HW3

1(a) $\|A\|_1 = |1| + |6| = 7$

$A^{-1} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{10} \end{pmatrix}$ $\|A^{-1}\|_1 = \frac{4}{5}$

So $\mathrm{cond}_1(A) = 7 \cdot \frac{4}{5} = \frac{28}{5}$

$$AA^T = \begin{pmatrix} 1 & -2 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 40 \end{pmatrix} \Rightarrow (5-\lambda)(40-\lambda) - 100 = 0$$

So $\lambda_1 = \frac{45+5\sqrt{65}}{2}$ $\lambda_2 = \frac{45-5\sqrt{65}}{2}$ so $\|A\|_2 = \sqrt{\frac{45+5\sqrt{65}}{2}}$

$$(A^{-1})^T A^T = \begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{10} \end{pmatrix}\begin{pmatrix} -\frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{2}{25} & \frac{7}{50} \\ \frac{7}{50} & \frac{37}{100} \end{pmatrix} \Rightarrow \lambda_1 = \frac{9+\sqrt{65}}{40} \quad \lambda_2 = \frac{9-\sqrt{65}}{40}$$

So $\|A^{-1}\|_2 = \sqrt{\frac{9+\sqrt{65}}{40}}$ $\Rightarrow \mathrm{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = 4.266$

$\|A\|_\infty = |6| + |-2| = 8$ $\|A^{-1}\|_\infty = |-\frac{3}{5}| + |\frac{1}{10}| = \frac{7}{10}$

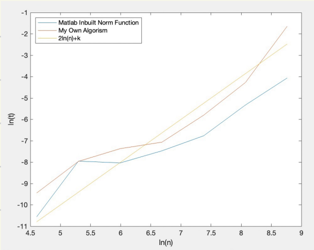
So $\mathrm{Cond}_\infty(A) = \frac{7}{10} \cdot 8 = \frac{28}{5}$ A is well conditioned.

(b) we need $n(n-1)$ additions for both $\|A\|_1$ and $\|A\|_\infty$

when $n = 2k$, $2k(2k-1) = 4k^2 - 2k$

Thus when n is large, calculation time will increase to about 4 times when the matrix size is doubled.

(c)



n = 100	200	400	800	1600	3200	6400
Matlab inbuilt result						
0.0000	0.0004	0.0003	0.0006	0.0012	0.0049	0.0172
My Own algorism result						
0.0001	0.0003	0.0006	0.0009	0.0030	0.0139	0.1936

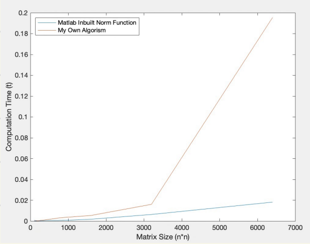
×4 ×4

(code attached) Also for part d

From part b, we know $t \propto n^2$ i.e. $\ln t = \ln k + 2\ln n$

So we plot $\ln(t)$ with respect to $\ln(n)$ for our own algorism and for comparison purpose, plot $2\ln(n) + k$. We can see the slope of the curves are really close. So I could confirm our result from part (b) with order two.

(d)



From my result, MATLAB's implementation is three times faster than my own algorism.

(code attached)

2(a) $\|AB\| = \max_{\vec{v} \in \mathbb{R}^n_*} \frac{\|AB\vec{v}\|}{\|\vec{v}\|}$ and $\|A\| = \max_{B\vec{v} \in \mathbb{R}^n_*} \frac{\|AB\vec{v}\|}{\|B\vec{v}\|} \Rightarrow \|AB\vec{v}\| \leq \|A\|\|B\vec{v}\|$

So $\|AB\| \leq \max_{\vec{v} \in \mathbb{R}^n_*} \frac{\|A\|\|B\vec{v}\|}{\|\vec{v}\|} = \|A\| \max_{\vec{v} \in \mathbb{R}^n_*} \frac{\|B\vec{v}\|}{\|\vec{v}\|} = \|A\|\|B\|$

(b) $\|I\| = \max_{\vec{v} \in \mathbb{R}^n_*} \frac{\|I\vec{v}\|}{\|\vec{v}\|} = \max_{\vec{v} \in \mathbb{R}^n_*} \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1$

(c) Take $B = A^{-1}$

$\mathrm{cond}(A) = \|A\|\|A^{-1}\| = \|A\|\|B\| \geq \|AB\| = \|I\| = 1$

(d) Take the identity matrix I_n. $\|I_n\|_F = (n \cdot 1^2)^{\frac{1}{2}} = \sqrt{n} \neq 1$ for $n \neq 1$

So it cannot be induced by a suitable vector norm.

