On the structure of Union-closed family and Poset

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Abstract

A finite family F is union-closed if and only if for any A, B ∈ F implies A ∪ B ∈ F. The Families defined like this are well-known for the union-closed conjecture.

In this paper, We mainly discuss the structure of Union-closed family based on several new definitions and viewpoints such as minimal element, maximal element, acyclic digraph in part II, III, IV. One of the results in this note is that we proof that an union-closed family is always built up by a fixed recurrence relation and several initial sets given at first (Therom3.1). We prove some results on the union-closed conjecture in part VI based on the structure of the union-closed family.

In part III, we also analyzed the structure of more general partially ordered set based on the method and definition used in part II and III.

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1. introduction

Definition 1.1

Let F be a family of sets which satisfied the following property: For any A, B $\in$ F ,there exists an element C=A $\cup$ B $\in$ F .Then we call this kind of family is union-closed.

Based on the definition1.1, Frankl writes down the following conjecture in 1979:

Conjecture 1.1

Frankl's conjecture states that: for any finite union-closed family of sets F$\neq${$\emptyset$} has an element that is contained in at least half of the member-sets.

Although this statement is simple and easy to understand and, it still wide open. About this system we have proofed some special cases and give several relevant estimations on this system. Generally speaking, there are three viewpoints on this conjecture, which are graph, sets, lattice, correspondingly, this conjecture can be translated into several major forms.

And there are also two mainly methods used to solve this method, which are set up a function between the sets which have the element a which is supposed to be the abundant one and those sets who don't have element a. If we prove this is an injection, then we prove the Frankl's conjecture. Another powerful way to solve is considering the averaging ,although it may not work for all the union-closed family, however, if we consider separated union-closed family, this tool seems to be able to solve all the cases ,furthermore to solve this conjecture

In set theory, the formulations can be stated as follows besides the form mentioned in conjecture one:

Conjecture 1.2

For any family of proper subsets of U closed under intersection, there must exist some element of U that belongs to at most half of the sets of F.

As for graph, the formulation is related to bipartite graph.it is stated as follows:

Conjecture1.3

Let G be a finite graph with at least one edge. Then there will be two adjacent vertices each belonging to at most half of the maximal stable sets.

It also can be translated into conjecture 4.

Conjecture 1.4

.Let G be a finite graph with at least one edge. Then there will be two adjacent vertices each belonging to at most half of the maximal stable sets.

Besides graph and sets theory, it can be also relevant to lattice, which has a formulation like this:

Conjecture1.5

Let L be a finite lattice with at least two elements. Then there is a join-irreducible element a with $|[a)| \leq 1/2\ |L|$.

No matter which form it states, one of the reasons on why this conjecture is difficult to solve is about the complex and varied forms of this kind of family. For example, when it comes to discuss the conjecture 5, some special cases have been solved: Reinhold proof that the conjecture 5 holds for

Lower semimodular lattice. Poonen shows that this conjecture is true for geometric, and then upper semimodular. Abe discuss the case for strong upper semimodular lattices. From the results on lattice form for this conjecture , we can find that although several cases are solved ,however , up to now it is only a subclass of lattices and so many cases are left to discuss, of course, we cannot rule out the possibility that this conjecture cannot be solved by just discussing more extra conditions, which means we finally find that all the lattices can be classified into several cases and every case is supposed to solve. The similiar gap between us and the answer to this conjecture also exists in graph and sets theory. Considering this, some researches have tried to estimate average. we just need to prove for every finite union-closed family F, the following posture is True:

$$\frac{1}{|\cup \mathrm{F}|} \cdot \sum_{a \in \cup F} |Fa| \geq \frac{1}{2}|\mathrm{F}| \quad (1)$$

In this posture, $\cup F$ refers to a set combined by all the elements exists in the family F. Fa refers to a set whose elements are the sets in F contain the element a. When we considering this principle, we don't have to point out which element who exists in over $\frac{1}{2}$ sets of the Family ,this is an advantage towards discussing some specific cases ,But this methods also have its limitations ,it is not always true. For example, we consider the following union-closed family:

{{a1},{a1,a2},{a1,a2,a3,a4,a5,a6,a7}}

The left side of (1) is smaller than right side but it does form a union-closed family, which means the average Value Principle doesn't always work for this system.

So it seems that unless we can find a powerful composite object hidden beside the system which works in all cases to tackle this conjecture or we can translate every complex case of union-closed family into several finite simple cases. It seems that one of the practical ways to research this conjecture is still that we make efforts to clarify every possible structure of union-closed family step by step.

The structure of the paper is as following:

In the second part of this paper, we will discuss the Layered decomposition of union-closed family, we will also see how a union-closed family is structure out on the viewpoints of minimal elements in poset. In the third part of the note, we will come back to the theory on poset, after we discuss some universal property on poset, we will know more clearly about the decomposition mentioned in part II. We discuss the structure of the union-closed family based on directed acyclic graph in part IV. We also give a few new results on the Union-closed family conjecture based on the structure we find out in part VI.

In order to discuss more convenient. Now we give several extra limitations on the union-closed family F, owing to the union-closed conjecture is equal to the following cases or some other obvious facts, although all these things mentioned below are well-known, However, for the completeness of the note, we present the proof of these cases below.

(I)

Firstly, we strengthen that we assume that |F| is a finite number, although we can build Union-closed family whose |F| is uncountable like $\{A|\forall x \in A \rightarrow x \in R\}$, however, considering our final purpose is to proof there exists an element x, which belongs to at least half of the union-closed family F,if F is uncountable, we can't define the concept half.

