## 8.3 ARMA, AR and MA-processes

The results in the previous section hold for arbitrary LTI-systems. In this section we will restrict ourselves to systems that can be described by rational frequency functions and whose input signals are white noise, so called ARMA-processes.

We study the causal BIBO-stable system with frequency function as in (8.9); that is,

$$H(\nu) = \frac{b_0 + b_1\, e^{-j2\pi\nu} + \cdots + b_\ell\, e^{-j2\pi\ell\nu}}{1 + a_1\, e^{-j2\pi\nu} + \cdots + a_k\, e^{-j2\pi k\nu}} \tag{8.21}$$

If the input signal $X(n)$ is a weakly stationary stochastic process with bounded autocorrelation function $r_X(k)$ then the output signal

$$Y(n) = h(n) \star X(n) = \sum_{\ell=0}^{\infty} h(\ell)\, X(n-\ell) \tag{8.22}$$

exists in mean square. According to Section 8.2 the process $Y(n)$ is weakly stationary. Analogously with the deterministic case we may write the input output relationship as (see Equation (8.8) on page 146)

$$\begin{aligned} &Y(n) + a_1\, Y(n-1) + \cdots + a_k\, Y(n-k) \\ &\qquad = b_0\, X(n) + b_1\, X(n-1) + \cdots + b_\ell\, X(n-\ell) \end{aligned} \tag{8.23}$$

When the input signal $X(n)$ is white noise with variance $\sigma_X^2$ then $Y(n)$ is referred to as an ARMA$(k,\ell)$-process. An implementation of the second order ARMA-process

$$Y(n) + a_1\, Y(n-1) + a_2\, Y(n-2) = X(n) + b_1\, X(n-1) + b_2\, X(n-2)$$

is given in Figure 8.2.

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A special case of an ARMA-process is an AR$(k)$-process:

$$Y(n) + a_1\, Y(n-1) + \cdots + a_k\, Y(n-k) = X(n) \tag{8.24}$$

and another special case is an MA$(\ell)$-process: 2

$$Y(n) = X(n) + b_1\, X(n-1) + \cdots + b_\ell\, X(n-\ell) \tag{8.25}$$

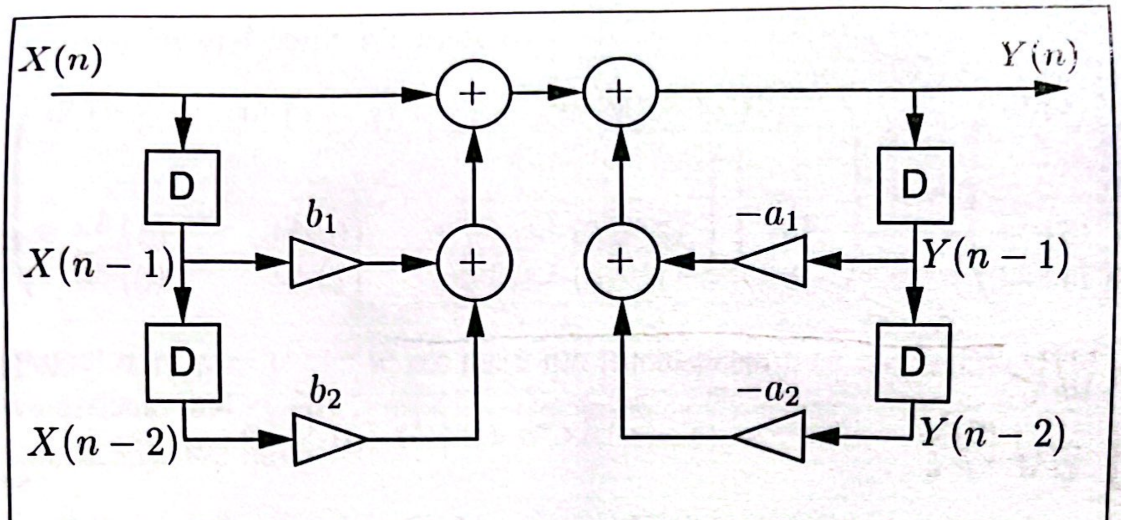


Figure 8.2: An implementation of a second order ARMA-process.

