Introduction to Analysis

Chen Gao

Spring 2025

Course:	MATH 104A
Instructor:	Professor [who]
Institution:	University of California, Berkeley
Semester:	Spring 2025

Contents

1 Course Overview

Note

These notes compile key concepts, theorems, and insights from the Mathematical Analysis course. They serve as a comprehensive reference for understanding advanced mathematical principles.

2 Preliminaries on Sets, Mappings, and Relations

Note

In these preliminaries, we describe some notions regarding sets, mappings, and relations that will be used throughout the book. Our purpose is descriptive and the arguments given are directed toward plausibility and understanding rather than rigorous proof based on an axiomatic basis for set theory.

2.1 Unions and Intersections of Sets

Definition 2.1: Set Membership

For a set A, the membership of an element x in A is denoted by $x \in A$ and the nonmembership of x in A is denoted by $x \notin A$.
Sets are often denoted by $\{x \mid \text{statement about } x\}$, representing all elements for which the statement is true. Two sets are the same provided they have the same members.

Definition 2.2: Subset

Let A and B be sets. A is a subset of B, denoted by $A \subseteq B$, provided each member of A is a member of B.
A subset A of B is called a proper subset of B provided $A \neq B$.

Definition 2.3: Set Operations

For sets A and B:

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (in the nonexclusive sense)
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- Complement: $B \setminus A = \{x \mid x \in B, x \notin A\}$

If all sets are subsets of a reference set X, $X \setminus A$ is simply called the complement of A.

Definition 2.4: Special Sets

- Empty set: A set with no members, denoted by ①
- Non-empty set: A set that is not equal to the empty set
- Singleton set: A set with a single member

- Power set: For a set X, $\mathcal{P}(X)$ or 2^X is the set of all subsets of X

Definition 2.5: Generalized Set Operations

Let $\mathcal{F}$ be a collection of sets:

- Union: $\bigcup_{F\in\mathcal{F}} F$ is the set of points that belong to at least one set in $\mathcal{F}$
- Intersection: $\bigcap_{F\in\mathcal{F}} F$ is the set of points that belong to every set in $\mathcal{F}$

A collection of sets $\mathcal{F}$ is disjoint if the intersection of any two distinct sets in $\mathcal{F}$ is empty.

Theorem 2.1: De Morgan's Identities

For a family $\mathcal{F}$ of sets and a reference set X:

$$X \setminus \left[\bigcup_{F\in\mathcal{F}} F\right] = \bigcap_{F\in\mathcal{F}} [X \setminus F]$$

$$X \setminus \left[\bigcap_{F\in\mathcal{F}} F\right] = \bigcup_{F\in\mathcal{F}} [X \setminus F]$$

2.2 Mappings between Sets

Definition 2.6: Mapping/Function

Given sets A and B, a mapping (or function) $f : A \longrightarrow B$ is a correspondence that assigns to each member of A a member of B.
For each $x \in A$, $f(x)$ denotes the member of B to which x is assigned.

Definition 2.7: Image and Domain

For a mapping $f : A \longrightarrow B$ and a subset $A' \subseteq A$:

- Domain: The set A
- Image of A': $f(A') = \{b \mid b = f(a) \text{ for some } a \in A'\}$
- Range/Image of f: $f(A)$

Definition 2.8: Function Types

- Onto (Surjective): $f(A) = B$
- One-to-One (Injective): For each $b \in f(A)$, there is exactly one $a \in A$ such that $b = f(a)$
- Invertible: Both one-to-one and onto, establishing a one-to-one correspondence

Definition 2.9: Inverse Mapping

For an invertible mapping $f : A \longrightarrow B$:

- $f^{-1}(b)$ is the unique $a \in A$ such that $f(a) = b$
- $f^{-1} : B \longrightarrow A$ is the inverse mapping

Two sets A and B are equipotent if there exists an invertible mapping from A to B.

Definition 2.10: Function Composition

For mappings $f : A \longrightarrow B$ and $g : C \longrightarrow D$ where $f(A) \subseteq C$:

- Composition $g \circ f : A \longrightarrow D$ defined by $[g \circ f](x) = g(f(x))$
- The composition of invertible mappings is invertible

Definition 2.11: Inverse Image

For a mapping $f : A \longrightarrow B$ and a set $E \subseteq B$:

- $f^{-1}(E) = \{a \in A \mid f(a) \in E\}$
- Useful properties:

$$f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2)$$
$$f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$$
$$f^{-1}(E_1 \setminus E_2) = f^{-1}(E_1) \setminus f^{-1}(E_2)$$

Definition 2.12: Function Restriction

For a mapping $f : A \longrightarrow B$ and $A' \subseteq A$, the restriction of f to A', denoted $f\big|_{A'}$, is the mapping from A' to B which assigns $f(x)$ to each $x \in A'$.

2.3 Equivalence Relations, Axiom of Choice, and Zorn's Lemma

Definition 2.13: Cartesian Product

For non-empty sets A and B, the Cartesian product $A \times B$ is the collection of all ordered pairs (a, b) where $a \in A$ and $b \in B$, with $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Definition 2.14: Relation

For a non-empty set X, a relation R on X is a subset of $X \times X$.

