Online Appendix – Not for Publication

A Proofs

A.1 Proof of proposition 1 (existence of Markov perfect equilibrium)

We prove existence of Markov perfect equilibrium constructively, following a fixed point procedure similar to the one typically used by the sovereign debt literature to find an equilibrium. Section A.1.1 defines a functional $\mathbb{B}(V)$ mapping value functions to default thresholds, and proves properties of this mapping. Section A.1.2 defines a functional $\mathbb{V}(B)$ mapping default thresholds to value functions, and proves properties of that mapping. Finally, Section A.1.2 shows that iterating on the operator $\mathbb{T} = \mathbb{B} \circ \mathbb{V}$, starting from thresholds identically equal to zero, produces a limit set of default thresholds that constitute an equilibrium.

A.1.1 Default thresholds for given V: $\mathbb{B}(V)$

Consider a set of S strictly decreasing, continuous functions $V(b,s)$. For each state s, define the threshold $b^*(s)$ as $-\infty$ if $\sup_b V(b,s) < V^d(s)$, or $+\infty$ if $\inf_b V(b,s) > V^d(s)$. In other cases, let $b^*(s)$ be equal to the unique solution to

$$V(b^*(s), s) = V^d(s)$$

This defines a functional $\mathbb{B}(V)$. The following shows that this is a monotone mapping, and provides conditions on V under which $\mathbb{B}(V)$ is positive and bounded.

Lemma 14. *The following propositions hold for every s.*

a) If $V(0,s) \geq V^{nb}(0,s)$, then $b^*(s) \geq 0$

b) If $V(b,s) = -\infty$, then $b^*(s) < b$

c) If $V^A(b,s) \geq V^B(b,s)$ for all b, then the respective default thresholds satisfy $b^{*A}(s) \geq b^{*B}(s)$

Proof. The proof follows because V is continuous and strictly decreasing. Assumption 5 guarantees that $V(0,s) \geq V^{nb}(0,s) \geq V^d(s) = V(b^*(s), s)$, so a) holds. Assumption 5 also guarantees that $V^d(s)$ is finite, so $V(b_s, s) < V(b^*(s), s)$, and b) holds. Finally, $V^B\left(b^{*B}(s), s\right) = V^d(s) = V^A\left(b^{*A}(s), s\right) \geq V^B\left(b^{*A}(s), s\right)$, so c) holds. □

A.1.2 Value functions V given default thresholds: $\mathbb{V}(B)$

Consider now a set of positive default thresholds $B = \{b^*(s)\}$, $b^*(s) \geq 0$. Define V as the solution to

$$\begin{aligned} V\left(b, s; \{b^*(s')\}\right) \equiv & \sup_{b(s^t), p(s^t) \in \{0,1\}} \left\{ \sum_{s^t} \beta^t \Pi\left(s^t\right) u\left(c\left(s^t\right), s_t\right) \mathbf{1}_{\{p(s^t)=1\}} \right. \\ & \left. + \sum_{s^t} \beta^t \Pi\left(s^t\right) V^d(s_t) \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}} \right\} \\ & \text{s.t. } c\left(s^t\right) = y\left(s_t\right) + \frac{b(s^t)}{R} \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}} \right) - b\left(s^{t-1}\right) \\ & p(s^t) \leq p(s^{t-1}) \\ & p\left(s_0\right) = 1 \\ & b\left(s^{-1}\right) = b \\ & s_0 = s \end{aligned} \tag{A.1}$$

This defines a mapping $\mathbb{V}(B)$ from default thresholds to value functions. We now prove properties of this mapping, including monotonicity and continuity.

Lemma 15. *The following propositions hold for every s.*

a) The supremum in (A.1) is attained for any b, and $V(b, s) < \infty$

b) $V(b, s)$ is strictly decreasing in b

c) $V(b, s)$ is continuous in b for every s

d) $V(b, s; \{b^*(s')\})$ is increasing in $\{b^*(s')\}$

e) $V(0, s) \geq V^{nb}(0, s)$

f) $V\left(\frac{\bar{b}}{R} + y(s), s\right) = -\infty$

g) $V(b, s; \{b^*(s')\})$ is continuous in $\{b^*(s')\}$

Proof. We prove each of the propositions in turn.

a) We restrict ourselves to cases where such that $V(b, s) > -\infty$, otherwise the proposition is trivial. We prove that the maximum is attained by showing that the problem in (A.1) is the maximization of an upper semicontinuous function on a compact set, and exhibit an upper bound to show $V(b, s) < \infty$. First, assumption 3 guarantees that $c\left(s^t\right) > 0$, which (given that assets receive the risk-free rate) bounds the rate of growth of assets: there exists $D > 0$ such that $b\left(s^t\right) \geq -DR^{t+1}$. Together with assumption 4, this guarantees that $b\left(s^t\right)$ must be chosen on a compact interval $\left[-DR^{t+1}, \bar{b}\right]$, and hence that the set of all arguments

$\{b(s^t), p(s^t)\}$ is compact. Second, these bounds on $b(s^t)$ place an upper bound on $c(s^t)$ which, together with assumption 2, yields a bound on flow utility, $\beta^t u(c(s^t), s_t) \leq (\beta R^\kappa)^t \overline{u}$ where $\overline{u} < \infty$. Third, the presence of the default option implies that flow utility along the no-default path is bounded below in all periods, $\beta^t u(c(s^t), s_t) \geq \beta^t \underline{u}$ for $\overline{u} > -\infty$. Summing up, we have bounds on flow utility:

$$\beta^t \underline{u} \leq \beta^t u(c(s^t), s_t) \leq (\beta R^\kappa)^t \overline{u} \tag{A.2}$$

Next, all partial sums in the maximand (A.1) are upper semicontinuous in the argument. This follows from the fact that they consist entirely of continuous functions except $\mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}$, which is upper semicontinuous. Inequality (A.2) together with $\beta < 1$ and $\beta R^\kappa < 1$ allows one to apply the Weierstrass M-test to conclude that the sum converges uniformly, and hence that the limit is also upper semicontinous in the argument. Hence the maximum in (A.1) is attained. Finally, (A.2) together with the fact that default values are finite guarantee that the objective in (A.1) is uniformly bounded from above, and hence the maximum $V(b, s) < \infty$ as well.

