Preconditioning of systems of partial differential equations[1]

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Outline

Krylov Space Methods

Assumption

Let X be a separable, real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$, and assume that $\mathscr{A} : X \to X$ is a symmetric isomorphism on X, i.e.

$$\mathscr{A}, \mathscr{A}^{-1} \in \mathscr{L}(X, X).$$

Krylov Space Methods

Linear System

Find $x \in X$ such that

$$\mathscr{A} x = f,$$

where the right-hand side $f \in X$ is given.

Definition

If the operater $\mathscr{A}$ and the right-hand side f are given, then the Krylov space of order m is given as

$$K_m = K_m(\mathscr{A}, f) = \mathrm{span}\{f, \mathscr{A} f, \ldots, \mathscr{A}^{m-1} f\}.$$

Krylov Space Methods

Solution and Approximation

If the operater $\mathscr{A}$ is SPD, the unique solution $x \in X$ of the linear system can be characterized as

$$x = \arg\min_{y \in X} E(y),$$

where

$$E(y) = \langle \mathscr{A} y, y \rangle - 2\langle f, y \rangle.$$

The approximation $x_m \in K_m$ is defined as

$$x_m = \arg\min_{y \in K_m} E(y),$$

Alternatively, $x_m \in K_m$ solves the Galerkin system

$$\langle \mathscr{A} x_m, y \rangle = \langle f, y \rangle, \qquad y \in K_m.$$

This leads to the conjugate gradient method.

Krylov Space Methods

Theorem

If the operater $\mathscr{A} : X \to X$ is a SPD isomorphism. If the sequence $\{x_m\}$ is generated by the conjugate gradient method, then

$$\|x - x_m\|_{\mathscr{A}} \leqslant 2\alpha^m \|x - x_0\|_{\mathscr{A}} ,$$

where $\|x\|_{\mathscr{A}} = \langle x, \mathscr{A} x \rangle$, and $\alpha = (\sqrt{\kappa(\mathscr{A})} - 1)/(\sqrt{\kappa(\mathscr{A})} + 1)$.

Krylov Space Methods

Solution and Approximation

If the operater $\mathscr{A}$ is symmetric, the unique solution $x \in X$ of the linear system can be characterized as

$$x = \arg\min_{y \in X} \|\mathscr{A} y - f\|^2 .$$

The approximation $x_m \in K_m$ is defined as

$$x = \arg\min_{y \in K_m} \|\mathscr{A} y - f\|^2 .$$

Krylov Space Methods

Theorem

If the operater $\mathscr{A} : X \to X$ is a symmetric isomorphism. If the sequence $\{x_m\}$ is generated by the minimum residual method, then

$$\|\mathscr{A}(x - x_{2m})\| \leqslant 2\alpha^m \|\mathscr{A}(x - x_0)\|,$$

where $\alpha = (\kappa(\mathscr{A}) - 1)/(\kappa(\mathscr{A}) + 1)$.

Example

Laplace equation

Let Ω be a bounded domain in $\mathbb{R}$. Consider the Laplace equation with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Example

Weak Formulation

Denote

$$X = H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) = X^*.$$

Define $\mathscr{A} : X \to X^*$ by

$$\langle \mathscr{A} u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \qquad u, v \in X.$$

The weak formulation for the problem is

$$\mathscr{A} u = f,$$

where the right-hand side $f \in X^*$ is given, and the unknown $u \in X$.

Preconditioning

- In fact, the Krylov space method can't be used directly when solving the problem above.
- Since the operator $\mathscr{A}$ map functions in X out of the space, and has unbounded spectrum.
- How can we solve the problem with the Krylov space method?
- Now we introduce a preconditioner.

Preconditioning

Assumptions

1. X is a Hilbert space, and X^* is the dual of X.
2. $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X, while $\langle \cdot, \cdot \rangle_X$ is the inner product on X.
3. The operator $\mathscr{A} \in \mathscr{L}(X, X^*)$ is an isomorphism mapping. Also $\mathscr{A}$ is symmetric in the sense that $\forall x, y \in X$, $\langle \mathscr{A} x, y \rangle = \langle \mathscr{A} y, x \rangle$.

Linear System

Find $x \in X$ such that

$$\mathscr{A} x = f,$$

where the right-hand side $f \in X^*$ is given.

Preconditioning

Defination

The preconditioner $\mathscr{B} : X^* \to X$ for $\mathscr{A}$ is an isomorphism mapping. Furthermore, $\mathscr{B}$ is SPD in sence that $\langle \cdot, \mathscr{B} \cdot \rangle$ is an inner product on X^*.

