TUTORIAL 1

INTRODUCTION TO TOPOLOGY IN AND VIA LOGIC - 2025

Basic Set Theory

Exercise 1. *The following results are used often in topology: Let X, Y be sets, $f : X \to Y$ a function, $S \subseteq X$, $\{S_i : i \in I\} \subseteq \mathcal{P}(X)$, $T \subseteq Y$ and $\{T_j : j \in J\} \subseteq \mathcal{P}(Y)$. Then*

(1) $f[\bigcup_{i\in I} S_i] = \bigcup_{i\in I} f[S_i]$.

(2) $f[\bigcap_{i\in I} S_i] \subseteq \bigcap_{i\in I} f[S_i]$.

(3) $f^{-1}[\bigcup_{j\in J} T_j] = \bigcup_{j\in J} f^{-1}[T_j]$.

(4) $f^{-1}[\bigcap_{j\in J} T_j] = \bigcap_{j\in J} f^{-1}[T_j]$.

(5) $f[S] \cap T = f[S \cap f^{-1}[T]]$.

Furthermore, if $f[\bigcap_{i\in I} S_i] = \bigcap_{i\in I} f[S_i]$ if f is injective. Prove them.

Basic Topology

Exercise 2. *Recall that the* Euclidean topology τ_{Euc} *on $\mathbb{R}$ is defined as follows:*

$$\text{for all } U \subseteq \mathbb{R},\ U \in \tau_{Euc} \text{ if and only if } \forall z \in U \exists x, y \in U (z \in (x, y) \subseteq U).$$

Verify that $(\mathbb{R}, \tau_{Euc})$ is a topological space.

Exercise 3. *Recall that the* Cantor set *is defined to be the set 2^ω of all binary sequences of length ω. Let $2^{<\omega}$ denote the set of all finite binary sequences. For all $s \in 2^{<\omega}$ and $t \in 2^\omega \cup 2^{<\omega}$, we write $s \lhd t$ if $t{\restriction}\mathsf{dom}(s) = s$. Intuitively, $s \lhd t$ means that s is an initial subsequence of t. For each $s \in 2^{<\omega}$, we define the set $C(s)$ by*

$$C(s) = \{t \in 2^\omega : s \lhd t\}.$$

Let $B = \{C(s) : s \in 2^{<\omega}\}$. Verify that the following statements hold:

(1) *B covers 2^ω, i.e., $\bigcup B = 2^\omega$.*

(2) *B is closed under finite intersections.*

By Proposition 2.2.5 in the lecture note, there is a unique topology τ_{Can} on the Cantor set for which B is a basis. The topological space $(2^\omega, \tau_{Can})$ is called the Cantor space.

Closure, Interior and Neighbourhoods

Definition 1. *Let (X, τ) be a topological space. We say that a set $U \in \mathcal{P}(X)$ is closed if its complement is open, i.e., if $(X \setminus U) \in \tau$.*

Exercise 4. *Let (X, τ) be a topological space and $S \subseteq X$. Show that the following hold:*

(1) *There exists an open set $int(S)$ such that (i) $int(S) \subseteq S$; and (ii) for all open set U, $U \subseteq S$ implies $U \subseteq int(S)$.*

(2) There exists an closed set $cl(S)$ such that (i) $S \subseteq cl(S)$; and (ii) for all closed set U, $S \subseteq U$ implies $cl(S) \subseteq U$.

Definition 2. *The sets $int(S)$ and $cl(S)$ in Exercise 4 are called the* interior *and the* closure *of S, respectively. Moreover, we see that a set S is closed if $S = cl(S)$, and open if $S = int(S)$. The operators*

$$int : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto int(S)$$

and

$$cl : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto cl(S)$$

are called the topological interior *and* topological closure*, respectively.*

Exercise 5. *Let (X, τ) be a topological space and $A, B \subseteq X$. Prove the following statements:*

(1) $A \subseteq cl(A)$ and $int(A) \subseteq A$.

(2) $cl(cl(A)) = cl(A)$ and $int(int(A)) = int(A)$.

(3) $cl(A) = X \setminus (int(X \setminus A))$ and $int(A) = X \setminus (cl(X \setminus A))$.

(4) $cl(A) \cup cl(B) = cl(A \cup B)$ and $int(A) \cap int(B) = int(A \cap B)$.

(5) If $A \subseteq B$, then $cl(A) \subseteq cl(B)$ and $int(A) \subseteq int(B)$.

Is $cl(A) \cap cl(B) = cl(A \cap B)$ or $int(A) \cup int(B) = int(A \cup B)$ true in general? Prove your answer.

Definition 3. *Given a topological space (X, τ) and a point $x \in X$, we say that $V \in \mathcal{P}(X)$ is a* neighbourhood of x *if there is an open set U such that $x \in U \subseteq V$.*

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[1]

Let $N(x)$ denote the set of all open neighbourhoods of x, i.e., $N(x) = \{U \in \tau : x \in U\}$.