(II)

In this note, we assume that F is a union-closed family which doesn't have the set ∅ on account of the following reason:

Fact1.1

It's obviously that F who has ∅ is union-closed if and only if $F\backslash\{\emptyset\}$ is union-closed, if we proof for every

$F\backslash\{\emptyset\}$ there exists an element a which $|Fa|>\frac{1}{2}|F|$, then F satisfies conjecture1.1

(III)

We strengthen that F is a set of non-repetitive elements on account of the following example:

Example1.1

{{a1} {a1} {a1} {a2} {a2} {a2} {a3} {a3} {a3} {a1,a2} {a1,a3},{a2,a3},{a1,a2,a3}}

F is a family which has repetitive elements, for every pair of x, y $\in$ F,x$\cup$ y $\in$ F,however, it does not satisfy conjecture1.1.

(IV)

We call a1, a2$\in\cup$ F(a1$\neq$ a2) are identical distributed, if for each x$\in$ F suggests that a1and a2$\in$ x or a1 and a2 $\notin$x. And we define a union-closed family F is simplified if and only if for each pair of a1,a2(a1$\neq$ a2), they are not identical distributed.In this note, we assume that union-closed family F is simplified on account of the following reason:

Fact1.2

If conjecture 1.1 holds for the family which are simplified, then the conjecture holds for all the union-closed family.

Proof:

Let F1 be a union-closed family, if it is simplified, then it's done.

If there exists at least one set $A_x=\{y \in \cup F1 \mid x,y \text{ are identical distributed}\}$ which $|A_x|>1$, let $W=\{A_x \mid x \in \cup F1\}$, for any $A_x \in W$, we delete all the elements except x in Ax from F1 and let the new smaller sets A*replace the A in the old union closed family F1,then we get a new familyF2. Notice that this operation doesn't create set $A^*=\emptyset$,and it doesn't create $A^*=B^*$, so F2 is still a union-closed family.

Now we get a new union-closed family F2. We select an element a which exists in at least half of the sets in the family, then dating back to F1,notice that in fact this operation between F1 and F2 is a bijection. Then we find the element a in F1which exists in at least half of the sets in F1.

(V)

We assume that $|F| \geq 1$ and $|\cup F|=m$, m is a positive ,integer on account of the following reason on account of the following fact:

Fact1.3

Let union-closed family F satisfied (i), (iii), and $|F| \geq 1$ then $|\cup F|=m$, m is a positive integer.

On account of $\emptyset$ doesn't exist in $|F|$ and $|F| \geq 1$, so m>=1.

If F is a union-closed family F the universe $\cup$ F has infinite elements there will always exists a pair of x, y $\in$ F which are identical distributed. Because |F| is finite, so the |W| which W is mentioned in Fact 2.2 is smaller than $2^{|F|}$,there must be a pair of x, y in $\cup$ F which are identical distributed.It's conflict with F is simplified.

Remark 1.1

If we want to prove conjecture1.1, we can consider proofing the case when F satisfied(i) (iii) (IV)

2 .The Layered structure of Union-closed family

Definition 2.1

F is a finite union-closed family, Let Ω1 be a subfamily of F, We define that Ω1 is the 1-th source of F if:

$\Omega 1=\{A \in F \mid \forall B \in F, B \neq A \Rightarrow B \not\subset A\}$

Definition 2.2

F is a finite union-closed family or just a poset, Let Ωk be a subfamily of F,we define that Ωk (k≥ 2) is the k-th source of F if:

$\Omega k=\{A\in F\backslash\bigcup_{i=1}^{k-1}\Omega i \ |\forall\ B\in F\backslash\bigcup_{i=1}^{k-1}\Omega i\ ,B\neq A\Rightarrow B\not\subset A\ \}$.We call this operation is the minimal element decomposition of a finite union-closed family or a poset F.

We can find out some basic facts about Ωk ($k\geq 1$) which are defined in definition 1.1 and definition 1.2:

Definition2.3

Similarly, we can define the Maximal element decomposition of a finite union- closed family F or a poset F:

We define that $\Phi 1=\{A\in F|\forall B\in F, B\neq A\Rightarrow A\not\subset B\}$

For $k\geq 2$:

$$\Phi k=\{A\in F\backslash\bigcup_{i=1}^{k-1}\Phi i\ \ |\forall\ B\in F\backslash\bigcup_{i=1}^{k-1}\Omega i\ ,B\neq A\Rightarrow A\not\subset B\ \}.$$

Definition 2.4

We define several symbols:

(I)

let F be a finite union-closed family,| F |=n,we use i=1,2,3…n to mark the sets A∈F,we define ∪F as follow :

$$\cup\mathbf{F}=\bigcup_{i=1}^{n}A\mathbf{i}$$

(II)

We call (A,B), whose A,B are sets , is a pair of set,and we define |(A,B)|=A∪ B

(III)

We define $A\backslash B=A-B=\{x\in A|x$ dosen't belong to $B\}$.

(IV)

We call W is a family of set pair if every x in w is a set pair and we define $|w|=\{|x| \mid x\in w\}$

(V)

We define $(A,B)(1)=A$, $(A,B)(2)=B$

(VI)

We define $w(1)=\{a \in\cup x(1) \mid x\in w\}$, $w(2)=\{a \in\cup x(2) \mid x\in w\}$

Lemma2.1

F is a finite union-closed family, then there exists a $K(K<+\infty)$ such that $\Omega K = \{\cup$ **F**$\}$.

Proof:

Let $An=\cup F$, then for any other Ai belongs to F, Ai is included in An. Because for $\forall Ai,Aj$ belong to same Ωi, $Ai\not\subset Aj$ and $Aj\not\subset Ai$, An must be alone in $\Omega K = \{\cup F\}$.

Lemma 2.2

F is a finite union-closed family, then $\mathbf{F}=\cup_{i=1}^{K} \Omega i$,and for $1\leq i<j\leq K$, $\Omega \mathbf{i} \cap \Omega \mathbf{j}=\emptyset$.

Proof:

According to the definition of Ωi, for 1≤i<j≤K, Ω**i** ∩ Ω**j**=∅. On account of F is a finite family and every operation to create the next Ωi at least use one Aj ∈ F .So **F=** $\bigcup_{i=1}^{K}$ Ωi ,and for 1≤i<j≤K, Ω**i** ∩ Ω**j**=∅.

