A General form of Residual Pairing Formula on Kähler Manifold

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Abstract

We give a general form of residual pairing formula which is proposed by Fan H. and Shen Y. in [FS16], Theorem 3.4 for a given noncompact complete Kähler manifold M and a strongly tame holomorphic function defined on M.

Keywords: Residual Pairing, Kähler Manifold

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1. Introduction

For a clear state of our main formula, we will go over some preconditions and settings of this formula. Firstly, we recall the $\bar{\partial}_f$-Poincaré lemma for Kähler case and an inference of this lemma([Fan11] and [Cai21]). Without special statement, in this paper we set all Kähler manifold M is a noncompact and complete manifold, and its dimension is N. We also let f keep being a tamed holomorphic function defined on M, then the operator $\bar{\partial}_f = \bar{\partial} + \partial f \wedge$.

Theorem 1.1($\bar{\partial}_f$-Poincaré lemma)

Let $U \subset M$ be a simply connected domain of the noncompact complete Kähler manifold. Let φ be $\bar{\partial}_f$-closed k-form on U. Then there exists a (k-1)-form ψ and a holomorphic (k,0)-form ϕ unique modulo $df \wedge \Omega^{k-1}(M)$ such that:

$$\varphi = \phi + \bar{\partial}_f \psi$$

This theorem leads to the following lemma about the existence of a compactly supported form representation of $\bar{\partial}_f$-cohomology class.

Lemma 1.1

Let f be a holomorphic tamed function on M, then for any given $\bar{\partial}_f$-cohomology class in $\Lambda^k(M)$, it can be represented by a compactly supported k-form.

Let α be a $\bar{\partial}_f$-closed N-form, then according to lemma 1.1, we can select a holomorphic N-form $A \in \Omega^N(M)/df \wedge \Omega^{N-1}(M)$ and a (k-1)-form β, that is $A = a\, dz_1 \wedge dz_2 \ldots \wedge dz_N$ and $\beta \in \Omega^{k-1}(M)$, such that $\alpha = A + \bar{\partial}_f \beta$ has a compactly support. Then if we consider two $\bar{\partial}_f$-closed forms α_1 and α_2, let $\alpha_1 = A_1 + \bar{\partial}_f \beta_1$, $\alpha_2 = A_2 + \bar{\partial}_{-f} \beta_2$. We just need to assume that one of each has a compact support, then we are going to prove the following formula:

Theorem 1.2 (Residual Pairing Formula For Kähler Case)

Let f be a holomorphic tamed function on M, set $\alpha_1 = A_1 + \bar{\partial}_f \beta_1$, $\alpha_2 = A_2 + \bar{\partial}_{-f} \beta_2$, where $A_i \in \Omega^N(M)/df \wedge \Omega^{N-1}(M)$, i=1,2. We assume that one of them has a compact support. Then we have the following formula:

$$\int_{\mathbf{M}} \alpha_1 \wedge \alpha_2 = \sum_{p_i \in (df)^{-1}(0)} \mathrm{Res}_{p_i}((2\pi i)^N \frac{A_1 A_2 dz_1 \wedge dz_2 \ldots \wedge dz_N}{\partial_1 f\, \partial_2 f \ldots \partial_N f})$$

The main goal of this paper is going to prove **Theorem1.2**.

2. Dolbeault Representative η_ω and The Global Residue Theorem

We simply recall Dolbeault Representative and The Global Residue Theorem in this section. (See[GH78]), for they will play an important role in the later proof.

Let M⊂ M′are given complex N-mainfolds, where M is relatively compact with smooth boundary. Now we suppose that D_1 ,..., D_N are effective divisors defined in some neighborhood U of $\overline{M}$ in M′ an whose intersection is a finite set. We let :

$$D=D_1 + D_2+\ldots+D_N$$
$$U^* = U - (D_1 \cap D_2 \ldots \cap D_N)$$
$$U_i = U - D_i$$

Since $\underline{U} = \{U_i\}$ is an open covering of U^*. We set $\omega \in H^0(U, \Omega^N(D))$ is a meromorphic n-form on U with polar divisor D. For each point P∈ $U_1 \cap U_2 \ldots \cap U_N$ we can restrict ω to a neighborhood U_p of P and define the residue $\mathrm{Res}_p\omega$. Now we give the definition of Dolbeault Representative under above symbols' setting:

