2.2. 目标轮廓曲线的表示（傅立叶展开）

$L: (x(\sigma), y(\sigma))$.

① $x(\sigma), y(\sigma)$ 满足 Dirichlet 条件 $x(0)=x(\pi)=0 \quad y(0)=y(\pi)=0$

$$x(\sigma)=\sum_{n=1}^{+\infty} X_n \sin(n\sigma). \qquad y(\sigma)=\sum_{n=1}^{+\infty} Y_n \sin(n\sigma). \quad \sigma\in[0,\pi]$$

$$X_n=\frac{2}{\pi}\int_0^{\pi} x(\sigma)\sin(n\sigma)d\sigma \qquad Y_n=\frac{2}{\pi}\int_0^{\pi} y(\sigma)\sin(n\sigma)d\sigma$$

② $x(\sigma), y(\sigma)$ 满足 Neuman 条件 $\left.\frac{dx}{d\sigma}\right|_{\sigma=0}=\left.\frac{dx}{d\sigma}\right|_{\sigma=\pi}=\left.\frac{dy}{d\sigma}\right|_{\sigma=0}=\left.\frac{dy}{d\sigma}\right|_{\sigma=\pi}=0$

$$x(\sigma)=\sum_{n=1}^{+\infty} X_n \cos(n\sigma) \qquad y(\sigma)=\sum_{n=1}^{+\infty} Y_n \cos(n\sigma)$$

$$X_n=\frac{2}{\pi}\int_0^{\pi} x(\sigma)\cos n\sigma d\sigma \qquad Y_n=\frac{2}{\pi}\int_0^{\pi} y(\sigma)\cos n\sigma d\sigma$$

③ $x(0)=X_s. \quad x(\pi)=X_e. \quad y(0)=Y_s \quad y(\pi)=Y_e$

变换 $\bar{x}(\sigma)=x(\sigma)-X_s-\frac{X_e-X_s}{\pi}\sigma.$
$\bar{y}(\sigma)=y(\sigma)-Y_e-\frac{Y_e-Y_s}{\pi}\sigma.$ $\quad \sigma\in[0,\pi]$

转化为①.

$$\bar{x}(\sigma)=\sum_{n=1}^{+\infty} X_n\sin(n\sigma) \qquad \bar{y}(\sigma)=\sum_{n=1}^{+\infty} Y_n\sin(n\sigma) \quad \sigma\in[0,\pi]$$

$$\therefore \quad x(\sigma)=X_s+\frac{X_e-X_s}{\pi}\sigma+\sum_{n=1}^{+\infty} X_n\sin(n\sigma).$$

$$y(\sigma)=Y_s+\frac{Y_e-Y_s}{\pi}\sigma+\sum_{n=1}^{+\infty} Y_n\sin(n\sigma).$$

$$x'(\sigma)=\frac{X_e-X_s}{\pi}+\sum_{n=1}^{+\infty} nX_n\cos(n\sigma). \qquad y'(\sigma)=\frac{Y_e-Y_s}{\pi}+\sum_{n=1}^{+\infty} nY_n\cos(n\sigma).$$

$$x''(\sigma)=-\sum_{n=1}^{+\infty} n^2X_n \sin(n\sigma). \qquad y''(\sigma)=-\sum_{n=1}^{+\infty} n^2Y_n\sin(n\sigma).$$

3.1 目标轮廓模型的建立

$$E_{tol} = E_{int} + E_{ext}$$

动能 $E_{int-1} = \frac{1}{2}\int_0^{\pi} \left((x'(\sigma))^2 + (y'(\sigma))^2\right) d\sigma$

势能 $E_{int-2} = \frac{1}{2}\int_0^{\pi} \left((x''(\sigma))^2 + (y''(\sigma))^2\right) d\sigma$

$E_{int} = \alpha E_{int-1} + \beta E_{int-2}$ $\alpha . \beta$ 权重系数.

