Geometry of transfer matrix

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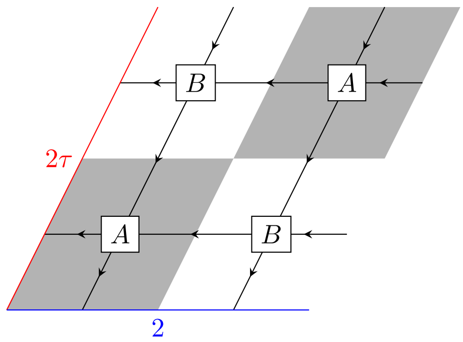
1 Transfer matrix with size $\sqrt{2}\times 2\sqrt{2}$ and normalization-independent calculation of the central charge

Based on the intuition that a fixed-point tensor represents a partition function on a block of system:



we expect that

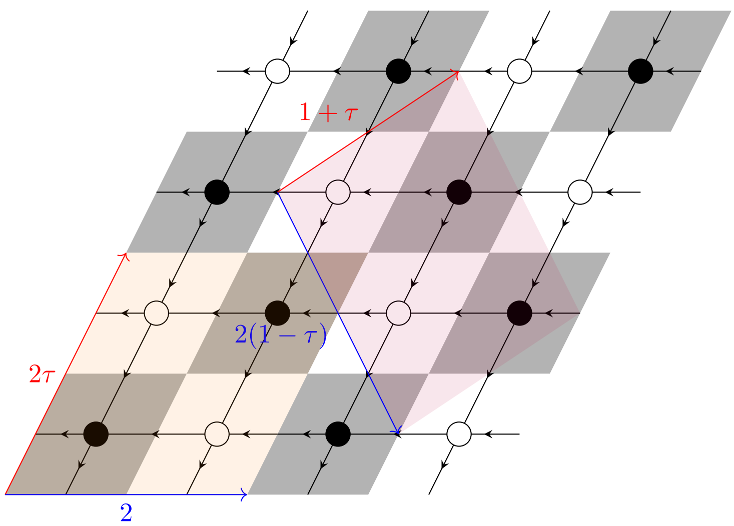
$$\lambda_{ij} = \exp\left[2\pi\mathrm{i}\tau\left(h_i - \frac{c}{24}\right) - 2\pi\mathrm{i}\bar{\tau}\left(h_j - \frac{c}{24}\right) + fA\right].$$

Here A is the area of the cell and f is proportional to the free-energy density.

There exists a canonical normalization factor corresponds to the case when $i = j = 0$:

$$\lambda_{00} = \exp\left[2\pi\mathrm{i}(\bar{\tau} - \tau)\frac{c}{24} + fA\right].$$

Now we choose another cell that has the same area as the first one [1, 5]



The new modular parameter of this cell is

$$\tau' = \frac{1+\tau}{2(1-\tau)}.$$

And the area of the new cell is the same as the previous one. Let λ'_{ij} be the eigenvalue of the new transfer matrix, then

$$\frac{\lambda'_{ij}}{\lambda_{00}} = \exp\left[2\pi\mathrm{i}\tau'\left(h_i - \frac{c}{24}\right) - 2\pi\mathrm{i}\bar{\tau}'\left(h_j - \frac{c}{24}\right) - 2\pi\mathrm{i}(\bar{\tau} - \tau)\frac{c}{24}\right].$$

In particular, the central charge can be obtained from

$$\frac{\lambda'_{00}}{\lambda_{00}} = \exp\left[2\pi\mathrm{i}\left(\tau - \tau' - \bar{\tau} + \bar{\tau}'\right)\frac{c}{24}\right] = \exp\left[-\pi\Im(\tau - \tau')\frac{c}{6}\right].$$

And the conformal dimensions can be obtained from

$$\frac{\lambda'_{ij}}{\lambda'_{00}} = \exp\left[-2\pi\Im\tau'(h_i + h_j) + 2\pi\mathrm{i}\Re\tau'(h_i - h_j)\right].$$

When $\tau = \mathrm{i}$, we have $\tau' = \frac{\mathrm{i}}{2}$. Then the conformal data are calculated via

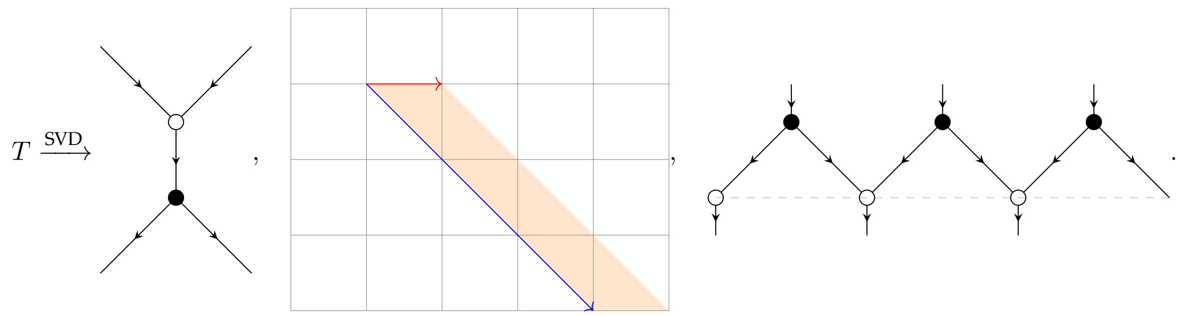
$$\frac{\lambda'_{00}}{\lambda_{00}} = \mathrm{e}^{-\frac{\pi c}{12}}, \quad \frac{\lambda'_{ij}}{\lambda'_{00}} = \mathrm{e}^{-\pi(h_i + h_j)}.$$