Lemma 2.3

F is a finite union-closed family, **F=** $\bigcup_{i=1}^{K}$ Ωi ,| Ωi |=li(li ∈ N *,lk=1),then for 1≤ m < n ≤li,Am,An∈ Ωi:

Am⊄An, and An⊄Am

Proof:

Because Ωi can be seen as the minimal elements in set F\\$\bigcup_{i=1}^{k-1}$ Ω**i**.So Am and An can't be compared.

Definition 2.4

W is a family of sets and let W={A1,A2…Am},We define L(W)={Ai∪Aj|1≤ i < j ≤ m}.

Definition 2.5

W is a family of sets and let W={A1,A2…Am},

We define Ω(w)={A ∈W|∀ **B**∈**w** ,**B**≠A⇒B⊄A}

Definition 2.6

F is a finite union-closed family, **F**$=\bigcup_{i=1}^{K}$ Ωi,| Ωi|=li(li∈ N*),

then we call A∈ Ωi (2 ≤i≤n) is a addition-new-element set of Ωi

if:

(i) A∉L(Ω i − 1)

(ii) ∃| B∈ Ω i − 1 such that B⊆ A

Definition 2.7

F is a finite union-closed family, **F**$=\bigcup_{i=1}^{K}$ Ωi,A∈ Ωi （2 ≤i≤n） is a addition-new-element set of Ωi .then let A=A\B∪ B(B∈ Ωi − 1).

(i) we call A is a real addition-new-element set of Ωi if there exists a x∈ A\B and x∉ $\bigcup_{j=1}^{i-1}$ Ω**j** ,

(ii) otherwise we call A is a fake real addition-new-element set of Ωi

Definition2.8

F is a finite union-closed family, **F**$=\bigcup_{i=1}^{K}$ Ωi,for1 ≤i≤n-1 ， Let:

Q （Ωi） ={(ΔC,B)|ΔC =A/B,A is a real addition-new-element set of Ωi and A=A\B∪ B(B∈ Ωi + 1)}

Definition2.9

F is a finite union-closed family, **F**=$\bigcup_{i=1}^{K}$ Ωi ,| Ωi|=li(li∈ N*) , then we call A∈ Ωi（2 ≤i≤n）is an old-generated set of Ωi,if:

$A \in \Omega(L(\Omega i - 1))$

Definition 2.10

F is finite union-closed family, **F**= $\bigcup_{i=1}^{K}$ Ωi .we define Ei={A∈ Ωi|A is a addition-new-element set of Ωi}we also define

$Oi=\{A \in \Omega i|A \text{ is an old} - \text{generated set of } \Omega i\}$

$Ni=\{A\in \Omega(L(\Omega i - 1))|\exists B \in Ei, \text{such that B belongs to A}\}$

Definition 2.11

F is finite union-closed family, **F**= $\bigcup_{i=1}^{K}$ Ωi .we define Ei={A∈ Ωi|A is a addition-new-element set of Ωi}we also define

$Oi=\{A \in \Omega i|A \text{ is an old} - \text{generated set of } \Omega i\}$

$Ni=\{A\in \Omega(L(\Omega i - 1))|\exists B \in Ei, \text{such that B belongs to A}\}$

Lemma2.3

F is a finite union-closed family, **F**=$\bigcup_{i=1}^{K}$ Ωi,| Ωi|=li(li∈ N*),then

(i)$\Omega（L（\Omega 1））\backslash N2 \subseteq \Omega 2$

(ii)$\Omega（L（\Omega k））\backslash Nk + 1 \subseteq \Omega k + 1$（2≤ k ≤n-1）

Proof:

When k=1

Let $B=Ai\cup Aj,(Ai,Aj\in \Omega 1)B\in \Omega(L(\Omega 1))\backslash N2$,we need to proof $B\in \Omega 2$.

If except $A1,A2\ldots.Ak\in \Omega 1$,There exists an A which belongs to $F/\Omega 1\cup B$, such that A belongs to B. Let A belongs to $\Omega k(k>1)$

Case 1:

There exists only one $Ai\in \Omega 1$,which belongs to A .Then in $\Omega 2$,there exists a new additional element in $\Omega 2$ such that belongs to B.

Case 2:

There exists over one elements in $\Omega 1$ which belong to A, let it be A1,A2,then $A1\cup A2$ belongs to A ,A belongs to $B(A\neq B)$,so $A1\cup A2$ belongs to B. It's conflicted withB dosen't belong to N2

Now we proof that case one and case two ,there exists one but only one case stand,which means for this A .There exists at least one set C in $\Omega 1$ which belongs to the set A. We assume that A belongs to Ωk.We see that there exists at least one element in $\Omega k-1$ which belongs to A, the rest may be Deduced by analogy, so we can find an C which belongs to A in $\Omega 1$.

Lemma 3.4

F is a finite union-closed family, **F=** $\bigcup_{i=1}^{k} \Omega i$ .Then For 2 ≤i≤n :在此处键入公式。

$$\forall A \in \Omega i \text{ the following two statements:}$$

$$\text{(I)A is a addition} - \text{new} - \text{element set of } \Omega i$$

$$\text{(II)A is an old} - \text{generated set of } \Omega i.$$

One and only one statement about this A∈ Ωi is True.

Proof:

Let B belong to Ωi,according to the definition of Ωi.There must exists an element A which belongs to $\Omega i - 1$ and A belongs to B:

Case 1

If there exists only one A. Then B is a new-additional-element .

Case 2

If there exists over two elements A1,A2..As,such that Ai(i>0,i<s)belongs to B .Then for 0<i<j<s+1,Ai∪ Aj belongs to B.if B doesn't belong to Ai∪ Aj Then except $\bigcup_{j=1}^{i-1} \Omega\mathbf{j}$,there exists a set C=Ai∪ Aj belongs toB(on account of Ai,Aj are not compareable, so Ai∪ Aj doesn't belong to $\Omega i - 1$.)it's conflicted with B belongs to Ωi.