b) Fix s and consider $\widetilde{b} > b$. Consider the optimal plan $\left\{\widetilde{b}(s^t), \widetilde{p}(s^t)\right\}$ starting at $\left(\widetilde{b}, s\right)$. Then the plan $\left\{\widetilde{b}(s^t), \widetilde{p}(s^t)\right\}$ is also feasible starting at (b, s), so that, letting $Q = \frac{\widetilde{b}(s^0)}{R} \sum_{\left\{s' : \widetilde{b}(s^0) \leq b^*(s')\right\}} \pi(s'|s)$, we have

$$\begin{aligned} V(b, s) - V\left(\widetilde{b}, s\right) &\geq u(y(s) + Q - b, s) - u\left(y(s) + Q - \widetilde{b}, s\right) \\ &> u\left(y(s) + Q - \widetilde{b}, s\right) - u\left(y(s) + Q - \widetilde{b}, s\right) = 0 \end{aligned}$$

c) Fix (b, s) and let $\epsilon > 0$. We show that (i) there exists $\delta_1 > 0$ such that for any $b < \widetilde{b} < b + \delta_1$, $V(\widetilde{b}, s) > V(b, s) - \epsilon$, and (ii) there exists $\delta_2 > 0$ such that for any $b > \widetilde{b} > b - \delta_2$, $V(\widetilde{b}, s) < V(b, s) + \epsilon$. Together with V being strictly decreasing, (i) and (ii) establish continuity.

For (i), consider the optimal plan $\{b(s^t), p(s^t)\}$ starting at (b, s). This plan is also feasible starting at $(\widetilde{b}, s)$ and delivers the same consumption at every point except $t = 0$, where consumption is $\widetilde{b} - b$ lower. Hence letting $c(s^0)$ be the $t = 0$ consumption level for the optimal plan starting at (b, s), we know

$$V(b, s) - V(\widetilde{b}, s) = u(c(s^0)) - u(c(s^0) - \delta_1)$$

will be $< \epsilon$ as desired if $\delta_1 > 0$ is defined via continuity of u such that $|u(c) - u(c(s^0))| < \epsilon$ for all $|c - c(s^0)| < \delta_1$.

For (ii), we must appeal to a uniform continuity argument to choose δ_2. We first find a compact set $[\underline{c}, \overline{c}]$ such that any optimal plan with $\widetilde{b} < b$ (and hence $V(\widetilde{b}, s) > V(b, s)$) has first period consumption $\widetilde{c}(s^0) \in [\underline{c}, \overline{c}]$. To do this, recall from A.1.2 that the sum of all terms in (A.1) for $t \geq 1$ is bounded from above by an upper bound $\overline{V} < \infty$. Hence the initial consumption level $\widetilde{c}(s^0)$ associated with an optimum $V(\widetilde{b},s) > V(b,s)$ must be such that

$$u(\widetilde{c}(s^0),s_0) + \overline{V} \geq V(b,s) \tag{A.3}$$

From assumption 3, $u(\widetilde{c}(s^0),s_0) \to -\infty$ as $\widetilde{c}(s^0) \to 0$, and hence for (A.3) to be satisfied we must have $\widetilde{c}(s^0) \geq \underline{c} > 0$ for some lower bound $\underline{c}$. We also know that $\widetilde{c}(s^0) \leq \overline{b}/R + \overline{y} - b(s^0) \equiv \overline{c}$, giving us an upper bound. Since u is continuous and $[\underline{c},\overline{c}]$ is a compact interval, we can pick a single $\delta_2 > 0$ such that $|u(c_A,s_0) - u(c_B,s_0)| < \epsilon$ for all $c_A \in [\underline{c},\overline{c}]$ and $|c_B - c_A| < \delta_2$.
Now consider the optimal plan $\{\widetilde{b}(s^t),\widetilde{p}(s^t)\}$ starting at $(\widetilde{b},s)$. This plan is also feasible starting at (b,s) and delivers the same consumption at every point except $t = 0$, where consumption is $b - \widetilde{b}$ lower. Hence we have

$$V(\widetilde{b},s) - V(b,s) = u(\widetilde{c}(s^0),s_0) - u(\widetilde{c}(s^0) - \delta_2,s_0) < \epsilon$$

as desired.

d) Since $b^*(s') \geq 0$, increasing $b^*(s')$ always weakly increases $\frac{b(s^t)}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t)\leq b^*(s_{t+1})\}}\right)$ when $b(s^t) \geq 0$ and leaves it unchanged when $b(s^t) \leq 0$, which completes the proof.

e) Follows from d), since $V^{nb}(0,s)$ is the value with default thresholds all equal to zero, as shown in section 3.2.

f) Assumption 4 ensures that for any $b' > 0$, $\frac{b'}{R}\sum_{\{s':b'\leq b^*(s')\}} \pi(s'|s) < \frac{\overline{b}}{R}$. Hence feasible consumption at date 0 is $c(s^0) < y(s) + \frac{\overline{b}}{R} - \left(\frac{\overline{b}}{R} + y(s)\right) = 0$. Given that the continuation value for any b' is finite, f) follows from assumption 3.

g) Fix b and s^0. Let $\epsilon > 0$ and let $\{b^*(s')\}$ be a set of default thresholds. In an argument similar to the proof of c), we show that (i) there exists δ_1 such that, for any alternative set of default thresholds $\{\widetilde{b}^*(s')\}$ such that $|b^*(s') - \widetilde{b}^*(s')| < \delta_1$ for all s', we have $V(b,s,\{\widetilde{b}^*(s')\}) > V(b,s,\{b^*(s')\}) - \epsilon$, and (ii) there exists δ_2 such that, for any $\{\widetilde{b}^*(s')\}$ such that $|b^*(s') - \widetilde{b}^*(s')| < \delta_2$ for all s', we have $V(b,s,\{b^*(s')\}) > V(b,s,\{\widetilde{b}^*(s')\}) - \epsilon$. Combining (i) and (ii) then proves continuity. In both cases, we use the fact that a government facing debt thresholds that are lower by at most δ can guarantee itself a consumption plan that is only δ below that of a government with reference debt thresholds at date 0—and above at every other date—using a mimicking strategy, as embodied in the following claim.

