CHAPTER I
Principal Theory of Logic

Institute of Technology of Cambodia
ITC

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Set

Definition 1 (Set)

A set is simply a collection of objects or members . These objects are called elements of the set.
If X is a finite set, the cardinality of X is number of elements in X,denoted $|X|$ or $n(X)$

Example 2

$A = \{1, 3, 5, 7\}$
$B = \left\{x | x \quad \text{is a positive integer less than 7, even integer}\right\} = \{2, 4, 6\}$
then $|A| = 4$ and $|B| = 3$

Some Common number sets

$$\mathbb{N} = \{1, 2, 3, \cdots\}, \text{natural number set.}$$

$$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}, \text{whole number set.}$$

$$\mathbb{Z} = \{\cdots - 3, -2, 0, 1, 2, 3, \cdots\}, \text{integer number set.}$$

$$\mathbb{Q} = \left\{\frac{p}{q} \quad | \quad p, q \text{ are integers}\right\}, \text{rational number set.}$$

$$\mathbb{I} = \text{non-rational number set.}$$

$$\mathbb{R} = \text{all of the above number sets.}$$

$$\mathbb{C} = \{a + ib \quad | \quad a, b \quad : \text{are real number}\}, \text{complex number set}$$

Empty and Equal Set

Definition 3 (Empty set)

The set with no elements is called the empty (or null or void) set and is denoted $\emptyset$. Thus $\emptyset = \{\}$

Definition 4 (Equal set)

Two sets X and Y are equal, denoted $X = Y$ if X and Y have the same elements.That is, there are two following condition are satisfied

- $\forall x \in X$ then $x \in Y$
- $\forall x \in Y$ then $x \in X$

Example 5

1. Given $A = \{1, 3, 3, 4\}$ and $\{1, 3, 4\}$
2. Let $A = \left\{x | x^2 - 3x + 2 = 0\right\}$ and $B = \{1, 2\}$. Show that $A = B$

Subset

Definition 6 (Subset)

Suppose that X and Y are sets. If every element of X is an element of Y, we say that X is a subset of Y, denoted $X \subseteq Y$. That is

$$\forall x \in X \Longrightarrow x \in Y$$

Example 7

1. $A = \{a, b, d\}$ and $B = \{a, b, c, d, e\}$
2. $B = \left\{x | x^2 - 6x + 5 = 0\right\}$. Show that B is the proper subset of $\mathbb{Z}$

Power Set

Definition 8 (Power Set)

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(\mathcal{X})$, is called the power set of X.

Example 9

If $A = \{a, b, c\}$, the members of $\mathcal{P}(A)$ are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$$

All but $\{a, b, c\}$ are proper subsets of A.

Union and Intersection

Definition 10 (Union and Intersection)

Let X and Y are two sets, there are various set operations involving X and Y that can produce a new set

- The Union of X and Y, denoted by $X \cup Y$, is defined by

$$X \cup Y = \{x | x \in X \quad \text{or} \quad x \in Y\}$$

- The Intersection of X and Y, denoted by $X \cap Y$, is defined by

$$X \cap Y = \{x | x \in X \quad \text{and} \quad x \in Y\}$$

Difference

Definition 11 (Difference)

Let X and Y are two sets, the difference (or relative complement) of X and Y, denoted by $X - Y$, is defined by

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

Example 12

1. Let $A = \{a, b, c, d\}$ and $B = \{1, 3, b, e\}$. Determine
   $A \cup B, \quad A \cap B, \quad A - B$ and $B - A$
2. Given the common set, the set of the rational number $\mathbb{Q}$ and the set of the real number $\mathbb{R}$, determine $\mathbb{R} \cup \mathbb{Q}, \quad \mathbb{R} \cap \mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$

Collection of Sets

Definition 13 (Collection of sets)

We call a set $\mathcal{S}$, whose elements are sets, a collection of sets or a family of sets.

Example 14

If

$$\mathcal{S} = \{\{1,2\},\{1,3\},\{1,7,10\}\}$$

then $\mathcal{S}$ is a collection or family of sets. The set $\mathcal{S}$ consists of the sets

$$\{1,2\},\{1,3\},\{1,7,10\}$$

Pairwise disjoint

Definition 15 (Pairwise disjoint)

Sets X and Y are disjoint if $X \cap Y = \varnothing$. A collection of sets $\mathcal{S}$ is said to be pairwise disjoint if, whenever X and Y are distinct sets in $\mathcal{S}$, X and Y are disjoint.

Example 16

The sets $\{1, 4, 5\}$ and $\{2, 6\}$ are disjoint. The collection of sets $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}\ \{7, 8\}\}$ is pairwise disjoint.