Definition 3.10

F is finite union-closed family, **F**=$\cup_{i=1}^{K}$ Ωi.

(I) we define Ei={A∈ Ωi|A is a addition-new-element set of Ωi}

(II) we also define Oi={A∈ Ωi|A is an old −generated set of Ωi}

(III) Ni={A ∈ Ω(L(Ωi − 1))|∃B ∈ Ei,such that B belongs to A}

Theorem 3.1

F is a finite union-closed family, **F**=$\cup_{i=1}^{K}$ Ωi. For 2 ≤i≤n :,then:

$$\Omega i \backslash Ei = \Omega(L(\Omega i - 1)) \backslash Ni$$

Proof:

According to Theorem3.1, $\Omega(L(\Omega i - 1))\backslash Ni$ belongs to Ωi. ,

And Ei belongs to Ωi, so $\Omega(L(\Omega i - 1))\backslash Ni \cup Ei$ belongs to Ωi.

On the other hand.∀A belongs to Ωi, According to Theorem 1.2, A belongs to Ei or A belongs to $\Omega(L(\Omega i - 1))\backslash Ni$ .So Ωi=$\Omega(L(\Omega i - 1))\backslash Ni \cup Ei$ .Because $Ei \cap \Omega(L(\Omega i - 1))\backslash Ni = \emptyset$ and the Ei belongs to Ωi,so we get that： $\Omega i\backslash Ei = \Omega(L(\Omega i - 1))\backslash Ni$.

Remark3.1

According to theorem1.1~1.3,we know that every finite union closed family of set F can be decided by $\Omega 1$ and Q(Ωi) ($2 \leq i \leq n$),

On the other hand ,if we give w (w is a finite family of set and every set in this family is none empty and different from the others,every $A \in x, A \subseteq X$) and let $\Omega 1 = w$, then we give Q1={ (ΔC,B)| $\Delta C \subseteq X$,$B \subseteq \Omega 1$) , then Let

$$\Omega 2 = \Omega(L(\Omega 1)) \backslash N2 \cup \mid Q1 \mid$$

We give Q2={ (ΔC,B)| $\Delta C \subseteq X$,$B \subseteq \Omega 2$}, then let

$$\Omega 3 = \Omega(L(\Omega 2)) \backslash N3 \cup \mid Q2 \mid$$

The rest Ωi and Qi can be given in the same manner.

And at some point for $i \geq K(K \in N*)$, we give Qi={($\emptyset$,B)| $B \subseteq \Omega i$} then $\Omega K + 1, \Omega k + 2$ ...can be decided, and at some point there exists a K such that $\Omega K = \{\Omega 1 \cup \bigcup_{j=1}^{K-1} |Qj|\}\}$

Remark3.2

So we can see that if we give the w, K,Q1,Q2…QK-1 , then we can decide a union-closed family **F.**

Remark3.3

In fact, let K-1 be 0,we see that the development of w w={A1,A2…An} in the process of iteration Ω(L(w)) is not depended on what specific element every Ai gets. It only depends on the truth-value in every independent area spilt up by w’s Venn Diagram.(If there exists an element in an area spilt up by its Venn Diagram, then the truth value is 1,else it’s 0.).

3.The Fractal structure of poset and union-closed family

In this part ,We start from studying the generalized structure of poset, which may result in a more specific conclusion on the structure of union-closed family.

Definition 3.1

Let X be a family of sets, we define that a widest incomparable subfamily W of X is the largest subfamily which satisfies that any sets x,y($x\neq y$) $\in$ W,$x\not\subset y$ and $y\not\subset x$.

Fact 3.1

Let X be a family of sets, then there are two facts about its’ widest incomparable subfamily W:

(i) W may have many different choices

(ii) Let W be one of those sets, for any other sets S in X-W, there exists a set S0 in W such that this s0 belong to S or S belong to S0.

Definition 3.2

Let X be a family of sets, W={a1,a2,a3…aK} is one of the widest incomparable subfamily of X. Then for every A∈X-W, there exists an ai which can be compared with A. If A belongs to ai, then for the other elements in W, A belongs to aj or A⊄aj and aj⊄A. We define that the set combined by all the sets in family X-W which satisfies this case is represented by W↓.As for the other case, which means ∀ai ∈ W →A∋ ai or ai and A are incomparable. We define the set is represented by W↑ in this case.

Lemma3.1

Let w be a widest in comparable subfamily of X, then it's obvious that w↑∪ w ↓=X-w, w↑∩ w ↓= ∅.

Definition3.3

(i)Let w be a widest in comparable subfamily of X, if w↑= ∅,we call this w is the top widest incomparable subfamily of X. Record it as X↑

(ii)Let w be a widest in comparable subfamily of X, if w↓= ∅,we call this w is the Low widest incomparable subfamily of X. Record it as X↓.

(iii) If w doesn't satisfies (i),(ii),we call this w is a medium widest incomparable subfamily of X. Record it as X→

Fact 3.2

Let X be a family of sets, then there exists a subfamily W of X which is the top widest incomparable subfamily of X if and only if the amount of the maximal members in X is equal the amount of elements in its widest incomparable subfamily. Similarly, It is also true when it comes to the low widest incomparable subfamily.

Fact 3.3

Let X be a family of sets, then X↑ and X↓ (if they do exist)are both unique.

Remark3.1

However, when we talk about X → ,it may exist several different X→.

Definition 3.4

Let X be a family of set, according to Fact3.2 and Fact 3.3,it can be classified into 4 cases:

(i)It doesn't have a X→,but it has the x↑ and x ↓.

(ii)It doesn't have a X→,it doesn't have the x↓,but it has the x↑.

(iii) It doesn't have a X→,it doesn't have the x ↑ .,but it has the x↓

(IV)It has a X→

Remark3.2

In case (i),this x only has two widest incomparable subfamily. In case (ii) (iii), this x only has one widest incomparable subfamily.

