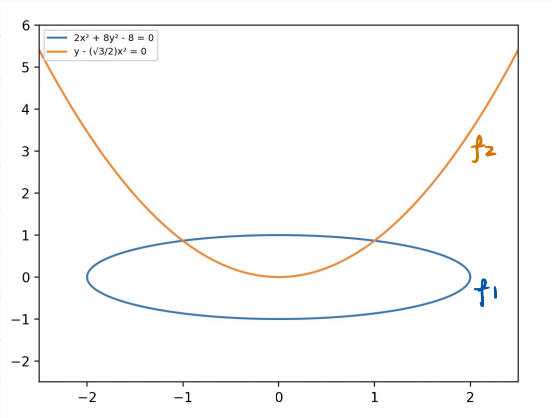
HW 2

1.



$2x^2+8y^2-8=0$

use parametric equation

$x = 2\cos t$, $y = \sin t$

S_1 is an ellipse

S_2 is a parabola

(b) $\begin{array}{l} 2x^2+8y^2-8=0 \\ y-\frac{\sqrt{3}}{2}x^2=0 \end{array} \Rightarrow \begin{array}{l} \frac{2}{\sqrt{3}}y = x^2 \\ \frac{4\sqrt{3}}{3}y+8y^2-8=0 \end{array} \Rightarrow \begin{array}{l} y=\frac{-\sqrt{3}\pm 7\sqrt{3}}{12} \\ y>0 \end{array} \Rightarrow \begin{array}{l} (x_1,y_1)=(1,\frac{\sqrt{3}}{2}) \\ (x_2,y_2)=(-1,\frac{\sqrt{3}}{2}) \end{array}$

(c) $J_f(x,y) = \begin{pmatrix} 4x & 16y \\ -\sqrt{3}x & 1 \end{pmatrix}$

(d) See code

2 (a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $(1\times 8-2\times 4=0)$

(b) Suppose $A = LU = L_1U_1$

$L_1^{-1}LUU^{-1} = L_1^{-1}L_1U_1U^{-1}$

$L_1^{-1}L = U_1U^{-1}$

L_1 is unit lower triangular matrix $\Rightarrow$ L_1^{-1} is still unit lower triangular

So $L_1^{-1}L$ is unit lower triangular

U is upper triangular matrix $\Rightarrow$ U^{-1} is still upper triangular

So U_1U^{-1} is upper triangular

So $L_1^{-1}L = U_1U^{-1} = I$

Thus $L_1L_1^{-1}L = L_1I \Rightarrow L = L_1$

$U_1U^{-1}U = IU \Rightarrow U_1 = U$

i.e. LU factorization of invertible A is unique.

3(a) So u is in the form $\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$

Could always be written as $\begin{bmatrix} u_{11} & 0 & \cdots & 0 \\ \vdots & u_{22} & & \vdots \\ & & \ddots & \\ 0 & \cdots & & u_{nn} \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} & \cdots & \frac{u_{1n}}{u_{11}} \\ \vdots & 1 & \frac{u_{23}}{u_{22}} & \cdots & \frac{u_{2n}}{u_{22}} \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$

So take $D = \begin{bmatrix} u_{11} & 0 & \cdots & 0 \\ \vdots & u_{22} & & \vdots \\ & & \ddots & \\ 0 & \cdots & & u_{nn} \end{bmatrix}$, $U' = \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} & \cdots & \frac{u_{1n}}{u_{11}} \\ \vdots & 1 & \frac{u_{23}}{u_{22}} & \cdots & \frac{u_{2n}}{u_{22}} \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$. $A = LDU'$ where U' is a unit upper triangular matrix

(b) $A = LU = LDU'$

$A^T = (LDU')^T = U'^T D^T L^T = U'^T (LD)^T$

U'^T is the transpose of unit upper triangular matrix, which is unit upper triangular. $(LD)^T$ is the transpose of a lower triangular matrix, which is unit upper triangular.

So take $L'' = U'^T$, $U'' = (LD)^T$ and thus we find $A = L''U''$

4(a) • $a_1 = [1\ 0 \cdots 0] \begin{bmatrix} e_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_1$ $\qquad e_1 = a_1$

$c_1 = [1\ 0 \cdots 0] \begin{bmatrix} f_1 \\ e_2 \\ \vdots \\ 0 \end{bmatrix} = f_1$ $\qquad f_1 = c_1$

First row of U is the same as the first row of A.

