On Measurable Median Value Convex Function in Open Interval

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Abstract:

In this paper, we prove that a (strict) measurable median value convex function in open interval is (strict) equal to a measurable convex function in open interval, hence we have that it is also a continuous function.

Keyword: Measurable, Median Value Convex Function

1. Introduction

Convex function is a kind of function with good properties. It is continuous and almost everywhere differentiable on the open interval, and Lipschitz continuous on any inner closed subinterval of the open interval. The definition of median convex function is much weaker than that of convex function, but the equivalence of the above two definitions can be proved as long as measurable conditions are added Two definitions are introduced first:

Definition 1 1.

Convex function: if f (x) is defined on (a, b) and satisfies $\forall$ x1, x2 $\in$ (a, b) , $\forall\lambda \in$ (0, 1), we all have that:

$$f(\lambda x1 + (1 - \lambda) x2) \leq \lambda f(x1) + (1 - \lambda) f(x2),$$

Then f (x) is called a convex function on (a, b). If the above inequality takes a strict inequality sign when x1 $\neq$ x2, then f (x) is called Strictly convex functions on (a, b).

Definition 1 2.

Median convex function: if f (x) is defined on (a, b) and satisfies $\forall$ x1, x2 $\in$ (a, b), we keep have that:

$$f\left(\frac{x1 + x2}{2}\right) \leq \frac{1}{2}(f(x1) + f(x2))$$

Then f (x) is called a median convex function on (a, b). If the above inequality takes a strict inequality sign when x1 $\neq$ x2, then f (x) is called a strict median convex function on (a, b).

2. The Equality of Median Value Convex Function and Convex Function on (a, b)

The following proof process is to first prove that the measurable convex function on the open interval is bounded on any closed sub interval, and then obtain that the bounded measurable convex function is continuous. From the

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continuity, the equivalence of the two definitions of convex function and median convex function is obtained. Finally, it shows that for measurable function, strict convex function and strict median convex function are also equivalent. We first list the main theorem of this section:

Theorem 2.1

If f(x) is a (strict) median convex function on (a, b) and it is measurable, then f(x) is also a (strict) convex function on (a, b).

Lemma 2.1

If f(x) is a median convex function on (a, b), then $\forall$ x1, x2 $\in$ (a, b) , $\forall\lambda \in$ (0, 1)$\cap$ Q, we all have that:

$$f(\lambda x1 + (1-\lambda) x2) \le \lambda f(x1) + (1-\lambda) f(x2),$$

Proof:

For $\forall n \in N$, $\forall x_1 \epsilon (a, b) (i = 1,2, \dots, 2^n)$, then we have the following Inequality by Induction:

$$f(\frac{x_1 + x_2 + \cdots x_{2^n}}{2n}) \leqslant \frac{1}{2^n}\sum_{i=1}^{2n} f(x_1),$$

Then we notice that for any n $\in$ N, n$< 2^n$, so we have that:

$$\forall n \in \mathbb{N}, \forall x_i \in (a,b)(i = 1,2,\cdots n),$$

$$f(\frac{x_1 + x_2 + \cdots x_n}{n})$$

$$= f\left(\frac{1}{2^n}(\underbrace{\frac{x_1 + x_2 + \cdots x_n}{n} + \cdots + \frac{x_1 + x_2 + \cdots x_n}{n}}_{n项}) + \frac{(2^n - n)}{2^n}\frac{x_1 + x_2 + \cdots x_n}{n}\right)$$

$$= f\left(\frac{1}{2^n}x_1 + \frac{1}{2^n}x_2 + \cdots + \frac{1}{2^n}x_n + \frac{(2^n - n)}{2^n}\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

$$\leqslant \frac{1}{2^n}(f(x_1) + f(x_2) + \cdots + f(x_n)) + \frac{(2^n - n)}{2^n}f(\frac{x_1 + x_2 + \cdots x_n}{n})$$

Hence, we have that:

$$f\left(\frac{x_1 + x_2 + \cdots x_n}{n}\right) \leqslant \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n))$$

For any $\lambda \in (0, 1)\cap Q$, $\lambda = \frac{m}{n}$, where m, n$\in$ N, so we get that:

$$\forall x_1, x_2 \in (a,b),$$

$$f(\frac{m}{n}x_1 + \frac{n-m}{n}x_2) = f(\frac{\overbrace{x_1 + x_1 + \cdots x_1}^{m项} + \overbrace{x_2 + x_2 + \cdots x_2}^{n-m项}}{n})$$

Remark 2.1

For any strict median convex function, if there exist $x_i \neq x_j$, then **Lemma 2.1** can strictly hold.

Lemma 2.2

We have following properties for any given median convex function on (a, b):

(1) Any strict bounded median convex function on (a, b) is a.e. continuous.

(2) For any $[c, d] \subset (a, b)$, Any median convex function on (a, b) is bounded.

(3) Any measurable median convex function on (a, b) is a.e. continuous.

(4) If f(x) is a measurable median convex function on (a, b), then f(x) is a convex function on (a, b)

Proof:

(1) See [GG81] page 288.