Theorem 3.1

X is a family of sets, then it can be divided into several subfamily of $X=X1\cup x2 \cup x3 ... \cup xk$:

Which satisfies the following rules:

(i)X2,X3…Xk-1 belong to case(i) mentioned in definition 3.4

(ii)$xi \cap xi+1 = Wi (1 \leq i < k)$ Wi is the low widest incomparable subfamily of Xi+1,and wi is also the top widest incomparable subfamily of xi.

(iii)X1 belongs to case(i) or case(ii) while Xk belongs to or case （i） or case(iii).

Proof:

We assume that X belongs to case(IV),then we can get a $x\rightarrow$,let $x\rightarrow' = x\rightarrow \cup x\uparrow$,$x\rightarrow'' = x \cup x\downarrow$.

Then $x\rightarrow'$and $x\rightarrow''$ take $x\rightarrow$ as its top widest incomparable subfamily of X. and low widest incomparable subfamily of X.

We do this operation to Y who belongs to case(IV) in $x\rightarrow'$ and $x\rightarrow''$,repeat this operation again and again, now that we assume this is a finite poset, so every time $x\rightarrow'$ and $x\rightarrow''$ *have less* elements than before, then there exists a time that all the sets separated from X are not in case(IV),let these sets be X1,X2…Xn,

Then we get X= X1∪X2…∪Xn.

Fact 3.1

Let X= X1 ∪ x2 ∪ x3... ∪ xk (x,x1,x2…xk are defined in Theorem 3.1) M=|x1 ∩ x2 |,then |X2|=|x3|=|x4|…=|xk-1|=2M, │xk-1∩ xk│=M , in fact M is the width of this poset whose elements are sets。

Definition3.5

We call finite poset X is a square , if it can be divided into several subfamilies of X:

X= X1∪ x2 ∪ x3... ∪ xk

Which satisfies the following rules:

(1)$|X1|=|X2|=\ldots|xk|=M$

(2)$\forall i,i+1(i>0,i<k)Xi$ is the low widest incomparable subfamily of $xi \cup xi+1$,while $xi+1$ is the top widest incomparable subfamily of $xi\cup xi+1$.

Remark3.2

From Theorem3.1,we can see that X is a square if and only if x1 belongs to case(i), xk also belongs to case(i).And M is also the width of X.

Definition3.6

We call X1,X2…Xk the 1st line ,the 2nd line…kth line of the square,kth floor is also defined as last floor. K is lenth of the square.

Corollary3.1

X can be decomposed into $S1 \cup S2 \ldots USN$;S1,S2…SN are squares mentioned in definition3.5.

Proof:

Let x be a finite family of sets, then let $x=x1\cup x2 \cup x3\ldots \cup xK$(x1,x2…xK are defined in theorem3.1) if x1 or xk does not belong to case(i),for example x1 belongs to case(ii)we let x1'=x1-x1 $\cap$ x2,and we let x1'=B1 $\cup$ B2 $\cup$ B3... $\cup$ Bk $-$ 1 $\cup$ Bk (B1,B2…Bk-1,Bk are defined in theorem3.1),at this time, the width of x1' is shorter than the with of X, so Bi $\cap$ Bi + 1 = M' （M' < M= ｜ A1 $\cap$ A2 ｜） The rest operation can be down in the same manner because of the finite of the F and its width M. At the same time, we Notice that when we decomposed x=x1 $\cup$ x2 $\cup$ x3... $\cup$ xK(x1,x2…xK are defined in theorem3.1) Let S1=X2 $\cap$ x1 $\cup$ x2 $\cup$ x3... $\cup$ xk $-$ 1 $\cup$ xk $-$ 1 $\cap$ xk,then S1 is square . We let s2= B2 $\cap$ B1 $\cup$ B2 $\cup$ B3... $\cup$ Bk $-$ 1 $\cup$ Bk $-$ 1 $\cap$ Bk,then S2 is a square. Now that |X| and M are both finite, so finally We proof that X can be decomposed into s1 $\cup$ s2...USN S1,s2…SN are squares mentioned before

Corollary3.2

Let X be a finite posets, then x can be decomposed into S1 $\cup$ s2...USN,(s1,s2…SN) are all squares)Then according to the operation which products these squares, we can find these simple properties as follow:

(i)Every operation leads to a new square Si built up. If this operation also leads to a new set x1'=x1-x1 $\cap$ x2 which can be decomposed. Then the Si's built from X1 width is strictly less than Si's width.

（ii） Let Si be a square which is product by the decomposition of X,di1,di2…dik is the 1^{st} floor,2^{nd} floor …kth floor of Si. Then for any s>t, ∀A ∈ ds,∃B ∈dt,B belongs to A; ∀A ∈ dt,∃B ∈ds,B belongs to A.

Definition 3.7

Let X be a poset and Let X= S1∪ S2 ∪ S3... ∪ Sk（S1，S2…Sk are defined in **Corollary3.1**）

We define:

(i)This decomposition is the Fractal decomposition of a poset

(ii) l(Si)=the lenth of Si

(iii) Sfirst={d|∃Si(i≥ 2） such that d is the first floor of Si}

(iv) Slast={d|∃Si(i≥ 1） such that d is the last floor of Si}

Remark3.3

Let X be a poset and Let X= S1∪ S2 ∪ S3... ∪ Sk ,if di is the last floor of Si, dj is first floor of Si+1. Then for ∀A ∈ dj ， ∃B ∈ di,A belongs to B, However,the statement ∀A ∈ di,∃B ∈dj,B belongs to A." is not True.

Definition 3.8

Let X be a poset and Let X= $S1 \cup S2 \cup S3 \ldots \cup Sk$ (S1, S2…Sk are defined in Corollary3.2.