结构

In (8.24) and (8.25) we have put $b_0 = 1$. This is not a restriction since it is always possible to scale the output variance by the variance $\sigma_X^2$ of the white noise $X(n)$.

The power spectrum according to (8.17) of an ARMA$(k, \ell)$-process with frequency function (8.21) follows directly

$$R_Y(\nu) = |H(\nu)|^2\,\sigma_X^2 = \left|\frac{1 + b_1\,e^{-j2\pi\nu} + \ldots + b_\ell\,e^{-j2\pi\ell\nu}}{1 + a_1\,e^{-j2\pi\nu} + \ldots + a_k\,e^{-j2\pi k\nu}}\right|^2 \sigma_X^2$$

It can be shown that $|H(\nu)|^2$ for an ARMA-process is a rational function in $\cos(\cdot)$; that is, there are constants $\beta_0, \ldots, \beta_\ell$, $\alpha_0, \ldots, \alpha_k$ such that

$$|H(\nu)|^2 = \frac{\beta_0 + \beta_1\,\cos(2\pi\,\nu) + \ldots + \beta_\ell\,\cos(2\ell\pi\,\nu)}{\alpha_0 + \alpha_1\,\cos(2\pi\,\nu) + \ldots + \alpha_k\,\cos(2k\pi\,\nu)} \tag{8.26}$$

It is common to use AR-, MA- and ARMA-models in parametric methods in model based signal processing. The most common models rely on the assumption that the signal is generated as white noise filtered by a causal stable linear filter with a rational transfer function where all poles and zeros are inside the unit circle (a so called *minimum phase system*; that is, even the system's inverse is BIBO-stable). Depending on the type of filter we have seen that these models have different names.

The interest in ARMA-processes is due to the fact that most spectral densities occurring in practice can be well described by a power spectrum as (8.26). Alternatively, the above power spectrum is sufficiently flexible to be able to describe most practical spectral densities. This means that almost all signals can be interpreted as being realizations of ARMA-processes when we limit ourselves to the study of their second order moment properties.

**ARMA-coefficients from the autocorrelation function** The question is how to compute the ARMA-coefficients from a given autocorrelation function? We begin by an example and then, in the next section, we study AR-processes in detail. Let $Y(n)$ be an MA(1)-process and assume that the autocorrelation function $r_Y(k)$ is given. The task is now to determine $b = b_1$ and the noise variance $\sigma_X^2$ in (8.25). The autocorrelation function of an MA(1)-process is given in (3.34) on page 64, from which we can formulate the following *nonlinear* system of equations

$$\begin{cases} r_Y(0) = (1+b^2)\sigma_X^2 \\ r_Y(1) = b\,\sigma_X^2 \end{cases}$$

From this we can solve for $b$ and $\sigma_X^2$.

Generally, for MA- and ARMA-processes we end up with a nonlinear systems of equations when we try to determine the filter coefficients given the autocorrelation function. On the contrary, for AR-models the filter coefficients may be determined from the autocorrelation sequence by solving a *linear* system of equations. This is one reason why AR-models are common in model based signal processing. In addition, often they describe the physical behavior sufficiently accurate.

## 8.4 Properties of AR-processes

For AR-processes the system of equations that relates the coefficients to the autocorrelation function becomes *linear*, which we will now prove.