$E_{ext} = \int_0^{\pi} f(x,y)\, d\sigma$ $f(x,y) = -\| \nabla (x(\sigma), y(\sigma)) \|^2$

$$E_{tol} = \frac{1}{2}\int_0^{\pi} \alpha\left((x'(\sigma))^2 + (y'(\sigma))^2\right) + \beta\left((x''(\sigma))^2 + (y''(\sigma))^2\right) d\sigma + \int_0^{\pi} f(x,y)\, d\sigma$$

$$\int_0^{\pi} (x'(\sigma))^2 d\sigma = \int_0^{\pi} \left(\frac{X_e - X_s}{\pi} + \sum_{n=1}^{+\infty} n X_n \cos(n\sigma)\right)^2 d\sigma = \frac{(X_e - X_s)^2}{\pi} + \frac{\pi}{2}\sum_{n=0}^{+\infty} n^2 X_n^2$$

$$\int_0^{\pi} (x''(\sigma))^2 d\sigma = \int_0^{\pi} \left(-\sum_{n=1}^{+\infty} n^2 X_n \sin(n\sigma)\right)^2 d\sigma = \frac{\pi}{2}\sum_{n=0}^{+\infty} n^4 X_n^2$$

$$\int_0^{\pi} (y'(\sigma))^2 d\sigma = \frac{(Y_e - Y_s)^2}{\pi} + \sum_{n=0}^{+\infty} \cancel{n^4 X_n} n^2 Y_n^2 \qquad \int_0^{\pi} (y''(\sigma))^2 d\sigma = \frac{\pi}{2}\sum_{n=0}^{+\infty} n^4 X_n^2$$

$$E_{tol} = \frac{\pi}{4}\sum_{n=1}^{+\infty} (\alpha n^2 + \beta n^4)(X_n^2 + Y_n^2) + \frac{\alpha\left[(X_e - X_s)^2 + (Y_e - Y_s)^2\right]}{2\pi} + \int_0^{\pi} f(x(\sigma), y(\sigma))\, d\sigma$$

离散化

$$E_{tol} \approx \frac{\pi}{4}\sum_{n=1}^{+\infty} (\alpha n^2 + \beta n^4)(X_n^2 + Y_n^2) + \frac{\alpha\left[(X_e - X_s)^2 + (Y_e - Y_s)^2\right]}{2\pi} + \frac{\pi}{M}\sum_{m}^{M} f\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right)$$

极小值条件 $\frac{\partial E_{tol}}{\partial X_n} = \frac{\partial E_{tol}}{\partial Y_n} = 0$ $n = 1, 2, \ldots N$

$$\begin{cases} (\alpha + \beta) X_1 + \frac{\pi}{M}\sum_{m=1}^{M} f_x\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right) \sin\frac{m\pi}{M} = 0 \\ \vdots \\ (\alpha n^2 + \beta n^4) X_n + \frac{\pi}{M}\sum_{m=1}^{M} f_x\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right) \sin\frac{mn\pi}{M} = 0 \\ (\alpha + \beta) Y_1 + \frac{\pi}{M}\sum_{m=1}^{M} f_y\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right) \sin\frac{m\pi}{M} = 0 \\ \vdots \\ (\alpha n^2 + \beta n^4) Y_n + \frac{\pi}{M}\sum_{m=1}^{M} f_y\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right) \sin\frac{mn\pi}{M} = 0 \end{cases}$$

3.2 模型求解

$$\begin{cases} g_1(X_1,\cdots X_n, Y_1,\cdots Y_n)=0 \\ g_2(X_1,\cdots X_n, Y_1,\cdots Y_n)=0 \\ \vdots \\ g_{2n}(X_1,\cdots Y_n)=0 \end{cases}$$

记 $G(z)=[g_1(z),\cdots g_n(z)]^T$

$z=(X_1,\cdots Y_n)^T$

$\Rightarrow G(z)=0$

迭代法 $z'=\varphi(z)$ $\varphi: D\subset R^n \to R^n$ $\varphi(z)=[\varphi_1(z),\cdots \varphi_n(z)]^T$

$z_{k+1}=\varphi(z_k)$ 需要 $\|\varphi(x)-\varphi(y)\|\le L\|x-y\|$ $\forall x,y\in R^n$ $L\in(0,1)$

Taylor $G(z)\approx G(z_k)+G'(z_k)(z-z_k)$ 代入 z_{k+1} $G(z_{k+1})=G(z_k)+G'(z_k)(z_{k+1}-z_k)$

在 $G(z_{k+1})\approx 0$ $z_{k+1}=z_k-(G'(z_k))^{-1}G(z_k)$ 需要 $G'(z_k)$ 可逆.