• $b_1 = [d_1\ 1\ 0 \cdots 0] \begin{bmatrix} e_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 e_1$ $\qquad d_1 = \frac{b_1}{e_1}$

With d_1, we can calculate e_2, f_2

$a_2 = [d_1\ 1\ 0 \cdots 0] \begin{bmatrix} f_1 \\ e_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 f_1 + e_2$ $\qquad e_2 = a_2 - d_1 f_1$

$c_2 = [d_1\ 1\ 0 \cdots 0] \begin{bmatrix} 0 \\ f_2 \\ e_3 \\ \vdots \\ 0 \end{bmatrix} = f_2$ $\qquad f_2 = c_2$

• $b_2 = [0\ d_2\ 1\ 0 \cdots 0] \begin{bmatrix} f_1 \\ e_2 \\ \vdots \\ 0 \end{bmatrix} = d_2 e_2$ $\qquad d_2 = \frac{b_2}{e_2}$

$$a_3 = [0\ d_2\ 1\ 0 \cdots 0]\begin{bmatrix}0\\f_2\\e_3\\\vdots\\0\end{bmatrix} = d_2 f_2 + e_3 \qquad e_3 = a_3 - d_2 f_2$$

$$c_3 = [0\ d_2\ 1\ 0 \cdots 0]\begin{bmatrix}0\\0\\f_3\\e_4\\\vdots\\0\end{bmatrix} = f_3$$

So generally, $d_i = \frac{b_i}{e_i}$

$e_i = a_i - d_{i-1} f_{i-1} \quad (e_1 = a_1)$

$f_i = c_i \qquad i \geq 2,\ i \in \mathbb{N}$

(b) Given A with $a_i = 2$, $b_i = -1$, $c_i = -1$

So $e_1 = 2 \quad d_1 = \frac{-1}{2} = -\frac{1}{2} \quad f_1 = -1$

$e_2 = 2 - \frac{1}{2} = \frac{3}{2} \quad d_2 = -\frac{2}{3} \quad f_2 = -1$

$e_3 = 2 - \frac{2}{3} = \frac{4}{3} \quad d_3 = -\frac{3}{4} \quad f_3 = -1$

$e_4 = 2 - \frac{3}{4} = \frac{5}{4} \quad d_4 = -\frac{4}{5} \quad f_4 = -1$

$\vdots$

$e_n = \frac{n+1}{n} \quad d_n = -\frac{n}{n+1} \quad f_n = -1$

$$\text{So } A = \begin{bmatrix}1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \ddots & \ddots & \\ & & & -\frac{n-1}{n} & 1\end{bmatrix}\begin{bmatrix}2 & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 & \\ & & & \ddots & -1 \\ & & & & \frac{n+1}{n}\end{bmatrix}$$

5(a) $A = A^T$ Take D' such that $D'D' = D$

$A = LD'D'U = (LD'D'U)^T = U^T D'^T D'^T L^T = (D'U)^T (LD')^T$

$(D'U)^{-T}(LD')(D'U) = (D'U)^{-T}(D'U)^T(LD')^T = (LD')^T$

Note that $(D'U)^T$ is lower triangular $\Rightarrow (D'U)^{-T}$ is lower triangular

The diagonal entries of $(D'U)^T$ are reciprocals of diagonal entries of $(D'U)^{-T}$

Also $(D'U)^T$ and (LD') have the same diagonal entries.

So the diagonal entries of (LD') are reciprocals of diagonal entries of $(D'U)^{-T}$

So $(D'U)^{-T}(LD')$ is unit lower triangular

Since $(D'U)^{-T}(LD')(D'U) = (LD')^T$ is upper triangular

Also $(D'U)$ is upper triangular

$(D'U)^{-T}(LD') = I$, i.e. $D'U = (LD')^T \Rightarrow (D'U)^T = ((LD')^T)^T = LD'$

So take $R = (D'U)^T$, $R^T = (LD')^T$, $A = (D'U)^T(LD')^T = RR^T$

(b) See code

6(a) $\begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & & & \\ l_{n1} & \cdots & & l_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ where $b_i = 0$ for $i = 1, \cdots k$

L non-singular $\Rightarrow l_{ii} \neq 0$ for $i = 1, \cdots n$

$l_{11} y_1 = b_1 = 0 \quad \Rightarrow y_1 = 0$

$l_{21} y_1 + l_{22} y_2 = l_{21} \cdot 0 + l_{22} y_2 = 0 \Rightarrow y_2 = 0$

$\vdots$

$l_{k1} y_1 + l_{k2} y_2 + \cdots l_{kk} y_k = 0 + l_{kk} y_k = b_k = 0 \quad \Rightarrow y_k = 0$

(b) $LL^{-1} = I$

We calculate L^{-1} column by column, denote L^{-1} by K.