- Reflexive: xRx for all $x \in X$
- Symmetric: $xRy \implies yRx$
- Transitive: xRy and $yRz \implies xRz$

Definition 2.15: Equivalence Relation

An equivalence relation R on a set X is a relation that is reflexive, symmetric, and transitive. For $x \in X$, the equivalence class $R_x = \{x' \in X \mid xRx'\}$. The collection of equivalence classes is denoted X/R.

Definition 2.16: Choice Function

Let $\mathcal{F}$ be a non-empty family of non-empty sets. A choice function f on $\mathcal{F}$ is a function $f : \mathcal{F} \to \bigcup_{F \in \mathcal{F}} F$ such that $f(F) \in F$ for each $F \in \mathcal{F}$.

Theorem 2.2: Axiom of Choice

For any non-empty collection of non-empty sets, there exists a choice function.

Definition 2.17: Partial Ordering

A relation R on a non-empty set X is a partial ordering if it is:

- Reflexive: xRx
- Antisymmetric: If xRy and yRx, then $x = y$
- Transitive: If xRy and yRz, then xRz

Theorem 2.3: Zorn's Lemma

Let X be a partially ordered set where every totally ordered subset has an upper bound. Then X has a maximal member.

Note

Zorn's Lemma is equivalent to the Axiom of Choice and will be crucial in proving important theorems such as the Hahn-Banach Theorem, the Tychonoff Product Theorem, and the Krein-Milman Theorem.

Definition 2.18: Generalized Cartesian Product

For a collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ parametrized by Λ, the Cartesian product $\prod_{\lambda \in \Lambda} E_\lambda$ is the set of functions $f : \Lambda \to \bigcup_{\lambda \in \Lambda} E_\lambda$ such that $f(\lambda) \in E_\lambda$ for each $\lambda \in \Lambda$.

Note

The Axiom of Choice is equivalent to the assertion that the Cartesian product of a non-empty family of non-empty sets is non-empty.

3 The Real Numbers: Sets, Sequences, and Functions

Note

We establish the fundamental properties of real numbers through axioms and derive key results about sets, sequences, and functions.

3.1 The Field, Positivity, and Completeness Axioms

Theorem 3.1: Field Axioms

The real numbers $\mathbb{R}$ form a field under addition and multiplication, satisfying:

(i) Commutativity: $a + b = b + a$, $ab = ba$

(ii) Associativity: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$

(iii) Identity: $a + 0 = a$, $a \cdot 1 = a$

(iv) Inverses: For each a, there exists $-a$ with $a + (-a) = 0$

(v) For each $a \neq 0$, there exists a^{-1} with $aa^{-1} = 1$

(vi) Distributivity: $a(b + c) = ab + ac$

(vii) $1 \neq 0$

Theorem 3.2: Positivity Axioms

There exists a set $\mathcal{P} \subset \mathbb{R}$ of positive numbers such that:

(i) If $a, b \in \mathcal{P}$, then $ab \in \mathcal{P}$ and $a + b \in \mathcal{P}$

(ii) For each $a \in \mathbb{R}$, exactly one of $a \in \mathcal{P}$, $-a \in \mathcal{P}$, or $a = 0$ holds

Definition 3.1: Order Relations

For real numbers a and b, we define:

$$a > b \iff a - b \in \mathcal{P}, \quad a \geq b \iff a > b \text{ or } a = b$$

Theorem 3.3: Properties of Order

For real numbers a, b, and c:

(i) If $a > b$ and $b > c$, then $a > c$ (transitivity)

(ii) If $a > b$, then $a + c > b + c$ (translation invariance)

(iii) If $a > b$ and $c > 0$, then $ac > bc$ (positive scaling)

Definition 3.2: Absolute Value

For a real number x, we define:

$$|x| = \max\{x, -x\}$$

Theorem 3.4: Triangle Inequality

For all real numbers a and b:

$$|a + b| \leq |a| + |b|$$

Definition 3.3: Bounded Sets

A set $E \subset \mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in E$. Similarly, E is bounded below if there exists $m \in \mathbb{R}$ such that $m \leq x$ for all $x \in E$. E is bounded if it is both bounded above and below.

Theorem 3.5: Completeness Axiom

Every non-empty set of real numbers that is bounded above has a least upper bound.

3.2 Natural and Rational Numbers

Definition 3.4: Inductive Set

A set $E \subset \mathbb{R}$ is inductive if $1 \in E$ and $x + 1 \in E$ whenever $x \in E$.

Definition 3.5: Natural Numbers

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive subsets of $\mathbb{R}$.

Theorem 3.6: Mathematical Induction

Let $S(n)$ be a statement for each $n \in \mathbb{N}$. If $S(1)$ is true and $S(k) \implies S(k+1)$ for all $k \in \mathbb{N}$, then $S(n)$ is true for all $n \in \mathbb{N}$.

Theorem 3.7: Theorem 1

Every non-empty set of natural numbers has a smallest member.

Definition 3.6: Integers

The set $\mathbb{Z}$ of integers consists of 0, the natural numbers, and their negatives.

