TOPOLOGY IN AND VIA LOGIC
HOMEWORK ASSIGNMENT 1

- Deadline: January 10 at 14:59.
- All exercises are worth the same points.
- Good luck!

Exercise 1. Consider the space $(\mathbb{R}, \tau_{Euc})$, with its Euclidean topology.

(1) Give an example of a set which is neither open nor closed.

(2) Show that the open intervals of the form (x, y) where $x, y \in \mathbb{Q}$ form a basis for this topology.

(3) Show that $\mathbb{Q}$ is a countable union of closed sets.

Exercise 2. Let X be a set. We say that an operation $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ is called an *interior operator* if it satisfies for each $U, V \in \mathcal{P}(X)$,

- (All set): $\Box X = X$;
- (Normality): $\Box(U \cap V) = \Box U \cap \Box V$;
- (Inflationarity): $\Box U \subseteq U$;
- (Idempotence): $\Box U \subseteq \Box\Box U$.

(1) Show that if (X, τ) is a topological space, the topological interior *int* is an interior operator in this sense.

(2) Given a set $(X, \Box)$ equipped with an interior operator, define a topology for which $\Box$ is the topological interior operator.

(3) We say that an interior operator $\Box$ is *completely multiplicative* if for each $(U_i)_{i \in I}$ we have that:

$$\Box(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} \Box U_i$$

Show that Alexandroff topologies are in 1-1 correspondence with completely multiplicative interior operators.

(4) Let $(X, \Box)$ be a set, equipped with a completely multiplicative interior operator, with the following property: if $x \neq y$, then there is some $U \subseteq X$ such that either $x \in \Box U$ and $y \notin \Box U$ or $y \in \Box U$ and $x \notin \Box U$. Show that then there is a poset $(P, \leq)$ such that the Alexandroff topology on P is the same as the topology induced on X by the interior operator.

Exercise 3. Let (X, τ) be a topological space. We say that a map $\nu : \tau \to \tau$ is a *nucleus* if it satisfies the following for all open subsets $U \subseteq X$:

(i) $U \subseteq \nu(U)$;

(ii) If $U \subseteq V$ then $\nu(U) \subseteq \nu(V)$;
(iii) $\nu(\nu(U)) = \nu(U)$;
(iv) $\nu(U \cap V) = \nu(U) \cap \nu(V)$.

(1) Show that if $K \subseteq X$ is any subset, then the map $\nu_K : \tau \to \tau$ given by setting

$$\nu_K(U) = int([X - K] \cup U)$$

for all opens U, is a nucleus, called the *induced nucleus of* K.

(2) (*) Note that the map:

$$j_{\neg\neg}(U) = int(cl(U))$$

is a nucleus as well. Show that there is a topological space (X, τ) such that $j_{\neg\neg}$ on this space is not the induced nucleus of any set $K \subseteq X$. *Hint: Consider the real line and show that the only K which could induce such a nucleus is the empty set, and that the empty set does not induce it.*