Definition 2.1 (Dolbeault Representative)

Let $\omega \in C^{N-1}(\underline{U}, \Omega^N)$, then it is obvious that $\delta\omega$ =0, which induces a class in $H^{n-1}(U^*, \Omega^N)$. We consider Dolbeault isomorphism $H^{N-1}(U^*, \Omega^N) \cong H_{\bar\delta}^{N,N-1}(U^*)$. Then the Dolbeault representative η_ω is the image of $\frac{\omega}{(2\pi i)^N}$ under Dolbeault isomorphism.

Then we can use Dolbeault Representative η_ω to calculate global residue by using the following theorem:

Theorem2.1 (Global Residue Theorem)

$$\sum_P \mathrm{Res}_p\omega=(2\pi i)^N \int_{\partial M} \eta_\omega$$

Proof:

See [GH78] P656.

Remark 2.1

By different definitions of residual, sometimes it may be different in a constant $(2\pi i)^N$ meaning.

3. Local Solution of external differential equation $\bar\partial_f \beta_1 = A_1$

As another lemma which will be used in the later proof of Residual Pairing Formula,

let $U_p(\varepsilon)$ be a ε -ball around $P \in M$, by a local coordinate representation z=$(z_1, z_2 \ldots z_N)$ covering $U_p(\varepsilon)$. We write fixed $A_1 = a_1(z)dz_1 \wedge dz_2 \ldots \wedge dz_N$. We will solve the external differential equation $\bar\partial_f\beta_1 = A_1$ on $U_p(\varepsilon)$ locally.

Lemma 3.1

The solution of $\bar\partial_f\beta_1 = A_1$ on $U_p(\varepsilon)$ is following (k-1)-form $(-a_1(z)\mu)$, where:

$$\begin{cases} \mu = \sum_{j=0}^{N-1}(-1)^j\gamma_j \\ \gamma_k = \sum_{i=1}^{N}(-1)^{i-1}\lambda_i \sum_{j_1\neq i}\dots\sum_{j_k\neq i} dz_1 \dots\wedge \widehat{dz_\iota}\wedge\dots\wedge\bar\partial\lambda_{j_1}\wedge\dots\wedge\bar\partial\lambda_{j_k}\wedge\dots\wedge dz_N \end{cases}$$

Proof:

This solution method is adapted by the method in [FS16] locally.

Firstly, we set $g=f_{|U_p(\varepsilon)}$, we assume that L_i is a divisor defined by $\partial_i g = 0$, $U_p(\varepsilon)\setminus\{z: dg(z)=0\} = \cup_i U_i$. On U_i, we define: $\lambda_i = \frac{\overline{\partial_i g}}{|dg|^2}$, $\rho_i = \lambda_i\,\partial_i g$ .Since $\sum_i \rho_i = 1, \rho_i > 0$, ρ_i is an unit decomposition of cover $\{U_i\}$.Now we let $\omega = dz_1\wedge dz_2 \dots\wedge dz_N$,We will firstly solve the equation $\overline{\partial_g}\mu = \bar\partial\mu + dg\wedge\mu = \omega$.Firstly, we set $\gamma_0 = \sum_j(-1)^{j-1}\lambda_j dz_1\dots\wedge\widehat{dz_j}\wedge\dots dz_N$.Then we get $dg\wedge\gamma_0 = \omega$.