Claim. Assume that $|b^*(s') - \widetilde{b}^*(s')| < \delta$. Let $\{b(s^t),p(s^t)\}$ be a plan that achieves consumption $c(s^t)$ subject to the default thresholds $\{b^*(s')\}$ starting from (b,s^0). Then there is another plan $\{\widetilde{b}(s^t),p(s^t)\}$ that achieves consumption $\widetilde{c}(s^t)$ subject to the default thresholds $\{\widetilde{b}^*(s')\}$ such that $\widetilde{c}(s^t) > c(s^t)$ for all $t \geq 1$ and $\widetilde{c}(s^0) > c(s^0) - \delta$.

Proof of claim. Define $\widetilde{b}(s^t) \equiv b(s^t) - \delta$ for all $t \geq 0$ and $\widetilde{b}(s^{-1}) \equiv b(s^{-1}) = b$. Then compute

$$\begin{aligned}\widetilde{c}(s^t) &= y\left(s_t\right) + \frac{b(s^t) - \delta}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\left\{b(s^t) - M \leq \widetilde{b}^*(s_{t+1})\right\}}\right) - \left(b(s^{t-1}) - \delta\right) \\ &\geq y\left(s_t\right) + \frac{b(s^t)}{R}\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}\right) - b(s^{t-1}) + \left(1 - \frac{1}{R}\right)\delta > c(s^t)\end{aligned}$$

and

$$\begin{aligned}\widetilde{c}(s^0) &= y\left(s_0\right) + \frac{b(s^0) - \delta}{R}\left(\sum_{s_1} \pi(s_1|s_0) \cdot \mathbf{1}_{\left\{b(s^0) - M \leq \widetilde{b}^*(s_1)\right\}}\right) - b(s^{-1}) \\ &\geq y\left(s_0\right) + b(s^0)\left(\sum_{s_1} \pi(s_1|s_0) \cdot \mathbf{1}_{\{b(s^0) \leq b^*(s_1)\}}\right) - b(s^{-1}) - \frac{\delta}{R} > c(s^0) - \delta\end{aligned}$$

□

To prove (i), consider the plan $\left\{c\left(s^t\right), b\left(s^t\right), p\left(s^t\right)\right\}$ that achieves $V(b, s, \{b^*(s')\})$. Using the continuity of $u\left(c, s^0\right)$, let δ_1 be such that $|u(c', s_0) - u(c(s^0), s_0)| < \epsilon$ for all $|c' - c(s^0)| < \delta_1$. Then, whenever the thresholds $\left\{\widetilde{b}^*(s')\right\}$ are such that $|b^*(s') - \widetilde{b}^*(s')| < \delta_1$ for all s', it follows from the claim that there is a consumption plan $\{\widetilde{b}(s^t), p(s^t)\}$ for these thresholds that achieves consumption above $c\left(s^0\right) - \delta$ in the first period and above $c\left(s^t\right)$ everywhere else, and hence value greater than $V(b, s, \{b^*(s')\}) - \epsilon$.
To prove (ii), suppose for some $\{\widetilde{b}^*(s')\}$ that $V(b, s, \{\widetilde{b}^*(s')\}) \geq V(b, s, \{b^*(s')\})$ (otherwise, the desired inequality is immediate), and let $\{\widetilde{b}(s^t), p(s^t)\}$ be the plan attaining the optimum for $V(b, s, \{\widetilde{b}^*(s')\})$. We can establish using the argument from the proof of c) we can pick a single $\delta_2 > 0$ such that $|u(c, s_0) - u(\widetilde{c}(s_0), s_0)| < \epsilon$ for $|c - \widetilde{c}(s_0)| < \delta_2$. It follows from the claim that there is a plan $\{\widetilde{\widetilde{b}}(s^t), p(s^t)\}$ that (subject to the default thresholds $\{b^*(s')\}$) achieves consumption $\widetilde{\widetilde{c}}(s^t)$ that is strictly greater than $\widetilde{c}(s^t)$ for all $t \geq 1$ and strictly greater than $\widetilde{c}(s^0) - \delta_2$ for $t = 0$. From the choice of δ_2 we know that $|u(\widetilde{\widetilde{c}}(s^0), s_0) - u(\widetilde{c}(s^0), s_0)| < \epsilon$, and hence that the proposed plan $\{\widetilde{\widetilde{b}}(s^t), p(s^t)\}$ gives value strictly greater than $V(b, s, \{\widetilde{b}^*(s')\}) - \epsilon$. It follows that $V(b, s, \{b^*(s')\}) > V(b, s, \{\widetilde{b}^*(s')\}) - \epsilon$ as desired.

□

A.1.3 Existence of equilibrium

Using the operators defined in Sections A.1.1 and A.1.2, we can define the operator $\mathbb{T} = \mathbb{B} \circ \mathbb{V}$.

Lemma 16. *The operator $\mathbb{T}$ is monotone increasing and maps the set $\prod_s \left[0, y\left(s\right) + \frac{\bar{b}}{R}\right]$ onto itself*

Proof. Monotonicity follows by combining lemmas 14c) and 15d). By combining lemmas 14a) and 15e), we obtain that $\mathbb{T}b^*\left(s\right) \geq 0$ whenever $b^*\left(s\right) \geq 0$. By combining lemmas 14b) and 15f), we obtain that $\mathbb{T}b^*\left(s\right) \leq y\left(s\right) + \frac{\bar{b}}{R}$ for each s. □

Let $b^{*0}(s) = 0$ for every s. For $n \geq 1$ define the sequence

$$b^{*n} = \mathbb{T} b^{*n-1}$$

By lemma 16, the sequences $b^{*n}(s)$ are increasing and bounded for every s. Hence they converge to form a set of thresholds $\{b^{*\infty}\}$. Define $V^n = \mathbb{V}(b^{*n})$ and $V^\infty = \mathbb{V}(b^{*\infty})$. From Lemma 15g) it follows that $V^\infty(b,s) = \lim_{n\to\infty} V^n(b,s)$. Next, because V^n is a sequence of continuous bijective functions with continuous inverses, whose limit V^∞ is continuous and bijective, and since by definition $\mathbb{B}(V^n)(s) = (V^n)^{-1}\left(V^d(s), s\right)$, we have that

$$\mathbb{B}\left(\lim_{n\to\infty} V^n\right) = \lim_{n\to\infty} \mathbb{B}(V^n)$$

and therefore

$$\mathbb{B}(V^\infty) = \lim_{n\to\infty} \mathbb{T} b^{*n} = b^{*\infty} \tag{A.4}$$

So $(V^\infty, b^{*\infty})$ constitutes an equilibrium, as we set out to prove. To map these objects to those in the main text, define $V = V^\infty$ and the bond revenue schedule Q as

$$Q(b', s) = \frac{b'}{R} \mathbb{P}_{s'|s}\left[b' \leq b^{*\infty}(s')\right] = \frac{b'}{R} \sum_{\{s' : b' \leq b^{*\infty}(s')\}} \pi(s'|s)$$

then (V, Q) is a Markov perfect equilibrium, since (1)-(2) is the recursive formulation of the problem in (A.1) for the schedule Q generated by the thresholds $\mathbb{B}(V)$, and (A.4) guarantees that (3) holds.