3.2.1 逆 Broyden 秩1法

$A_k=G'(z_k)$. $z_{k+1}=z_k-A_k^{-1}G(z_k)$

用差商代替求解导数 $A_{k+1}(z_{k+1}-z_k)=G(z_{k+1})-G(z_k)$. ①

设 $A_{k+1}=A_k+\Delta A_k$ $\mathrm{rank}(\Delta A_k)=1$. 设 $\Delta A_k=u_kV_k^T$ 代入①

$u_k=\frac{1}{V_k^TS_k}(y_k-A_kS_k)$. $S_k=z_{k+1}-z_k$. $y_k=G(z_{k+1})-G(z_k)$.

令 $V_k=S_k$. $u_k=\frac{1}{\|S_k\|^2}(y_k-A_kS_k)$. 记 $B_k=A_k^{-1}$

由 sherman-Morrison 公式 对非奇异矩阵 A_k 与 $1+V_kA_ku_k\neq0$

$$(A_k+u_kV_k^T)^{-1}=A_k^{-1}-\frac{A_k^{-1}u_kV_k^TA_k^{-1}}{1+V_k^TA_k^{-1}u_k}$$

$$\therefore B_{k+1}=(A_k+u_kV_k^T)^{-1}=B_k-\frac{1}{S_kB_k^Ty_k}(B_ky_k-S_k)B_k^T$$

至此 $z_{k+1}=z_k-B_kG(z_k)$.

B_0 可取 E 或 $G'(z_0)^{-1}$.

4.1 建立目标轮廓能量泛函模型

$E_{tol} = E_{int} + E_{ext}$. $E_{ext} = \int_L f(x,y)\,ds$. $f(x,y)$为势函数

取 $f(x,y) = e^{-c\|\nabla(x(\sigma), y(\sigma))\|^2}$

$$E_{tol} = \frac{\pi}{4}\sum_{n=1}^{+\infty}(\alpha n^2+\beta n^4)(X_n^2+Y_n^2) + \frac{\alpha[(x_e-x_s)^2+(y_e-y_s)^2]}{2\pi} + \int_L f(x,y)\,ds.$$

离散化.

$$E_{tol} = \frac{\pi}{4}\sum_{n=1}^{+\infty}(\alpha n^2+\beta n^4)(X_n^2+Y_n^2) + \frac{\alpha[(x_e-x_s)^2+(y_e-y_s)^2]}{2\pi} + \sum_{m=0}^{M-1} f\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right)\Delta s_m$$

$$\Delta s_m = \sqrt{\left(x\left(\frac{(m+1)\pi}{M}\right) - x\left(\frac{m\pi}{M}\right)\right)^2 + \left(y\left(\frac{(m+1)\pi}{M}\right) - y\left(\frac{m\pi}{M}\right)\right)^2}.$$

4.2 简模型求解

4.2.1 响应函数

$h_{\sigma_0}(\sigma)$: (1) $h_{\sigma_0}(\sigma) \geq 0$. (2) $h_{\sigma_0}(0) = h_{\sigma_0}(\pi) = 0$. $h_{\sigma_0}(\sigma_0) = 1$.

(3) $h_{\sigma_0}(\sigma)$ 在 $[0,\sigma_0]$ 上单增 $[\sigma_0,\pi]$ 递减 (4) $h''_{\sigma_0}(\sigma)$ 连续

取 $h_{\sigma_0}(\sigma) = e^{-r(\sigma-\sigma_0)^2} - a(\sigma-\sigma_0)^{2n}$ $n \geq 2$.

