Space is a Soup

Region-based theories of space in Point-Free Topology and Mereotopology

Matteo Celli, Marco De Mayda, Timo Franssen

ILLC, University of Amsterdam

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Overview

Frames

Topological spaces
(X, τ)
X is a set
$\tau \subseteq \mathcal{P}(X)$ s.t.:

- Contains X and $\emptyset$
- Closed under arbitrary unions
- Closed under finite intersections

Frames
Let $\Omega(X) = (\tau, \cap^\circ, \cup)$
Then $\Omega(X)$ forms a complete lattice.
In addition it satisfies the general distributive law:

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$$

Frames and locales

A map $h : L \to M$ is a frame morphism if:

- $h(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} h(a_i)$
- $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$

$f : X \to Y$ continuous map
f^{-1} maps opens to opens
$f^{-1} : \Omega(Y) \to \Omega(X)$ turns out to be satisfy same conditions

Locales = same objects, but morphisms in other direction (categorically speaking $Loc = Frm^{op}$)

Locales $\leftrightarrow$ Topological spaces

If there is (X, τ) s.t. $L \cong \Omega(X)$, we call L *spatial*.

Not every locale corresponds to a toplogical space

We can prove that every spatial locale corresponds to a sober space
(we can prove categorical equivalence)

Sober spaces

$C \subseteq X$ a closed subset is *join-irreducible* if:
$C = C_1 \cup C_2$ (with C_1, C_2 closed) implies $C = C_1$ or $C = C_2$

We call (X, τ) sober if the only non-empty join-irreducible closed sets are the closure of singletons

Many nice spaces are sober:
all Hausdorff spaces are sober

We can study sober spaces by studying locales and thus focusing on open sets ("regions") and their connectivity instead of points

Example

Top

$X = \{0, 1\}$

$\tau = \{\emptyset, \{1\}, \{0, 1\}\}$

Frame/locale

The lattice of open subsets looks like:

$$
\begin{array}{c}
\{0,1\} \\
\uparrow \\
\{1\} \\
\uparrow \\
\emptyset
\end{array}
$$

Prime filter

If a frame/locale is spatial, we can recover the points by looking at the prime filter:

- $F_1 = \{\{1\}, \{0, 1\}\}$
- $F_0 = \{\{0, 1\}\}$

Take-away: open sets come first, points second

Region Based Topology (Roeper 1997)

Regions form the foundational elements of this approach. Unlike traditional topology, we begin with regions as primitive entities. These regions:

- Can be of any dimension
- May consist of disconnected parts
- Form a Boolean algebra representing mereological structure
- Serve as the primary bearers of spatial properties

Basic definitions

The space Ω is modeled as a non-degenerate Boolean algebra. Then we get nice spatial reasoning from the operators:

- maximal $\mathbf{1}$: the entire region
- $\alpha \leq \beta$ denotes that α is a subregion of β.
- the join $\alpha \vee \beta$: smallest region having α, β as parts
- the meet $\alpha \wedge \beta$: the largest region common to bothα, β
- the complement: all of the space except α

Basic definitions

Note that:

- Normally we wouldn't want a null region mereologically, but for simplicity of definitions, we add it.
- Furthermore, we assume that Ω is complete, so that arbitrary meets and joins exist.

Connectedness

Connection (∞). Philosophically captures our intuitive spatial notion of spatial contact between regions.
In particular, it occurs when regions:

- Overlap
- Touch at boundaries
- Are infinitesimally close

If $\alpha \infty \beta$, then they either overlap or there is a point coincident with both.

Notions from Connectedness

With the notion of connectedness, we can further define:

- Component: $\alpha \ll \beta$ iff $\alpha \not\infty -\beta$
- Coherent: for α iff for all (non-null) regions β, γ, if $\beta \vee \gamma = \alpha$ then $\beta \infty \gamma$
- Convex: for α: iff for all (non-null) regions β, γ, if $\beta \vee \gamma = \alpha$ then there is a region α' such that $\alpha' \ll \alpha$ and: $(\alpha' \wedge \beta) \infty (\alpha' \wedge \gamma)$

Intuitively

- Component: Tells us (circa) that α is an interior part of β, or completely included in it. Sharing no boundary points.
- Coherent: tells us that the region is "navigable" without exiting it. That it does not consist of disconnected regions.
- Convex: tells us that the region is "navigable" without crossing any boundary at all.