$L \begin{bmatrix} k_{1j} \\ k_{2j} \\ \vdots \\ k_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{b}$ j-th row entry is 1.

Since $b_i = 0$ for $i = 1, \cdots j-1$, $k_{ij} = 0$ for $i = 1, \cdots j-1$

i.e. for the first column of L^{-1}, since $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, none of the entries has to be 0.

for the second column of L^{-1}, $b_1 = 0$, so the first entry k_{ij} is 0.

$\vdots$

for the n-th column of L^{-1}, $b_i = 0$ for $i = 1, \cdots n-1$, so all the entries except k_{nn} are zero.

Putting the columns of L^{-1} together, we see that it's again lower triangular.

7(a) $AA^{-1} = I \Rightarrow LUA^{-1} = I$

We find A^{-1} column by column.

$LU \begin{bmatrix} A_{11}^{-1} \\ \vdots \\ A_{n1}^{-1} \end{bmatrix} = b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Let $U \begin{bmatrix} A_{11}^{-1} \\ \vdots \\ A_{n1}^{-1} \end{bmatrix} = \vec{y}$. So $L\vec{y} = b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

so we can solve $\vec{y}$ by forward substitution

Now we have $\vec{y}$, we can solve $U \begin{bmatrix} A_{11}^{-1} \\ \vdots \\ A_{n1}^{-1} \end{bmatrix} = \vec{y}$ for $\begin{bmatrix} A_{11}^{-1} \\ \vdots \\ A_{n1}^{-1} \end{bmatrix}$ by backward substitution.

And we repeat the process to find all columns of A^{-1}.

In particular $y_{ij} = I_{ij} - \sum_{k=1}^{i-1} l_{ik} y_{kj}$

$A_{ij}^{-1} = \frac{1}{u_{ii}} \left(y_{ij} - \sum_{k=i+1}^{n} u_{ik} A_{kj}^{-1} \right)$

(b) Having L and U, we first need to calculate $\vec{y}$. $y_i = b_i - \sum_{k=1}^{i-1} l_{kj} y_k$, $i = 1, \cdots n$

So we have $i-1$ multiplications and $i-1$ subtracts for each i.

$\Rightarrow \sum_{i=2}^{n} 2(i-1) = n(n-1)$

Now we want to find the first column of A^{-1}, $A_{i1}^{-1} = \frac{y_i - \sum_{j=i+1}^{n} u_{ij} A_{j1}}{u_{ii}}$, $i = 1 \cdots n$

So we have $n-i$ multiplications, $n-i$ subtracts and 1 division for each i.

$\Rightarrow \sum_{i=1}^{n} 2(n-i) + 1 = n^2$

Finally we repeat this calculation n times, so in total $(n^2 + n(n-1))n = 2n^3 - n^2$ operations ($\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n + 2n^3 - n^2 = \frac{8}{3}n^3 - \frac{3}{2}n^2 - \frac{1}{6}n$ operations if LU decomposition included)

(c) When calculating the j-th column of A^{-1}, $b_i = 0$ for $i = 1, \cdots j-1$

So $y_i = \begin{cases} 0 & \text{for } i = 1, \cdots j-1 \\ 1 & \text{for } i = j \end{cases}$

During step one we could simplify the calculation, only considering $y_{j+1} \cdots y_n$

So we need $\sum_{i=j+1}^{n} 2(i-1)$ steps

Also, when calculate any of $y_{j+1} \cdots y_n$ by the formula $y_i = b_i - \sum_{k=1}^{i-1} l_{kj} y_k$, $i = 1, \cdots n$

we can avoid $j-1$ multiplications and $j-1$ subtractions with $y_1 = \cdots = y_{j-1} = 0$, and one more multiplication with $y_j = 1$.

So we only need $\sum_{i=j+1}^{n} 2(i-1) - 2(j-1) - 1 = \sum_{i=j+1}^{n} 2i - 2j - 1$

In total, we will need $\sum_{j=1}^{n} \left(\left(\sum_{i=j+1}^{n} 2i - 2j - 1 \right) + n^2 \right) = \frac{4n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$

$\left(\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n + \frac{4n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = 2n^3 \right.$ operations if LU decomposition included)