For j>1,we defineP1={A| A ∈the Last floor of S1 and for any x belongs to the first floor of S2 ,A doesn't belongs to x}

Fj={A|A∈the first floor of Sj and for any x∈the last floor of Sj-1 x dosen't belong to A

Lj={A| A ∈the Last floor of Sj and for any x belongs to the first floor of Sj+1 ,A doesn't belongs to x}

We define the strange set V of X=$(\bigcup_{j=2}^{k} Lj \cup Fj).\cup P1$

Definition3.9

When we say that two decomposition on the poset X are equal ,we mean that the family of sets which is composed by the separated sets from each decomposition are same.

Now we come to discussion The relationship between those three decomposition mentioned part two and part three

Theorem 3.2

Let X be a poset, then its minimal element decomposition is equal to its Fractal decomposition ,and its minimal element decomposition is also equal to its Maximal element decomposition **if and only if** $V=\emptyset$.

Proof:

Let $V=\emptyset$ .Then the first floor of S1 is Ω1 the second floor of Ω1 is S2 ,the rest can be checked by the definition of Ω.

Let those decomposition are equal, then for any element A in the last floor of Si(i>1),Because this is also a minimal element composition ,there exists an B which belongs to the last floor of Si-1 such that B belongs to A, So $Fj=\emptyset$ .On the other hand,this is also a maximal element decomposition so Lj and $P1=\emptyset$.So we get $V=\emptyset$.

Remark3.4

If F is a union-closed family, then it also must be a poset which has these three decompositions. Some useful decompositions may be used to help us to understand the structure of union-closed family like minimal element decomposition.

Remark: It's easy to see that if F is a union-closed family then the top of its decomposition must be $\cup$ F.

It's also easy to see that if the union-closed families satisfies conjecture 1.1 if and only if it's True for all the union-closed family whose every floor d or Ω, or Φ has more than one set.

4.the topological sort of the union-closed family

In this part, we discuss about the topological sort on finite-union closed family F, which leads to several results on subfamilies of the union-closed family. In order to state our results, we need to introduce some definition.

Definition4.1

(i)Let F be a union-closed family,| Let G(V,E) be a directed graph, |F|=N=|V| , F={A1,A2…,AN},V={a1,a2…aN},Let function Q:F→ V: Ai→ai.Then Q is a Bijection.

And we define E as follow:

if Ai, Aj ∈ F,Ai ∪ Aj = Ak(i ≠ j,k ≠ i,k ≠ j) ,then edge $\overrightarrow{aiak}$ and edge $\overrightarrow{ajak}$ ∈ E,if Ai,Aj ∈ F ,(i ≠ j) ,then edge $\overrightarrow{aiaj}$ ∈ E.We mark this G(V,E) decided by F by G(F).

(ii)We also define that two finite union-closed family F1,F2are equal if and only if |F1|=|F2|,and there exists a function between F1 and F2 which is a bijection and injection such that it keeps the union operation.

(iii)For a directed graph G(V,E),V={a1,a2…an},when we write ai1ai2ai3…aik(i1,i2…ik belong to{1,2…n}),it means $\forall \ \leq 1 \leq \mathrm{u} \leq \mathrm{k}-1, \overrightarrow{\mathrm{aiuaiu}+1}$ belongs to E.

Fact 4.1

If Ai∪ Aj = Ak, then Ai belong to Ak, Aj belong to Ak, so $\overrightarrow{Q\ \ (Ai)\ \ Q\ \ (Ak)}\ \ and\ \overrightarrow{Q(Aj)Q(Ak)}$ belong to E.

If Ai∪ Aj = Aj or Ai then $\overrightarrow{Q\ \ (Ai)\ \ Q\ \ (Aj)}$ belong to E.

Remark4.1

We notice that for two union-closed family F1 and F2 G(F1)=G(F2),it doesn't mean that F1=F2,because if Ai ∪ Aj = Ak ((i-k)(i-j)(j-k)≠ 0)then we only describe Ai,Aj belong to Ak in G(F),however, it does not mean that we describe Ak is just right the union of Ai and Aj in G(F)

Lemma 4.1

G(F) is an acyclic directed digraph.

Proof:

if has a cycle ai1ai2ai3ai4…aiu (aiu=ai1) then Ai1belong to Ai2,Ai2 belong to Ai3 …Aiu-1belong to Aiu=A1, then Ai1=Ai2=Ai3…=Aiu, it is conflict with F is a none-repetitive set.

Lemma 4.2

We can give an order in union-closed family F, Let it be A1,A2…AN. For 1 $\le i<j\le$ NLet Ai$\cup$ Aj = Ak,then k$\ge${i,j}max

Proof:

G(F) is an acyclic digraph, then it has a topological order, considering the Initialization from F to G(F).Let the topological order in G(F) be i1<i2<i3…<iN, then we order the sets in F as Ai1,Ai2…AiN, remark the Aik(Aik belong to{Ai1,Ai2…AiN}as Ak, then the new order A1,A2…AN is the topological order in G(F).

Definition 4.2

Let G(V,E) be a Graph, |V|=N,V={a1,a2,a3…aN}.Let W={(ai,ai)|i=1,2,3…N},R=V$\times$V\W, if there exists a S belong to R and there exists a function f between S to {a1,a2,a3…aN},we call this function f is the extra relation on G(V,E)

Definition 4.3

Let F be a union-closed family, G(F) be the directed graph translated by F, we define an extra relation f on G(F) as follow:

Let S={(ai,aj)∈ W|Ai∪ Aj = Ak,((i − j)(i − k)(k − j) ≠ 0}

And function f : (ai,aj) ∈ S → ak. We use f[G(F)] to represent this extra relation on G(F)。

Lemma 4.3

Let F be a union-closed family ,we define F{N}={F||F|=N}.

We define:

f{N}={f|∃F ∈ F{N} such that f is the extra relation on G(F)}

We also define:

(F, f){N}={(G(F),f|F ∈ F{N},f is the extra relation on G(F)}

Then function φ defined as follow:

φ: F{N} →(F,f){N}

F→ (G(F),f[G（F）])

Then φ is a bijection.

Proof:

F1 ≠ F2 if and only if there exist a pair of Ai,Aj(i ≠ j)Ai ∪ Aj = Al(l ≥ {i,j} max)such that Ai∪ Aj = Ak(k ≥ {i,j}) but k≠ l.

