HW6

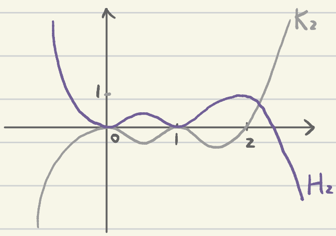
1(a) $\mathcal{L}_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -x(x-2) = -x^2+2x$

$\mathcal{L}_1'(x_1) = -2x+2 = 0$

$H_1(x) = \mathcal{L}_1^2(x)\left(1-2\mathcal{L}_1'(x_1)(x-x_1)\right) = \mathcal{L}_1^2(x) = x^4-4x^3+4x^2$

(b) $K_1(x) = \mathcal{L}_1^2(x)(x-x_1) = (x^4-4x^3+4x^2)(x-1) = x^5-5x^4+8x^3-4x^2$

(c)



2(a) $\mathcal{L}_0 = \frac{x-a}{-a}$ $\quad \mathcal{L}_1 = \frac{x}{a}$ $\quad y_0 = 0$ $\quad y_1 = a^3$

$p_1(x) = \frac{x}{a}\cdot a^3 = a^2x$

(b) $f'(x) = 3x^2$ $\quad f''(x) = 6x$

$\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x-x_j) = \frac{f''(\xi)}{2!}x(x-a) = \frac{1}{2}\cdot 6\xi\cdot x(x-a) = 3\xi x(x-a)$

$f(x)-p_1(x) = x^3-a^2x$

$\Rightarrow x(x+a)(x-a) = 3\xi x(x-a)$

$\xi = \frac{1}{3}(x+a) \in (0,a)$ for $x\in[0,a]$

(c) $\mathcal{L}_0 = \frac{x-a}{-a}$ $\quad \mathcal{L}_1 = \frac{x}{a}$ $\quad y_0 = a^4$ $\quad y_1 = a^4$

$p_1(x) = \frac{x-a}{-a}\cdot a^4 + \frac{x}{a}\cdot a^4 = a^4$

$f'(x) = 8(2x-a)^3$ $\quad f''(x) = 48(2x-a)^2$

$f(x)-p_1(x) = (2x-a)^4-a^4$

$\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x-x_j) = 24(2\xi-a)^2x(x-a)$

$((2x-a)^2+a^2)((2x-a)^2-a^2) = 24(2\xi-a)^2x(x-a)$

$4x(x-a)(4x^2+2a^2-4xa) = 24x(x-a)(2\xi-a)^2$

$4x^2+2a^2-4xa = 6(2\xi-a)^2$

$2\xi-a = \pm\sqrt{\frac{2x^2+a^2-2ax}{3}}$

$\xi_1 = \frac{1}{2}\sqrt{\frac{2x^2+a^2-2ax}{3}}+\frac{a}{2}$ $\quad \xi_2 = -\frac{1}{2}\sqrt{\frac{2x^2+a^2-2ax}{3}}+\frac{a}{2}$

3(a) $|E_2(f)| \leq \frac{(b-a)^5}{2880} M_4$, $M_4 = \max_{x\in[a,b]} f^{(4)}(x)$

Since $f^{(4)}(x)$ will be zero if $f(x)$ is of degree lower or equal to 3

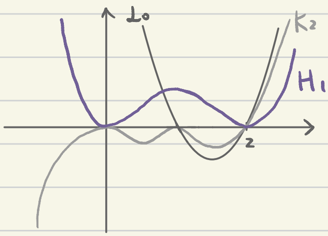
So polynomials up to degree 3 could be integrated exactly by Simpson's rule.

(b) $f'(x) = x^3 + \cos x$ $f''(x) = 3x^2 - \sin x$ $f^{(3)}(x) = 6x - \cos x$ $f^{(4)}(x) = 6 + \sin x$

$M_4 = \max_{x\in[0,\pi]} |f^{(4)}(x)| = \max_{x\in[0,\pi]} |6+\sin x| = 7$

$|E_2(f)| \leq \frac{(\pi-0)^5}{2880} \cdot 7 \approx 0.0076$

(c)



4(a) See the code.

(b) See the code. Then use $A^T A x = A^T \vec{b}$

$$A = \begin{bmatrix} 1 & \log 10 \\ 1 & \log 20 \\ 1 & \log 40 \\ 1 & \log 80 \end{bmatrix}$$

Theoretically, $\varepsilon \leq \frac{(b-a)^3}{12m^2} M_2$

$$\log \varepsilon \leq \log\left(\frac{(b-a)^3}{12m^2} M_2\right) = \log\left(\frac{(b-a)^3 M_2}{12}\right) - 2\log m$$

$k = -2$

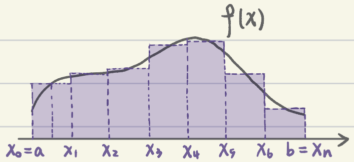
The numerical method gives $k = -1.99$, so the estimation is quite precise.

