Introduction to Analysis

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1 Course Overview

Note

These notes compile key concepts, theorems, and insights from the Mathematical Analysis course. They serve as a comprehensive reference for understanding advanced mathematical principles.

2 Preliminaries on Sets, Mappings, and Relations

Definition 2.1: Set Membership

Let A be a set. For an element x:

- $x \in A$ denotes x is a member of A
- $x \notin A$ denotes x is not a member of A

Sets are typically denoted by $\{x \mid \text{condition}\}$, representing all elements satisfying the condition.

Definition 2.2: Set Equality and Subset

Two sets are equal if and only if they have exactly the same members.
For sets A and B:

- $A \subseteq B$ means every member of A is a member of B
- A is a proper subset of B if $A \subseteq B$ and $A \neq B$

Definition 2.3: Set Operations

For sets A and B:

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- Complement: $B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$

Theorem 2.1: De Morgan's Laws

For a family of sets $\mathcal{F}$ and a reference set X:

$$X \setminus \left(\bigcup_{F \in \mathcal{F}} F \right) = \bigcap_{F \in \mathcal{F}} (X \setminus F)$$

$$X \setminus \left(\bigcap_{F \in \mathcal{F}} F \right) = \bigcup_{F \in \mathcal{F}} (X \setminus F)$$

2.1 Mappings and Functions

Definition 2.4: Mapping

A mapping (or function) $f : A \longrightarrow B$ is a correspondence that assigns to each $x \in A$ a unique $y \in B$, denoted $f(x)$.

- Domain: The set A
- Image/Range: $f(A) = \{b \in B \mid \exists a \in A, f(a) = b\}$

Definition 2.5: Function Types

- Onto (Surjective): $f(A) = B$
- One-to-One (Injective): $f(a_1) = f(a_2) \implies a_1 = a_2$
- Invertible (Bijective): Both one-to-one and onto

Definition 2.6: Function Composition

For mappings $f : A \longrightarrow B$ and $g : B \longrightarrow C$, the composition $g \circ f : A \longrightarrow C$ is defined by $(g \circ f)(x) = g(f(x))$.

2.2 Relations

Definition 2.7: Relation

A relation R on a set X is a subset of $X \times X$.

- Reflexive: $\forall x \in X, xRx$
- Symmetric: $xRy \implies yRx$
- Transitive: xRy and $yRz \implies xRz$

Definition 2.8: Equivalence Relation

An equivalence relation is a relation that is reflexive, symmetric, and transitive.
For an equivalence relation R on X:

- Equivalence class of x: $[x]_R = \{y \in X \mid xRy\}$
- Quotient set: $X/R = \{[x]_R \mid x \in X\}$

2.3 Axiom of Choice and Zorn's Lemma

Definition 2.9: Choice Function

A choice function on a family $\mathcal{F}$ of non-empty sets is a function $f : \mathcal{F} \longrightarrow \bigcup_{F \in \mathcal{F}} F$ such that $f(F) \in F$ for all $F \in \mathcal{F}$.

Theorem 2.2: Axiom of Choice

For any non-empty collection of non-empty sets, there exists a choice function.

Definition 2.10: Partial Ordering

A relation $\preceq$ on a set X is a partial ordering if it is:

- Reflexive: $x \preceq x$
- Antisymmetric: $x \preceq y$ and $y \preceq x \implies x = y$
- Transitive: $x \preceq y$ and $y \preceq z \implies x \preceq z$

Theorem 2.3: Zorn's Lemma

If a partially ordered set has an upper bound for every totally ordered subset, then it contains a maximal element.

Note

Zorn's Lemma is equivalent to the Axiom of Choice and will be crucial in proving several important theorems in advanced mathematics.

2.4 Cardinality and Set Comparisons

Definition 2.11: Cardinality

Two sets A and B are said to be equipotent (or have the same cardinality) if there exists a bijective mapping $f : A \to B$.
Key properties:

- Equipotence is an equivalence relation on the collection of sets
- $|A| = |B|$ if and only if there exists a bijection between A and B
- Finite sets have cardinality equal to the number of their elements

Theorem 2.4: Cantor-Bernstein Theorem

If there exist injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijection between A and B.
Symbolically, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

2.5 Countable and Uncountable Sets

Definition 2.12: Countable Sets

A set A is:

- Finite if $|A| < \aleph_0$
- Countably infinite if $|A| = \aleph_0$

- Countable if it is either finite or countably infinite
- Uncountable if it is not countable

Theorem 2.5: Countability Properties

- The set of natural numbers $\mathbb{N}$ is countably infinite
- The set of integers $\mathbb{Z}$ is countably infinite
- The set of rational numbers $\mathbb{Q}$ is countably infinite
- The set of real numbers $\mathbb{R}$ is uncountable

2.6 Generalized Cartesian Product

Definition 2.13: Generalized Cartesian Product

For a family of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ indexed by Λ, the Cartesian product is defined as:

$$\prod_{\lambda\in\Lambda} E_\lambda = \left\{ f : \Lambda \to \bigcup_{\lambda\in\Lambda} E_\lambda \mid \forall\lambda \in \Lambda, f(\lambda) \in E_\lambda \right\}$$

Theorem 2.6: Cartesian Product Properties

- If Λ is finite and each E_λ is non-empty, then $\prod_{\lambda\in\Lambda} E_\lambda$ is non-empty
- The Axiom of Choice is equivalent to the statement that the Cartesian product of a non-empty family of non-empty sets is non-empty

2.7 Well-Ordering Principle

Theorem 2.7: Well-Ordering Theorem

Every set can be well-ordered. That is, for every set X, there exists a total order $\leq$ on X such that every non-empty subset of X has a least element.
This theorem is equivalent to the Axiom of Choice.

2.8 Foundations of Set Theory

Definition 2.14: Zermelo-Fraenkel Axioms

Set theory is formally grounded in the Zermelo-Fraenkel (ZF) axiom system, which provides a rigorous foundation for mathematical reasoning about sets:

- **Axiom of Extensionality:** Two sets are equal if and only if they have exactly the same elements
- **Axiom of Pairing:** For any two sets, there exists a set containing exactly those two sets

- **Axiom of Union:** For any collection of sets, there exists a set that contains all elements that belong to at least one set in the collection
- **Axiom of Power Set:** For any set, there exists a set containing all possible subsets of that set

The Axiom of Choice (AC) can be added to ZF to form ZFC, the standard foundation for most mathematical reasoning.

Definition 2.15: Parametrization of Sets

For a set Λ and a family of sets $\{E_\lambda\}_{\lambda\in\Lambda}$, the parametrization provides a systematic way to index sets:

- Λ is called the *index set*
- Each E_λ is associated with a unique $\lambda \in \Lambda$
- Different parametrizations of the same family may yield different mathematical structures

Example: Let $\Lambda = \{1,2\}$ and define $E_1 = \{a,b\}$, $E_2 = \{c,d\}$. Then $\{E_\lambda\}_{\lambda\in\Lambda} = \{\{a,b\},\{c,d\}\}$.

Theorem 2.8: Parametrized Cartesian Product Properties

For a parametrized family $\{E_\lambda\}_{\lambda\in\Lambda}$:

- The Cartesian product depends critically on the specific parametrization
- Two different indexings of the same underlying sets can produce different Cartesian products
- The choice of index set Λ is crucial in defining the product structure

Note

The subtleties of set parametrization highlight the depth and complexity inherent in foundational set theory, demonstrating how seemingly simple concepts can reveal profound mathematical structures.