HW3

1. $v_1 = y_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad V_2 = y_2 - \frac{V_1^T y_2}{V_1^T V_1} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{(1\,0\,0)\begin{pmatrix}1\\1\\0\end{pmatrix}}{(1\,0\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$V_3 = y_3 - \left(\frac{V_2^T y_3}{V_2^T V_2} V_2\right) - \left(\frac{V_1^T y_3}{V_1^T V_1} V_1\right) = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{(0\,1\,0)\begin{pmatrix}1\\1\\1\end{pmatrix}}{(0\,1\,0)\begin{pmatrix}0\\1\\0\end{pmatrix}}\begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

2. $R^T = B^{-1}$ so $\left(\begin{array}{ccc|ccc}-1&1&1&1&0&0\\1&1&1&0&1&0\\0&-2&0&0&0&1\end{array}\right) = (B|I) = \begin{pmatrix}0&2&2&1&1&0\\1&1&1&0&1&0\\0&-2&0&0&0&1\end{pmatrix} = \begin{pmatrix}0&2&2&1&1&0\\1&1&1&0&1&0\\0&0&2&1&1&1\end{pmatrix}$

$= \begin{pmatrix}0&1&1&\frac12&\frac12&0\\1&1&1&0&1&0\\0&0&0&\frac12&\frac12&\frac12\end{pmatrix} = \begin{pmatrix}0&1&0&0&0&-\frac12\\1&1&0&-\frac12&\frac12&-\frac12\\0&0&1&\frac12&\frac12&\frac12\end{pmatrix} = \begin{pmatrix}0&1&0&0&0&-\frac12\\1&0&0&-\frac12&\frac12&0\\0&0&1&\frac12&\frac12&\frac12\end{pmatrix} = \begin{pmatrix}1&0&0&-\frac12&\frac12&0\\0&1&0&0&0&-\frac12\\0&0&1&\frac12&\frac12&\frac12\end{pmatrix}$

so $B^{-1} = \begin{pmatrix}-\frac12&\frac12&0\\0&0&-\frac12\\\frac12&\frac12&\frac12\end{pmatrix} \Rightarrow r_1 = \begin{pmatrix}-\frac12\\\frac12\\0\end{pmatrix}$ $r_2 = \begin{pmatrix}0\\0\\-\frac12\end{pmatrix}$ $r_3 = \begin{pmatrix}\frac12\\\frac12\\\frac12\end{pmatrix}$ $x = \begin{pmatrix}1\\2\\2\end{pmatrix}$

$(-\frac12\ \frac12\ 0)\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac12$ $\quad (0\ 0\ -\frac12)\begin{pmatrix}1\\2\\2\end{pmatrix} = -1$ $\quad (\frac12\ \frac12\ \frac12)\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac52$ $\quad$ so $x^v = \begin{pmatrix}\frac12\\-1\\\frac52\end{pmatrix}$

3.(1) $\begin{cases} f_1 = 1 & (0,1) \\ f_2 = \begin{cases}1 & (0,\frac14)\\ -1 & (\frac14,1)\end{cases} \\ f_3 = \begin{cases}1 & (0,\frac34)\\ -1 & (\frac34,1)\end{cases}\end{cases} \Rightarrow \begin{cases} [0,\frac14] & a+b+c = af_1+bf_2+cf_3 = 0 \\ [\frac14,\frac34] & a-b+c=0 \\ [\frac34,1] & a-b-c=0 \end{cases}$ $\Rightarrow a=b=c=0$

so linear independent.

(2) f_1, f_2, f_3 are linear independent, so let

$V_1 = f_1$ $\quad V_2 = f_2 - \frac{\langle f_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1$ $\quad V_3 = f_3 - \frac{\langle f_3, V_1\rangle}{\langle V_1\, V_1\rangle} V_1 - \frac{\langle f_3\, V_2\rangle}{\langle V_2, V_2\rangle} V_2$

$v_1 = f_1 = 1.\quad t \in [0,1]$

$\langle f_2, f_1\rangle = \int_0^1 f_1 f_2\, dt = \int_0^{1/4} 1\, dt + \int_{1/4}^1 -1\, dt = -\frac12$

$\langle V_1\, V_1\rangle = \int_0^1 1\, dt = 1$ $\quad$ so $V_2 = f_2 + \frac12 V_1 \Rightarrow V_2 = \begin{cases}\frac32 & t\in[0,\frac14]\\ -\frac12 & t\in(\frac14,1]\end{cases}$

$\langle f_3\, V_1\rangle = \int_0^{3/4} 1\, dt + \int_{3/4}^1 -1\, dt = \frac34 - (1-\frac34) = \frac12$

$\langle f_3, V_2\rangle = \int_0^{1/4} \frac32 dt + \int_{1/4}^{3/4} (-\frac12) dt + \int_{3/4}^1 \frac12 dt = \frac32 \cdot \frac14 - \frac12(\frac34 - \frac14) + \frac12(1-\frac34) = \frac14$

$\langle V_2 \cdot V_2\rangle = \int_0^{1/4} \frac94 dt + \int_{1/4}^1 \frac14 dt = \frac{9}{16} + \frac{3}{16} = \frac34$

$t \in [0,\frac14]$ $\quad V_3 = 1 - \frac{\frac12}{1}\cdot 1 - \frac{\frac14}{\frac34}\cdot\frac32 = 0$