Complement

Definition 17 (Complement)

Given a universal set U and a subset X of U, the set $U - X$ is called the complement of X and is written $\bar{X}$.

Example 18

Given the universal set $U = \{1, 2, 3, a, b\}$ and $A = \{1, a, 3\}$. Determine the complement set of A

Venn diagrams

Definition 19 (Venn diagrams)

Venn diagrams provide pictorial views of sets. In a Venn diagram, a rectangle depicts a universal set . Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set.

Example 20

Let $U = \{1, 2, 3, 4\}$, $A = \{2, 3\}$ and $B = 3, 4$. Use Vann diagram to find $A \cup B$, $A \cap B$, $\bar{A}$ and $\overline{A \cup B}$

Example 21

Among a group of 165 students, 8 are taking calculus, psychology, and computer science; 33 are taking calculus and computer science; 20 are taking calculus and psychology; 24 are taking psychology and computer science; 79 are taking calculus; 83 are taking psychology; and 63 are taking computer science. How many are taking none of the three subjects?

Properties

Let U be a universal set and let A, B, and C be subsets of U. The following properties hold.

(a) Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) Identity laws:

$$A \cup \varnothing = A, \quad A \cap U = A$$

Properties

(e) Complement laws:

$$A \cup \bar{A} = U, \quad A \cap \bar{A} = \varnothing$$

(f) Idempotent laws:

$$A \cup A = A, \quad A \cap A = A$$

(g) Bound laws:

$$A \cup U = U, \quad A \cap \varnothing = \varnothing$$

(h) Absorption laws:

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) Involution law:

$$\bar{\bar{A}} = A^{\dagger}$$

(j) 0/1 laws:

$$\bar{\varnothing} = U, \quad \bar{U} = \varnothing$$

Properties

(k) De Morgan's laws for sets:

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}, \quad \overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

Definition 22 (union of a collection of sets)

We define the union of a collection of sets S to be those elements x belonging to at least one set X in S. Formally,

$$\cup \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}$$

Similarly, we define the intersection of a collection of sets S to be those elements x belonging to every set X in S. Formally,

$$\cap \mathcal{S} = \{x \mid x \in X \text{ for all } X \in S\}$$

Example

Example 23

Let $\mathcal{S} = \{\{1,2\},\{1,3\},\{1,7,10\}\}$. Then $\cup S = \{1,2,3,7,10\}$. Also $\cap\mathcal{S} = \{1\}$ since only the element 1 belong to every set in $\mathcal{S}$.

Short form

Definition 24 (Shortform)

If
$$\mathcal{S} = \{A_1, A_2, \ldots, A_n\}$$
we write
$$\bigcup S = \bigcup_{i=1}^{n} A_i, \quad \bigcap S = \bigcap_{i=1}^{n} A_i$$
and if
$$\mathcal{S} = [A_1, A_2, \ldots\}$$
we write
$$\bigcup S = \bigcup_{i=1}^{\infty} A_i, \quad \bigcap S = \bigcap_{i=1}^{\infty} A_i$$

Example

Example 25

For $i \geq 1$, define $A_i = \{i, i+1, \ldots\}$ and $S = \{A_1, A_2, \ldots\}$. As examples, $A_1 = \{1, 2, 3, \ldots)$ and $A_2 = \{2, 3, 4, \ldots\}$. Then

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i = \{1, 2, \ldots\}, \quad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i = \varnothing$$

Definition 26

Let X and Y are two sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the Cartesian product of X and Y.

Generally, The Cartesian product of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$ for $i = 1, \ldots, n$; it is denoted $X_1 \times X_2 \times \cdots \times X_n$

Example

Cardinality

Notice that $|X \times Y \times Z| = |X| \cdot |Y| \cdot |Z|$. In general,

$$|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdots |X_n|$$

Example 27 (Cartesian product)

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$\begin{aligned}
X \times Y &= \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\} \\
Y \times X &= \{(a,1), (b,1), (a,2), (b,2), (a,3), (b,3)\} \\
X \times X &= \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\} \\
Y \times Y &= \{(a,a), (a,b), (b,a), (b,b)\}
\end{aligned}$$

example

Example 28 (Example)

If $X = \{1, 2\}$, $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then
$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha)$

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Proposition

Definition 29 (Proposition)

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Example 30 (Proposition)

All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Example

Example 31 (Example)

Some sentences that are not propositions are given in Example
Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$
4. $x + y = z$

Definition 32 (OR, AND)

Let p and q be two propositions.

