on $[0,1]$ but not uniformly on $[0,1]$. For $0 \le x < 1$, we have*

$$|f_n(x) - f(x)| = x^n.$$

If $0 < \epsilon < 1$, we cannot make $x^n < \epsilon$ for all $0 \le x < 1$ however large we choose n. The problem is that x^n converges to 0 at an arbitrarily slow rate for x sufficiently close to 1 . There is no difficulty in the rate of convergence at 1 itself, since $f_n(1) = 1$ for every $n \in \mathbb{N}$. As we will show, the uniform limit of continuous functions is continuous, so since the pointwise limit of the continuous functions f_n is discontinuous, the sequence cannot converge uniformly on $[0,1]$. The sequence does, however, converge uniformly to 0 on $[0,b]$ for every $0 \le b < 1$; given $\epsilon > 0$, we take N large enough that $b^N < \epsilon$.

Example 2.2. *The functions in Example 1.3 converge uniformly to 0 on $\mathbb{R}$, since*

$$|f_n(x)| = \frac{|\sin nx|}{n} \le \frac{1}{n},$$

so $|f_n(x) - 0| < \epsilon$ for all $x \in \mathbb{R}$ if $n > 1/\epsilon$.

3 Cauchy condition for uniform convergence (Section 25)

The Cauchy condition provides a necessary and sufficient condition for a sequence of real numbers to converge. There is an analogous uniform Cauchy condition that provides a necessary and sufficient condition for a sequence of functions to converge uniformly.

Definition 3.1. *A sequence (f_n) of functions $f_n : A \to \mathbb{R}$ is uniformly Cauchy on A if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$m, n > N \text{ implies that } |f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in A.$$

Theorem 3.1. *A sequence (f_n) of functions $f_n : A \to \mathbb{R}$ converges uniformly on A if and only if it is uniformly Cauchy on A.*

Proof. Suppose that (f_n) converges uniformly to f on A. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in A \text{ if } n > N.$$

It follows that if $m, n > N$ then

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \epsilon \quad \text{for all } x \in A,$$

which shows that (f_n) is uniformly Cauchy.

Conversely, suppose that (f_n) is uniformly Cauchy. Then for each $x \in A$, the real sequence $(f_n(x))$ is Cauchy, so it converges by the completeness of $\mathbb{R}$. We define $f : A \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x),$$

and then $f_n \to f$ pointwise.

To prove that $f_n \to f$ uniformly, let $\epsilon > 0$. Since (f_n) is uniformly Cauchy, we can choose $N \in \mathbb{N}$ (depending only on ϵ) such that

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in A \text{ if } m, n > N.$$

Let $n > N$ and $x \in A$. Then for every $m > N$ we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + |f_m(x) - f(x)|.$$

Since $f_m(x) \to f(x)$ as $m \to \infty$, we can choose $m > N$ (depending on x, but it doesn't matter since m doesn't appear in the final result) such that

$$|f_m(x) - f(x)| < \frac{\epsilon}{2}.$$

It follows that if $n > N$, then

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in A,$$

which proves that $f_n \to f$ uniformly.

Alternatively, we can take the limit as $m \to \infty$ in the uniform Cauchy condition to get for all $x \in A$ and $n > N$ that

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

□

4 Properties of uniform convergence (Section 24, 25)

In this section we prove that, unlike pointwise convergence, uniform convergence preserves boundedness and continuity. Uniform convergence does not preserve differentiability any better than pointwise convergence. Nevertheless, we give a result that allows us to differentiate a convergent sequence; the key assumption is that the derivatives converge uniformly.

4.1 Boundedness

First, we consider the uniform convergence of bounded functions.

Theorem 4.1. *Suppose that $f_n : A \to \mathbb{R}$ is bounded on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Then $f : A \to \mathbb{R}$ is bounded on A.*

Proof. Taking $\epsilon = 1$ in the definition of the uniform convergence, we find that there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1 \quad \text{for all } x \in A \text{ if } n > N.$$

Choose some $n > N$. Then, since f_n is bounded, there is a constant $M \geq 0$ such that

$$|f_n(x)| \leq M \quad \text{for all } x \in A.$$

It follows that

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 1 + M \quad \text{for all } x \in A,$$

meaning that f is bounded on A. □

4.2 Continuity

One of the most important properties of uniform convergence is that it preserves continuity.