A.2 Proof of proposition 4 (uniqueness of subgame perfect equilibrium)

This appendix proves uniqueness of the subgame perfect equilibrium in the game of section 2. In order to define the game explicitly, we assume that there exist overlapping generations of two-period lived international investors. The set of investors born at time t is denoted by $\mathcal{I}_t$. We assume that $\mathcal{I}_t$ is finite, that $|\mathcal{I}_t| \geq 2$, and that all investors are risk-neutral with preferences given by

$$-q_t a^i_{t+1} + \frac{1}{R} \mathbb{E}_t\left[a^i_{t+1} p_{t+1}\right] \tag{A.5}$$

where $R > 1$. We next describe the sequence of actions.

Every period, with incoming history h^{t-1}, after Nature realizes the exogenous state s_t, the government chooses repayment p_t. If it chooses $p_t = 0$ (default), it obtains value $V^d(s_t)$, investors receive zero, and the game ends.

If it chooses $p_t = 1$, the government receives income $y(s_t) \geq 0$ and chooses next period debt b_{t+1}. Next, every investor i simultaneously bids a price $q^i_t \geq 0$ for the government's debt. Given bids q_t^i, an auctioneer allocates the bonds a_{t+1}^i according to the following rule:

$$a_{t+1}^i = \begin{cases} \frac{b_{t+1}}{J} & \text{if} \quad q_t^i = \max_{i'} q_t^{i'} \\ 0 & \text{otherwise} \end{cases}$$

where J is the number of investors bidding the maximum price. History for period t is now $h^t = \left(h^{t-1}, s_t, b_{t+1}, \{q_t^i\}\right)$.

The government receives $Q_t = q_t b_{t+1}$ where $q_t = \max_{i'} q_t^{i'}$ and repays debt b_t to previous investors. Its consumption is then

$$c_t = y\left(s_t\right) - b_t + q_t b_{t+1}$$

for which it receives flow utility $u\left(c_t, s_t\right)$, and expected value

$$V\left(h^{t-1}, s_t\right) = \begin{cases} u\left(c_t, s_t\right) + \beta \mathbb{E}_t\left[V\left(h^t, s_{t+1}\right)\right] & \text{if} \quad p_t = 1 \\ V^d\left(s_t\right) & \text{if} \quad p_t = 0 \end{cases} \tag{A.6}$$

Definition 2. A government strategy is $p\left(h^{t-1}, s_t\right)$, $b'\left(h^{t-1}, s_t\right)$ specifying the repayment and next period debt decision after each history h^{t-1} and state s_t. A strategy of investor i born at time t is a price bid $q^i\left(h^{t-1}, s_t, b_{t+1}\right)$.

Together, investor strategies imply a bond revenue function $Q\left(h^{t-1}, s_t, b_{t+1}\right)$.

Definition 3. A subgame perfect equilibrium consists of strategies for the government and investors such that at each $\left(h^{t-1}, s_t\right)$:

a) $p\left(h^{t-1}, s_t\right)$, $b'\left(h^{t-1}, s_t\right)$ maximize (A.6)

b) For all $i \in \mathcal{I}_t$, $q^i\left(h^{t-1}, s_t, b_{t+1}\right)$ maximizes (A.5)

In any subgame perfect equilibrium, investor maximization leads to

$$q\left(h^{t-1}, s_t, b_{t+1}\right) = \frac{1}{R}\mathbb{E}_t\left[p\left(h^t, s_{t+1}\right)\right] \tag{A.7}$$

We retain the other assumptions from the model in section 2 on u, V^d, and the no-Ponzi bound on debt $\overline{b}$. These include assumption 1 and assumptions 1 through 5. Importantly, assumption 5 continues to imply that a government with debt $b < 0$ never finds it optimal to default, so $q\left(h, s, b'\right) = \frac{1}{R}$ for any $b' < 0$.

The following lemma is crucial to the proof of unique equilibrium. It shows that in equilibrium, regardless of the history of play, a government with a strictly lower level of debt can always achieve a weakly higher value than a government with more debt in the same state, and is also weakly more likely to repay. Like the proof of lemma 2, it uses a mimicking-based argument, although here the proof is written in a recursive setting and must deal with technical complications that arise from the more general notion of equilibrium.

Lemma 17. *Consider two subgame perfect equilibria A and B. For any* (h_A, h_B, s)*, if* $b(h_A) > b(h_B)$ *then* $V_A(h_A, s) \leq V_B(h_B, s)$*, and* $p_B(h_B, s) = 1$ *if* $p_A(h_A, s) = 1$*.*

Proof. Define

$$M \equiv \sup_{h_A, h_B, s} \{b(h_A) - b(h_B) \quad s.t. \quad V_A(h_A, s) \geq V_B(h_B, s) \quad and \quad p_A(h_A, s) = 1\}$$

Assume $M > 0$.[13] Let $0 < \epsilon < \frac{R-1}{R+1} M$, and let (h_A, h_B, s) be such that $V_A(h_A, s) \geq V_B(h_B, s)$, $p_A(h_A, s) = 1$ and $b(h_A) > b(h_B) + M - \epsilon$. Define

$$\widetilde{b}'_B = b'_A(h_A, s) - M - \epsilon \tag{A.8}$$

and continuation histories

$$\begin{aligned} h'_A &= \left(h_A, s, b'_A(h_A, s), \left\{q'^{i}_A\right\}\right) \\ \widetilde{h}'_B &= \left(h_B, s, \widetilde{b}'_B, \left\{\widetilde{q}'^{i}_B\right\}\right) \end{aligned}$$

This is a feasible choice for the B government at (h_B, s) because we assume that debt can be chosen at any level below some upper bound. We aim to prove that through this choice of $\widetilde{b}'_B$, the government in the B equilibrium achieves expected utility strictly greater than $V_A(h_A, s)$, thus establishing that $V_B(h_B, s) > V_A(h_A, s)$, a contradiction. We first establish that continuation utility for B is weakly greater in each future state, and then that current consumption is strictly greater, than their corresponding values for A.