- The conjunction of p and q, denoted $p \wedge q$, is the proposition p and q
- The disjunction of p and q, denoted $p \vee q$, is the proposition p or q

Example 33

If p : It is raining, and q : It is cold, then the conjunction of p and q is $p \wedge q$: It is raining and it is cold. The disjunction of p and q is

True value Table

Definition 34

Let p, q be two propositions and the negative of proposition p, denoted by $\overline{p}$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure: AND "$\wedge$"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure: OR "$\vee$"

p	$\neg p$
T	F
F	T

Figure: Negative "$^{-}$" or "$\neg$"

Example

Example 35 (Example)

1. IF p: A decade is 10 years, and q: A millennium is 100 years. What is the conjuction $p \wedge q$?
2. If p: A millennium is 100 years, and q: A millennium is 1000 years. What is the the disjunction $p \vee q$?
3. If p: π was calculated to $1,000,000$ decimal digits in1954. What is the negative of the proposition p.
4. Given that proposition p is false, proposition q is true, and proposition r is false, determine whether the proposition

$$\overline{p} \vee q \wedge r \quad \text{is true or false?}$$

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Conditional Proposition

Definition 36 (Conditional Proposition)

If p and q are propositions, the proposition if p then q is called a conditional proposition and is denoted

$$p \Longrightarrow q$$

The proposition p is called the hypothesis (or antecedent), and the proposition q is called the conclusion (or consequent)

Example 37

Example If we define p: The Mathematics Department gets an additional $60,000$
q: The Mathematics Department will hire one new faculty member.
What is the truth value of the dean's statement?

Definition 38

The truth value of the conditional proposition $p \Longrightarrow q$ is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure: True value Table

Example

Example 39 (Example)

1. Let $p : 1 > 2$ and $q : 4 < 8$. Determine the proposition $p \implies q$, and $q \implies p$.
2. Assuming that p is true, q is false, and r is true, find the truth value of each proposition.
    1. $p \wedge q \implies r$
    2. $p \vee q \implies \neg r$
    3. $p \wedge (q \implies r)$
    4. $p \to (q \implies r)$
3. Write the conditional proposition, If Jerry receives a scholarship, then he will go to college, and its converse symbolically and in words. Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

If and only if

Definition 40 (If and only if)

If p and q are propositions, the proposition p if and only if q is called a biconditional proposition and is denoted

$$p \Longleftrightarrow q \quad \text{or} \quad p \longleftrightarrow q$$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure: The true value table

Logically equivalent

The proposition

$$1 < 5 \text{ if and only if } 2 < 8$$

Definition 41 (Logically equivalent)

Suppose that the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$. We say that P and Q are logically equivalent and write

$$P \equiv Q$$

provided that, given any truth values of $p_1, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

Example 42

1. $\overline{p \cap q} \equiv \overline{p} \cup \overline{q}$ and $\overline{p \cup q} \equiv \overline{p} \cap \overline{q}$
2. $p \iff q \equiv (p \implies q) \cap (q \implies p)$

Definition 43 (Contrapositive)

The contrapositive (or transposition) of the conditional proposition $p \rightarrow q$ is the proposition $\overline{q} \rightarrow \overline{p}$.

Example 44

Write the conditional proposition, If the network is down, then Dale cannot access the internet, symbolically. Write the contrapositive and the converse symbolically and in words. Also, assuming that the network is not down and Dale can access the internet, find the truth value of the original proposition, its contrapositive, and its converse.

Contrapositive

Theorem 45 (Contrapositive)

The conditional proposition $p \Longrightarrow$ *and its contrapositive* $\overline{q} \Longrightarrow \overline{p}$ *are logically equivalent.*

Deductive reasoning

This process of drawing a conclusion from a sequence of propositions is called deductive reasoning. The given propositions, called hypotheses or premises, and the proposition that follows from the hypotheses,is called the conclusion. A (deductive) argument consists of hypotheses together with a conclusion. Many proofs in mathematics and computer science are deductive arguments. Any argument has the form

$$\text{IF} \quad p_1 \quad \text{and} \quad p_2 \quad \text{and} \cdots \quad \text{and} \quad p_n, \quad \textit{then} \quad q$$

This argumen is said to be valid if the conclusion follows from the hypotheses; that is, if p_1 and p_2 and $\cdots$ and pn are true, then q must also be true. This discussion motivates the following definition.

Definition

Definition 46 (In short)

An argument is a sequence of propositions written

$$\begin{array}{r} p_1 \\ p_2 \\ \vdots \\ \underline{\;\; p_n} \\ \therefore q \end{array}$$

Example

Example 47

Determine whether the argument

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

is valid.

