HW4

1(a) - Consider eigenvector $c\vec{v}$ in the direction of $\vec{v}$, c constant

By definition $(I - \frac{2}{\vec{v}^T\vec{v}}\vec{v}\vec{v}^T)c\vec{v} = \lambda \cdot c\vec{v}$

$$c\vec{v} - \frac{2c}{\vec{v}^T\vec{v}}\vec{v}(\vec{v}^T\vec{v}) = c\lambda\vec{v}$$

$$c\vec{v} - 2c\vec{v} = c\lambda\vec{v} \quad \Rightarrow \lambda = -1$$

- Consider eigenvector $\vec{u}_i$ for $i \in \{1, 2, \dots n-1\}$ such that $\vec{u}_i^T\vec{v} = 0$

(i.e. $\vec{u}_i$ lies in the n-1 directions that are orthogonal to $\vec{v}$)

By definition $(I - \frac{2}{\vec{v}^T\vec{v}}\vec{v}\vec{v}^T)\vec{u}_i = \lambda\vec{u}_i$

$$\vec{u} - \frac{2}{\vec{v}^T\vec{v}}\vec{v}(\vec{u}_i^T\vec{v})^T = \lambda\vec{u}_i$$

$$\vec{u}_i = \lambda\vec{u}_i \quad \Rightarrow \lambda = 1$$

So $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = \dots = \lambda_n = 1$

(b) $\det H = \prod_{i=1}^{n} \lambda_i = -1$

(c) Since H, H_1 both orthogonal, HH_1 is still orthogonal

$\det HH_1 = \det H \cdot \det H_1 = (-1)\cdot(-1) = 1$

However determinant of Householder matrix is -1, HH_1 cannot be a Householder matrix for any vector $\vec{w}$

2(a)

$\vec{x} = \begin{bmatrix} 9 \\ 12 \\ 0 \end{bmatrix}$ $\quad \beta = 9 \quad c = \|\vec{x}\|_2 = \sqrt{9^2+12^2} = 15$

$\vec{v} = \begin{bmatrix} 9 \\ 12 \\ 0 \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 12 \\ 0 \end{bmatrix}$ $\quad H_1 = I - \frac{2}{24^2+12^2}\begin{bmatrix} 24 \\ 12 \\ 0 \end{bmatrix}\begin{bmatrix} 24 & 12 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$H_1A = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 9 & -6 \\ 12 & -8 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} -15 & 10 \\ 0 & 0 \\ 0 & 20 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$ $\quad \beta = 0 \quad c = \sqrt{0 + 20^2} = 20$

$\vec{v} = \begin{bmatrix} 0 \\ 20 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$ $\quad H_2' = I - \frac{2}{20^2+20^2}\begin{bmatrix} 20 \\ 20 \end{bmatrix}\begin{bmatrix} 20 & 20 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ $\quad R = H_2H_1A = \begin{bmatrix} -15 & 10 \\ 0 & -20 \\ 0 & 0 \end{bmatrix}$ $\quad Q = H_1H_2 = \begin{bmatrix} -\frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & -1 & 0 \end{bmatrix}$

$\hat{Q} = \begin{bmatrix} -\frac{3}{5} & 0 \\ -\frac{4}{5} & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \hat{R} = \begin{bmatrix} -15 & 10 \\ 0 & -20 \end{bmatrix}$

(b) $H\vec{x}$, $\vec{x}$ and $\vec{v}$ are coplaner and we seek for $\vec{v} \in \mathbb{R}^n$ as a linear combination of $\vec{x}$ and $\vec{y}$.

$$\vec{v} = \vec{x} + c\vec{y}$$

$$H = I - \frac{2}{\vec{v}^T\vec{v}}\vec{v}\vec{v}^T$$

$$\vec{v}^T\vec{x} = (\vec{x}^T + c\vec{y}^T)\vec{x} = \vec{x}^T\vec{x} + c\vec{y}^T\vec{x}$$

$$\vec{v}^T\vec{v} = (\vec{x}^T + c\vec{y}^T)(\vec{x} + c\vec{y}) = \vec{x}^T\vec{x} + c\vec{x}^T\vec{y} + c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y}$$