3(a) $A^TA\vec{x} = \lambda\vec{x} \quad \Rightarrow (A^T)^{-1}A^TA\vec{x} = (A^T)^{-1}\lambda\vec{x} = \lambda(A^T)^{-1}\vec{x}$

$\Rightarrow A^{-1}A\vec{x} = A^{-1}\lambda(A^T)^{-1}\vec{x} = \lambda A^{-1}(A^T)^{-1}\vec{x}$

$\Rightarrow \frac{1}{\lambda}\vec{x} = A^{-1}(A^{-1})^T\vec{x}$ i.e $\frac{1}{\lambda}$ is the eigenvalue of $A^{-1}(A^{-1})^T$ (AB)

By theorem, $\frac{1}{\lambda}$ is also eigenvalue of $(A^{-1})^TA^{-1}$. (BA)

$\|A\|_2 = \max_{i=1}^{n} \sqrt{\lambda} = \sqrt{\lambda_{max}}$, $\|A^{-1}\|_2 = \max_{i=1}^{n} \lambda^{-\frac{1}{2}} = \frac{1}{\sqrt{\lambda_{min}}}$

So $\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}}$

(b) Want $\frac{\|\Delta X\|_2}{\|X\|_2} = \kappa(A)\frac{\|\Delta b\|_2}{\|b\|_2} = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \cdot \frac{\|\Delta b\|_2}{\|b\|_2}$

We consider $\vec{b}$ when the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ is an eigenvector of A^TA with corresponding eigenvalue λ_{max}. Similarly, take $\Delta\vec{b}$ s.t. the solution $\Delta\vec{x}$ to $A\Delta\vec{x} = \Delta\vec{b}$ is an eigenvector of A^TA with corresponding eigenvalue λ_{min}.

Thus $A^TA\vec{x} = \lambda_{max}\vec{x}$, $A^TA\Delta\vec{x} = \lambda_{min}\Delta\vec{x}$

$A\vec{x} = \vec{b} \quad \Rightarrow (A\vec{x})^TA\vec{x} = \vec{b}^T\vec{b} \quad \Rightarrow \vec{x}^TA^TA\vec{x} = \vec{b}^T\vec{b} \quad \Rightarrow \vec{x}^T\lambda_{max}\vec{x} = \vec{b}^T\vec{b}$

$\Rightarrow \lambda_{max}\vec{x}^T\vec{x} = \vec{b}^T\vec{b} \quad \Rightarrow \lambda_{max}\|\vec{x}\|_2^2 = \|\vec{b}\|_2^2$

Similarly, $\lambda_{min}\|\Delta\vec{x}\|_2^2 = \|\Delta\vec{b}\|_2^2$

Thus $\frac{\lambda_{min}}{\lambda_{max}} \cdot \frac{\|\Delta\vec{x}\|_2^2}{\|\vec{x}\|_2^2} = \frac{\|\Delta\vec{b}\|_2^2}{\|\vec{b}\|_2^2} \quad \Rightarrow \frac{\|\Delta X\|_2}{\|X\|_2} = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \cdot \frac{\|\Delta b\|_2}{\|b\|_2} = \kappa_2(A)\frac{\|\Delta b\|_2}{\|b\|_2}$

4.

$$A^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 0 & \cdots & 0 \\ & & 1 & \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \quad \text{So } A^TA = \begin{bmatrix} n & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$A^TA\vec{x} = \lambda\vec{x} \Rightarrow \begin{aligned} nx_1 + x_2 + x_3 + \cdots + x_n &= \lambda x_1 \\ x_1 + x_2 &= \lambda x_2 \\ x_1 + x_3 &= \lambda x_3 \\ &\vdots \\ x_1 + x_n &= \lambda x_n \end{aligned}$$

When $x_1 = 0$, $x_2 + x_3 + \cdots + x_n = 0$, the equation is satisfied, so $\vec{x}$ will be an eigenvector.

- If $x_2 = x_3 = \cdots = x_n = \bar{x}$, the system we want to solve is equivalent to:

$$\begin{cases} nx_1 + (n-1)\bar{x} = \lambda x_1 \\ x_1 + \bar{x} = \lambda\bar{x} \quad \Rightarrow \quad x_1 = (\lambda - 1)\bar{x} \end{cases}$$

$(\lambda^2 - (n+1)\lambda + 1)\bar{x} = 0$

If $\bar{x} = 0$, $x_1 = 0$, $\vec{x} = 0$, rejected

So $\lambda^2 - (n+1)\lambda + 1 = 0$, we can find two more eigenvectors when $x_2 = x_3 = \cdots = x_n$