Case 1

l=j, k≠ i,j, then (i,j) belongs to the original image set of f[G{F2}] while (i，j) doesn‘t belong to the original image set of f[G{F1}],so $\varphi(F1) \neq \varphi(F2)$

Case2

l>{i, j}max, k>{i, j}max, then f[G(F1)](i, j)=l while f[G(F2)](i, j)=k

In summary, φ is a bijection.

Corollary4.1

If F is a union-closed family, for ∀k(k>0 andk<|F|+1),it has a subfamily F such that |F|=k

Proof:

According to Lemma2.2 and Lemma2.3,for any F={A1,A2…AN} ,let V(G(F))={a1,a2…aN},a1a2…aN is a topological sort for G（F）.For $\forall k > 0$ and $k < |F| + 1$,we Built subF={Q^{-1}（aj）| j=N,N-1…N-k+1} .Then according to the definition of G（F）and a1a2…aN is a topological sort for G(F), subF is a subfamily of F such that |subF|=k

Definition 4.4

Let G (V, E) be a directed Graph with a extra relation on F,it is built as follow:

(i)|V|=N,V={a1,a2…aN}

(ii)E={e |for ∀ 1 ≤i<j≤ N， we add $\overrightarrow{aiaj}$ ， otherwise， we choose k （k>i,k>j） randomly and add $\overrightarrow{aiak}$ and $\overrightarrow{ajak}$}

(iii) the extra relation f defined as:

S = {(ai,aj)| edge $\overrightarrow{\mathrm{aiaj}}$ doesn't belong to E}

then we must choose the k for ai,aj defined in (ii)

f: S→{a1,a2,a3…an}

(ai,aj)→ak

We call the graph G and the extra relation f on this graph an union-closed graph (G,f).

Fact 4.2

The edges $\overrightarrow{\mathrm{a1an}}$, $\overrightarrow{\mathrm{a2an}}$, $\overrightarrow{\mathrm{a3an}}$… $\overrightarrow{\mathrm{an-1an}}$ belongs to E

Remark 4.2

If edge $\overrightarrow{\mathrm{aiaj}}$ belongs to E then (ai,aj) doesn't belong to S, if $\overrightarrow{\mathrm{aiaj}}$ doesn't belong to E then∃| k>i,k>j such that (ai,aj) belongs to S and f((ai,aj))=ak

Lemma4.4

Let S1={F|F is a union-closed family},Let S2={(G, f)| (G, f) is a union-closed graph}

Then function ω :

$F \in S1 \rightarrow (G(F), \ f(G(F)))$ is an injection from S1 to S2.

Proof:

(1) Compare definition 2.2 and definition 2.1 for ∀F F is a union -closed family,(G(F),f(G(F)) $\in S2$ so ω is a function from S1 to S2.

(2) From Lemma 2.3 we find that if |F1|=|F2| but F1≠ F2, then (G(F1), f(G(F1)) ≠ (G(F2), f(G(F2))) if |F1|≠ |F2|, then it's obviously that ' (G (F1), f (G (F1)) ≠ (G (F2), f (G (F2))) .so ω is an injection。

Remark4.3

ω is not a bijection , for example , let G=(V,E) be a union-closed graph ,Let V={a1,a2,a3,a4}, and we let a1a2belong to E,a2a3belong to E ,f((a1,a3))=a4 ,if $\omega(F) = V$(F is an union-closed family) then A1 belongs to A2,A2 belongs to A3,$A1 \cup A3 = A4 = A3$,,it's conflicted with the definition of F.

Lemma4.5

if G(V,E) is a union-closed graph and there exists a F such that $\omega(F) = G$,it's easy to see that this system must satisfy the following rules:

(i) if $\overrightarrow{aiaj}$ belongs to E, $\overrightarrow{ajak}$ belongs to E,then aiak belongs to E.

(ii) If (ai,aj) belongs to S, let as=f((ai,aj)), $\overrightarrow{ajak}$ (k<N) belongs to E, then (ai,ak)belongs to S and f((ai,ak))=at (t≥ s) .

Now we give a conjecture based on directed acyclic graph which can reason out conjecture 1.1

Definition4.5

(i)Let(G(V,E),f) be an union-closed graph, V={a1,a2…aN}.For ai∈V:we define Wi={aj|∃ak0ak1ak2ak3..akt such that ak0=ai, akt=aN, $\overrightarrow{akpakp+1} \in E$, ai∈{ak0,ak1…akt}}

(ii)Let G(V,E) be a graph,then we (ai,aj) and (ai',aj')are separated(ai,aj,ai',aj' belong to V) if and only if $\{ai,aj\} \cap \{ai',aj'\} = \emptyset$

Definition4.6

Let (G(V,E),f) be an union-closed graph,V={a1,a2…aN} For ai ∈ V,we define Ki={(ai,aj)|(ai,aj) ∈ S,and f((ai,aj)) ∈ Wi,∀a,b ∈ Ki,a,b are seperated}We call this Ki a special set on ai.

Definition4.7

Let (G(V,E),f) be an union-closed graph,V={a1,a2…aN}Pi={Ki|Ki is a special set on ai},we defined [Pi]={|Ki|}max

Conjecture 4.1

Let (G(V,E),f) be an union-closed graph, V={a1,a2…aN},then $\{[Pi]+|Wi|\}^{1\le i\le N}_{max} \ge \frac{N}{2}$

Remark4.4

If Conjecture 4.1 is True, then the Conjecture 1.1 is True.

5. some results on Frankl's conjecture based on the minimal decomposition of union-closed family

Lemma 5.1

Let X= $\bigcup_{i=1}^{K}$ Ωi ,if A belongs to Ωi (i>1) is an old generated element, let B={A|A belongs to Ωi and A also belongs to B}Then each pair of C,D (C≠ D)belong to B,C∪ D=B.