(c) No. $\begin{bmatrix} D \\ K \end{bmatrix} = \begin{bmatrix} -4.50 \\ 0.15 \end{bmatrix}$

$M_2 = \max_{x\in[0,1]} |f''(x)| = \max_{x\in[0,1]} \frac{1}{4\sqrt{x^3}} = \infty$ when $x = 0$

So the error estimate $\varepsilon \leq \frac{(b-a)^3}{12m^2} M_2$ is a trivial one in this case and does not give accurate result.

5(a)



(b) If f is either constant or linear in each subinterval, let $f_i(x) = k_i x + m_i$

$$\int_{x_{i-1}}^{x_i} f_i(x)\,dx = \int_a^b f(x)\,dx = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} k_i x + m_i\,dx = \sum_{i=1}^{n}\left[\frac{k_i}{2}x^2 + m_i x\right]_{x_{i-1}}^{x_i}$$

$$= \sum_{i=1}^{n}\left(\frac{k_i(x_i^2 - x_{i-1}^2)}{2} + m_i(x_i - x_{i-1})\right) = \sum_{i=1}^{n}\left(\frac{hk_i(x_i + x_{i-1})}{2} + m_i h\right)$$

$$= h\sum_{i=1}^{n}\left(k\left(\frac{x_i + x_{i-1}}{2}\right) + m_i\right) = h\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)$$

So the formula is exact.

(c) Consider each subinterval, denote $x_{\frac{i+1}{2}}$ with $\bar{x}$ for each i for simplicity

Since $f \in C^2([a,b])$, $f(x) \approx f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2 + \frac{1}{6}f'''(\bar{x})(x-\bar{x})^3 + \cdots$

$$= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2$$

$$\int_{x_{i-1}}^{x_i} f(x)\,dx \approx \int_{x_{i-1}}^{x} f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2\,dx$$

$$= \left[f(\bar{x})x + f'(\bar{x})\left(\frac{1}{2}x^2 - \bar{x}\cdot x\right) + \frac{1}{2}f''(\bar{x})\frac{1}{3}(x-\bar{x})^3\right]_{x_{i-1}}^{x_i}$$

$$= f(\bar{x})h + f'(\bar{x})\left(\frac{1}{2}(x_i^2 - x_{i-1}^2) - \bar{x}h\right) + \frac{1}{6}f''(\bar{x})\left((x_i-\bar{x})^3 - (x_{i-1}-\bar{x})^3\right)$$

$$= f(\bar{x})h + f'(\bar{x})h\left(\frac{1}{2}(x_i + x_{i-1}) - \bar{x}\right) + \frac{1}{6}f''(\bar{x})\left(\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right)$$

$$= f(\bar{x})h + \frac{1}{24}f''(\bar{x})h^3$$

So $\int_{x_{i-1}}^{x_i} f(x)\,dx - hf(\bar{x}) = \frac{1}{24}f''(x)h^3$

Then add up the error for each interval, $\sum_{i=1}^{n}\left(\int_{x_{i-1}}^{x_i} f(x)\,dx - hf(\bar{x}_i)\right)$

$$= \sum_{i=1}^{n}\frac{1}{24}f''(\bar{x}_i)h^3$$

$$\leq \max_{i\in\{1,\cdots n\}} n\left(\frac{1}{24}f''(\bar{x}_i)h^3\right)$$

$$= \max_{i\in\{1,\cdots n\}} \frac{1}{24}f''(\bar{x}_i)(b-a)h^2$$

So the midpoint rule is second order accurate.

b(a)

$$a_0 = \frac{\int_0^1 x^3 dx}{\int_0^1 1^2 dx} = \frac{1}{4} \qquad a_1 = \frac{\int_0^1 (x-\frac{1}{2}) x^3 dx}{\int_0^1 (x-\frac{1}{2})^2 dx} = \frac{[\frac{1}{5}x^5 - \frac{1}{8}x^4]_0^1}{[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x]_0^1} = \frac{9}{10}$$

$$a_2 = \frac{\int_0^1 (x^2-x+\frac{1}{6}) x^3 dx}{\int_0^1 (x^2-x+\frac{1}{6})^2 dx} = \frac{[\frac{1}{6}x^6 - \frac{1}{5}x^5 + \frac{1}{24}x^4]_0^1}{[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x]_0^1} = \frac{3}{2}$$