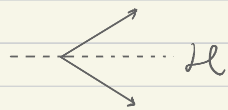
$$H\vec{x} = I\vec{x} - \frac{2}{\vec{v}^T\vec{v}}\vec{v}(\vec{v}^T\vec{x}) = \vec{x} - \frac{2(\vec{x}+c\vec{y})(\vec{x}^T\vec{x} + c\vec{y}^T\vec{x})}{\vec{x}^T\vec{x} + c\vec{x}^T\vec{y} + c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y}} \qquad (\text{Let } \vec{y}^T\vec{x} = \vec{x}^T\vec{y} = \alpha)$$

$$H\vec{x} = \frac{(\vec{x}^T\vec{x} + 2c\alpha + c^2\vec{y}^T\vec{y})\vec{x} - 2\vec{x}^T\vec{x}\,\vec{x} - 2c\alpha\vec{x} - 2c\vec{x}^T\vec{x}\,\vec{y} - 2c^2\alpha\vec{y}}{\vec{x}^T\vec{x} + 2c\alpha + c^2\vec{y}^T\vec{y}} = \frac{(c^2\vec{y}^T\vec{y} - \vec{x}^T\vec{x})\vec{x} - 2c(\vec{x}^T\vec{x} + c\alpha)\vec{y}}{\vec{x}^T\vec{x} + 2c\alpha + c^2\vec{y}^T\vec{y}}$$

$$\begin{cases} c^2\vec{y}^T\vec{y} - \vec{x}^T\vec{x} = 0 \quad \Rightarrow \quad c = \pm\sqrt{\dfrac{\vec{x}^T\vec{x}}{\vec{y}^T\vec{y}}} \\ \vec{x}^T\vec{x} + 2c\alpha + c^2\vec{y}^T\vec{y} \neq 0 \end{cases}$$

Since $\vec{x}^T\vec{x} + c\vec{x}^T\vec{y} + c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y} = (\vec{x}^T + c\vec{y}^T)(\vec{x} + c\vec{y}) = \|\vec{x} + c\vec{y}\|_2^2 \geq 0$

So take $c = \begin{cases} \operatorname{sign}(\vec{y}^T\vec{x})\sqrt{\dfrac{\vec{x}^T\vec{x}}{\vec{y}^T\vec{y}}}, & \vec{y}^T\vec{x} \neq 0 \\ \sqrt{\dfrac{\vec{x}^T\vec{x}}{\vec{y}^T\vec{y}}}, & \vec{y}^T\vec{x} = 0 \end{cases}$



$$H\vec{x} = \frac{(c^2\vec{y}^T\vec{y} - \vec{x}^T\vec{x})\vec{x} - 2c(\vec{x}^T\vec{x} + c\alpha)\vec{y}}{\vec{x}^T\vec{x} + 2c\alpha + c^2\vec{y}^T\vec{y}} = \frac{-2c(\vec{x}^T\vec{x} + c\alpha)\vec{y}}{2(\vec{x}^T\vec{x} + c\alpha)} = -c\vec{y}$$

Given the scalar c, the matrix H is unique.

3(a) $A^TA = (QR)^T(QR) = R^TQ^TQR = R^TR$

$k_2(A^TA) = \|A^TA\|_2\,\|(A^TA)^{-1}\|_2 = \|R^TR\|_2\,\|(R^TR)^{-1}\|_2 = \max\sqrt{\lambda(R^TR)} \cdot \max\sqrt{\lambda((R^TR)^{-1})}$

(b) $A = QR \Rightarrow QR\vec{x} = \vec{b} \Rightarrow R\vec{x} = Q^T\vec{b}$

Step one: calculate $\vec{y} = Q^T\vec{b}$, $Q \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$

We need n multiplication, $n-1$ addition for each row of Q^T

$\Rightarrow$ In total $n(2n-1)$ steps

Step two: Solve $R\vec{x} = \vec{y}$ for $\vec{x}$

$x_i = \frac{1}{r_{ii}}\left(y_i - \sum_{j=i+1}^{n} r_{ij}x_j\right)$ for $i = 1, \cdots n$, each need $2(n-i)+1$ steps

$\Rightarrow$ In total $\sum_{i=1}^{n} 2(n-i)+1 = n^2$ steps

$\Rightarrow$ In total $2n^2 - n + n^2 \sim 3n^2$ steps

LU decomposition needs $2n^2$ steps, so QR factorization needs 1.5 times the computation time for LU decomposition.