Remark

1. The preconditioner is a Riesz operator mapping X^* to X.
2. $\langle \mathscr{B}^{-1} \cdot, \cdot \rangle$ is an inner product on X.
3. The operater $\mathscr{B}\mathscr{A} \in \mathscr{L}(X, X)$ is an isomorphism mapping X to itself.
4. The operater $\mathscr{B}\mathscr{A} \in \mathscr{L}(X, X)$ is symmetric in the inner product $\langle \cdot, \mathscr{B} \cdot \rangle$ on X, but may not be symmetric in the original inner product $\langle \cdot, \cdot \rangle_X$.
5. If the operater $\mathscr{A}$ is SPD. Then $\mathscr{B}\mathscr{A}$ is SPD with respect to the inner product $\langle \mathscr{B}^{-1} \cdot, \cdot \rangle$ and $\langle \mathscr{A} \cdot, \cdot \rangle$.
6. The preconditioner $\mathscr{B}$ is not unique.

Example of Second-order Elliptic operaters

Example of Second-order Elliptic operaters

The elliptic operator $\mathscr{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle \mathscr{A} u, v \rangle = a(u, v) = \int_\Omega (A \nabla u) \cdot \nabla v \,\mathrm{d}x, \qquad u, v \in H_0^1(\Omega).$$

Here, we assume that $A = A(x) \in \mathbb{R}^{n \times n}$ is uniformly SPD, i.e. there are positive constants c_0 and c_1 such that

$$c_0 \left|\xi\right|^2 \leqslant \xi^T A(x) \xi \leqslant c_1 \left|\xi\right|^2, x \in \Omega, \xi \in \mathbb{R}^n.$$

Then the preconditioner could be

$$\mathscr{B} = (-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega).$$

The condition number of $\mathscr{B}\mathscr{A}$ satisfies $\kappa(\mathscr{B}\mathscr{A}) \leqslant c_1 / c_0$.

Example of Parameter-dependent Problem

Reaction-diffusion equation

Consider the boundary value problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = f, & \text{in } \Omega. \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$ is a small parameter.

- Our goal is to produce a preconditioner $\mathscr{B}$ for the operater $\mathscr{A}$ such thst the condition number $\kappa(\mathscr{B}\mathscr{A})$ could be uniformly bounded with respect to the parameters.
- The natural norm for the solution u is

$$\|u\|_{l^2 \cap \varepsilon H_0^1} = (\|u\|_0^2 + \varepsilon^2 \|\nabla u\|_0^2)^{1/2},$$

where $\|\cdot\|$ denotes the norm in $H^s(\Omega)$.

Example of Parameter-dependent Problem

- We want to find a norm for f, such that

$$\|u\|_{L^2\cap\varepsilon H_0^1} \leqslant c\,\|f\|_{?}\,,$$

where the constant c is independent of ε. Formally we have

$$u = (I-\varepsilon^2\Delta)^{-1}f, \qquad \|u\|_{L^2\cap\varepsilon H_0^1} = \langle (I-\varepsilon^2\Delta)u, u\rangle.$$

- Notice

$$\begin{aligned}\langle (I-\varepsilon^2\Delta)u, u\rangle =& \langle f, (I-\varepsilon^2\Delta)^{-1}f\rangle \\ =& \langle (I-\varepsilon^2\Delta)^{-1}f, (I-\varepsilon^2\Delta)^{-1}f\rangle \\ &+ \varepsilon^2\langle (-\Delta)(I-\varepsilon^2\Delta)^{-1}f, (I-\varepsilon^2\Delta)^{-1}f\rangle.\end{aligned}$$

Example of Parameter-dependent Problem

- We can define an ε-dependent norm on f by

$$\begin{aligned}\|f\|_{*\varepsilon}^2 &= \langle f, (I - \varepsilon^2\Delta)^{-1} f\rangle \\ &= \inf_{f=f_0+f_1, f_0\in L^2, f_1\in H^{-1}} (\|f_0\|_0^2 + \varepsilon^{-2}\|f_1\|_{-1}^2).\end{aligned}$$

- Then

$$\|u\|_{I^2\cap\varepsilon H_0^1} = \|f\|_{*\varepsilon}^2 .$$

- The preconditioner is $(I - \varepsilon^2\Delta)^{-1}$.

Reference

K. A. Mardal and R. Winther, “Preconditioning discretizations of systems of partial differential equations,” *Numerical Linear Algebra with Applications*, vol. 18, no. 1, pp. 1–40, 2011.