$t \in (\frac14, \frac34]$ $\quad V_3 = 1 - \frac12 \cdot 1 - \frac{\frac14}{\frac34}\cdot(-\frac12) = 1 - \frac12 + \frac16 = \frac23$

$t \in (\frac34, 1]$, $\quad V_3 = -1 - \frac12 \cdot 1 - \frac{\frac14}{\frac34}(-\frac12) = -1 - \frac12 + \frac16 = -\frac86 = -\frac43$

so. $V_1 = 1$. $[0,1]$

$$V_2 = \begin{cases} \frac{3}{2} & t \in [0, \frac{1}{4}] \\ -\frac{1}{2} & t \in (\frac{1}{4}, 1] \end{cases}$$

$$V_3 = \begin{cases} 0 & t \in [0, \frac{1}{4}] \\ \frac{2}{3} & t \in [\frac{1}{4}, \frac{3}{4}] \\ -\frac{4}{3} & t \in [\frac{3}{4}, 1] \end{cases}$$

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4. $(\|x-ay\|) = ((x-ay)^2)^{\frac{1}{2}}$ we want $\min((x-ay)^2)^{\frac{1}{2}} \Rightarrow \min(x-ay)^2$

$((x-ay)^2)' = 2(x-ay)(-y) = 0$ $\cancel{a = \frac{2x+y}{2y}}$ $(-xy) + ay^2 = 0$ $a = \frac{xy}{y^2} = \frac{\langle x, y\rangle}{\langle y, y\rangle}$

as $(x-ay)^2 \geq 0$ $a = \frac{\langle x, y\rangle}{\langle y, y\rangle}$ $\|x-ay\|$ will be min

then ① $\langle x-ay, y\rangle = (x-ay)^T y = x^T y - a y^T y = x^T y - \frac{x^T y}{y^T y} \cdot y^T y = 0$

so they're orthogonal

② $\langle x-ay, x-ay\rangle + \langle ay, ay\rangle = \langle x, x\rangle$: $(x-ay)^T(x-ay) + ay^T \cdot ay$

$= (x^T - ay^T)(x-ay) + a^2 y^T y = x^T x - a x^T y - a y^T x + a^2 y^T y + a^2 y^T y$

$\cancel{= x^T x - a\langle -ay + x, y\rangle = x^T x - a\langle x-ay, y\rangle = x^T x = \langle x, x\rangle}$

$= x^T x - a x^T y - a y^T x + 2a \cdot x^T y = x^T x + \overset{=0}{a}(x^T y - y^T x)$

a $x^T y = y^T x$ so $x^T x + a(x^T y - y^T x) = x^T x = \langle x, x\rangle$ proved,

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5, (i) $\cancel{\begin{pmatrix} 1+j & 0 \\ 0 & 1-j \end{pmatrix} (1+j) = \begin{pmatrix} (1+j)^2 & 0 \\ 0 & 1-j^2 \end{pmatrix}}$ $(1+j)^2 = 2j = ((1+j) - (1-j)) = 1S_1 + (-1)S_2$

$1 - j^2 = 2 = (1+j) + (1-j) = 1S_1 + 1S_2$ so $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

(ii). $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = -1$ $\lambda = 1 \pm i$

① $\lambda = 1+i$ $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$ $\begin{matrix} -ai + b = 0 \\ -a - bi = 0 \end{matrix}$ $b = ai$ $\begin{pmatrix} 1 \\ i \end{pmatrix}$

② $\lambda = 1-i$ $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$ $\begin{matrix} ai + b = 0 \\ b = -ai \end{matrix}$ $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

(iii). $\begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} 1+i & 1-i \\ -1+i & -1-i \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2}(1+i) + \frac{1}{2i}(i-1) & \frac{1}{2}(1-i) - \frac{1}{2i}(1+i) \\ \frac{1}{2}(1+i) - \frac{1}{2i}(i-1) & \frac{1}{2}(1-i) + \frac{1}{2i}(1+i) \end{pmatrix} = \begin{pmatrix} 1+j & 0 \\ 0 & 1-j \end{pmatrix}$

HW3.

6. (i) $D(\sin t) = \cos t = 0s_1 + 1s_2$
$D(\cos t) = -\sin t = -1s_1 + 0s_2$ $\Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A$

(ii). $\begin{vmatrix} -\lambda & 1 \\ 1 & \lambda \end{vmatrix} = 0$ $\lambda = \pm i$

① $\lambda = i$. $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ $b = -ia$ $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$

② $\lambda = -i$ $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ $b = ia$ $\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\sin t - i\cos t$
$\sin t + i\cos t$

(iii). $B^{-1}AB = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2i} \\ \frac{1}{2} & \frac{1}{2i} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} -\frac{1}{2i} & -\frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$
$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

7. (i) $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $a_1 + a_2 = 1$, $a_3 + a_4 = 1$. $a_2 = 1$. $a_1 = 0$, $a_4 = 3$ $a_3 = -2$ $\Rightarrow A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $a_1 + 2a_2 = 2$, $a_3 + 2a_4 = 4$.

(ii). $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

8 (i). $\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = R$ so $r_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $r_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

(ii). $Av_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $(2\,1)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1$ $(-1\,-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$ $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

(iii) $Av_2 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ $(2\,1)\begin{pmatrix} -2 \\ 4 \end{pmatrix} = 0$ $(-1, -1)\begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2$ $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

(iv) $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$