Theorem 4.2. *If a sequence (f_n) of continuous functions $f_n : A \to \mathbb{R}$ converges uniformly on $A \subset \mathbb{R}$ to $f : A \to \mathbb{R}$, then f is continuous on A.*

Proof. Suppose that $c \in A$ and let $\epsilon > 0$. Then, for every $n \in \mathbb{N}$,

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|.$$

By the uniform convergence of (f_n), we can choose $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in A,$$

and for such an n it follows that

$$|f(x) - f(c)| < |f_n(x) - f_n(c)| + \frac{2\epsilon}{3}.$$

(Here, we use the fact that f_n is close to f at both x and c, where x is an an arbitrary point in a neighborhood of c; this is where we use the uniform convergence in a crucial way.)

Since f_n is continuous on A, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(c)| < \frac{\epsilon}{3} \quad \text{if } |x - c| < \delta \text{ and } x \in A,$$

which implies that

$$|f(x) - f(c)| < \epsilon \quad \text{if } |x - c| < \delta \text{ and } x \in A.$$

This proves that f is continuous.

□

This result can be interpreted as justifying an "exchange in the order of limits"

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x).$$

Exercise 1. *Prove*

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x).$$

if f_n uniformly converges to f.

4.3 Differentiability

The uniform convergence of differentiable functions does not, in general, imply anything about the convergence of their derivatives or the differentiability of their limit (Example 2.2 for example).

However, we do get a useful result if we strengthen the assumptions and require that the derivatives converge uniformly, not just pointwise.

Theorem 4.3. *Suppose that (f_n) is a sequence of differentiable functions $f_n : (a,b) \to \mathbb{R}$ such that $f_n \to f$ pointwise and $f_n' \to g$ uniformly for some $f, g : (a,b) \to \mathbb{R}$. Then f is differentiable on (a,b) and $f' = g$.*

Proof. Let $c \in (a,b)$, and let $\epsilon > 0$ be given. To prove that $f'(c) = g(c)$, we estimate the difference quotient of f in terms of the difference quotients of the f_n :

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} - g(c) \right| &\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \\ &+ \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + |f_n'(c) - g(c)| \end{aligned}$$

where $x \in (a,b)$ and $x \neq c$. We want to make each of the terms on the right-hand side of the inequality less than $\epsilon/3$. This is straightforward for the second term (since f_n is differentiable) and the third term (since $f_n' \to g$). To estimate the first term, we approximate f by f_m, use the mean value theorem, and let $m \to \infty$.

Since $f_m - f_n$ is differentiable, the mean value theorem implies that there exists ξ between c and x such that

$$\begin{aligned} \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} &= \frac{(f_m - f_n)(x) - (f_m - f_n)(c)}{x - c} \\ &= f_m'(\xi) - f_n'(\xi). \end{aligned}$$

Since (f_n') converges uniformly, it is uniformly Cauchy by Theorem 9.13 Therefore there exists $N_1 \in \mathbb{N}$ such that

$$|f_m'(\xi) - f_n'(\xi)| < \frac{\epsilon}{3} \quad \text{for all } \xi \in (a,b) \text{ if } m, n > N_1,$$

which implies that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| < \frac{\epsilon}{3}.$$

Taking the limit of this equation as $m \to \infty$, and using the pointwise convergence of (f_m) to f, we get that

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \le \frac{\epsilon}{3} \quad \text{for } n > N_1.$$

Next, since (f_n') converges to g, there exists $N_2 \in \mathbb{N}$ such that

$$|f_n'(c) - g(c)| < \frac{\epsilon}{3} \quad \text{for all } n > N_2.$$

Choose some $n > \max(N_1, N_2)$. Then the differentiability of f_n implies that there exists $\delta > 0$ such that

$$\left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| < \frac{\epsilon}{3} \quad \text{if } 0 < |x - c| < \delta.$$

Putting these inequalities together, we get that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon \quad \text{if } 0 < |x - c| < \delta,$$

which proves that f is differentiable at c with $f'(c) = g(c)$.

□

Like Theorem 9.16, Theorem 9.18 can be interpreted as giving sufficient conditions for an exchange in the order of limits:

$$\lim_{n\to\infty} \lim_{x\to c} \left[\frac{f_n(x) - f_n(c)}{x - c} \right] = \lim_{x\to c} \lim_{n\to\infty} \left[\frac{f_n(x) - f_n(c)}{x - c} \right].$$

Exercise 2. *Prove the above claim.*

5 Series (Section 25)

The convergence of a series is defined in terms of the convergence of its sequence of partial sums, and any result about sequences is easily translated into a corresponding result about series.