$\lambda = \frac{n+1 \pm \sqrt{n^2+2n-3}}{2}$ are the two corresponding eigenvalues

$$K_2(A) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \sqrt{\frac{\frac{1}{2}(n+1+\sqrt{n^2+2n-3})}{\frac{1}{2}(n+1-\sqrt{n^2+2n-3})}} = \sqrt{\frac{(n+1+\sqrt{n^2+2n-3})^2}{(n+1-\sqrt{n^2+2n-3})(n+1+\sqrt{n^2+2n-3})}}$$

$$= \sqrt{\frac{(n+1)^2 + (n^2+2n-3) + 2(n+1)\sqrt{n^2+2n-3}}{(n+1)^2 - (n^2+2n-3)}} = \sqrt{\frac{2n^2+4n-2+2(n+1)\sqrt{n^2+2n-3}}{4}}$$

$$= \frac{1}{2}\sqrt{2(n+1)^2 - 4 + 2(n+1)\sqrt{n^2+2n-3}} = \frac{1}{2}(n+1)\sqrt{2 - \frac{4}{(n+1)^2} + \frac{2\sqrt{n^2+2n-3}}{n+1}}$$

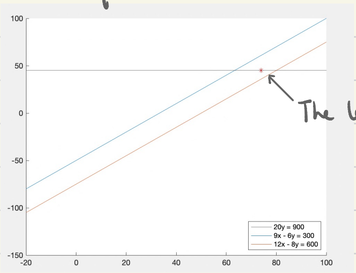
$$= \frac{1}{2}(n+1)\sqrt{1 + 2\sqrt{1 - \frac{4}{(n+1)^2}} + \left(1 - \frac{4}{(n+1)^2}\right)}$$

$$= \frac{1}{2}(n+1)\left(1 + \sqrt{1 - \frac{4}{(n+1)^2}}\right)$$

When n is very large, $K_2(A) \gg 1$, A ill-conditioned, so I won't trust the result.

5. $A = \hat{Q}\hat{R} = \begin{bmatrix} -0.6 & 0 \\ -0.8 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -15 & 10 \\ 0 & -20 \end{bmatrix}$

Least square solution $(x^*, y^*) = (74, 45)$



The least square solution

6. $0 = ae^0 + 0^2 b + 0 \cdot c + d$

$0.2 = ae^{0.5} + 0.5^2 b + 0.5c + d$

$0.27 = ae^1 + 1^2 b + c + d$

$0.3 = ae^{1.5} + 1.5^2 b + 1.5c + d$

$0.32 = ae^2 + 2^2 b + 2c + d$

$0.35 = ae^2 + 2^2 b + 2c + d$

$0.27 = ae^{2.5} + 2.5^2 b + 2.5c + d$

$$\underbrace{\begin{bmatrix} e^0 & 0 & 0 & 1 \\ e^{0.5} & 0.5^2 & 0.5 & 1 \\ e^1 & 1^2 & 1 & 1 \\ e^{1.5} & 1.5^2 & 1.5 & 1 \\ e^2 & 2^2 & 2 & 1 \\ e^2 & 2^2 & 2 & 1 \\ e^{2.5} & 2.5^2 & 2.5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0.2 \\ 0.27 \\ 0.3 \\ 0.32 \\ 0.35 \\ 0.27 \end{bmatrix}}_{\vec{b}}$$

$$\begin{bmatrix} e^0 & e^{0.5} & e^1 & e^{1.5} & e^2 & e^2 & e^{2.5} \\ 0 & 0.5^2 & 1^2 & 1.5^2 & 2^2 & 2^2 & 2.5^2 \\ 0 & 0.5 & 1 & 1.5 & 2 & 2 & 2.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^0 & 0 & 0 & 1 \\ e^{0.5} & 0.5^2 & 0.5 & 1 \\ e^1 & 1^2 & 1 & 1 \\ e^{1.5} & 1.5^2 & 1.5 & 1 \\ e^2 & 2^2 & 2 & 1 \\ e^2 & 2^2 & 2 & 1 \\ e^{2.5} & 2.5^2 & 2.5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} e^0 & e^{0.5} & e^1 & e^{1.5} & e^2 & e^2 & e^{2.5} \\ 0 & 0.5^2 & 1^2 & 1.5^2 & 2^2 & 2^2 & 2.5^2 \\ 0 & 0.5 & 1 & 1.5 & 2 & 2 & 2.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.2 \\ 0.27 \\ 0.3 \\ 0.32 \\ 0.35 \\ 0.27 \end{bmatrix}$$

$(A^T A \vec{x} = A^T \vec{b})$

Solution $\begin{bmatrix} a^* \\ b^* \\ c^* \\ d^* \end{bmatrix} = \begin{bmatrix} 0.0127 \\ -0.1310 \\ 0.3774 \\ -0.0024 \end{bmatrix}$