4(a)

- Prove $\vec{q}_i^T \vec{q}_i = 1$ for all $i \in \{1, \cdots n\}$

$$\vec{q}_1^T \vec{q}_1 = \left(\frac{\vec{a}_1}{\|\vec{a}_1\|_2}\right)^T \frac{\vec{a}_1}{\|\vec{a}_1\|_2} = \frac{\vec{a}_1^T \vec{a}_1}{\vec{a}_1^T \vec{a}_1} = 1$$

$$\vec{q}_i = \frac{\vec{b}_i}{\|\vec{b}_i\|_2}, \quad \|\vec{b}_i\|_2 = \sqrt{\vec{b}_i^T \vec{b}_i} \quad \Rightarrow \quad \vec{q}_i^T \vec{q}_i = \frac{\vec{b}_i^T \vec{b}_i}{\vec{b}_i^T \vec{b}_i} = 1$$

- Prove $\vec{q}_i^T \vec{q}_j = 0$ for $i \neq j$, $i, j \in \{1, \cdots n\}$ (by induction)

Base case: $\vec{q}_1^T \vec{q}_2 = \frac{1}{\|\vec{b}_2\|_2} \vec{q}_1^T (\vec{a}_2 - \vec{q}_1^T \vec{a}_2 \vec{q}_1) = \frac{1}{\|\vec{b}_2\|_2} (\vec{q}_1^T \vec{a}_2 - \vec{q}_1^T \vec{a}_2 \underbrace{\vec{q}_1^T \vec{q}_1}_{=1}) = 0$

$$\vec{q}_1^T \vec{q}_3 = \frac{1}{\|\vec{b}_3\|_2} \vec{q}_1^T (\vec{a}_3 - \vec{q}_1^T \vec{a}_3 \vec{q}_1 - \vec{q}_2^T \vec{a}_3 \vec{q}_2) = \frac{1}{\|\vec{b}_3\|_2} (\vec{q}_1^T \vec{a}_3 - \vec{q}_1^T \vec{a}_3 \underbrace{\vec{q}_1^T \vec{q}_1}_{=1} - \vec{q}_2^T \vec{a}_3 \underbrace{\vec{q}_1^T \vec{q}_2}_{=0}) = 0$$

$$\vec{q}_2^T \vec{q}_3 = \frac{1}{\|\vec{b}_3\|_2} \vec{q}_2^T (\vec{a}_3 - \vec{q}_1^T \vec{a}_3 \vec{q}_1 - \vec{q}_2^T \vec{a}_3 \vec{q}_2) = \frac{1}{\|\vec{b}_3\|_2} (\vec{q}_2^T \vec{a}_3 - \vec{q}_1^T \vec{a}_3 \underbrace{\vec{q}_2^T \vec{q}_1}_{=0} - \vec{q}_2^T \vec{a}_3 \underbrace{\vec{q}_2^T \vec{q}_2}_{=1}) = 0$$

Induction: Suppose $\vec{q}_1, \vec{q}_2 \cdots \vec{q}_i$ are orthogonal to each other

$$\vec{q}_1^T \vec{q}_{i+1} = \frac{1}{\|\vec{b}_{i+1}\|_2} \vec{q}_1^T (\vec{a}_{i+1} - \sum_{k=1}^{i} \vec{q}_k^T \vec{a}_{i+1} \vec{q}_k)$$

$$= \frac{1}{\|\vec{b}_{i+1}\|_2} (\vec{q}_1^T \vec{a}_{i+1} - \vec{q}_1^T \vec{a}_{i+1} \underbrace{\vec{q}_1^T \vec{q}_1}_{=1} - \vec{q}_2^T \vec{a}_{i+1} \underbrace{\vec{q}_1^T \vec{q}_2}_{=0} - \cdots - \vec{q}_i^T \vec{a}_{i+1} \underbrace{\vec{q}_1^T \vec{q}_i}_{=0})$$