Proof:

Now that C∪ D belongs to B, if C∪ D ≠ B,then B must belong to $\Omega k(k \geq i+1)$,it's conflicted with B belong to Ωi.

Definition5.1

If B is a union-closed family without any new- addition element, then we call B is a normal union-closed family.

Lemma 5.2

Let F be a union-closed family F, F=$\bigcup_{i=1}^{K} \Omega i$ .if K≤ 2,then for this case the conjecture is True.

Proof:

For K=1 : then $|\Omega 1|=1=|F|$, it's true for this case.

For K=2 : Let $\Omega 1=\{A1,A2\ldots An\}$ $\Omega 2=\{A\}|A|=m$

Then:

$$\sum_{1\leq i<j\leq n} |Ai| + |Aj| \geq \sum_{1\leq i<j\leq n} |Ai \cup Aj|$$
$$\geq \sum_{1\leq i<j\leq n} |A|$$
$$=\frac{n(n-1)}{2}|A|$$

$$\sum_{1\leq i<j\leq n} |Ai| + |Aj| = (n-1)\sum_{i=1}^{n} |Ai| \geq \frac{n(n-1)}{2}|A|$$

$\rightarrow |A|+\sum_{i=1}^{n} |Ai| \geq \frac{n+2}{2}|A|$

$\rightarrow \exists$ x∈ at least $\frac{n+2}{2}$ sets in this family.

Definition5.2

Let F be a finite family of sets F={A1, A2….An} (in this family Ai, Aj with different mark can be one same set)

We define that $||F||=\sum_{i=1}^{n}|Ai|$

Lemma5.2

Let X=$\bigcup_{i=1}^{k}$ Ωi ,if A1 belongs to Ωi(i>1) is an old generated element, let B1={A|A belongs to Ωi and A also belongs to A1}.Let A2 (A2≠ A1) belongs to Ωi(i>1) be an old generated element, let B2={A|A belongs to Ωi and A also belongs to A2}.

Then | B1∩ B2|≤ 1

Proof:

If $|B1 \cap B2| > 1$, then we can two different sets C1, C2 selected from B1 ∩ B2, such that A1=C1 ∪ C2 = A2, but A1≠ A2.

Theorem 5.1

If B is a normal union-closed family, F= $\bigcup_{i=1}^{k}$ Ωi, Let $|\Omega 1| = n1, |\Omega 2| = n2, |\Omega 3| = n3 ..., |\Omega k-1| = nk-1 = p, |\Omega k| = 1$,then there exists an element x ∈ F belongs to at least$1+\frac{p}{2}+\frac{1}{p-1}*$ c(n1,n2…nk-2,k)sets in this family. c is a constant which is decided by n1,n2…nk-2,k:

$$c=1+\sum_{j=1}^{k-3}2^{j}\prod_{i=k-1-j}^{k-2}\frac{1}{ni(ni-1)}$$

Proof:

For A which belongs to $\Omega i(i > 2)$, Let WA={B |B belongs to $\Omega i - 1$ and B belongs to A}

Let Ω2={A1,A2…An2}

Let $||\Omega k||$=m=||∪F||

Let |wA1|=k1,|wA2|=K2…|WAn2|=Kn2

Then according to Theorem4.1 and Lemma4.1:

||L(wA1)||≥ $\frac{k1(k1-1)}{2}$|A1|

||L(wA2)|| ≥ $\frac{k2(k2-1)}{2}$|A2|

…

||L(wAn2)||≥ $\frac{kn2(kn2-1)}{2}$|An2|

Notice that:

||L(wA1)||=(k1-1)||w(A1)||

||Lw(A2)||=(k2-1)||w(A2)||

…

||Lw(An2)||=(kn2-1)||w(An2)||

So we get:

||w(Ai)||≥ $\frac{ki}{2}$||Ai|| (1 ≤i≤ n2)

$\sum_{i=1}^{n2} ||w(Ai)|| \geq \sum_{i=1}^{n2} \frac{ki}{2}$||Ai||

(ki≥2)

On the other hand, according to lemma5.2 we notice that:

$\frac{n2(n2-1)}{2}||\Omega 1|| \geq \sum_{i=1}^{n2} ||w(Ai)|| \geq \sum_{i=1}^{n2} \frac{ki}{2}$||Ai||

On account of ki>1 So:

$$||\Omega 1|| \geq \frac{2}{n2(n2-1)}\sum_{i=1}^{n2}||Ai|| = \frac{2}{n2(n2-1)}||\Omega 2||$$

The rest may be deduced by analogy:

$$||\Omega 1|| \geq 2^{k-2}\prod_{i=2}^{k-1}\frac{1}{ni(ni-1)} \quad ||\Omega k-1|| \geq 2^{k-3}\mathrm{m} * \mathrm{p} * \prod_{i=1}^{k-1}\frac{1}{ni(ni-1)}$$

….

$$||\Omega k-2|| \geq 2^{0}m * p * \frac{1}{nk-1(nk-1-1)}$$

$$||\Omega k-1|| \geq 2^{-1} * \mathrm{m} * \mathrm{p}$$

So we get:

$$\sum_{A\in F}\frac{|A|}{m} = \sum_{i=1}^{k}||\Omega i|| * \frac{1}{\mathrm{m}} \geq$$

$$1+p*(2^{-1}+2^{0}\frac{1}{p(p-1)}+2\frac{1}{nk-2p(p-1)(nk-2-1)}+\ldots+2^{k-3}\prod_{i=2}^{k-1}\frac{1}{ni(ni-1)})$$

$$=1+\frac{p}{2}+\frac{1}{p-1}(1+\sum_{j=1}^{k-3}2^{j}\prod_{i=k-1-j}^{k-2}\frac{1}{ni(ni-1)})$$

$$=1+\frac{p}{2}+\frac{1}{p-1}*C$$

Remark 5.1

This estimation is not the strongest one, however, we think it may provide us a possible way to estimate that average according to the structure of an union-closed family given in theorem 3.1.

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