(2) See [Zhou04] page 146.

(3) For any $x \in (a, b)$, we always have that $[\frac{a+x}{2}, \frac{x+b}{2}] \subset [a, b]$. So it is a corollary from (1), (2).

(4) According to (3), f(x) is a continuous function in (a,b), so we have that:

$$\forall \lambda \in (0,1), \exists \{\lambda_n\}_{n=1}^{\infty} \in \subset (0,1) \text{ 使得} \lim_{n\to\infty} \lambda_n = \lambda$$

We also have that for $\forall x_1, x_2 \in (a, b)$:

$$f(\lambda_n x_1 + (1 - \lambda_n) x_2) \leqslant \lambda_n f(x_1) + (1 - \lambda_n) f(x_2)$$

Now that, we notice this limitation equality:

$$\lim_{n\to\infty} \lambda_n x_1 + (1 - \lambda_n) x_2 = \lambda x_1 + (1 - \lambda) x_2$$

So according to f(x) is a continuous function, we finally have that:

$$\begin{aligned} & f(\lambda x_1 + (1 - \lambda) x_2) \\ & = \lim_{n\to\infty} f(\lambda_n x_1 + (1 - \lambda_n) x_2) \\ & \leqslant \lim_{n\to\infty} (\lambda_n f(x_1) + (1 - \lambda_n) f(x_2)) \\ & = \lambda f(x_1) + (1 - \lambda) f(x_2) \end{aligned}$$

Lemma 2.3

If f(x) is a strict measurable median convex function on (a, b), then f(x) is a strict convex function on (a, b)

Proof:

According to Lemma 2.2 (4), f(x) is a convex function on (a, b). Now we assume that it is not a strict convex function on (a, b), then:

$$\exists x, y \in (a,b)(x < y), \exists z \in (x,y), z \neq \frac{x+y}{2}, \text{使得}$$

$$f(z) = \frac{y-z}{y-x} f(x) + \frac{z-x}{y-x} f(y),$$

Hence, for $\forall x_1 \in (x, z)$ and $\forall x_2 \in (z, y)$, we have that:

$$\begin{aligned} f(z) & \leqslant \frac{y_1 - z}{y_1 - x_1} f(x_1) + \frac{z - x_1}{y_1 - x_1} f(y_1) \\ & \leqslant \frac{y_1 - z}{y_1 - x_1} \left(\frac{z - x_1}{z - x} f(x) + \frac{x_1 - x}{z - x} f(z) \right) + \frac{z - x_1}{y_1 - x_1} \left(\frac{y - y_1}{y - z} f(z) + \frac{y_1 - z}{y - z} f(y) \right) \end{aligned}$$

$$x_1)(y-z)+(y_1-z)(z-x_1)(z-x)]$$
$$=f(z)$$
$$+\left[\frac{(y_1-z)(z-x_1)}{(y_1-x_1)(z-x)}f(x)+\frac{(z-x_1)(y_1-z)}{(y_1-x_1)(y-z)}f(y)\right]$$
$$=\frac{f(z)}{(y_1-x_1)(z-x)(y-z)}[(y_1-z)(x_1-x)(y-z)+(z-x_1)(y-y_1)(z-x)+(y_1-z)(z-$$

Where by direct computing, we get that the

$$(y_1-z)(x_1-x)(y-z)+(z-x_1)(y-y_1)(z-x)+(y_1-z)(z-x_1)(y-z)+(y_1-z)(z-x_1)(z-x)$$

is in fact equal to $(\mathbf{z}-\mathbf{x})(\mathbf{y}-\mathbf{z})(\mathbf{y_1}-\mathbf{x_1})$.

That means for any $\mathbf{x_1}\in(\mathbf{x},\mathbf{z}),\mathbf{x_2}\in(\mathbf{z},\mathbf{y})$, we always have that:

$$f(x_1)=\frac{z-x_1}{z-x}f(x)+\frac{x_1-x}{z-x}f(z)=\frac{y-x_1}{y-x}f(x)+\frac{x_1-x}{y-x}f(y)$$
$$f(y_1)=\frac{y-y_1}{y-z}f(z)+\frac{y_1-z}{y-x}f(y)=\frac{y-y_1}{y-x}f(x)+\frac{y_1-x}{y-x}f(y)$$

Hence, we have that for any $\mathbf{w}\epsilon(\mathbf{x},\mathbf{y})$

$$f(w)=\frac{y-w}{y-x}f(x)+\frac{w-x}{y-x}f(y)$$

If we let $w=\frac{x+y}{2}$, then we have that $f(\frac{x+y}{2})=\frac{1}{2}(f(x)+f(y))$, this makes a contradiction to f(x) is a strict measurable median convex function on (a, b). We finish the proof.

To put it in a nut shell, from **lemma 2.3**, **lemma 2.2(4)**, we finally prove **theorem 2.1**.

Remark 2.2

The other direction of theorem 2.1 is an obvious according to the definition.

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Reference

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