(b)

$$p_3 = x^3 - \frac{3}{2}(x^2 - x + \frac{1}{6}) - \frac{9}{10}(x - \frac{1}{2}) - \frac{1}{4} = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20} = 0$$

$$\frac{1}{20}(2x-1)(10x^2-10x+1) = 0 \rightarrow x_0 = \frac{1}{2} \quad x_1 = \frac{5+\sqrt{15}}{10} \quad x_2 = \frac{5-\sqrt{15}}{10}$$

$$L_0 = \frac{(x - \frac{5+\sqrt{15}}{10})(x - \frac{5-\sqrt{15}}{10})}{(\frac{1}{2} - \frac{5+\sqrt{15}}{10})(\frac{1}{2} - \frac{5-\sqrt{15}}{10})} \qquad W_0 = \frac{\int_0^1 (x^2 - x + \frac{1}{10})^2 dx}{(\frac{1}{2} - \frac{5+\sqrt{15}}{10})^2 (\frac{1}{2} - \frac{5-\sqrt{15}}{10})^2} = \frac{4}{9}$$

$$L_1 = \frac{(x-\frac{1}{2})(x - \frac{5-\sqrt{15}}{10})}{(\frac{5+\sqrt{15}}{10} - \frac{1}{2})(\frac{5+\sqrt{15}}{10} - \frac{5-\sqrt{15}}{10})} \qquad L_2 = \frac{(x-\frac{1}{2})(x - \frac{5+\sqrt{15}}{10})}{(\frac{5-\sqrt{15}}{10} - \frac{1}{2})(\frac{5-\sqrt{15}}{10} - \frac{5+\sqrt{15}}{10})}$$

$$W_1 = \int_0^1 [L_1(x)]^2 dx = \frac{5}{18} \qquad W_2 = \int_0^1 [L_2(x)]^2 dx = \frac{5}{18}$$

(c)

$$I_1 = \int_0^1 x + x\,dx = 1 \qquad I_2 = \int_0^1 x^2 + x\,dx = \frac{5}{6} \qquad I_3 = \int_0^1 x^3 + x\,dx = \frac{3}{4}$$

$$I_4 = \frac{1}{5} + \frac{1}{2} = \frac{7}{10} \qquad I_5 = \frac{1}{6} + \frac{1}{2} = \frac{2}{3} \qquad I_6 = \frac{1}{7} + \frac{1}{2} = \frac{9}{14} \qquad I_7 = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

See the code

(d)

As we discussed in class, Gauss quadrature rule was obtained by exact integration of the Hermite interpolation polynomial of degree $2n+1$ for f. Here we are using $n=2$, and our graph confirmed this by remaing at zero when $k \le 5$, and increases after that.

For simpson's rule, $|E(f)| \le \frac{(b-a)^5}{2880} M_4$, where M_4 denotes $\max_{\xi \in [a,b]} |f^{(4)}(\xi)|$. Note that $f^{(4)}(\xi) = 0$ when f is of degree of three or lower, and undergoes polynomial growth when n goes larger, our error graph also proved such tendency. Despite that, we also notice that Gauss quadrature gives a more precise approximation than Simpson's rule.

7 (a) Let $\ell_0 = 1$

$$a_0 = \frac{\int_0^\infty e^{-x} x \, dx}{\int_0^\infty e^{-x} dx} = 1 \qquad \ell_1 = x - 1$$

$$a_0 = \frac{\int_0^\infty e^{-x} x^2 dx}{\int_0^\infty e^{-x} dx} = 2, \quad a_1 = \frac{\int_0^\infty e^{-x} (x-1) x^2 dx}{\int_0^\infty e^{-x} (x-1)^2 dx} = 4$$

$$\ell_2 = x^2 - 2 - 4(x-1) = x^2 - 4x + 2$$

$$a_0 = \frac{\int_0^\infty e^{-x} x^3 dx}{\int_0^\infty e^{-x} dx} = 6, \quad a_1 = \frac{\int_0^\infty e^{-x} (x-1) x^3 dx}{\int_0^\infty e^{-x} (x-1)^2 dx} = 18, \quad a_2 = \frac{\int_0^\infty e^{-x} (x^2-4x+2) x^3 dx}{\int_0^\infty e^{-x} (x^2-4x+2)^2 dx} = 9$$

$$\ell_3 = x^3 - 6 - 18(x-1) - 9(x^2-4x+2) = x^3 - 9x^2 + 18x - 6$$

Plot in the code.

(b) See the code.

(c) See the code.