We have, for all $s' \in \mathcal{S}$,

$$V_B\left(\widetilde{h}'_B, s'\right) \geq V_A(h'_A, s') \tag{A.9}$$

Indeed, if $p_A(h'_A, s') = 0$, then immediately $V_B\left(\widetilde{h}'_B, s'\right) \geq V^d(s') = V_A(h'_A, s')$. Moreover, if $p_A(h'_A, s') = 1$ then, since $b(h'_A) - b\left(\widetilde{h}'_B\right) > M$ by (A.8), we must have $V_B\left(\widetilde{h}'_B, s'\right) > V_A(h'_A, s') \geq V^d(s')$.

This last observation also implies that $p_B\left(\widetilde{h}'_B, s'\right) = 1$ whenever $p_A(h'_A, s') = 1$. Hence, using the pricing condition (A.7), we also have

$$q_B\left(h_B, s, \widetilde{b}'_B\right) \geq q_A(h_A, s, b'_A) \tag{A.10}$$

Using (A.10), we now show that the consumption achieved by B from the choice of $\widetilde{b}'_B$ is strictly greater than that achieved by A. Indeed, using the flow budget constraints of both governments,

[13] One can rule out the case $M = \infty$ through a more direct mimicking argument: whenever $b(h_A) - b(h_B) > \overline{b}$, where $\overline{b}$ is the upper bound on debt, then a government at (h_B, s) can mimic at distance $\overline{b}$ the strategy of a government at (h_A, s), with weakly more favorable prices (and hence strictly higher consumption due to its lower b) guaranteed because it will never be in debt.

and dropping dependence on history for ease of notation:

$$\begin{aligned}\widetilde{c}_B &= c_A + b_A - b_B + \widetilde{q}_B \widetilde{b}'_B - q_A b'_A \\ &\geq c_A + M - \epsilon + \left(\widetilde{q}_B - q_A\right) b'_A + \widetilde{q}_B \left(\widetilde{b}'_B - b'_A\right) \end{aligned} \tag{A.11}$$

where the inequality follows from the definition of A and B.

Now if $b'_A < 0$ then, since $\widetilde{b}'_B \leq b'_A < 0$ as well we have $q_A = \widetilde{q}_B = \frac{1}{R}$, and hence $\left(\widetilde{q}_B - q_A\right) b'_A = 0$. If $b'_A \geq 0$ then using (A.10), $\left(\widetilde{q}_B - q_A\right) b'_A \geq 0$.

Moreover, from (A.8), $\widetilde{b}'_B - b'_A = -M - \epsilon$, and using $\widetilde{q}_B \leq \frac{1}{R}$, $\widetilde{q}_B \left(\widetilde{b}'_B - b'_A\right) \geq -\frac{1}{R}\left(M + \epsilon\right)$. Using these inequalities in (A.11),

$$\begin{aligned}\widetilde{c}_B &\geq c_A + M - \epsilon - \frac{1}{R}\left(M + \epsilon\right) \\ &\geq c_A + \left(1 - \frac{1}{R}\right) M - \epsilon \left(1 + \frac{1}{R}\right) \\ &> c_A \end{aligned} \tag{A.12}$$

where the last line follows from the choice of ϵ.

Since the utility from choosing $\widetilde{b}'_B$ provides a lower bound on $V_B\left(h_B, s\right)$, we have

$$\begin{aligned}V_B\left(h_B, s\right) &\geq u\left(\widetilde{c}_B, s\right) + \beta \sum_{s'} V_B\left(\widetilde{h}'_B, s'\right) \\ &> u\left(c_A, s\right) + \beta \sum_{s'} V_A\left(h'_A, s'\right) = V_A\left(h_A, s\right)\end{aligned}$$

where the second line follows from (A.9) and (A.12). This contradicts $M > 0$. Hence $M \leq 0$. We have proved that for $\left(h_A, h_B, s\right)$, if $V_A\left(h_A, s\right) \geq V_B\left(h_B, s\right)$ and $p_A\left(h_A, s\right) = 1$ then $b\left(h_A\right) \leq b\left(h_B\right)$.

So if $b\left(h_A\right) > b\left(h_B\right)$, either $p_A\left(h_A, s\right) = 0$ so that $V^d\left(s\right) = V_A\left(h_A, s\right) \leq V_B\left(h_B, s\right)$, or $p_A\left(h_A, s\right) = 1$ and $V_A\left(h_A, s\right) < V_B\left(h_B, s\right)$. The lemma is proved. □

With lemma 17 in hand, the proof of proposition 4 requires only one additional step. We need to show that the value function is uniquely determined by b and s. If two governments start with the same levels of b and s, either one of them can mimic the other but choose ϵ less debt in the next period; lemma 17 implies that from this point forward, the mimicking government is weakly better off. The utility loss from paying down ϵ debt in the initial period can be made arbitrarily small by choosing arbitrarily small ϵ, and hence the mimicking government's value must be weakly higher. Since this argument works in both directions, we conclude that the value is indeed uniquely determined by b and s.

