ECE/CS 584: Embedded and cyberphysical system verification	Fall 2025
Your name, netid	*Homework 4: Hybrid Systems and CTL model*
	Due Novermber 23rd 11:59pm

Upload your writeup (.pdf) and code (.py) to Gradescope. In the writeup, put each answer on a separate page and label it with the correct number. Code for this homework can be downloaded here.:
https://uofi.box.com/s/ugwhbjik45exxbn10ekz9n8g12pwllel

160 points total

Problem 1. (15 points). Consider a *deterministic* hybrid automaton $\mathcal{A} = (V, \Theta, A, \mathcal{D}, \mathcal{T})$. For every action $a \in A$, the transition is specified by a guard (**pre**condition) $pre_a \subseteq val(V)$ and an **eff**ect function $\mathit{eff}_a : val(V) \to val(V)$. Every trajectory $\tau \in \mathcal{T}$ is solution of some differential equation.

Let $\alpha(\mathbf{x}_0, t)$ be the state reached by the execution of $\mathcal{A}$ from the initial state $\mathbf{x}_0$ at time t. *Assume that all effect maps eff_a are continuous, that trajectories starting from any admissible initial state follow the same sequence of actions/modes, and that each switching time varies continuously with respect to the initial state.* Under these assumptions, show $\alpha(x_0, t)$ is a continuous function of x_0. Start by writing the definition of an execution α in terms of pre_a, eff_a, and solutions.

Problem 2. (25 points). Consider an idealized billiard table of length $a > 0$ and width $b > 0$. The table has no pockets; its surface has no friction; and its boundary bounces the balls perfectly. The balls have some initial velocities. A ball bounces off a wall when its position is at the boundary. Balls collide whenever $|x_1 - x_2| \leq \epsilon$ and $|y_1 - y_2| \leq \epsilon$ and their velocity vectors are pointing towards each other. Here $\epsilon > 0$ is a small positive constant. Whenever a bounce occurs, the appropriate velocity changes sign. Whenever a collision occurs, the balls exchange their velocity vectors.

(a) Write a hybrid automaton model $\mathcal{A}$ of the position of **two** balls of equal mass on this table. That is, write the code specifying the automaton using the language we have been using in lectures. Wall bounces are modeled by an action called bounce, and each collision is modeled by an action called collision.

(b) State and prove conservation of speed along each axis as an invariant property of the automaton $\mathcal{A}$.

Problem 3. (20 points). (a) Consider two satellites orbiting Earth in circular orbits with constant angular speeds ω_1 and ω_2. Write a hybrid automaton model of the position of the satellite pair with state in $[0, 2\pi]^2$. When the angular position of one of the satellites hit 0 or 2π, its position has to be reset appropriately. Model that using an action called jump.

(b) The orbit of the satellite pair can be seen as a billiard ball on a frictionless $[0, 2\pi] \times [0, 2\pi]$ table following the usual rules of wall-reflection. What conclusions can you reach from your answer to (a) regarding the relative position of satellites?

Problem 4. (15 points) Construct the region automaton corresponding to the timed automaton illustrated below. Assume that model with $x = 0$ is the starting state.

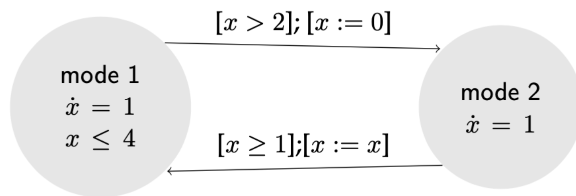

Figure 1: Timed Automaton to Problem 5

Problem 5. (25 points) (a) Convert the Initialized Rectangular Hybrid Automaton illustrated below to a timed automaton. Clocks may be initialized to constant or constant intervals. For each arrow, a label of the form $[A][B]$ corresponds to a set of transitions, where $[A]$ is a precondition/-guard and $[B]$ is the corresponding effect or the reset function. No reset [] implies that the state variables are not reset in the poststate of the transition. Draw a circle-arrow diagram using any software tool or scan and attach a hand-drawn figure.

(b) Plot an execution of the original hybrid automaton and the corresponding execution of the timed automaton.

Problem 6. (25 points). (a) Model the following hysteresis-based switching system as a hybrid automaton. The automaton $\mathcal{A}$ has (at least) n continuous variables $x_1, \ldots, x_n$ and a discrete variable called m that takes the values in the set $m_1, \ldots, m_n$. For a state $\mathbf{x}$ of the automaton, we say that the system is in *mode* m_i, if $\mathbf{x}.m = m_i$. There are $n \times (n-1)$ actions $\mathsf{switch}(i, j)$, where $i, j \in [n]$ and $j \neq i$. When the system is in mode m_i,

$$\dot{x}_i = a_i x_i,$$

where $a_i > 0$ is a positive constant and $\dot{x}_j = 0$ for all $j \neq i$. Further, when the system is in mode m_i, for any $j \neq i$, if x_i becomes greater than $(1 + h)x_j$, then the automaton switches to mode m_j; otherwise, it continues in mode m_i. Here, $h > 0$ is a parameter of the model. Is your model deterministic?