Exercise 6. *Suppose (X, τ) is a topological space and $S \subseteq X$. Then for all $x \in X$, the following are equivalent:*

- *x is in the closure of S, i.e., $x \in cl(S)$.*
- *All open neighbourhoods U of x have non-empty intersection with S, i.e.,*

$$\forall U \in N(x)(U \cap S \neq \varnothing).$$

There is a proof of this proposition in the note, but try to prove it yourself first :)

Topological Semantics of Modal Logic

McKinsey and Tarski [1] proposed interpretations of the unary modal operator $\Diamond$ as the closure operator cl and the derived set operator d of a topological space. These interpretations give the topological C-semantics and d-semantics for modal logic.

In this section, we take a closer look at the topological C-semantics of modal logic.

Definition 4. *A topological model is a triple $\mathfrak{M} = (X, \tau, \nu)$ where (X, τ) is a topological space and $\nu : Var \to \mathcal{P}(X)$ a function called a valuation for X.*

A valuation ν is extended to the set Fm of all modal formulas by the following rules:

$$\nu(\bot) = \varnothing,\ \nu(\varphi \to \psi) = (X \setminus \nu(\varphi)) \cup \nu(\psi) \text{ and } \nu(\Diamond\varphi) = cl(\nu(\varphi)).$$

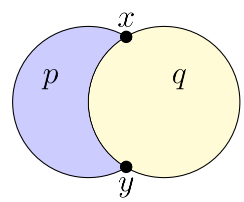
[1] In the literature, you will sometimes find that a neighbourhood is already required to be open. We do not adopt that convention, but simply speak of 'open neighbourhoods' when needed.

A formula φ is true at x in $\mathfrak{M}$, notation $\mathfrak{M}, x \models \varphi$, if $x \in \nu(\varphi)$. Note that by definition of the operator $cl : \mathcal{P}(X) \to \mathcal{P}(X)$, the following statements hold:

(a) $\mathfrak{M}, x \models \Box\varphi$ if and only if there is $U \in \tau$ such that $x \in U$ and $\mathfrak{M}, y \models \varphi$ for all $y \in U$;
(b) $\mathfrak{M}, x \models \Diamond\varphi$ if and only if for all $U \in \tau$, $x \in U$ implies $\mathfrak{M}, y \not\models \varphi$ for some $y \in U$.

Exercise 7. *Show that (a) and (b) in Definition 4 hold.*

Exercise 8. *Consider the following topological model, where p is true in the blue area and its border line, and q is true in the yellow area and its border line.*



(1) Draw the region defined by the modal formula $\Diamond p \vee \Box q$.
(2) Find modal formulas that define the set $\{x, y\}$.

Recall that modal logic $\mathsf{S4}$ is defined to be $\mathsf{K} \oplus \{\Diamond\Diamond p \to \Diamond p, p \to \Diamond p\}$.

Exercise 9. *Prove that every theorem of $\mathsf{S4}$ is valid. That is, given any $\varphi \in \mathsf{S4}$, $\mathfrak{M}, x \models \varphi$ for all topological model $\mathfrak{M} = (X, \tau, \nu)$ and $x \in X$.*

In fact, the converse of Exercise 9 also holds. Thus we have

Theorem 5. *$\mathsf{S4}$ is the modal logic of all topological spaces.*

Exercise 10. *Let (X, τ) be a topological space and $A \subseteq X$. We say that A is regular open if $int(cl(A)) = A$. Prove the following statements:*

(1) if A is open, then $A \subseteq int(cl(A))$.
(2) $int(cl(A))$ is regular open.

More on topological semantics: d-semantics

There are more than one semantics of modal logic based on topological spaces. The one we have seen in class is also called C-semantics. We now introduce the d-semantics as follows:

Definition 6. *Let $\mathcal{X} = (X, \tau)$ be a topological space and $x \in X$. A subset $Y \subseteq X$ is an* open neighborhood *of x if $x \in Y \in \tau$. Let $N(x)$ be the set of all open neighborhoods of x. For every subset $A \subseteq X$, let $\mathsf{d}(A)$ be the* derived set *of A, i.e.,*

$$\mathsf{d}(A) = \{x \in X : \forall U \in N(x)(U \cap (A \setminus \{x\}) \neq \varnothing)\}.$$

A topological model *is a triple $\mathcal{M} = (X, \tau, \nu)$ where $\mathcal{X} = (X, \tau)$ is a topological space and $\nu : \mathsf{Prop} \to \mathcal{P}(X)$ is a function which is called a* valuation *in $\mathcal{X}$. A valuation ν is extended to all modal formulas $\mathcal{L}$ as follows:*

$$\nu(\neg\varphi) = X \setminus \nu(\varphi),\ \nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi) \text{ and } \nu(\Diamond\varphi) = \mathsf{d}(\nu(\varphi)).$$

For each formula φ, φ is d-true *at w in $\mathcal{M}$ (notation: $\mathcal{M}, w \models_{\mathsf{d}} \varphi$) if $w \in \nu(\varphi)$. We say that φ is d-valid if $\mathcal{M}, w \models_{\mathsf{d}} \varphi$ for all topological model $\mathcal{M}$ and point w in $\mathcal{M}$.*

Exercise 11. *Try to understand the d-semantics given above and show:*

(1) $\Diamond\Diamond p \to \Diamond p$ *is not* d-*valid.*
(2) $\Diamond\Diamond p \to \Diamond p \vee p$ *is* d-*valid.*

References

[1] McKinsey J. C. C., Tarski A., The algebra of topology, Annals of Mathematics **45**, 141–191 (1944)