$$= \frac{1}{\|\vec{b}_{i+1}\|_2} (\vec{q}_1^T \vec{a}_{i+1} - \vec{q}_1^T \vec{a}_{i+1}) = 0$$

$$\vec{q}_j^T \vec{q}_{i+1} = \frac{1}{\|\vec{b}_{i+1}\|_2} \vec{q}_j^T (\vec{a}_{i+1} - \sum_{k=1}^{i} \vec{q}_k^T \vec{a}_{i+1} \vec{q}_k)$$

$$= \frac{1}{\|\vec{b}_{i+1}\|_2} (\vec{q}_j^T \vec{a}_{i+1} - \vec{q}_1^T \vec{a}_{i+1} \underbrace{\vec{q}_j^T \vec{q}_1}_{=0} - \cdots - \vec{q}_j^T \vec{a}_{i+1} \underbrace{\vec{q}_j^T \vec{q}_j}_{=1} - \cdots - \vec{q}_i^T \vec{a}_{i+1} \underbrace{\vec{q}_j^T \vec{q}_i}_{=0})$$

$$= \frac{1}{\|\vec{b}_{i+1}\|_2} (\vec{q}_j^T \vec{a}_{i+1} - \vec{q}_j^T \vec{a}_{i+1}) = 0 \quad \text{for any } j \in \{1, \cdots i\}$$

So $\vec{q}_i^T \vec{q}_j = 0$ for $i \neq j$, $i, j \in \{1, \cdots n\}$

(b)

$$\vec{a}_{i+1} = \vec{b}_{i+1} + \sum_{k=1}^{i} \vec{q}_k^T \vec{a}_{i+1} \vec{q}_k = \sqrt{\vec{b}_{i+1}^T \vec{b}_{i+1}}\, \vec{q}_{i+1} + \sum_{k=1}^{i} \vec{q}_k^T \vec{a}_{i+1} \vec{q}_k = \vec{q}_1^T \vec{a}_{i+1} \vec{q}_1 + \cdots + \vec{q}_i^T \vec{a}_{i+1} \vec{q}_i + \sqrt{\vec{b}_{i+1}^T \vec{b}_{i+1}}\, \vec{q}_{i+1}$$

Since $\sqrt{\vec{b}_{i+1}^T \vec{b}_{i+1}}$ and $\vec{q}_k^T \vec{a}_k$ are scalar values, above is a linear combination of $\vec{q}_1 \cdots \vec{q}_{i+1}$

(c)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & & & \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & r_{2n} \\ & & & \vdots \\ & & & r_{nn} \end{bmatrix}$$

- $a_{11} = q_{11} r_{11}$, $a_{21} = q_{21} r_{11}$, $a_{31} = q_{31} r_{11}$, $\cdots$ $a_{n1} = q_{n1} r_{11}$ $\Rightarrow \vec{a}_1 = \vec{q}_1 \vec{r}_1 \Rightarrow \vec{q}_1^T \vec{a}_1 = \vec{r}_1$

- $a_{12} = q_{11} r_{12} + q_{12} r_{22}$, $a_{22} = q_{21} r_{12} + q_{22} r_{22}$, $a_{32} = q_{31} r_{12} + q_{32} r_{22}$

$$\vec{a}_2 = \vec{q}_1 r_{12} + \vec{q}_2 r_{22} \Rightarrow \vec{q}_1^T \vec{a}_2 = \vec{q}_1^T \vec{q}_1 r_{12} + \vec{q}_1^T \vec{q}_2 r_{22} = r_{12}$$

$$\vec{q}_2^T \vec{a}_2 = \vec{q}_2^T \vec{q}_1 r_{12} + \vec{q}_2^T \vec{q}_2 r_{22} = r_{22}$$

Similarly, compute the rest entries of R.