Definition 3.7: Rational Numbers

The set $\mathbb{Q}$ of rational numbers consists of quotients $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$.

Theorem 3.8: Archimedean Property

For any positive real numbers a and b, there exists $n \in \mathbb{N}$ such that $na > b$.

Theorem 3.9: Theorem 2

The rational numbers are dense in $\mathbb{R}$. That is, between any two distinct real numbers lies a rational number.

Theorem 3.10: Irrationality of $\sqrt{2}$

The number $\sqrt{2}$ is irrational.

3.3 Countable and Uncountable Sets

Definition 3.8: Equipotence

Sets A and B are equipotent if there exists a bijection $f : A \to B$.

Definition 3.9: Countability

A set is countable if it is either finite or equipotent to $\mathbb{N}$. A set that is not countable is called uncountable.

Theorem 3.11: Theorem 3

A subset of a countable set is countable.

Theorem 3.12: Corollary 4

The following sets are countably infinite:

(i) For each natural number n, the Cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (n times)

(ii) The set of rational numbers $\mathbb{Q}$

Theorem 3.13: Theorem 5

A non-empty set is countable if and only if it is the image of a function whose domain is a non-empty countable set.

Theorem 3.14: Corollary 6

The union of a countable collection of countable sets is countable.

Theorem 3.15: Theorem 7

A non-degenerate interval of real numbers is uncountable.

3.4 Open Sets, Closed Sets, and Borel Sets

Definition 3.10: Intervals

For real numbers $a < b$, we define:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$
$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$$
$$[a, b) = \{x \in \mathbb{R} : a \le x < b\}$$
$$(a, b] = \{x \in \mathbb{R} : a < x \le b\}$$

Definition 3.11: Open Set

A set O of real numbers is called open provided for each $x \in O$, there is an $r > 0$ for which $(x - r, x + r) \subset O$.

Theorem 3.16: Properties of Open Sets

(i) $\mathbb{R}$ and $\emptyset$ are open

(ii) The intersection of any finite collection of open sets is open

(iii) The union of any collection of open sets is open

Definition 3.12: Closure Points

A point x is a point of closure of a set E if every open interval containing x also contains a point of E. The closure of E, denoted $\overline{E}$, is the set of all points of closure of E.

Definition 3.13: Closed Set

A set E is closed if it contains all its points of closure (i.e., if $E = \overline{E}$).

Theorem 3.17: Properties of Closed Sets

(i) $\mathbb{R}$ and $\emptyset$ are closed

(ii) The union of any finite collection of closed sets is closed

(iii) The intersection of any collection of closed sets is closed

Theorem 3.18: Heine-Borel

A set of real numbers is compact if and only if it is closed and bounded.

3.5 Sequences and Series

Definition 3.14: Convergence

A sequence $\{a_n\}$ converges to a if for every $\epsilon > 0$ there exists N such that:

$$\text{if } n \ge N \text{ then } |a_n - a| < \epsilon$$

Theorem 3.19: Properties of Limits

For convergent sequences $\{a_n\}$ and $\{b_n\}$ and real numbers α and β:

$$\lim_{n\to\infty} [\alpha a_n + \beta b_n] = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n$$

Moreover, if $a_n \leq b_n$ for all n, then:

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Theorem 3.20: Monotone Convergence

A monotone sequence of real numbers converges if and only if it is bounded.

Theorem 3.21: Bolzano-Weierstrass

Every bounded sequence of real numbers has a convergent subsequence.

Definition 3.15: Cauchy Sequence

A sequence $\{a_n\}$ is Cauchy if for each $\epsilon > 0$ there exists an index N such that:

$$\text{if } n, m \geq N \text{ then } |a_m - a_n| < \epsilon$$

Theorem 3.22: Cauchy Criterion

A sequence of real numbers converges if and only if it is Cauchy.

3.6 Continuous Functions

Definition 3.16: Continuity

A function f is continuous at $x \in E$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\text{if } x' \in E \text{ and } |x' - x| < \delta \text{ then } |f(x') - f(x)| < \epsilon$$

Theorem 3.23: Characterization of Continuity

For a function f defined on E, the following are equivalent:

(i) f is continuous on E

(ii) For each open set $\mathcal{O}$, $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$ where $\mathcal{U}$ is open

(iii) For any sequence $\{x_n\}$ in E converging to $x \in E$, $\{f(x_n)\}$ converges to $f(x)$

Theorem 3.24: Extreme Value

A continuous real-valued function on a non-empty closed, bounded set takes a minimum and maximum value.

Theorem 3.25: Intermediate Value

Let f be continuous on $[a,b]$ with $f(a) < c < f(b)$. Then there exists $x_0 \in (a,b)$ such that $f(x_0) = c$.

Definition 3.17: Uniform Continuity

A function f on E is uniformly continuous if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for all x, x' in E:

$$\text{if } |x - x'| < \delta \text{ then } |f(x) - f(x')| < \epsilon$$

Theorem 3.26: Uniform Continuity on Compact Sets

A continuous real-valued function on a closed, bounded set is uniformly continuous.