Proof of proposition 4. Consider $\left(h_A, h_B\right)$ such that $b\left(h_A\right) = b\left(h_B\right) = b$. At $\left(h_B, s\right)$ consider the feasible choice

$$\widetilde{b}'_B = b'_A\left(h_A, s\right) - \epsilon$$

for some $\epsilon > 0$. Define continuation histories

$$
\begin{aligned}
h'_A &= \left(h_A, s, b'_A\left(h_A, s\right), \left\{q^i_A\right\}\right) \\
\widetilde{h}'_B &= \left(h_B, s, \widetilde{b}'_B, \left\{\widetilde{q}^i_B\right\}\right)
\end{aligned}
$$

From Lemma 17,

$$
V_B\left(\widetilde{h}'_B, s'\right) \geq V_A\left(h'_A, s'\right) \tag{A.13}
$$

and B repays if A repays. Hence $\widetilde{q}_B \geq q_A$ by the pricing condition (A.7). Moreover,

$$
\begin{aligned}
\widetilde{c}_B &= c_A + \widetilde{q}_B \widetilde{b}'_B - q_A b'_A \\
&= c_A + \left(\widetilde{q}_B - q_A\right) b'_A + \widetilde{q}_B\left(\widetilde{b}'_B - b'_A\right) \\
&\geq c_A - \frac{1}{R}\epsilon
\end{aligned} \tag{A.14}
$$

where the inequality follows as in (A.12) in the proof of Lemma 17.

Now,

$$
\begin{aligned}
V_B\left(h_B, s\right) - V_A\left(h_A, s\right) &\geq u\left(\widetilde{c}_B\right) - u\left(c_A\right) + \beta \sum_{s'}\left(V_B\left(\widetilde{h}'_B, s'\right) - V_A\left(h'_A, s'\right)\right) \\
&\geq u\left(c_A - \frac{1}{R}\epsilon\right) - u\left(c_A\right)
\end{aligned}
$$

where inequality follows form (A.13) and (A.14). Taking the limit as $\epsilon \to 0$ and using continuity of u, we obtain $V_B\left(h_B, s\right) \geq V_A\left(h_A, s\right)$. The symmetric argument implies that $V_B\left(h_B, s\right) \leq V_A\left(h_A, s\right)$, which concludes the proof. □

A.3 Proof of proposition 7 (Bulow-Rogoff)

Proof. We first verify that when $V^d(s) = V^{nb}(0, s)$, there exists an equilibrium where the government will default for any positive amount of debt $b > 0$. This equilibrium is (V^{nb}, Q^{nb}), where V^{nb} is given in (4) and the government faces

$$
Q^{nb}(b', s) = \begin{cases} \frac{b'}{R} & b' \leq 0 \\ 0 & b' \geq 0 \end{cases} \tag{A.15}
$$

which is the revenue schedule induced by default thresholds identically equal to zero.

First, Q^{nb} generates V^{nb}. The budget constraint in (4) is effectively the same as the constraint in (1) given prices (A.15); although (4) does not allow $b' > 0$ while (1) does, positive borrowing $b' > 0$ will never be optimal given prices (A.15) because it raises no revenue. Moreover, proposition 1 shows that the value function generated by the prices in (A.15) is decreasing in b; hence whenever $b \leq 0$, we have $V^{nb}\left(b, s\right) \geq V^{nb}\left(0, s\right) = V^d\left(s\right)$ for all s, so that default is never optimal.

Second, the default thresholds corresponding to V^{nb} are identically equal to zero, thereby generating Q^{nb}. This also follows from the monotonicity of V^{nb} in b (proposition 1): since

$$V^{nb}(b,s) \geq V^d(s) = V^{nb}(0,s) \iff b \leq 0 \quad \forall s$$

we have $b^*(s) = 0$ for all s.

Proposition 3 then implies that (V^{nb}, Q^{nb}) must be the unique Markov perfect equilibrium, and hence that there is no distinct equilibrium in which debt can be sustained. In particular, there is no equilibrium where the expectation of being able to borrow in the future is enough to discourage default and sustain some positive debt. This is the incomplete markets version of the Bulow and Rogoff (1989) result. □

A.4 Proof of lemma 8

Proof. This equilibrium can be verified almost immediately. If $b^*(s) = 0$ for all s, then $Q(b', s) = 0$ for all s and feasible b' by (7). A decision rule $b'(b,s) \equiv 0$, where b' is identically equal to zero, is weakly optimal, since higher b' offers no revenue and lower b' is not feasible. It follows that $V(0,s) = u(y(s),s) + \beta\mathbb{E}[V(0,s')|s']$, and hence that $V(0,s)$ equals $V^{aut}(s) = V^d(s)$. For $b > 0$, then,

$$V(b,s) = u(y(s)-b,s) + \beta\mathbb{E}[V^{aut}(s')|s'] < u(y(s),s) + \beta\mathbb{E}[V^{aut}(s')|s'] = V^{aut}(s) = V^d(s)$$

and it is indeed optimal to default whenever $b > 0$, verifying the equilibrium with debt thresholds $b^*(s) = 0$. □

A.5 Proof of proposition 9

Proof. Suppose first that the government can borrow at the risk-free rate R. Assuming that the lower bound $\underline{b} = 0$ on b is not binding, then since $\beta R = 1$ it is optimal to consume at a constant level c in all periods.[14]

If the government faces $b = 0$ and $s = s_L$, then this level of consumption is

$$c = \frac{y(s_L) + \beta y(s_H)}{1+\beta}$$

and to achieve it the government must borrow $b'(0,s_L) = R(c - y(s_L)) = \frac{y(s_H)-y(s_L)}{1+\beta}$. Then, when the government enters the next period with this debt and $s = s_H$, its endowment $y(s_H)$ is exactly enough to repay the debt and consume c: $y(s_H) = c + b'(0,s_L)$. It follows that the optimal plan

[14]Otherwise, if $c_t < c_{t+1}$ for any t, then perturbing consumption to $(c_t+\epsilon, c_{t+1} - R\epsilon)$ provides higher utility

$$v(c_t+\epsilon) + \beta v(c_{t+1} - R\epsilon) > v(c_t) + \beta v(c_{t+1})$$

due to strict concavity of v and $\beta R = 1$, and similarly for $c_t > c_{t+1}$.

involves alternating between $(b,s)=(0,s_L)$ and $(b,s)=\left(\frac{y(s_H)-y(s_L)}{1+\beta},s_H\right)$.