Rules of Inference for Propositions

Rule of Inference	*Name*	*Rule of Inference*	*Name*
$p \to q$ $\underline{p \quad\quad}$ $\therefore q$	Modus ponens	p $\underline{q \quad\quad}$ $\therefore p \wedge q$	Conjunction
$p \to q$ $\underline{\neg q \quad\quad}$ $\therefore \neg p$	Modus tollens	$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	Hypothetical syllogism
$\underline{p \quad\quad}$ $\therefore p \vee q$	Addition	$p \vee q$ $\underline{\neg p \quad}$ $\therefore q$	Disjunctive syllogism
$\underline{p \wedge q}$ $\therefore p$	Simplification		

Figure: Rules of Inference for Propositions

Example

Example 48

Represent the argument

$$\begin{array}{l} \text{If} \quad 2 = 3 \\ \quad \text{then I ate my hat.} \\ \hline \\ \text{I ate my hat.} \\ \therefore 2 = 3 \end{array}$$

symbolically and determine whether the argument is valid.

Contents

Proposition function

Definition 49 (Proposition function)

Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a propositional function or predicate (with respect to D) if for each $x \in D, P(x)$ is a proposition. We call D the domain of discourse of P.
In Definition , the domain of discourse specifies the allowable values for x

Example 50 (Example)

Let $P(n)$ be the statement n is an odd integer.

Universally quantified

Definition 51 (universally quantified)

Let P be a propositional function with domain of discourse D. The statement for every $x, P(x)$ is said to be a universally quantified statement. The symbol $\forall$ means "for every." Thus the statement for every $x, P(x)$ may be written

$$\forall x,\ P(x)$$

The symbol $\forall$ is called a Universal quantifier. The statement is true if $P(x)$ is true for every x in D. The statement is false if $P(x)$ is false for at least one x in D.

Counter Example

Example 52

Consider the universally quantified statement

$$\forall x (x^2 \geq 0)$$

The domain of discourse is R. The statement is true because, for every real number x, it is true that the square of x is positive or zero.

Remark 1

According to Definition, the universally quantified statement is false if for at least one x in the domain of discourse, the proposition $P(x)$ is false. A value x in the domain of discourse that makes $P(x)$ false is called a counterexample to the statement .

existentially quantified

Definition 53 (existentially quantified)

Let P be a propositional function with domain of discourse D. The statement there exists $x, P(x)$ is said to be an existentially quantified statement. The symbol $\exists$ means "there exists". Thus the statement there exists $x, P(x)$ may be written

$$\exists\, xP(x)$$

The symbol $\exists$ is called an existential quantifier. The statement is true if $P(x)$ is true for at least one x in D. The statement is false if $P(x)$ is false for every x in D.

Example

1. Verify tha the existentially quantified statement

$$\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right)$$

   is true.

2. Verify that the existentially quantified statement

$$\exists x \in \mathbf{R} \left(\frac{1}{x^2+1} > 1 \right)$$

   is false.

Generalized De Morgan's Laws for Logic

Theorem 54 (Generalized De Morgan's Laws for Logic)

If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values (i.e., either both are true or both are false).

(a) $\neg(\forall x P(x)); \exists x \neg P(x)$

(b) $\neg(\exists x P(x)); \forall x \neg P(x)$

Example 55

Write the statement" Some birds cannot fly", symbolically. Write its negation symbolically and in words.

Rules of Inference for Quantified Statements

Rule of Inference	*Name*
$\dfrac{\forall x\, P(x)}{\therefore P(d) \text{ if } d \in D}$	Universal instantiation
$\dfrac{P(d) \text{ for every } d \in D}{\therefore \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore P(d) \text{ for some } d \in D}$	Existential instantiation
$\dfrac{P(d) \text{ for some } d \in D}{\therefore \exists x\, P(x)}$	Existential generalization

† The domain of discourse is D.

Figure: Rules of Inference for Quantified Statements

Example

Example 56 (Example)

Given that for every positive integer $n, n^2 \geq n$ is true, we may use universal instantiation to conclude that $542 \geq 54$ since 54 is a positive integer (i.e., a member of the domain of discourse).

Nested Quantifiers

Example 57

1. Write the assertion Everybody loves somebody, symbolically, letting $L(x, y)$ be the statement "x loves y".
2. Tell whether the statement
$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$
is true.
3. Tell whether the statement
$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0))$$
is true.

Example

Example 58

Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. The following pseudocode determines whether $\forall x \forall y P(x, y)$ is true or false:

```
for i = 1 to n
    if (¬ exists_dj(i))
        return false
return true
exists_dj(i) {
    for j = 1 to n
        if (P(d_i, d_j))
            return true
    return false
}
```

Figure