$$R = \begin{bmatrix} \vec{q}_1^T \vec{a}_1 & \vec{q}_1^T \vec{a}_2 & \vec{q}_1^T \vec{a}_3 & \cdots & \vec{q}_1^T \vec{a}_n \\ & \vec{q}_2^T \vec{a}_2 & \vec{q}_2^T \vec{a}_3 & \cdots & \vec{q}_2^T \vec{a}_n \\ & & \vec{q}_3^T \vec{a}_3 & \cdots & \vec{q}_3^T \vec{a}_n \\ & & & \cdots & \vec{q}_4^T \vec{a}_n \\ & & & & \vdots \\ & & & & \vec{q}_n^T \vec{a}_n \end{bmatrix}$$

(d)

$$\vec{q}_1 = \frac{1}{\sqrt{(1+\varepsilon)^2+1^2+1^2}}\begin{bmatrix}1+\varepsilon\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}}\\\frac{1}{\sqrt{2\varepsilon+3}}\\\frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix}1\\1+\varepsilon\\1\end{bmatrix} - \begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix}\begin{bmatrix}1\\1+\varepsilon\\1\end{bmatrix}\begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}}\\\frac{1}{\sqrt{2\varepsilon+3}}\\\frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix} = \begin{bmatrix}-\varepsilon\\\varepsilon\\0\end{bmatrix}$$

$$\vec{q}_2 = \frac{1}{\sqrt{(-\varepsilon)^2+\varepsilon^2+0^2}}\begin{bmatrix}-\varepsilon\\\varepsilon\\0\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix}$$

$$\vec{b}_3 = \begin{bmatrix}1\\1\\1+\varepsilon\end{bmatrix} - \begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix}\begin{bmatrix}1\\1\\1+\varepsilon\end{bmatrix}\begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}}\\\frac{1}{\sqrt{2\varepsilon+3}}\\\frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix} - \begin{bmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\end{bmatrix}\begin{bmatrix}1\\1\\1+\varepsilon\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix} = \begin{bmatrix}-\varepsilon\\0\\\varepsilon\end{bmatrix}$$

$$\vec{q}_3 = \frac{1}{\sqrt{(-\varepsilon)^2+0^2+\varepsilon^2}}\begin{bmatrix}-\varepsilon\\0\\\varepsilon\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix}$$

$$\vec{q}_1^{\,T}\vec{q}_2 = \begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix} = \frac{-1-\varepsilon+1}{\sqrt{4\varepsilon+6}} = \frac{-\varepsilon}{\sqrt{4\varepsilon+6}} \approx 0$$

$$\vec{q}_1^{\,T}\vec{q}_3 = \begin{bmatrix}\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}} & \frac{1}{\sqrt{2\varepsilon+3}}\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix} = \frac{-1-\varepsilon+1}{\sqrt{4\varepsilon+6}} = \frac{-\varepsilon}{\sqrt{4\varepsilon+6}} \approx 0$$

$$\vec{q}_2^{\,T}\vec{q}_3 = \begin{bmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{bmatrix} = \frac{1}{2}$$

$$|\vec{q}_1| = \sqrt{\left(\frac{1+\varepsilon}{\sqrt{2\varepsilon+3}}\right)^2+\left(\frac{1}{\sqrt{2\varepsilon+3}}\right)^2+\left(\frac{1}{\sqrt{2\varepsilon+3}}\right)^2} = \sqrt{\frac{\varepsilon^2+2\varepsilon+3}{2\varepsilon+3}} \approx 1$$

$$|\vec{q}_2| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+0^2} = 1 \qquad |\vec{q}_3| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2+0^2+\left(\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\vec{q}_1^{\,T}\vec{q}_2 = |\vec{q}_1||\vec{q}_2|\cos\theta \Rightarrow \cos\theta = 0 \quad \theta = \frac{\pi}{2}$$

$$\vec{q}_1^{\,T}\vec{q}_3 = |\vec{q}_1||\vec{q}_3|\cos\theta \Rightarrow \cos\theta = 0 \quad \theta = \frac{\pi}{2}$$

$$\vec{q}_2^{\,T}\vec{q}_3 = |\vec{q}_2||\vec{q}_3|\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3} \neq \frac{\pi}{2}$$

Ignoring trivial error in A leads to huge problem in Q (Q is not orthogonal)

i.e. Gram Schmidt orthogonalization is sensitive to rounding errors.