Connection Axioms

(A1) Symmetry: $\alpha \propto \beta \Rightarrow \beta \propto \alpha$.

(A2) Reflexivity: $\alpha \neq 0 \Rightarrow \alpha \propto \alpha$.

(A3) Disjointness: $0 \not\propto \alpha$.

(A4) Monotonicity: $\alpha \propto \beta$ and $\beta \leq \gamma \Rightarrow \alpha \propto \gamma$.

(A5) Additivity: $\alpha \propto (\beta \vee \gamma) \Rightarrow \alpha \propto \beta$ or $\alpha \propto \gamma$.

Limitedness

(Δ) **Limitedness** Another primitive we add, is a region enclosed in all directions by a boundary.

- Represents bounded regions
- Capture the notion of finite spatial entities
- Plays a role in recovering points

Not Enoguh

There is no obvious way to define limitedness in terms of connection with the axioms given thus far. Instead of adding axioms for connectedness that would allow so, we keep Limitedness as another primitive and axiomatize it directly.

Limitedness Axioms

(A6) 0 is limited.

(A7) If α is limited and $\beta \leq \alpha$, then β is limited.

(A8) The join of two limited regions is limited.

Mixed Axioms

We further want some axioms constraining the interplay between connection and limitedness:

(A9) If $\alpha \infty \beta$, then there exists a limited subregion $\beta' \leq \beta$ such that $\alpha \infty \beta'$.

(A10) If α is limited and $\alpha \ll \beta$ (α is an interior part of β), then there exists a limited γ such that $\alpha \ll \gamma \ll \beta$.

The former, telling us that a connection between two regions constitutes a boundary.
The latter telling us (roughly) that space is infinitely divisible. For every region, there is a proper subregion.

Some Results

Lemma (1.1)

If $\alpha \wedge \beta \neq 0$ *then* $\alpha \infty \beta$

Lemma (1.2)

If $\alpha \ll \beta$ *then* $\alpha \leq \beta$

Lemma (1.3)

For any region α*:*

1. *If* $\alpha \neq 0$ *then there exists a limited region* $\beta \neq 0$ *with* $\beta \ll \alpha$
2. *If* $\alpha \neq 0$ *then there exists a limited region* $\beta \neq 0$ *with* $\beta \leq \alpha$
3. $\alpha = \bigvee \{\beta \mid \beta$ *is limited, and* $\beta \leq \alpha\}$ *(by* 2. *and* 0 *is limited).*

Some Results

Lemma (1.4)

If α is limited and $\alpha \not\infty \beta$ then there is a region γ such that $\alpha \not\infty -\gamma$ and $\gamma \not\infty \beta$

Lemma (1.5)

1. *if α is limited, then there is a limited γ with $\alpha \ll \gamma$*
2. *if α is limited, β unlimited and $\alpha \leq \beta$ then there is a limite γ such that $\alpha < \gamma < \beta$*
3. *if α is limited and 1 is unlimited then there is a limite γ such that $\alpha \ll \gamma$ and $\alpha < \gamma$.*

Lemma (1.8)

1 is coehrent iff for every region α $(0 \neq \alpha \neq 1,)$ $\alpha \infty -\alpha$

Filter Operations

Given a filter Φ on Ω, we say $\Phi \infty \alpha$ iff for every $\beta \in \Phi$, we have $\beta \infty \alpha$.