Let us compare the value of this plan to the value of autarky/default, starting at s_H. We obtain

$$\begin{aligned}
& \frac{v(c)}{1-\beta}-V^{aut}(s_H) \\
= \quad & v(c)+\beta v(c)+\ldots-v(y(s_H))-\beta v(y(s_L))-\ldots \\
= \quad & (v(c)-v(y(s_H)))+\beta\left(v(c)+\beta u(c)+\ldots-v(y(s_L))-\beta v(y(s_H))-\ldots\right) \\
= \quad & (v(c)-v(y(s_H)))+\frac{\beta}{1-\beta^2}\left(v(c)+\beta v(c)-v(y(s_L))-\beta v(y(s_H))\right) \qquad \text{(A.16)}
\end{aligned}$$

In (A.16), the first term in parentheses is strictly negative, while the second term in parentheses is strictly positive by strict concavity of v. Since $\beta/(1-\beta^2)\to\infty$ as $\beta\to 1$, however, for β sufficiently close to 1 the second term will always dominate the first, and $v(c)/(1-\beta)$ will exceed $V^{aut}(s_H)$. In this case, it is strictly suboptimal for the government to default at $(b,s)=\left(\frac{y(s_H)-y(s_L)}{1+\beta},s_H\right)$. It also follows that $v(c)/(1-\beta)>V^{aut}(s_H)>V^{aut}(s_L)$, and therefore that it is also strictly suboptimal for the government to default at $(b,s)=(0,s_L)$.

We have shown that if the government can borrow at the risk-free rate R up to $\frac{y(s_H)-y(s_L)}{1+\beta}$ when in the low state, its optimal plan starting from either $(b,s)=(0,s_L)$ or $(b,s)=\left(\frac{y(s_H)-y(s_L)}{1+\beta},s_H\right)$ is to alternate between the two. In this case, it is strictly suboptimal to default at either point, and therefore creditors will indeed lend at rate R up to an amount strictly greater than $\frac{y(s_H)-y(s_L)}{1+\beta}$ in the low state.

To finish constructing the equilibrium, we need to find the thresholds $b^*(s_L)$ and $b^*(s_H)$ associated with that equilibrium. For this, we rely on the machinery developed in the existence proof in A.1.3. In terms of the $\mathbb{T}$ operator defined in that proof, we have already shown that if $b^{*0}=(b^{*0}(s_L),b^{*0}(s_H))\equiv\left(0,\frac{y(s_H)-y(s_L)}{1+\beta}\right)$, then $\mathbb{T}b^{*0}>b^{*0}$. Iterating forward as in A.1.3 to obtain $b^{*\infty}=\mathbb{T}^\infty b^{*0}$, it follows from monotonicity of $\mathbb{T}$ that $b^{*\infty}>b^{*0}\geq 0$, with these thresholds being part of a Markov perfect equilibrium. This finishes our construction. □

A.6 Proof of lemma 10

Proof. First, note that for any x and s, we have

$$\begin{aligned}
\widetilde{Q}(\lambda x+\underline{b},s) &= \frac{(\lambda x+\underline{b})}{R}\sum_{\{s':\lambda x\leq\tilde{b}^*(s')-\underline{b}\}}\pi(s'|s) \\
&\geq (1-\lambda)\underline{b}+\frac{\lambda(x+\underline{b})}{R}\sum_{\{s':x\leq b^*(s')-\underline{b}\}}\pi(s'|s)=\lambda Q(x+\underline{b},s)+(1-\lambda)\underline{b} \qquad \text{(A.17)}
\end{aligned}$$

where there is strict inequality if $\underline{b}<0$.

Now we can formally define the *mimicking at a distance* policy. Suppose that at time 0 we have state s and debt level b. The equilibrium (V,Q) induces an allocation $\left\{c\left(s^t\right),b\left(s^{t-1}\right),p\left(s^t\right)\right\}_{s^t\succeq s^0}$ at all histories following s^0. We construct a policy for the government in the equilibrium $\left(\widetilde{V},\widetilde{Q}\right)$

starting at s^0 as follows. For every history $s^t \succeq s^0$, let $\widetilde{p}\left(s^t\right) = p\left(s^t\right)$, and whenever $p(s^t) = 1$ define a plan for debt

$$\widetilde{b}\left(s^{t-1}\right) - \underline{b} \quad = \quad \lambda(b\left(s^{t-1}\right) - \underline{b})$$

The resulting consumption path, again for $p(s^t) = 1$, satisfies

$$\begin{aligned}
\widetilde{c}(s^t) &= y(s_t) - \widetilde{b}(s^{t-1}) + \widetilde{Q}(\widetilde{b}(s^t), s_t) \\
&= y(s_t) - \lambda b(s^{t-1}) - (1-\lambda)\underline{b} + Q(\lambda(b(s^t) - \underline{b}) + \underline{b}, s_t) \\
&\geq y(s_t) - \lambda b(s^{t-1}) - (1-\lambda)\underline{b} + \lambda Q(b(s^t), s_t) + (1-\lambda)\underline{b} \\
&= (1-\lambda)y(s_t) + \lambda(y(s_t) - b(s^{t-1}) + Q(b(s^t), s_t)) \\
&= (1-\lambda)y(s_t) + \lambda c(s^t)
\end{aligned}$$

Using the concavity of u, whenever $p(s^t) = 1$ we have

$$u(\widetilde{c}(s^t), s_t) \geq (1-\lambda)u(y(s_t), s_t) + \lambda u(c(s^t), s_t) \geq (1-\lambda)v^d(s_t) + \lambda u(c(s^t), s_t) \tag{A.18}$$

where the strict inequality from (A.17) persists in (A.18) whenever $\underline{b} < 0$, and by assumption, $u(y(s_t), s_t) > v^d(s_t)$ gives strict inequality whenever $\underline{b} = 0$. Summing (A.18) across all times and states where $p(s^t) = 1$, together with $v^d(s_t) = v^d(s_t)$ across all times and states where $p(s^t) = 0$, we obtain the result

$$\tilde{V}(\tilde{b}, s) > (1-\lambda)V^d(s) + \lambda V(b, s)$$

□

A.7 Proof of proposition 12

Proof. Suppose to the contrary that there exist two distinct equilibria (V, Q) and $\left(\widetilde{V}, \widetilde{Q}\right)$, with associated default thresholds $\{b^*(s)\}_{s\in\mathcal{S}}$ and $\left\{\widetilde{b}^*(s)\right\}_{s\in\mathcal{S}}$, such that $Q(b,s) \geq \widetilde{Q}(b,s)$ for all b and s. It follows that $V(b,s) \geq \widetilde{V}(b,s)$ for all b and s as well, since a government facing the weakly higher revenue schedule Q can always replicate the policy of the government facing $\widetilde{Q}$, achieving weakly higher consumption in the process.[15]

Since Q and $\widetilde{Q}$ are distinct, there exists some s' such that $b^*(s') > \widetilde{b}^*(s')$, and we define

$$M = \max_s b^*(s) - \widetilde{b}^*(s) > 0 \tag{A.19}$$

We first seek to prove that, for any s and $b \leq b^*(s)$

$$\widetilde{V}(b - M, s) - \widetilde{V}^d(s) > V(b, s) - V^d(s) \tag{A.20}$$

[15]Explicitly, it can set $b = \widetilde{b}$, $p = \widetilde{p}$, $c = \widetilde{c} + Q(b,s) - \widetilde{Q}(b,s) \geq 0$.