5 (a) The exact result for a is 1×10^{-16}, for b is -1×10^{-16}

But the computer result is $a = 1\times10^{-6}$, $b = 0$

Rounding error does not affect a too much because $1-1$ is first calculated.

Since $1+1\times10^{-16}$ is calculated first in b, and is rounded to 1, the result becomes zero.

(b) The exact result $= \|H_n(H_n^{-1}e_n) - e_n\| = \|I\cdot e_n - e_n\| = 0$

```
The norm is
0.00000000000803893619
Condition number is
476607.25024224253138527274
```

$n=5$

```
The norm is
0.00078085096536644719
Condition number is
16024909625167.58007812500000000000
```

$n=10$

```
The norm is
67.29006916708335950261
Condition number is
1531575599359190272.00000000000000000000
```

$n=20$

(c)

```
The best estimation is at iteration
     8
```

when $h = 10^{-8}$

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The error is
   5.4467e-09
```

Error will become larger when h goes smaller

(Plot in pdf form and code attached)

6 (a) True. $(A+\alpha I)\vec{x} = A\vec{x} + \alpha I\vec{x} = \lambda\vec{x} + \alpha\vec{x} = (\lambda+\alpha)\vec{x}$

(b) True. $\alpha A\vec{x} = \alpha\lambda\vec{x} = (\alpha\lambda)\vec{x}$

(c) True. $\vec{x} = A^{-1}A\vec{x} = A^{-1}\lambda\vec{x} = \lambda A^{-1}\vec{x} \Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

(d) True. By induction, base case $A^2\vec{x} = A\cdot A\vec{x} = A\cdot\lambda\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$

Induction: $A^{k+1}\vec{x} = A^k A\vec{x} = A^k\lambda\vec{x} = \lambda A^k\vec{x} = \lambda\cdot\lambda^k\vec{x} = \lambda^{k+1}\vec{x}$

(e) True $B = SAS^{-1} \Rightarrow S^{-1}BS = A$

$S^{-1}BS\vec{x} = A\vec{x} = \lambda\vec{x} \Rightarrow BS\vec{x} = S\lambda\vec{x} = \lambda S\vec{x}$

Eigenvector of B is $S\vec{x}$.

(f) False. Counterexample: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(g) False. Counterexample: $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(h) True. A singular $\Leftrightarrow \det A = 0 = \sum_{i=1}^{n} \lambda_i$ where λ_i are the eigenvalues of A.

So $\exists\ \lambda_i$ for $i \in \{1, \cdots n\}$ such that $\lambda_i = 0$

(i) False. Counterexample: $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(j) True. If A has comple eigenvalues, they must appear in conjugate pairs

Suppose for the sake of contradiction that $\lambda \neq \bar{\lambda}$ are eigenvalues of A

$A\vec{x} = \lambda\vec{x}, \quad A\vec{x} = \bar{\lambda}\vec{x}$

$\vec{x}^T\lambda\vec{x} = \vec{x}^T A\vec{x} = (\vec{x}^T A^T \vec{x})^T = (\vec{x}^T A\vec{x})^T = (\vec{x}^T\bar{\lambda}\vec{x})^T = \vec{x}^T\bar{\lambda}\vec{x}$

$\Rightarrow \lambda = \bar{\lambda}$ contradict

So all eigenvalues are real

(k) True. $A\vec{x}_1 = \lambda_1\vec{x}_1, \quad A\vec{x}_2 = \lambda_2\vec{x}_2$

$\lambda_1\vec{x}_2^T\vec{x}_1 = \vec{x}_2^T\lambda_1\vec{x}_1 = \vec{x}_2^T A\vec{x}_1 = (\vec{x}_1^T A^T\vec{x}_2)^T = (\vec{x}_1^T A\vec{x}_2)^T = (\vec{x}_1^T\lambda_2\vec{x}_2)^T = \lambda_2\vec{x}_2^T\vec{x}_1$

$\lambda_1 \neq \lambda_2 \Rightarrow \vec{x}_2^T\vec{x}_1 = 0$