由(2) $\Rightarrow$ $a = \begin{cases} \dfrac{e^{-r\sigma_0^2}}{\sigma_0^{2n}} & 0<\sigma<\sigma_0 \\ \dfrac{e^{-r(\pi-\sigma_0)^2}}{(\sigma_0-\pi)^{2n}} & \sigma_0<\sigma<\pi. \end{cases}$

$\because$ $h_{\sigma_0}(\sigma)$ 满足 Dirichlet 条件 $h_{\sigma_0}(\sigma) = \sum_{n=1}^{+\infty} H_{\sigma_0}(n)\sin n\sigma$.

$$H_n = \frac{2}{\pi}\int_0^\pi h_{\sigma_0}(\sigma)\sin n\sigma\, d\sigma \approx \frac{2}{M}\sum_{m=0}^{M-1} h_{\sigma_0}\left(\frac{m\pi}{M}\right)\sin\left(\frac{nm\pi}{M}\right).$$

$$\begin{cases} x_1(\sigma) = x_0(\sigma) + d\,h_{\sigma_0}(\sigma) \\ y_1(\sigma) = y_0(\sigma) + d\,h_{\sigma_0}(\sigma). \end{cases} \quad \begin{cases} x_1(\sigma) = x_s + \frac{x_e-x_s}{\pi}\sigma + \sum_{n=1}^{+\infty}(X_n + dH_{\sigma_0}(n))\sin(n\sigma) \\ y_1(\sigma) = y_s + \frac{y_e-y_s}{\pi}\sigma + \sum_{n=1}^{+\infty}(Y_n + dH_{\sigma_0}(n))\sin(n\sigma) \end{cases}$$

变化前参数 $X_{01}\cdots X_{0n}$, $Y_{01}\cdots Y_{0n}$.

$$\begin{cases} X_{1i} = X_{0i} + dH_{\sigma_0}(i) \\ Y_{1i} = Y_{0i} + dH_{\sigma_0}(i) \end{cases}$$

初始 $X_i = Y_i = 0$

$$\begin{cases} x(\sigma) = x_s + \frac{x_e-x_s}{\pi}\sigma + \sum_{m=1}^{M} x_i h_{\sigma_i}(\sigma) \\ y(\sigma) = y_s + \frac{y_e-y_s}{\pi}\sigma + \sum_{m=1}^{M} y_i h_{\sigma_i}(\sigma). \end{cases} \quad \begin{cases} X_n = \sum_{m=1}^{M} x_i H_{\sigma_i}(n) \\ Y_n = \sum_{m=1}^{M} y_i H_{\sigma_i}(n) \end{cases}$$

4.2.2 模型求解

(1) 选取初始状态

$$\begin{cases} x(\sigma) = x_s + \frac{x_e - x_s}{\pi}\sigma \\ y(\sigma) = y_s + \frac{y_e - y_s}{\pi}\sigma \end{cases} \qquad \begin{matrix} X_m = Y_m = 0 \\ X_n = Y_n = 0 \end{matrix}$$

$$E_{int} = \frac{\alpha[(x_e - x_s)^2 + (y_e - y_s)^2]}{2\pi} \qquad E_{ext} = \sum_{m=0}^{M-1} f\left(\left(x\left(\frac{m\pi}{M}\right), y\left(\frac{m\pi}{M}\right)\right)\right)\Delta s_m$$

$$E_{tol} = E_{int} + E_{ext}$$

(2) 设置可变范围

$\Delta x, \Delta y \in [-d, d]$

找 $$E_{min} = \min_{\substack{1 \le m_0 \le M \\ -d \le \Delta x \le d \\ -d \le \Delta y \le d}} E\left(x(\sigma) + \Delta x\, h_{\sigma_{m_0}},\ y(\sigma) + \Delta y\, h_{\sigma_{m_0}}(\sigma)\right)$$

(3) 若 $E_{min} < E(x(\sigma), y(\sigma))$ 取 $E_{tol} = E_{min}$

更新 X_n, Y_n, 转回(2) 否则终止.