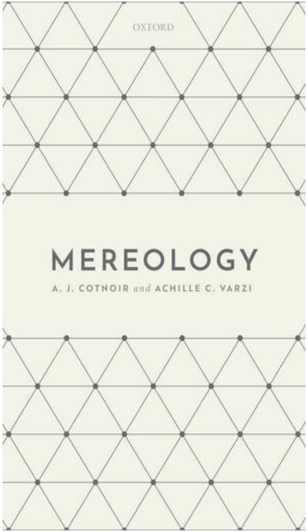
Given two filters Φ, Ψ on Ω we say $\Phi \infty \Psi$ if for each $\alpha \in \Phi, \beta \in \Psi$ we have $\alpha \infty \beta$.

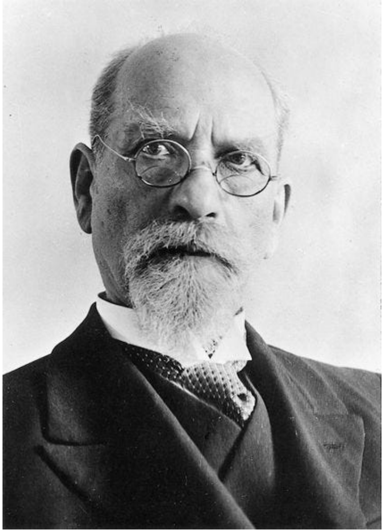
Recovering Points

Φ is limited if some $\alpha \in \Phi$ is limited.

Given a limited ultrafilter ∇ on Ω, then our points are $\{\nabla' \mid \nabla' \infty \nabla\}$, i.e. $[\nabla]_\infty$, the equivalence class under connection to ∇.

Mereology: a brief introduction (Cotnoir and Varzi 2021)





GEM - Axioms of General Extensional Mereology

P1, Reflexive:	Pxx
P2, Antisymmetric:	$(Pxy \wedge Pyx) \rightarrow x = y$
P3, Transitive:	$(Pxy \wedge Pyz) \rightarrow Pxz$
P4, Weak Supplementation:	$PPxy \rightarrow \exists z[Pzy \wedge \neg Ozx]$
P5, Strong Supplementation:	$\neg Pyx \rightarrow \exists z[Pzy \wedge \neg Ozx]$
P5', Atomistic Supplementation:	$\neg Pxy \rightarrow \exists z[Pzx \wedge \neg Ozy \wedge \neg \exists v[PPvz]]$
Top:	$\exists W \forall x[PxW]$
Bottom:	$\exists N \forall x[PNx]$
P6, Sum:	$Uxy \rightarrow \exists z \forall v[Ovz \leftrightarrow (Ovx \vee Ovy)]$
P7, Product:	$Oxy \rightarrow \exists z \forall v[Pvz \leftrightarrow (Pvx \wedge Pvy)]$
P8, Unrestricted Fusion:	$\exists x[\phi(x)] \rightarrow \exists z \forall y[Oyz \leftrightarrow \exists x[\phi(x) \wedge Oyx]]$

Mereology: Limitations (Casati and Varzi 1997)

Mereology provides a theory of how *parts* are related to one another, but cannot account for *wholes*. In particular, there is no mereological way to distinguish scattered wholes -like archipelagos - and unitary, cohesive ones, like a sphere. Whitehead's difficulties in his Mereology provide an example. Consider the following formula, where the binary predicates J, O, P mean, respectively, *Join, Overlap* and *Part* and x, y, z are parts:

$$J(x,y) = \exists z\big(O(z,x) \wedge O(z,y) \wedge \forall w(P(w,z) \rightarrow (O(w,x) \vee O(w,y)))\big)$$

The only way that this definition of "Join" rules out scattered wholes is to implicitly assume that the part z is already connected.

Adding Topology to the Picture (Varzi 1998)

The limitations of mereology call for an interplay with topology to develop an account of spatial reasoning that can formalise the concept of a *whole* with a satisfactory granularity. The preferred approach is to treat mereology and topology as independent, but mutually beneficial and complementary theories.
We introduce the Mereotopological axioms from the theory GEMTC of (Casati and Varzi 1999). Axioms labelled with "C" - for *connection* - are topological in nature, while those labelled with "P" - for *parthood* - are mereological.