Definition 5.1. *Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$. Let (S_n) be the sequence of partial sums $S_n : A \to \mathbb{R}$, defined by*

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then the series

$$S(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges pointwise to $S : A \to \mathbb{R}$ *on* A *if* $S_n \to S$ *as* $n \to \infty$ *pointwise on* A*, and uniformly to* S *on* A *if* $S_n \to S$ *uniformly on* A*.*

We illustrate the definition with a series whose partial sums we can compute explicitly.

Example 5.1. *The geometric series*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

has partial sums

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Thus, $S_n(x) \to 1/(1-x)$ *as* $n \to \infty$ *if* $|x| < 1$ *and diverges if* $|x| \geq 1$*, meaning that*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{pointwise on } (-1, 1).$$

Since $1/(1-x)$ *is unbounded on* $(-1,1)$*, Theorem 4.1 implies that the convergence cannot be uniform.*

The series does, however, converge uniformly on $[-\rho, \rho]$ *for every* $0 \leq \rho < 1$*. To prove this, we estimate for* $|x| \leq \rho$ *that*

$$\left| S_n(x) - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{1-x} \leq \frac{\rho^{n+1}}{1-\rho}.$$

Since $\rho^{n+1}/(1-\rho) \to 0$ *as* $n \to \infty$*, given any* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$*, depending only on* ϵ *and* ρ*, such that*

$$0 \leq \frac{\rho^{n+1}}{1-\rho} < \epsilon \qquad \text{for all } n > N$$

It follows that

$$\left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right| < \epsilon \qquad \text{for all } x \in [-\rho, \rho] \text{ and all } n > N,$$

which proves that the series converges uniformly on $[-\rho, \rho]$*.*

The Cauchy condition for the uniform convergence of sequences immediately gives a corresponding Cauchy condition for the uniform convergence of series.

Theorem 5.1. *Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$. The series*

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on A if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \epsilon \quad \text{for all } x \in A \text{ and all } n > m > N.$$

Proof. Let

$$S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + f_2(x) + \cdots + f_n(x).$$

From Theorem 9.13 the sequence (S_n), and therefore the series $\sum f_n$, converges uniformly if and only if for every $\epsilon > 0$ there exists N such that

$$|S_n(x) - S_m(x)| < \epsilon \quad \text{for all } x \in A \text{ and all } n, m > N.$$

Assuming $n > m$ without loss of generality, we have

$$S_n(x) - S_m(x) = f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x) = \sum_{k=m+1}^{n} f_k(x),$$

so the result follows. □

The following simple criterion for the uniform convergence of a series is very useful. The name comes from the letter traditionally used to denote the constants, or "majorants," that bound the functions in the series.

Theorem 5.2 (Weierstrass M-test)**.** *Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$, and suppose that for every $n \in \mathbb{N}$ there exists a constant $M_n \geq 0$ such that*

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \quad \sum_{n=1}^{\infty} M_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} f_n(x).$$

converges uniformly on A.

Proof. The result follows immediately from the observation that $\sum f_n$ is uniformly Cauchy if $\sum M_n$ is Cauchy.

In detail, let $\epsilon > 0$ be given. The Cauchy condition for the convergence of a real series implies that there exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} M_k < \epsilon \quad \text{for all } n > m > N.$$

Then for all $x \in A$ and all $n > m > N$, we have

$$\begin{aligned} \left| \sum_{k=m+1}^{n} f_k(x) \right| &\le \sum_{k=m+1}^{n} |f_k(x)| \\ &\le \sum_{k=m+1}^{n} M_k \\ &< \epsilon \end{aligned}$$

Thus, $\sum f_n$ satisfies the uniform Cauchy condition in Theorem 9.21, so it converges uniformly.

$\square$

Example 5.2. *Returning to Example* 5.1*, we consider the geometric series*

$$\sum_{n=0}^{\infty} x^n.$$

If $|x| \le \rho$ *where* $0 \le \rho < 1$*, then*

$$|x^n| \le \rho^n, \quad \sum_{n=0}^{\infty} \rho^n < 1.$$

The M-test, with $M_n = \rho^n$*, implies that the series converges uniformly on* $[-\rho, \rho]$.