Now we decompose μ into a series of unknow (N-j-1,j)-form γ_j:

$$\mu = \textstyle\sum_{j=0}^{N-1}(-1)^j\gamma_j, \gamma_N = 0$$

Hence the original equation is equal to :

$$dg\wedge\gamma_j = \bar\partial\gamma_{j-1}, \qquad j = 1,2\dots N$$

We select a vector field X on $U_p(\varepsilon)\setminus\{z: dg(z)=0\}$, $X=\sum_{i=1}^N\lambda_i\frac{\partial}{\partial z_i}$, τ_X is the contraction operator induced by X. We assume that $\tau_X\gamma_j = 0$ then we can get:

$$\gamma_j = \tau_X\left(\bar\partial\gamma_{j-1}\right) = \sum_{i=1}^{N}\lambda_i\tau_{\frac{\partial}{\partial z_i}}(\bar\partial\gamma_{j-1})$$

Obviously, $\tau_X\gamma_j = \tau_X\circ\tau_X\left(\bar\partial\gamma_{j-1}\right) = 0$, so we get the following sequence of iterations $\{\gamma_j\}$:

$$\begin{cases} \gamma_0 = \sum_j(-1)^{j-1}\lambda_j dz_1\dots\wedge\widehat{dz_j}\wedge\dots dz_N, \gamma_N = 0 \\ \gamma_j = \sum_{i=1}^{N}\lambda_i\tau_{\frac{\partial}{\partial z_i}}(\bar\partial\gamma_{j-1}) \end{cases}$$

When j=1:

We have $\gamma_1 = \sum_{i=1}^N\lambda_i\tau_{\frac{\partial}{\partial z_i}}\left(\bar\partial\gamma_0\right)$

$=\sum_{i=1}^N\lambda_i\tau_{\frac{\partial}{\partial z_i}}\left(\sum_j(-1)^{j-1}\bar\partial\lambda_j dz_1\dots\wedge\widehat{dz_j}\wedge\dots dz_N\right)$

$=\sum_{i=1}^N\lambda_i\tau_{\frac{\partial}{\partial z_i}}\left(\sum_j dz_1\dots\wedge\bar\partial\lambda_j\wedge\dots dz_N\right)$

$=\sum_{i=1}^N(-1)^{i-1}\lambda_i\sum_{j\neq i}dz_1\dots\wedge\widehat{dz_\iota}\wedge\dots\wedge\bar\partial\lambda_j\wedge\dots dz_N$

According to mathematical induction, we finally get a group of solution of this equation:

$$\gamma_k = \sum_{i=1}^{N}(-1)^{i-1}\lambda_i\sum_{j_1\neq i}\dots\sum_{j_k\neq i}dz_1\dots\wedge\widehat{dz_\iota}\dots\wedge\bar\partial\lambda_{j_1}\wedge\dots\wedge\bar\partial\lambda_{j_k}\wedge\dots\wedge dz_N$$

Where $\bar\partial\lambda_{j_s}$ represents that it is located at the $j_s{}^{th}$ location, hence $\beta_1 = -a_1(z)\mu$.

4.Local Residual Pairing Formula for $U_p(\varepsilon)$ on Kähler Manifold M

In this section. We prove the Local Residual Pairing Formula for $U_p(\varepsilon)$ on a Kähler Manifold M, which is proved as the following **lemma 3.1,** where $U_p(\varepsilon)$ is an ε-ball around P.

Lemma 4.1

We still keep assuming that one of α_1 and α_2 has a compact support. Then locally we have the following residual pairing formula for $U_p(\varepsilon)$:

$$\int_{U_p(\varepsilon)} \alpha_1 \wedge \alpha_2 = (2\pi i)^N \int_{\partial U_p(\varepsilon)} \eta_\omega$$

Proof:

Without lose of generality. We assume that $\alpha_1 = A_1 + \bar{\partial}_f \beta_1$ has a compact support. Then by Kähler case Hodge theory for Schrödinger operator in [Fan11]

We have:

$$\int_{U_p(\varepsilon)} \alpha_1 \wedge \overline{\partial_{-f}}\beta_2 = (-1)^N \int_{U_p(\varepsilon)} \alpha_1 \wedge * \overline{\partial_{-f}}\beta_2 = (-1)^{N+1} \int_{U_p(\varepsilon)} \alpha_1 \wedge * \partial_f^{+} * \beta_2$$

$=(-1)^{N+1} \int_{U_p(\varepsilon)} \bar{\partial}_f \alpha_1 \wedge * \beta_2$, since α_1 is a (N,0)-form, o we get:

$\int_{U_p(\varepsilon)} \alpha_1 \wedge \overline{\partial_{-f}}\beta_2 = 0$, hence, by Stokes' Theorem:

$$\int_{U_p(\varepsilon)} \alpha_1 \wedge \alpha_2 = \int_{U_p(\varepsilon)} [(A_1 + \bar{\partial}_f \beta_1)] \wedge A_2 = \int_{\partial U_p(\varepsilon)} (A_2 \wedge \beta_1^{0,N-1})$$

Fixed A_1, now we compute β_1. By **lemma 3.1**:
we can directly get $\beta_1 = -a_1(z)\mu$, where:

$$\begin{cases} \mu = \sum_{j=0}^{N-1} (-1)^j \gamma_j \\ \gamma_k = \sum_{i=1}^{N} (-1)^{i-1} \lambda_i \sum_{j_1 \neq i} \dots \sum_{j_k \neq i} dz_1 \dots \wedge \widehat{dz_i} \wedge \dots \wedge \bar{\partial}\lambda_{j_1} \wedge \dots \wedge \bar{\partial}\lambda_{j_k} \wedge \dots \wedge dz_N \end{cases}$$

Hence we can get $\beta_1^{0,N-1} = (-1)^N a_1(z)\gamma_{N-1}$,on the other hand, we directly compute:

$$\gamma_{N-1} \wedge dz_1 \wedge dz_2 \dots \wedge dz_N = \frac{(N-1)!}{c_N} F^* K_{BM}^{N,N-1}$$

Where $F: U_p(\varepsilon) \to U_p(\varepsilon) \times U_p(\varepsilon)$, writing by local coordinates z=$(z_1, z_2 \dots z_N)$, $F(z) = (z + \nabla f(z), z)$ and $c_N = (-1)^N \frac{(N-1)!}{(2\pi i)^N}$, as well as $K_{BM}^{N,N-1}$ is the Bochner-Martinelli kernel.

Hence, we finally have that:

$$\int_{U_p(\varepsilon)} \alpha_1 \wedge \alpha_2 = \int_{\partial U_p(\varepsilon)} (-1)^N a_1(z) a_2(z) \gamma_{N-1} \wedge dz_1 \wedge dz_2 \dots \wedge dz_N$$

$$= \int_{\partial U_p(\varepsilon)} (-1)^N a_1(z) a_2(z) \frac{(N-1)!}{c_N} F^* K_{BM}^{N,N-1}$$

$$= (2\pi i)^N \int_{\partial U_p(\varepsilon)} \eta_\omega$$

5.Global Residual Pairing Formular on Kähler Manifold M

Now we finish the whole proof of Theorem 1.2:

Theorem 1.2 (Residual Pairing Formula For Kähler Case)

Let f be a holomorphic tamed function on M, set $\alpha_1 = A_1 + \bar{\partial}_f\beta_1$, $\alpha_2 = A_2 + \bar{\partial}_{-f}\beta_2$, where $A_i \in \Omega^N(M)/df \wedge \Omega^{N-1}(M)$, i=1,2. We assume that one of them has a compact support. Then we have the following formula:

$$\int_{\mathbf{M}} \alpha_1 \wedge \alpha_2 = \sum_{p_i \in (df)^{-1}(0)} \mathrm{Res}_{p_i}\left((2\pi i)^N \frac{A_1 A_2 dz_1 \wedge dz_2 \ldots \wedge dz_N}{\partial_1 f\, \partial_2 f \ldots\, \partial_N f}\right)$$

Proof:

Let $\omega = (2\pi i)^N \frac{A_1 A_2 dz_1 \wedge dz_2 \ldots \wedge dz_N}{\partial_1 f\, \partial_2 f \ldots\, \partial_N f}$, then we have:

Right=$\sum_P \mathrm{Res}_p \omega$

According to **Theorem 2.1**, we have:

$=(2\pi i)^N (\int_{\partial M} \eta_\omega)$

$=(2\pi i)^N (\sum_P \int_{\partial U_p(\varepsilon)} \eta_\omega)$

By **Lemma 3.1**:

$=\sum_P \int_{U_p(\varepsilon)} \alpha_1 \wedge \alpha_2$

$=\int_{\mathbf{M}} \alpha_1 \wedge \alpha_2 =$ Left

4. Reference

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