Suppose that the system is given by a causal BIBO-stable AR-process

$$Y(n) + a_1\,Y(n-1) + \ldots + a_s\,Y(n-s) = X(n)$$

差分方程

If the above difference equation is multiplied by $Y(n-k)$ we get

$$Y(n)\,Y(n-k) + \ldots + a_s\,Y(n-s)\,Y(n-k) = X(n)\,Y(n-k) \tag{8.27}$$

Letting $k > 0$ and computing the expected value we have

$$r_Y(k) + a_1\,r_Y(k-1) + \ldots + a_s\,r_Y(k-s) = 0$$

The zero in the right hand side is due to the fact that old output signals are uncorrelated with the current input signal (the white noise). That is,

$$\begin{aligned} \mathrm{E}[X(n)\,Y(n-k)] &= \mathrm{E}\left[X(n)\sum_{\ell=0}^{\infty} h(\ell)\,X(n-k-\ell)\right] \\ &= \sum_{\ell=0}^{\infty} h(\ell)\,\mathrm{E}[X(n)\,X(n-k-\ell)] = 0 \end{aligned}$$

The last equality follows since $k > 0$. Furthermore, we have used that the system is causal and can be written as the convolution sum (8.22).

In particular, for $k = 1$ we have

$$r_Y(1) + a_1\,r_Y(0) + \ldots + a_s\,r_Y(1-s) = 0$$

If we use the symmetry that $r_Y(-k) = r_Y(k)$ we can rewrite the equation as

$$a_1\,r_Y(0) + \ldots + a_s\,r_Y(s-1) = -r_Y(1)$$

只有AR process 才有的性质

Hence, with $k = 1, \ldots, s$, we have the linear system of equations

$$\begin{pmatrix} r_Y(0) & r_Y(1) & \cdots & r_Y(s-1) \\ r_Y(1) & r_Y(0) & \cdots & r_Y(s-2) \\ \vdots & \vdots & \cdots & \vdots \\ r_Y(s-1) & r_Y(s-2) & \cdots & r_Y(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix} = - \begin{pmatrix} r_Y(1) \\ r_Y(2) \\ \vdots \\ r_Y(s) \end{pmatrix} \tag{8.28}$$

If we introduce the column vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix}, \qquad \mathbf{r_Y} = \begin{pmatrix} r_Y(1) \\ r_Y(2) \\ \vdots \\ r_Y(s) \end{pmatrix} \tag{8.29}$$

and the matrix

$$\mathbf{R_Y} = \begin{pmatrix} r_Y(0) & r_Y(1) & \cdots & r_Y(s-1) \\ r_Y(1) & r_Y(0) & \cdots & r_Y(s-2) \\ \vdots & \vdots & \cdots & \vdots \\ r_Y(s-1) & r_Y(s-2) & \cdots & r_Y(0) \end{pmatrix} \tag{8.30}$$

then we can write the system of equations (8.28) in matrix form as

$$\mathbf{R_Y}\,\mathbf{a} = -\mathbf{r_Y} \tag{8.31}$$

This system of equations is known as the *Yule-Walker equations.* The matrix $\mathbf{R_Y}$ has the special structure that the elements along the different diagonals all are identical. Such a matrix is called a *Toeplitz matrix*.

If $\mathbf{R_Y}$ is non-singular, then the AR-coefficients are obtained by

$$\mathbf{a} = -\mathbf{R_Y}^{-1}\,\mathbf{r_Y} \tag{8.32}$$

The noise variance of the driving noise $X(n)$ can be calculated in the following way: Let $k = 0$ in (8.27) and take the expected value. This gives

$$r_Y(0) + a_1\, r_Y(1) + \ldots + a_s\, r_Y(s) = \sigma_X^2 \tag{8.33}$$

Below we give an example on how to compute the AR-coefficients given the acf.

针对AR(2)特殊情况

**The Yule-Walker equations for an AR(2)-process** We want to determine the model parameters $a_1$, $a_2$ and $\sigma_X^2$ given the autocorrelation coefficients of an AR(2)-process $Y(n)$. From the autocorrelations $r_Y(0)$, $r_Y(1)$ and $r_Y(2)$ we can form the corresponding Yule-Walker equations as in (8.28); that is,

$$\begin{pmatrix} r_Y(0) & r_Y(1) \\ r_Y(1) & r_Y(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} r_Y(1) \\ r_Y(2) \end{pmatrix}$$