To do so, we use the same mimicking at a distance argument as in lemma 2, although the calculation becomes somewhat more complicated. Writing $s^0 \equiv s$, we continue to set $\widetilde{b}(s^t) = b(s^t) - M$ and $\widetilde{p}(s^t) = p(s^t)$, along with the consumption policy $\widetilde{c}(s^t)$ in (11). This strategy places a lower bound on $\widetilde{V}(b-M,s)$:

$$\widetilde{V}(b-M,s) \geq \sum_{p(s^t)=1} \beta^t \Pi\left(s^t\right) u\left(\widetilde{c}\left(s^t\right)\right) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi\left(s^t\right) \widetilde{V}^d(s_t) \tag{A.21}$$

Subtracting the corresponding expression for $V(b,s)$, and using $\widetilde{c}(s^t) > c(s^t)$, we have

$$\widetilde{V}(b-M,s) - V(b,s) > \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t)\left(\widetilde{V}^d(s_t) - V^d(s_t)\right) \tag{A.22}$$

Subtracting $\widetilde{V}^d(s) - V^d(s)$ from both sides we obtain

$$\begin{aligned}\left(\widetilde{V}(b-M,s) - \widetilde{V}^d(s)\right) &- \left(V(b,s) - V^d(s)\right) \\ &> -(\widetilde{V}^d(s) - V^d(s)) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t)\left(\widetilde{V}^d(s_t) - V^d(s_t)\right)\end{aligned} \tag{A.23}$$

and to prove (A.20) it suffices to show that the right side of (A.23) is nonnegative.

Expanding $\widetilde{V}^d(s_t) - V^d(s_t)$ gives

$$\widetilde{V}^d(s_t) - V^d(s_t) = \sum_{s_\tau} \beta^{\tau-t}(1-\lambda)\lambda^{\tau-t-1}\Pi(s_\tau|s_t)\left(\widetilde{V}^o(0,s_\tau) - V^o(0,s_\tau)\right) \tag{A.24}$$

Now, using (A.24), we can rewrite the right side of (A.22) as

$$\begin{aligned}&-\sum_{s^\tau \succ s^0} \beta^\tau(1-\lambda)\lambda^{\tau-1}\Pi(s^\tau)\left(\widetilde{V}^o(0,s_\tau) - V^o(0,s_\tau)\right) \\ &\qquad + \sum_{p(s^t)=0, p(s^{t-1})=1} \sum_{s^\tau \succ s^t} \beta^\tau(1-\lambda)\lambda^{\tau-t-1}\Pi(s^\tau)\left(\widetilde{V}^o(0,s_\tau) - V^o(0,s_\tau)\right)\end{aligned}$$

which can be rearranged as

$$\lambda^{-1}(1-\lambda)\sum_{s^\tau \succ s^0} \beta^\tau \Pi(s^\tau)\left(V^o(0,s_\tau) - \widetilde{V}^o(0,s_\tau)\right) \cdot \left(1 - \sum_{s^\tau \succ s^t \succ s^0} \lambda^{\tau-t} \cdot \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}}\right) \tag{A.25}$$

Since for any s^τ there exists at most one s^t such that $p(s^t) = 0$ and $p(s^{t-1}) = 1$, the rightmost factor in parentheses is nonnegative. Since in addition $V(0,s_\tau) \geq \widetilde{V}(0,s_\tau)$, the preceding factor is nonnegative as well, and hence (A.25) is nonnegative. (A.20) therefore follows.

Finally, suppose that the maximum in (A.19) is attained at $\bar{s}$, so that $b^*(\bar{s}) = \widetilde{b}^*(\bar{s}) + M$. Applying (A.20), we have

$$0 = \widetilde{V}(\widetilde{b}^*(\bar{s}),\bar{s}) - \widetilde{V}^d(\bar{s}) > V(b^*(\bar{s}),\bar{s}) - V^d(\bar{s}) = 0$$

which is a contradiction. □

A.8 Proof of proposition 13

Proof. Write $V^{re} = \mathbb{E}_{s'}[V^o(0, s')]$, and similarly $\widetilde{V^{re}}$ for a conjectured alternative equilibrium. First, observe that if $\widetilde{V}^{re} = V^{re}$, then the two equilibria have the same expected value from default V^d, and we can apply proposition 3 taking V^d as given to conclude that the two equilibria must be the same.

Otherwise, assume without loss of generality that $V^{re} > \widetilde{V}^{re}$. It cannot be that $\widetilde{Q}(b') \geq Q(b')$ for all b', since in that case a government starting with zero debt and facing the weakly higher debt schedule $\widetilde{Q}$ could always replicate the policy of the government facing Q, achieving weakly higher consumption in the process. This would imply $\widetilde{V}^{re} \geq V^{re}$, a contradiction. Hence $Q(b') > \widetilde{Q}(b')$ for some b'. From this point on, the proof is the same as the proof for proposition 12 starting with the definition of M in (A.19), except that we can replace (A.24) with simply

$$\widetilde{V}^d(s_t) - V^d(s_t) = \sum_{\tau > t} \beta^{\tau - t}(1-\lambda)\lambda^{\tau - t - 1}\left(\widetilde{V}^{re} - V^{re}\right) \tag{A.26}$$

allowing us to replace (A.25) with

$$\lambda^{-1}(1-\lambda)\sum_{\tau > 0}\beta^{\tau}(V^{re} - \widetilde{V}^{re}) \cdot \left(1 - \sum_{\tau > t > 0}\lambda^{\tau - t} \cdot \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}}\right) \tag{A.27}$$

again concluding that the expression is nonnegative, from which a contradiction follows. □