(b) Write a simulator for this model and plot two executions starting from two different initial states: one where all the x_i's are same, and another where they are different. For each execution, plot the duration that $\mathcal{A}$ stays in each mode.

(c) Do you observe something interesting about how long $\mathcal{A}$ stays in each mode? Can you conjecture an invariant property of the automaton? No need to prove anything.

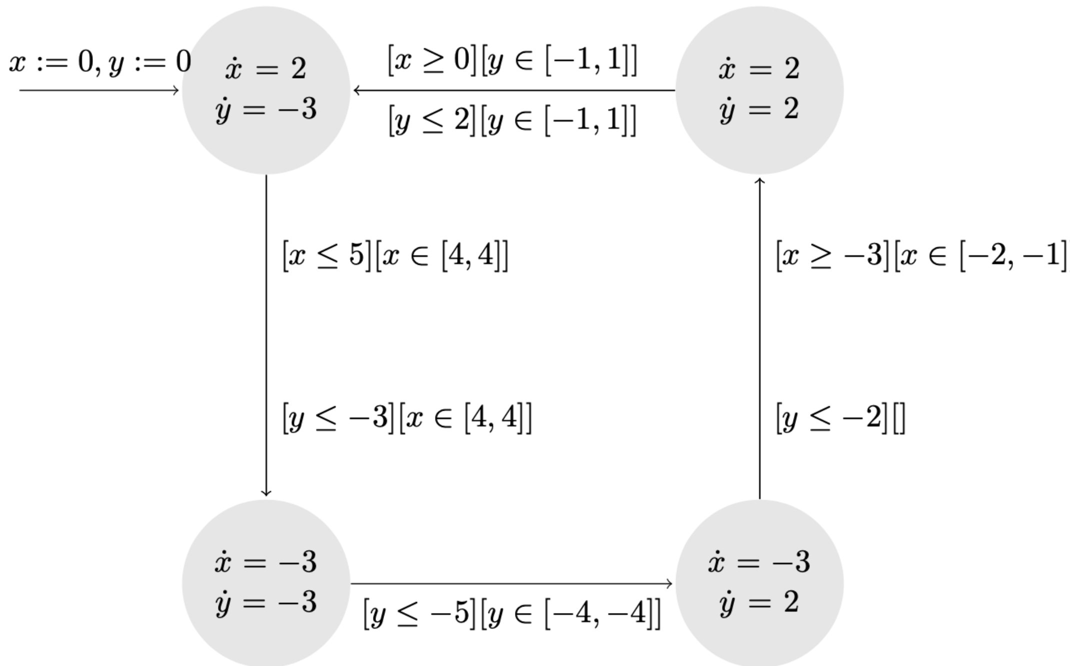

Figure 2: Initialized Rectangular Hybrid Automaton of Problem 4

Problem 7. (15 points). Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite-state automata that are bisimilar with a bisimulation relation $R \subseteq \text{val}(X_1) \times \text{val}(X_2)$. Recall that this means R is a forward simulation relation from $\text{val}(X_1)$ to $\text{val}(X_2)$, and R^{-1} is a forward simulation relation from $\text{val}(X_2)$ to $\text{val}(X_1)$. Show that $\text{Reach}_{\mathcal{A}_2} \subseteq R(\text{Reach}_{\mathcal{A}_1})$.

Problem 8. (20 points). Do this only if CTL model checking is covered in class. Consider the following automaton $\mathcal{A} = \langle Q, Q_0, T, L \rangle$. The set of states $Q = \{s_0, \cdots, s_4\}$, initial states $Q_0 = \{s_0, s_3\}$, the set of atomic propositions $AP = \{a, b\}$, transitions T, and the state labels L are shown in the figure. Consider the following CTL formulas:

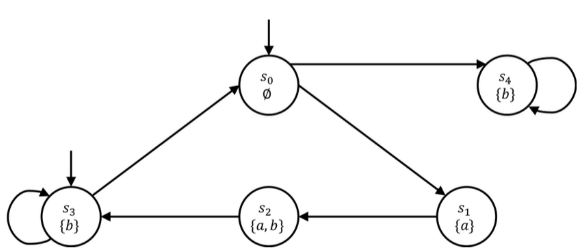

Figure 3: Automaton $\mathcal{A}$ with state $Q = \{s_1, \cdots, s_4\}$. State labels (atomic propositions) are shown. under each state.

(a) $\phi_1 = \mathbf{A}(a\mathbf{U}b) \vee \mathbf{EX}(\mathbf{AG}b)$

(b) $\phi_2 = \mathbf{AGA}(a\mathbf{U}b)$

For each formula ϕ_i, determine the set of states that satisfy it, and state whether $\mathcal{A}$ satisfies it.