GEMTC: Axiomatic Mereotopology (Casati and Varzi 1999) I

Connection Axioms

C1	Cxx (Reflexivity)
C2	$Cxy \rightarrow Cyx$ (Symmetry)

Let E be the binary relation of *Enclosure*, Exy, s.t. y encloses x iff $Czx \rightarrow Czy$ for some $z \in \Omega$. From the definition of enclosure, we can also define Extensionality:

$$(Exa \leftrightarrow Exb) \leftrightarrow a = b$$

GEMTC: Axiomatic Mereotopology (Casati and Varzi 1999) II

Making E transitive, reflexive and antisymmetric, and thus a partial order.

$$\text{C3} \qquad\qquad Pxy \rightarrow Exy$$

C3 is a nice bridge between the mereological concept of parthood and topological enclosure

GEMTC: Useful Definitions

Let the binary relation Oxy, as featured in **P6**, denote "x and y overlap" be defined as:

$$Oxy \leftrightarrow \exists z\,[Pzx \wedge Pzy].$$

Then, by $C3$ we get that $Oxy \rightarrow Cxy$.

Furthermore, Let $IPxy$ denote that "x is an internal part of y", such that the binary predicate IP is defined as:

$$IPxy \leftrightarrow (Pxy \wedge (Czx \rightarrow Ozy)).$$

GEMTC: Mereological and Topological operators

Let $\sigma(x)\,\varphi(x)$ denote the mereological sum (by fusion) of all individuals in the domain satisfying the property $\varphi(x)$.
By σ and the relation IP, we can define the interior of x, $\mathbf{i}x$, as the mereological sum of all interior parts z of x:

$$\mathbf{i}x := \sigma z[IPzx].$$

As a consequence, if W is taken to be the whole:

$$\mathbf{i}W = W.$$

GEMTC: Axioms of closure and self-connectedness

C5. $P(\mathbf{c}x)x$. (Inclusion)
C6. $\mathbf{c}(\mathbf{c}x) = \mathbf{c}x$. (Idempotence)
C7. $\mathbf{c}(x + y) = \mathbf{c}x + \mathbf{c}y$ (Additivity)

Furthermore, let x be *self-connected* if it satisfies the following unary predicate SCx:

$$SCx \leftrightarrow ((Owx \leftrightarrow (Owy \vee Owz)) \rightarrow Cyz).$$

Our theory can be further complemented with the axiom C8:

$$\text{C8} \qquad Cxy \rightarrow \exists z[SCz \wedge Ozx \wedge (Pwz \rightarrow (Owx \vee Owy))].$$

This axiom makes Whitehead's principle possible, as the right-to left direction is a theorem of GEMTC, while the left to right has counterexamples unless taken as an axiom (Casati and Varzi 1999).

Thank You!

Any Questions?

Bibliography I

Casati, Roberto, and Achille C. Varzi. 1997. “Spatial Entities.” In *Spatial and Temporal Reasoning,* edited by Oliviero Stock, 73–96. Kluwer Academic Publishers.

———. 1999. *Parts and Places: The Structures of Spatial Representation.* MIT Press.

Cotnoir, A. J., and Achille C. Varzi. 2021. *Mereology.* Oxford University Press.

Johnstone, Peter T. 1983. “The point of pointless topology.” *Bulletin of the American Mathematical Society* 8:41–53.

Picado, Jorge, and Aleš Pultr. 2012. *Frames and Locales: Topology without points.* Frontiers in Mathematics. Birkhäuser. https://doi.org/10.1007/978-3-0348-0154-6.

Roeper, Peter. 1997. “Region-Based Topology.” *Journal of Philosophical Logic* 26 (3): 251–309. https://doi.org/10.1023/a:1017904631349.

Bibliography II

van der Laan, Josje. 2024. "Constructivism and the Continuum." Bachelor's thesis, Radboud Universiteit Nijmegen.

Varzi, Achille C. 1998. "Basic Problems of Mereotopology." In *Formal Ontology in Information Systems,* edited by Nicola Guarino, 29–38. IOS Press.

Vickers, Steven. 1996. *Topology via Logic.* Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.