2 A method to compute the conformal spin

To compute the conformal spin, it is better to separate the calculation of the spin from the Loop-TNR part, as the non-homogeneous tensor can increase the truncation error during Loop-TNR [3, 1].

Moreover, since the conformal spin is determined by fixing the phase $\mathrm{e}^{2\pi\mathrm{i}\theta s}$, the smaller θ we choose, the more conformal spins can be determined unambiguously [2, 4]. We will choose $\theta = 1/6$, such that s can be determined from -3 to 3, as in [2, 4]. However, due to the fact that the Loop-TNR has two sublattices, the transfer matrix cannot be taken to be wide enough. We thus perform an extra step of TRG to go back to the tensor network with only one sublattice [4].

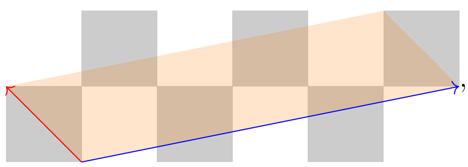
Based on the intuition that an SVD separate a square into two isosceles right triangles, we choose our transfer matrix as the following



This transfer matrix has the modular parameter

$$\tau = \frac{1}{3(1 - \mathrm{i})} = \frac{1}{6}(1 + \mathrm{i}).$$

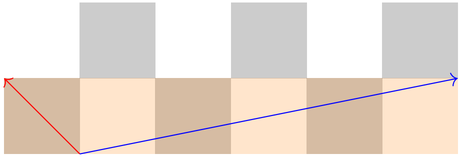
It would be nice if one can directly compute conformal spins without another the TRG step. There are possibly other choices of the unit cell on the checkerboard:



whose modular parameter is

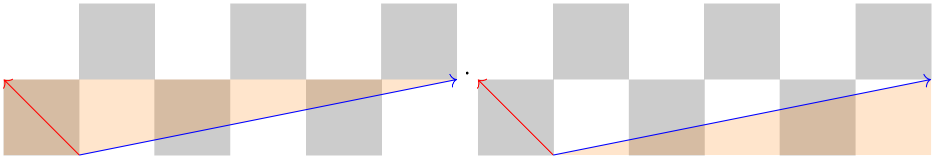
$$\tau = \frac{-1 + \mathrm{i}}{5 + \mathrm{i}} = \frac{-2 + 3\mathrm{i}}{13}.$$

Similar to the previous approach, we find another transfer matrix that has the same spectrum. We regard each block as a tensor. If



$$\cdot\, \psi = \lambda \psi,$$

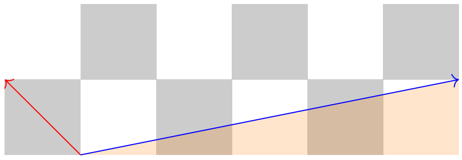
where edges are matched in the zig-zag direction. Then



$$\cdot\, \psi = \lambda \psi$$

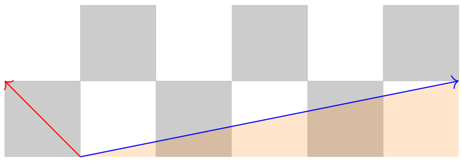
Let

$$\phi =$$

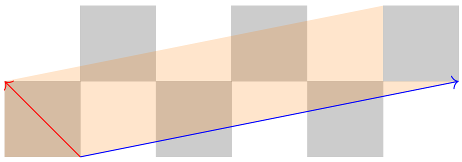


$$\cdot\, \psi,$$

then multiplying

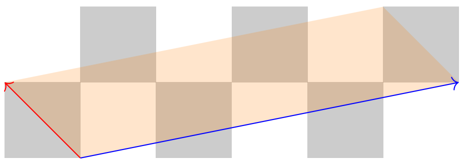


to the both hand sides, we have

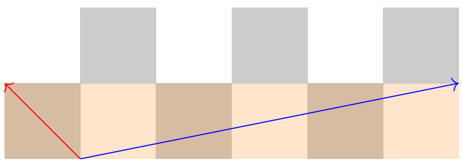


$$\cdot\, \phi = \lambda \phi.$$

A similar argument can be applied to the small triangle in the left bottom side. Thus



and



have the same set of eigenvalues.

References

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